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The Transmissibility of Vibration Isolators with a Nonlinear Anti-Symmetric Damping Characteristic

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Abstract: In the present study, the concept of the Output Frequency Response Function (OFRF), recently proposed by the authors, is applied to theoretically investigate the transmissibility of SDOF passive vibration isolators with a nonlinear anti-symmetric damping curve. The results reveal that a nonlinear anti-symmetric damping characteristic has almost no effect on the transmissibility of SDOF vibration isolators over both low and high frequency ranges where the frequencies are much lower or higher than the isolator's resonant frequency. On the other hand, the introduction of a nonlinear anti-symmetric damping can significantly reduce the transmissibility of the vibration isolator over the resonant frequency region. The results indicate that nonlinear vibration isolators with an anti-symmetric damping characteristic have great potential to overcome the dilemma encountered in the design of passive linear vibration isolators, that is, increasing the level of damping to reduce the transmissibility at the resonance could increase the transmissibility over the range of higher frequencies. These important theoretical conclusions are then verified by simulation studies.

1. Introduction

A vibration isolator is a device that is often inserted between a support base and equipment to reduce the vibration energy transmission from the support base so as to protect the equipment from undesired disturbances [1]. For a conventional passive vibration isolator design, there are two well-known trade-offs regarding the design of stiffness and damping [2]. In order to obtain a low transmissibility over a wide frequency range, the elastic stiffness of the isolator should be as small as possible. However, if the elastic stiffness is too small, this will lead to large static and quasi-static displacements which are likely to be detrimental to the supported equipment. In addition, to reduce transmissibility at the resonance, it is better to introduce a higher damping in the isolator. This may cause deterioration to the transmissibility over the higher frequency range. To overcome these limitations of conventional passive isolators, recent developments involve using the active control techniques, which generally fall into three categories: adaptive-passive [3], semi-active [4][5] and fully active [6]. A fully active isolator system turns out to be very complex. More effort has been made in the development of adaptive-passive and semi-active methods, among which the most popular method is the *skyhook* technique whose name is derived from the fact that it is a passive damper hooked to an imaginary inertial reference point. In skyhook controlled semi-active isolators, the damping effect can be automatically switched off to produce a desired damping characteristic that conventional passive isolators can not achieve so as to minimize the transmissibility level over a wide region of frequencies [2][7][8]. A comparison between different semi-active damping control strategies has been carried out by Liu and colleagues [5].

To improve the performance of conventional passive isolators, several authors have developed different types of nonlinear vibration isolators and have investigated the unique dynamic behaviours, which cannot be studied based on linear theories [9]~[12]. A very comprehensive survey of recent developments of nonlinear vibration isolators has been contributed by Ibrahim [13], in which many cited studies [14]-[21] reveal that the introduction of nonlinear damping and stiffness are of great benefit in vibration isolation. More recently, using the concept of the Output Frequency Response Functions (OFRFs) [22][23], the authors [24] have revealed that, for a single degree of freedom (SDOF) vibration isolator, a cubic nonlinear damping characteristic can produce an ideal vibration isolation such that only the transmissibility over the resonant region of frequencies is modified by the damping effect and the transmissibility over the non-resonant regions of frequencies remain almost unaffected. In the present study, these results are extended to investigate the analytical relationship between the transmissibility and the nonlinear damping characteristic parameters of SDOF vibration isolators with a nonlinear anti-symmetric damping curve. This analysis theoretically proves that the introduction of a nonlinear anti-symmetric damping characteristic can produce the ideal vibration isolation, that is, "*There is little damping in the isolation region but considerable damping around the isolator's natural frequency*" [25] so as to achieve a required transmissibility over the isolation range of frequencies and reduced amplification at the resonance at the same time. Numerical simulation studies are carried out to verify the theoretical analysis and demonstrate the considerable engineering significance of the conclusions reached in this study. The revelation that the isolators with a nonlinear anti-symmetric damping characteristic possess ideal vibration isolation properties provides an important basis for the development of novel passive solution to vibration isolation problems.

2. SDOF Linear Passive Isolators

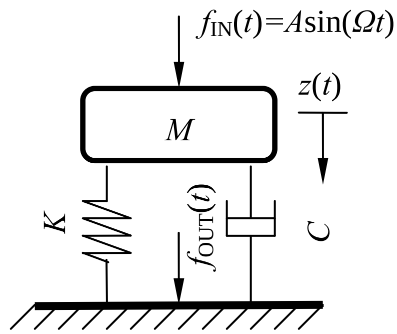

Fig. 1, SDOF linear passive isolator

Consider the SDOF linear passive isolator shown in Fig 1, where

$$f_{IN}(t) = A\sin(\Omega t) \tag{1}$$

is the harmonic force acting on the system with frequency Ω and magnitude A, $f_{OUT}(t)$ is the force transmitted to the supporting structure or base, and $z(t)$ is the displacement of mass M. The equations of motion of the SDOF vibration isolator system are given by

$$\begin{cases} M\ddot{z}(t) + C\dot{z}(t) + Kz(t) = f_{IN}(t) = A\sin(\Omega t) \\ f_{OUT}(t) = Kz(t) + C\dot{z}(t) \end{cases} \tag{2}$$

where K and C are the spring and damping characteristic parameters of the system respectively.

Eq. (2) can be described in a dimensionless form as follows

$$\begin{cases} \ddot{y}_1(\tau) + \xi\dot{y}_1(\tau) + y_1(\tau) = \sin(\overline{\Omega}\tau) \\ y_2(\tau) = \xi\dot{y}_1(\tau) + y_1(\tau) \end{cases} \tag{3}$$

where $\tau = \Omega_0 t$, $\Omega_0 = \sqrt{K/M}$ is the resonant frequency of the system, $\overline{\Omega} = \Omega/\Omega_0$, $\xi = C/\sqrt{KM}$, $y_1(\tau) = Kz(\tau/\Omega_0)/A$, $y_2(\tau) = f_{OUT}(\tau)/A$.

From Eq. (3), it can be shown that

$$\frac{f_{OUT}(t)}{A} = \frac{Kz(t) + C\dot{z}(t)}{A} = y_1(\tau) + \xi\dot{y}_1(\tau) = y_2(\tau) \tag{4}$$

Denote $T(\overline{\Omega})$ as the force transmissibility of the SDOF isolator system (2) in terms of the normalized frequency $\overline{\Omega}$, it is easy to deduce from Eq. (3) that

$$T(\overline{\Omega}) = \left|Y_2(j\overline{\Omega})\right| = \left|\frac{1 + j\xi\overline{\Omega}}{-\overline{\Omega}^2 + 1 + j\xi\overline{\Omega}}\right| \tag{5}$$

where $Y_2(j\overline{\Omega})$ is the spectrum of $y_2(\tau)$ described by $Y_2(j\omega)$ evaluated at frequency $\omega = \overline{\Omega}$.

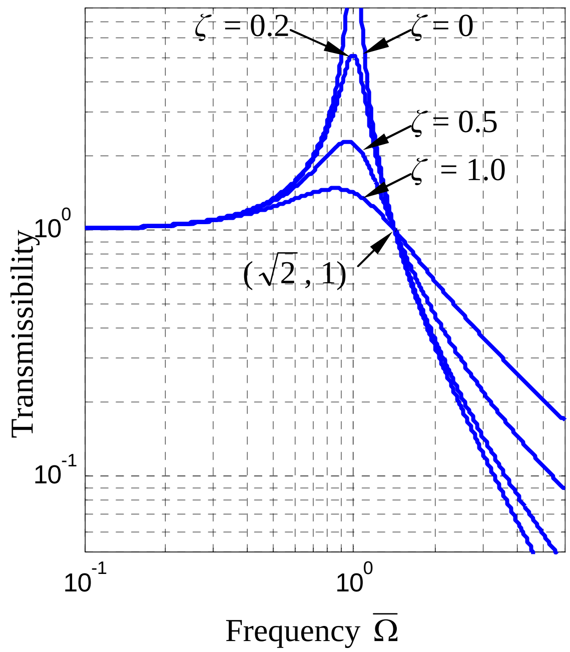

Fig. 2, Effect of damping on the force transmissibility of system (3)

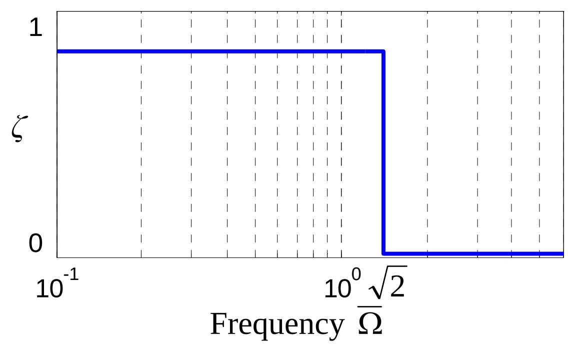

Fig. 3, The damping required by a ideal isolator

From Eq. (5), the effects of damping on the force transmissibility can be evaluated. The results are shown in Fig. 2, which clearly indicate that although the introduction of a higher damping effect reduces the transmissibility around the resonant frequencies, the higher damping effect, at the same time, increases the transmissibility where the normalized frequencies are higher than $\sqrt{2}$ Hz. The damping required by an ideal vibration isolator is shown in Fig. 3, which is frequency-dependent and the basis of the adaptive passive isolation systems [2]. However, such a requirement can obviously not be met simply by a linear passive isolator.

3. SDOF Passive Isolators with a Nonlinear Anti-symmetric Damping Characteristic

In addition to active control solutions, it has been realized that specific nonlinear passive isolators have the potential to overcome the limitations of linear passive isolators [13]. The objective of the present study is to theoretically investigate the effect of nonlinear damping characteristic parameters of SDOF vibration isolators with a nonlinear anti-symmetric damping curve on the transmissibility so as to extend the analysis results in [24] to a more general situation.

3.1 The Model of SDOF Nonlinear Passive Isolators

The considered SDOF nonlinear passive isolators are shown in Fig. 4.

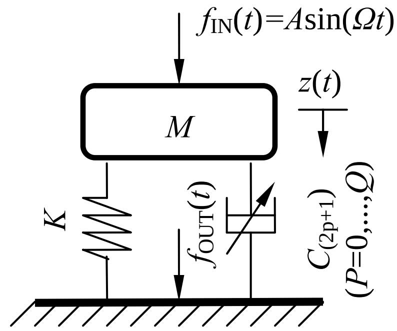

Fig. 4, SDOF passive isolator with a nonlinear anti-symmetric damping characteristic

For linear passive isolators the damping force F_d is equal to $C\dot{z}$, but the damping force of the nonlinear passive isolator is described by

$$F_d = C\dot{z}(t) + \sum_{p=1}^{Q} C_{(2p+1)} \left[\dot{z}(t)\right]^{2p+1} \tag{6}$$

where $C_{(2p+1)}, (p = 1, \cdots, Q)$ are the nonlinear damping characteristic parameters of the system. Therefore, the equations of motion of the SDOF nonlinear isolators are given by

$$\begin{cases} M\ddot{z}(t) + C\dot{z}(t) + \sum_{p=1}^{Q} C_{(2p+1)} \left[\dot{z}(t)\right]^{2p+1} + Kz(t) = A\sin(\Omega t) \\ f_{OUT}(t) = Kz(t) + C\dot{z}(t) + \sum_{p=1}^{Q} C_{(2p+1)} \left[\dot{z}(t)\right]^{2p+1} \end{cases} \tag{7}$$

Denote

$$\xi_{(2p+1)} = \frac{C_{(2p+1)} A^{2p}}{\sqrt{(KM)^{2p+1}}} \qquad (p = 1,\ldots,Q) \quad (8)$$

Then, the SDOF nonlinear isolator system (7) can be described as a dimensionless, one input two output system as

$$\begin{cases} \ddot{y}_1(\tau) + y_2(\tau) = u(\tau) \\ y_2(\tau) = y_1(\tau) + \xi\dot{y}_1(\tau) + \sum_{p=1}^{Q} \xi_{(2p+1)} [\dot{y}_1(t)]^{2p+1} \end{cases} \qquad (9)$$

From Eqs. (7) and (9), it can be shown that the force transmissibility of the nonlinear passive isolator is determined by

$$\frac{f_{OUT}(t)}{A} = y_1(\tau) + \xi\dot{y}_1(\tau) + \sum_{p=1}^{Q} \xi_{(2p+1)} [\dot{y}_1(t)]^{2p+1} = y_2(\tau) \qquad (10)$$

The force transmissibility $T(\overline{\Omega})$ of the SDOF nonlinear isolator (9) can also be studied by investigating the spectrum of $y_2(\tau)$ of system (9), that is,

$$T(\overline{\Omega}) = \left| Y_2(j\overline{\Omega}) \right| \qquad (11)$$

However, unlike the case for linear passive isolators there is currently no simple explicit analytical expression like Eq. (5) available which can be used to describe the relationship between the force transmissibility and system parameters for nonlinear passive isolators.

3.2 Representation of the Force Transmissibility of Nonlinear SDOF Isolators Using the OFRF

The OFRF is a concept recently proposed by the authors in [22][23] for the study of the output frequency response of nonlinear Volterra systems.

Nonlinear Volterra systems represent a wide class of nonlinear systems whose input output relationship can be described by a Volterra series model over the regime around a stable equilibrium [26][27]. For nonlinear Volterra systems which can equally be described by a polynomial type nonlinear differential equation model which has been widely used for the modeling of practical physical systems, it has been shown in [22][23] that the system output spectrum can be represented by an explicit polynomial function of the model parameters which define the system nonlinearity. This result is referred to as the OFRF, and provides a significant analytical link between the output frequency response and nonlinear characteristic parameters for a wide range of practical nonlinear systems.

In the following, the OFRF concept will be applied to the case of the one input two output system (9) to produce an analytical polynomial relationship between the spectrum $Y_2(j\omega)$ and the system's nonlinear characteristic parameters $\xi_{(2p+1)}, (p = 1, \cdots, Q)$.

Because $Y_2(j\omega)$ is related to the force transmissibility $T(\overline{\Omega})$ of system (9) via Eq. (11), the result will, in fact, provide an OFRF based analytical expression for $T(\overline{\Omega})$.

According to [28], it is known that when subject to a sinusoidal input

$$u(\tau)=\sin(\overline{\Omega}\tau)=\cos(\overline{\Omega}\tau-\pi/2) \tag{12}$$

the spectra of the outputs of system (9) are given by

$$Y_J(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_1+\cdots+\omega_n=\omega}H_n^{(J)}(j\omega_1,\cdots,j\omega_n)\overline{A}(\omega_1)\cdots\overline{A}(\omega_n) \qquad (J\text{=1, 2}) \tag{13}$$

where

$$\overline{A}(\omega_i)=\begin{cases}e^{-j\pi/2} & when\ \omega_{\text{i}}=\overline{\Omega}\\ e^{j\pi/2} & when\ \omega_{\text{i}}=-\overline{\Omega}\\ 0 & \text{otherwise}\end{cases} \qquad (i = 1,\ldots,n) \tag{14}$$

N is the maximum order of nonlinearity in the Volterra series expansion of the system outputs given by

$$y_J(\tau)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n^{(J)}(\tau_1,...,\tau_n)\prod_{i=1}^{n}u(\tau-\tau_i)d\tau_i \qquad (J = 1, 2) \tag{15}$$

with $h_n^{(J)}(\tau_1,...,\tau_n)$ (J = 1,2), denoting the n^{th} order Volterra kernel, and

$$H_n^{(J)}(j\omega_1,...,j\omega_n)=\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}h_n^{(J)}(\tau_1,...,\tau_n)e^{-(\omega_1\tau_1+,...,+\omega_n\tau_n)j}d\tau_1...d\tau_n \qquad (J = 1, 2) \tag{16}$$

defines the n^{th} order Generalised Frequency Response Function (GFRF) [29] between the input and the first and second system outputs respectively.

By using the harmonic probing method [30], the specific expression of $H_n^{(J)}(j\omega_1,...,j\omega_n)$ (J = 1,2) of the one input two output nonlinear differential model (9) can be determined to yield

$$H_1^{(1)}(j\omega_1)=\frac{1}{1+j\xi\omega_1-\omega_1^2} \tag{17}$$

$$H_1^{(2)}(j\omega_1)=(1+j\xi\omega_1)H_1^{(1)}(j\omega_1) \tag{18}$$

$$H_{(2n+1)}^{(2)}(j\omega_1,\cdots,j\omega_{(2n+1)})=-(j\omega_1+\cdots+j\omega_{(2n+1)})^2H_{(2n+1)}^{(1)}(j\omega_1,\cdots,j\omega_{(2n+1)}) \qquad (n=1,\cdots,\lfloor (N-1)/2\rfloor) \tag{19}$$

$$H_{2n}^{(2)}(j\omega_1,\cdots,j\omega_{2n})=H_{2n}^{(1)}(j\omega_1,\cdots,j\omega_{2n})=0 \qquad (n=1,\cdots,\lfloor N/2\rfloor) \tag{20}$$

where $\lfloor N/2 \rfloor$ is the floor function indicating the largest integer no less than $N/2$. Moreover, according to the results recently revealed by the authors [22][23], the high order GFRFs $H_{(2n+1)}^{(1)}(j\omega_1,\cdots,j\omega_{(2n+1)})$ for the nonlinear passive isolator (9) can be expressed as the following form

$$H_{(2n+1)}^{(1)}(j\omega_1,\cdots,j\omega_{(2n+1)}) = \frac{\prod_{i=1}^{2n+1}\left[j\omega_i H_1^{(1)}(j\omega_i)\right]}{L\left(j\omega_1+\cdots+j\omega_{(2n+1)}\right)} \sum_{(j_3,\cdots,j_{(2Q+1)})\in J_{(2n+1)}} \xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}}\Theta_{(2n+1)}^{(j_3\cdots j_{(2Q+1)})}\left(j\omega_1,\cdots,j\omega_{(2n+1)}\right)$$

$$(n=1,\cdots,\lfloor (N-1)/2 \rfloor) \quad (21)$$

where

$$L(j\omega_1+\cdots+j\omega_n) = -\left[-(\omega_1+\cdots+\omega_n)^2 + j(\omega_1+\cdots+\omega_n)\xi+1\right] \quad (22)$$

and $\Theta_{(2n+1)}^{(j_3\cdots j_{(2Q+1)})}\left(j\omega_1,\cdots,j\omega_{(2n+1)}\right)$ represents a function of frequency variables $\omega_1,\cdots,$ $\omega_{(2n+1)}$ and the system's linear characteristic parameters, and $J_{(2n+1)}$ is a set of Q dimensional nonnegative integer vectors which contains the exponents of those monomials $\xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}}$ which are present in the polynomial representation (21).

For example, applying the recursive algorithm proposed by the authors in [23], which is introduced in the Appendix, to system (9) for n = 1, 2, 3 respectively yields

$$J_3=\{(1,0)\},\ J_5=\{(2,0),(0,1)\},\ J_7=\{(3,0),(1,1)\}.$$

and

$$\Theta_3^{(1,0)}\left(j\omega_1,\cdots,j\omega_3\right)=0;\ \Theta_5^{(2,0)}\left(j\omega_1,\cdots,j\omega_5\right)=B_3;\ \Theta_5^{(0,1)}\left(j\omega_1,\cdots,j\omega_5\right)=1;$$

$$\Theta_7^{(3,0)}\left(j\omega_1,\cdots,j\omega_7\right)=B_3B_3+B_5B_3;\ \Theta_7^{(1,1)}\left(j\omega_1,\cdots,j\omega_7\right)=B_5+B_3$$

where

$$B_z = \begin{cases} 1 & \text{if } Z=1 \\ \dfrac{j\omega_{l(1)}+\cdots+j\omega_{l(Z)}}{\left[L\left(j\omega_{l(1)}+\cdots+j\omega_{l(Z)}\right)\right]} & \text{if } Z\geq 2 \end{cases}$$

$$\omega_{l(i)},(i=1,\cdots,Z)\in\left\{\omega_1,\cdots,\omega_{(2n+1)}\right\} \quad (23)$$

Therefore, the GFRFs up to 7th order for system (9) with Q = 2 can, for example, be determined as follows,

$$H_3^{(1)}(j\omega_1,\cdots,j\omega_3) = \frac{\prod_{i=1}^{3}\left[j\omega_i H_1^{(1)}(j\omega_i)\right]}{L\left(j\omega_1+\cdots+j\omega_3\right)}\xi_3 \quad (24)$$

$$H_5^{(1)}(j\omega_1,\cdots,j\omega_5)=\frac{\prod_{i=1}^{5}\left[j\omega_i H_1^{(1)}(j\omega_i)\right]}{L\left(j\omega_1+\cdots+j\omega_5\right)}\left[\xi_3^2 B_3+\xi_5\right] \tag{25}$$

$$H_7^{(1)}(j\omega_1,\cdots,j\omega_7)=\frac{\prod_{i=1}^{7}\left[j\omega_i H_1^{(1)}(j\omega_i)\right]}{L\left(j\omega_1+\cdots+j\omega_7\right)}\begin{bmatrix}\xi_3^3\left(B_3B_3+B_5B_3\right)\\+\xi_3\xi_5(B_5+B_3)\end{bmatrix} \tag{26}$$

Although the procedure introduced in the Appendix seems quite simple, the generated expression can be extremely complicated when the order of the GFRF becomes higher. However, it is easy to notice from the above procedure and the example that, for system (9), $\Theta_{(2n+1)}^{(j_3\cdots j_{(2Q+1)})}$ in Eq. (21) can be uniformly expressed as the following form

$$\Theta_{(2n+1)}^{(j_3\cdots j_{(2Q+1)})}\left(j\omega_1,\cdots,j\omega_{(2n+1)}\right)=\sum_{Z=1}^{\bar{n}}\prod_{i=1}^{Z}B_{l(2Z+1)}=\sum_{Z=1}^{\bar{n}}\prod_{i=1}^{Z}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(2Z+1)}}{\left[L\left(j\omega_{l(1)}+\cdots+j\omega_{l(2Z+1)}\right)\right]} \tag{27}$$

where $\bar{n}$ is an integer dependent on n.

From (13) and the expression for $H_n^{(2)}(j\omega_1,...,j\omega_n)$ given by (18), (21) and (27), the OFRF representation of $Y_2(j\omega)$ of system (9) can be written as

$$Y_2(j\omega)=P_1(j\omega)+P_3(j\omega)+\sum_{n=2}^{\lfloor (N-1)/2\rfloor}P_{2n+1}(j\omega) \tag{28}$$

where

$$P_1(j\omega)=H_1^{(2)}(j\omega)\overline{A}(\omega) \tag{29}$$

$$P_3(j\omega)=\frac{\xi_3\omega^2}{2^3L[j\omega]}\sum_{\omega_1+\cdots+\omega_3=\omega}\left[\prod_{i=1}^{3}H_1^{(1)}(j\omega_i)(j\omega_i)\overline{A}(\omega_i)\right] \tag{30}$$

$$P_{2n+1}(j\omega)=\frac{\omega^2}{2^{2n+1}L[j\omega]}\sum_{\omega_1+\cdots+\omega_{2n+1}=\omega}\left[\prod_{i=1}^{2n+1}H_1^{(1)}(j\omega_i)(j\omega_i)\overline{A}(\omega_i)\right]\times$$
$$\sum_{\left(j_3,\cdots,j_{(2Q+1)}\right)\in J_{(2n+1)}}\xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}}\sum_{Z=1}^{\bar{n}}\prod_{i=1}^{Z}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(2Z+1)}}{\left[L\left(j\omega_{l(1)}+\cdots+j\omega_{l(2Z+1)}\right)\right]}$$
$$(n=2,\cdots,\lfloor (N-1)/2\rfloor) \tag{31}$$

The OFRF (28) represents the spectrum of the second output of system (9) as an explicit polynomial function of the system's nonlinear characteristic parameters, which, obviously, can considerably facilitate the analysis of the effect of system nonlinearity on the output frequency responses.

By using Eq. (28), the transmissibility of the SDOF isolator system (9) as given by Eq. (11) can further be expressed as

$$T(\overline{\Omega}) = \left| P_1(j\overline{\Omega}) + \sum_{n=1}^{\lfloor (N-1)/2 \rfloor} P_{2n+1}(j\overline{\Omega}) \right| \tag{32}$$

where

$$P_1(j\overline{\Omega}) = -\frac{(1 + j\xi_1\overline{\Omega})}{L(j\overline{\Omega})} \tag{33}$$

$$P_{2n+1}(j\overline{\Omega}) = \frac{\overline{\Omega}^{2n+3} H_1^{(1)}(j\overline{\Omega}) \left| H_1^{(1)}(j\overline{\Omega}) \right|^{2n}}{2^{2n+1} L[j\overline{\Omega}]} \times$$
$$\sum_{\omega_1+\cdots+\omega_{2n+1}=\overline{\Omega}} \sum_{(j_3,\cdots,j_{(2Q+1)}) \in J_{(2n+1)}} \xi_3^{j_3} \cdots \xi_{(2Q+1)}^{j_{(2Q+1)}} \sum_{Z=1}^{\bar{n}} \prod_{i=1}^{Z} \frac{j\omega_{l(1)} + \cdots + j\omega_{l(2Z+1)}}{\left[L\left(j\omega_{l(1)} + \cdots + j\omega_{l(2Z+1)}\right)\right]}$$
$$= -\frac{\overline{\Omega}^{2n+3}}{2^{2n+1} \left\{L[j\overline{\Omega}]\right\}^2 \left| L[j\overline{\Omega}] \right|^{2n}} \times$$
$$\sum_{\omega_1+\cdots+\omega_{2n+1}=\overline{\Omega}} \sum_{(j_3,\cdots,j_{(2Q+1)}) \in J_{(2n+1)}} \xi_3^{j_3} \cdots \xi_{(2Q+1)}^{j_{(2Q+1)}} \sum_{Z=1}^{\bar{n}} \prod_{i=1}^{Z} \frac{j\omega_{l(1)} + \cdots + j\omega_{l(2Z+1)}}{\left[L\left(j\omega_{l(1)} + \cdots + j\omega_{l(2Z+1)}\right)\right]}$$
$$\left(n = 1, 2, \cdots, \lfloor (N-1)/2 \rfloor\right) \tag{34}$$

and $\omega_k \in \left\{-\overline{\Omega}, \overline{\Omega}\right\},\ k = 1, \ldots, 2n+1$.

From equations (32) and (34), it is known that when $\xi_{(2p+1)} = 0, (p = 1, \cdots, Q)$ i.e. there is no nonlinear damping, the transmissibility is determined as follows,

$$T(\overline{\Omega}) = \left| P_1(j\overline{\Omega}) \right| = \left| \frac{1 + j\xi\overline{\Omega}}{1 + j\xi\overline{\Omega} - \overline{\Omega}^2} \right| \tag{35}$$

which is the same as Eq. (5) and is the expression of transmissibility widely used in engineering practice for the design of linear SDOF vibration isolators.

When nonlinear damping is introduced, i.e., $\xi_{(2p+1)} \neq 0, (p = 1, \cdots, Q)$, Eq. (32) indicates that the transmissibility will be different from the well-known result given by Eq. (35) and, given the linear damping characteristic parameter ξ, the difference as described by the second term in Eq. (32) is a function of both the nonlinear anti-symmetric damping characteristic parameters $\xi_{(2p+1)}$, $(p = 1, \cdots, Q)$ and the frequency $\overline{\Omega}$. In the next section, $T(\overline{\Omega})$ given by (32) over the frequency ranges of $\overline{\Omega} << 1$ and $\overline{\Omega} >> 1$, and the effect of $\xi_{(2p+1)}$, $(p = 1, \cdots, Q)$ on the value of $T(\overline{\Omega})$ over the frequency range of $\overline{\Omega} \approx 1$ will be analyzed to reveal the significant benefits of nonlinear anti-symmetric damping characteristic on vibration isolation.

3.3. Effects of Nonlinear Anti-symmetric Damping on Transmissibility

Consider the SDOF vibration isolator subject to a sinusoidal force excitation as described by Eq. (2), and assume that the outputs of the isolator is dimensionless, one input two output system representation given by Eq. (9) can be described by the nonlinear Volterra series model (15) around zero equilibrium. The effect of a nonlinear anti-symmetric damping characteristic on the transmissibility of the vibration isolator is investigated over the resonant and non-resonant frequency ranges respectively in the following sections.

3.3.1 Transmissibility over the Non-Resonant Frequency Ranges

Over the non-resonant frequency ranges, $\overline{\Omega} << 1$ or $\overline{\Omega} >> 1$.

Substituting (22) into (34) yields

$$\begin{aligned}\left|P_{2n+1}(j\overline{\Omega})\right| &= \frac{\overline{\Omega}^{2n+3}}{2^{2n+1}\left|-\overline{\Omega}^2 + j\xi\overline{\Omega}+1\right|^{2n+2}}\left|\sum_{\omega_1+\cdots+\omega_{2n+1}=\overline{\Omega}}\ \sum_{(j_3,\cdots,j_{(2Q+1)})\in J_{(2n+1)}}\xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}}\right.\\ &\left.\times\sum_{Z=1}^{\bar{n}}\prod_{i=1}^{Z}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(2Z+1)}}{\left[-\left(\omega_{l(1)}+\cdots+\omega_{l(2Z+1)}\right)^2 + j\xi\left(\omega_{l(1)}+\cdots+\omega_{l(2Z+1)}\right)+1\right]}\right|\\ &\le \frac{\overline{\Omega}^{2n+3}}{2^{2n+1}\left|-\overline{\Omega}^2 + j\xi\overline{\Omega}+1\right|^{2n+2}}\sum_{\omega_1+\cdots+\omega_{2n+1}=\overline{\Omega}}\ \sum_{(j_3,\cdots,j_{(2Q+1)})\in J_{(2n+1)}}\xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}}\\ &\times\sum_{Z=1}^{\bar{n}}\prod_{i=1}^{Z}\frac{\left|j\omega_{l(1)}+\cdots+j\omega_{l(2Z+1)}\right|}{\left|-\left(\omega_{l(1)}+\cdots+\omega_{l(2Z+1)}\right)^2 + j\xi\left(\omega_{l(1)}+\cdots+\omega_{l(2Z+1)}\right)+1\right|}\end{aligned} \tag{36}$$

Therefore, when $\overline{\Omega} << 1$

$$\begin{aligned}\left|P_{2n+1}(j\overline{\Omega})\right| &\le \frac{\overline{\Omega}^{2n+3}}{2^{2n+1}\left|-\overline{\Omega}^2 + j\xi\overline{\Omega}+1\right|^{2n+2}}\sum_{\omega_1+\cdots+\omega_{2n+1}=\overline{\Omega}}\ \sum_{(j_3,\cdots,j_{(2Q+1)})\in J_{(2n+1)}}\xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}}\\ &\times\sum_{Z=1}^{\bar{n}}\prod_{i=1}^{Z}\frac{\left|j\omega_{l(1)}+\cdots+j\omega_{l(2Z+1)}\right|}{\left|-\left(\omega_{l(1)}+\cdots+\omega_{l(2Z+1)}\right)^2 + j\xi\left(\omega_{l(1)}+\cdots+\omega_{l(2Z+1)}\right)+1\right|}\\ &\approx \frac{\overline{\Omega}^{2\bar{n}+3}}{2^{2\bar{n}+1}}\sum_{\omega_1+\cdots+\omega_{2n+1}=\overline{\Omega}}\ \sum_{(j_3,\cdots,j_{(2Q+1)})\in J_{(2n+1)}}\xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}}\sum_{Z=1}^{\bar{n}}\prod_{i=1}^{Z}\left|j\omega_{l(1)}+\cdots+j\omega_{l(2Z+1)}\right|\\ &\le \frac{\overline{\Omega}^{2\bar{n}+3}}{2^{2\bar{n}+1}}\sum_{\omega_1+\cdots+\omega_{2n+1}=\overline{\Omega}}\ \sum_{(j_3,\cdots,j_{(2Q+1)})\in J_{(2n+1)}}\xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}}\sum_{Z=1}^{\bar{n}}\prod_{i=1}^{Z}\frac{\left|j\omega_{l(1)}+\cdots+j\omega_{l(2Z+1)}\right|}{\overline{\Omega}}\\ &= \frac{\overline{\Omega}^{2\bar{n}+3}}{2^{2\bar{n}+1}}\Gamma_{(2n+1)}^{(1)}\end{aligned} \tag{37}$$

where

$$\Gamma_{(2n+1)}^{(1)} = \sum_{\omega_1+\cdots+\omega_{2n+1}=\overline{\Omega}} \sum_{(j_3,\cdots,j_{(2Q+1)})\in J_{(2n+1)}} \xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}} \frac{\left|j\omega_{l(1)}+\cdots+j\omega_{l(2Z+1)}\right|}{\overline{\Omega}}$$

is a bounded constant which is dependent on n but independent of $\overline{\Omega}$. So that, when $\overline{\Omega} << 1$

$$\left|P_{2n+1}(j\overline{\Omega})\right| \le \frac{\overline{\Omega}^{2n+2}}{2^{2n+1}}\Gamma_{(2n+1)}^{(1)} \approx 0 \qquad (n = 1,2,\cdots,\lfloor N/2-1\rfloor) \quad (38)$$

When $\overline{\Omega} >> 1$, it is known from (36) that

$$\begin{aligned}
\left|P_{2n+1}(j\overline{\Omega})\right| &\le \frac{\overline{\Omega}^{2n+3}}{2^{2n+1}\left|-\overline{\Omega}^2 + j\xi\overline{\Omega}+1\right|^{2n+2}} \sum_{\omega_1+\cdots+\omega_{2n+1}=\overline{\Omega}} \sum_{(j_3,\cdots,j_{(2Q+1)})\in J_{(2n+1)}} \xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}} \\
&\quad \times \sum_{Z=1}^{\bar{n}}\prod_{i=1}^{Z} \frac{\left|j\omega_{l(1)}+\cdots+j\omega_{l(2Z+1)}\right|}{\left|-\left(\omega_{l(1)}+\cdots+\omega_{l(2Z+1)}\right)^2 + j\xi\left(\omega_{l(1)}+\cdots+\omega_{l(2Z+1)}\right)+1\right|} \\
&\le \frac{1}{2^{2n+1}\overline{\Omega}^{2n+1}} \sum_{\omega_1+\cdots+\omega_{2n+1}=\overline{\Omega}} \sum_{(j_3,\cdots,j_{(2Q+1)})\in J_{(2n+1)}} \xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}} \sum_{Z=1}^{\bar{n}}\prod_{i=1}^{Z} \frac{1}{\left|\omega_{l(1)}+\cdots+\omega_{l(2Z+1)}\right|} \\
&\le \frac{1}{2^{2n+1}\overline{\Omega}^{2n+1}} \sum_{\omega_1+\cdots+\omega_{2n+1}=\overline{\Omega}} \sum_{(j_3,\cdots,j_{(2Q+1)})\in J_{(2n+1)}} \xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}} \sum_{Z=1}^{\bar{n}} 1 \\
&= \frac{1}{2^{2n+1}\overline{\Omega}^{2n+1}}\Gamma_{(2n+1)}^{(2)} \qquad (39)
\end{aligned}$$

where

$$\Gamma_{(2n+1)}^{(2)} = \sum_{\omega_1+\cdots+\omega_{2n+1}=\overline{\Omega}} \sum_{(j_3,\cdots,j_{(2Q+1)})\in J_{(2n+1)}} \xi_3^{j_3}\cdots\xi_{(2Q+1)}^{j_{(2Q+1)}} \sum_{Z=1}^{\bar{n}} 1$$

is another bounded constant which is dependent on n but independent of $\overline{\Omega}$. So that, when $\overline{\Omega} >> 1$

$$\left|P_{2n+1}(j\overline{\Omega})\right| \le \frac{1}{2^{2n+1}\overline{\Omega}^{2n+1}}\Gamma_{(2n+1)}^{(2)} \approx 0 \qquad (\bar{n} = 1,2,\cdots,\lfloor N/2-1\rfloor) \quad (40)$$

Consequently, $\left|P_{2n+1}(j\overline{\Omega})\right| \approx 0$ for both $\overline{\Omega} << 1$ and $\overline{\Omega} >> 1$. Therefore, over the non-resonance frequency ranges

$$T(\overline{\Omega}) \approx \left|P_1(j\overline{\Omega})\right| \quad (41)$$

This conclusion shows that a nonlinear anti-symmetric damping characteristic has almost no effect on the transmissibility of SDOF vibration isolators over the frequency ranges where the frequencies are much lower or much higher than the isolator's resonant frequency.

3.3.2 Transmissibility over the Resonant Frequency Range

This case is more complicated than the non-resonance case studied in the last sub-section. For convenience of analysis and without loss of generality, it is assumed that only the $\overline{Q}^{\text{th}}$ term of the damping nonlinearity in Eq. (7) is nonzero, that is,

$$\xi_{(2n+1)}\begin{cases}\neq 0 & when \quad n=\overline{Q}\\ =0 & when \quad n\neq\overline{Q}\end{cases} \qquad (p=1,\cdots,Q) \quad (42)$$

For this case, $P_{2n+1}(j\omega)$ in Eq. (34) can be rewritten as

$$P_{2n+1}(j\omega)=\frac{\omega^2}{2^{2n+1}L[j\omega]}\sum_{\omega_1+\cdots+\omega_{2n+1}=\omega}\left[\prod_{i=1}^{2n+1}H_1^{(1)}(j\omega_i)(j\omega_i)\overline{A}(\omega_i)\right]$$
$$\times\xi_{(2\overline{Q}+1)}^{n/\overline{Q}}\sum_{Z=1}^{n/\overline{Q}-1}\prod_{i=1}^{Z}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(Z\overline{Q}+1)}}{\left[L\left(j\omega_{l(1)}+\cdots+j\omega_{l(Z\overline{Q}+1)}\right)\right]}$$
$$\left(n=\overline{Q}\times\left\{1,\cdots,\left\lfloor N/(2\overline{Q})-1\right\rfloor\right\}\right) \quad (43)$$

Denote

$$\Lambda_1(j\overline{\Omega})=P_1(j\overline{\Omega}) \quad (44)$$

$$\Lambda_{2n+1}(j\omega)=\frac{\omega^2}{2^{2n+1}L[j\omega]}\sum_{\omega_1+\cdots+\omega_{2n+1}=\omega}\left[\prod_{i=1}^{2n+1}H_1^{(1)}(j\omega_i)(j\omega_i)\overline{A}(\omega_i)\right]$$
$$\times\sum_{Z=1}^{n/\overline{Q}-1}\prod_{i=1}^{Z}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(Z\overline{Q}+1)}}{\left[L\left(j\omega_{l(1)}+\cdots+j\omega_{l(Z\overline{Q}+1)}\right)\right]} \quad (45)$$

Then $P_{2n+1}(j\omega)$ in Eq. (43) can be rewritten as

$$P_{2n+1}(j\omega)=\xi_{(2\overline{Q}+1)}^{n/\overline{Q}}\Lambda_{2n+1}(j\omega) \qquad \left(n=\overline{Q}\times\left\{1,\cdots,\left\lfloor N/(2\overline{Q})-1\right\rfloor\right\}\right) \quad (46)$$

Using Eq. (45), $\left[T(\overline{\Omega})\right]^2$ can be expressed as

$$\left[T\left(\overline{\Omega}\right)\right]^2=\left[\Lambda_1(j\overline{\Omega})+\sum_{n=1}^{\left\lfloor N/(2\overline{Q})-1\right\rfloor}\Lambda_{2\overline{Q}n+1}(j\overline{\Omega})\xi_{(2\overline{Q}+1)}^n\right]\left[\Lambda_1(-j\overline{\Omega})+\sum_{n=1}^{\left\lfloor N/(2\overline{Q})-1\right\rfloor}\Lambda_{2\overline{Q}n+1}(-j\overline{\Omega})\xi_{(2\overline{Q}+1)}^n\right]$$
$$=\sum_{n=0}^{2\left\lfloor N/(2\overline{Q})-1\right\rfloor}\xi_{(2\overline{Q}+1)}^n\sum_{q=0}^{n}\Lambda_{2\overline{Q}q+1}\left(j\overline{\Omega}\right)\Lambda_{2\overline{Q}(n-q)+1}\left(-j\overline{\Omega}\right) \quad (47)$$

Evaluate $\dfrac{d\left[T(\overline{\Omega})\right]^2}{d\xi_{(2\overline{Q}+1)}}$ from (47) to yield

$$\frac{d\left[T(\overline{\Omega})\right]^2}{d\xi_{(2\overline{Q}+1)}} = \mathrm{Re}\left[\Lambda_1(j\overline{\Omega})\Lambda_{2\overline{Q}+1}(-j\overline{\Omega})\right] + \xi_{(2\overline{Q}+1)} \sum_{n=2}^{2\lfloor N/(2\overline{Q})-1\rfloor} \xi_{(2\overline{Q}+1)}^{n-2} \sum_{q=0}^{n} \Lambda_{2\overline{Q}q+1}\left(j\overline{\Omega}\right)\Lambda_{2\overline{Q}(n-q)+1}\left(-j\overline{\Omega}\right) \tag{48}$$

When $\overline{\Omega} \approx 1$,

$$\frac{d\left[T(\overline{\Omega})\right]^2}{d\xi_{(2\overline{Q}+1)}} \approx \mathrm{Re}\left[\Lambda_1(j)\Lambda_{2\overline{Q}+1}(-j)\right] + \xi_{(2\overline{Q}+1)} \sum_{n=2}^{2\lfloor N/(2\overline{Q})-1\rfloor} \xi_{(2\overline{Q}+1)}^{n-2} \sum_{q=0}^{n} \Lambda_{2\overline{Q}q+1}(j)\Lambda_{2\overline{Q}(n-q)+1}(-j) \tag{49}$$

From (33) and (43), it can be obtained that, when $\overline{\Omega} \approx 1$

$$\Lambda_1(j\overline{\Omega}) \approx \frac{j\xi+1}{j\xi} = \frac{-j+\xi}{\xi} \tag{50}$$

$$\Lambda_{2\overline{Q}+1}(-j\overline{\Omega}) \approx \frac{-j}{2^{2\overline{Q}+1}\xi^{2\overline{Q}+2}} \tag{51}$$

so that

$$\mathrm{Re}\left[\Lambda_1(j)\Lambda_{2\overline{Q}+1}(-j)\right] = -\frac{1}{2^{2\overline{Q}+1}\xi^{2\overline{Q}+3}} < 0 \tag{52}$$

Therefore, when $\overline{\Omega} \approx 1$,

$$\frac{d\left[T(\overline{\Omega})\right]^2}{d\xi_{(2\overline{Q}+1)}} \approx -\frac{1}{2^{2\overline{Q}+1}\xi^{2\overline{Q}+3}} + \xi_{(2\overline{Q}+1)} \sum_{n=2}^{2\lfloor N/(2\overline{Q})-1\rfloor} \xi_{(2\overline{Q}+1)}^{n-2} \sum_{q=0}^{n} \Lambda_{2\overline{Q}q+1}(j)\Lambda_{2\overline{Q}(n-q)+1}(-j) \tag{53}$$

Eq. (53) implies that, when $\overline{\Omega} \approx 1$, there must exist a $\overline{\xi}_{(2\overline{Q}+1)} > 0$ such that if $0 < \xi_{(2\overline{Q}+1)} < \overline{\xi}_{(2\overline{Q}+1)}$,

$$\frac{d\left[T(\overline{\Omega})\right]^2}{d\xi_{(2\overline{Q}+1)}} \approx -\frac{1}{2^{2\overline{Q}+1}\xi^{2\overline{Q}+3}} + \xi_{(2\overline{Q}+1)} \sum_{n=2}^{2\lfloor N/(2\overline{Q})-1\rfloor} \xi_{(2\overline{Q}+1)}^{n-2} \sum_{q=0}^{n} \Lambda_{2\overline{Q}q+1}(j)\Lambda_{2\overline{Q}(n-q)+1}(-j) < 0 \tag{54}$$

The important conclusion described as Eq. (54) indicates that an increase in the nonlinear anti-symmetric damping characteristic can reduce the transmissibility over the resonant frequency range.

Next, assume the first two terms of the damping nonlinearity in Eq. (7) are positive and nonzero, that is, ξ_3 and ξ_5>0. Denote

$$\Delta_3(\xi_3,\xi_5) = \frac{\partial\left[T(\overline{\Omega})\right]^2}{\partial\xi_3} \text{ and } \Delta_5(\xi_3,\xi_5) = \frac{\partial\left[T(\overline{\Omega})\right]^2}{\partial\xi_5} \tag{55}$$

According to Eq. (54), there exist $\overline{\xi}_3$ and $\overline{\xi}_5 > 0$ such that if $\xi_3 \in (0,\overline{\xi}_3)$ and $\xi_5 \in (0,\overline{\xi}_5)$, then $\Delta_3(\xi_3,0) < 0$ and $\Delta_5(0,\xi_5) < 0$.

Moreover, as the *Sign-Preserving Property* [31] states, there is a $\delta_5 > 0$ such that if $\xi_5 \in (0, \delta_5)$, then $\Delta_3(\xi_3, \xi_5)$ has the same sign as $\Delta_3(\xi_3, 0)$. Similarly, there is a $\delta_3 > 0$ such that if $\xi_3 \in (0, \delta_3)$, then $\Delta_5(\xi_3, \xi_5)$ has the same sign as $\Delta_5(0, \xi_5)$. This means that, if $\xi_3 \in (0, \delta_3) \cap (0, \bar{\xi}_3)$ and $\xi_5 \in (0, \delta_5) \cap (0, \bar{\xi}_5)$, then the increase of ξ_3 and ξ_5 can reduce the transmissibility over the resonant frequency range. This conclusion can be extended to the more general case where all terms of the damping nonlinearity in equation (54) are nonzero. Therefore, when $\bar{\Omega} \approx 1$, there must exist $\bar{\delta}_{(2\bar{Q}+1)} > 0$ ($\bar{Q} = 1, \cdots, Q$) such that if $0 < \xi_{(2\bar{Q}+1)} < \bar{\delta}_{(2\bar{Q}+1)}$,

$$\frac{\partial [T(\bar{\Omega})]^2}{\partial \xi_{(2\bar{Q}+1)}} < 0 \tag{56}$$

The conclusions reached in Section 3.3 reveal that the vibration isolator with a nonlinear anti-symmetric damping characteristic has great potential to overcome the limitations of linear vibration isolators, and an effective exploitation of the capability of the nonlinear vibration isolator can provide a novel passive solution to the aforementioned well-known dilemma associated with the design of passive linear vibration isolators.

4. Numerical Verification and Discussions

4.1 Numerical Studies

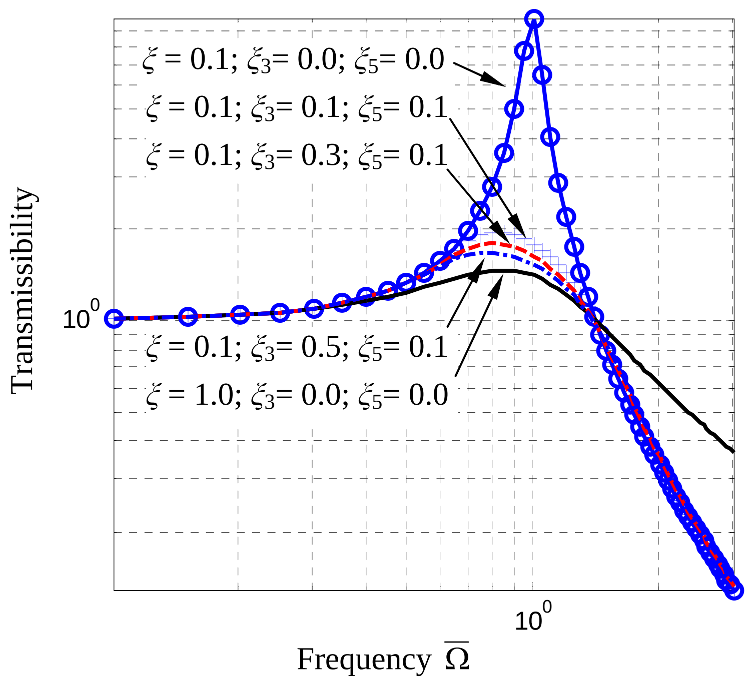

Fig. 5, the transmissibility of the nonlinear isolator with different ξ_3 and a constant ξ_5

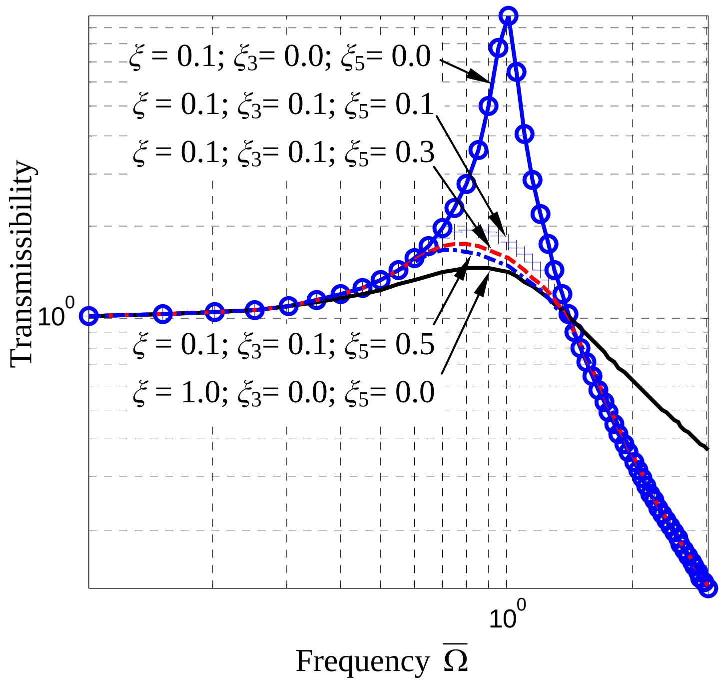

Fig. 6, The transmissibility of the nonlinear isolator with different ξ_5 and a constant ξ_3

In order to verify the significant effects of a nonlinear anti-symmetric damping characteristic on vibration isolation, which has been theoretically analysed above, numerical simulation studies were conducted by applying the *Runge-Kutta* method to the dimensionless, one input two output system (9) with Q = 2 to evaluate the transmissibility $T(\overline{\Omega})$. Two sets of results are shown in Fig. 5 and Fig. 6 respectively.

In the results shown in Fig. 5, ξ_5 is taken as a constant 0.1 and the other nonlinear damping characteristic parameter ξ_3 is varied from 0.1 to 0.5 in steps of 0.2. In Fig. 6, ξ_3 is kept constant at 0.1 and ξ_5 is varied. Moreover, for a better comparison with the linear isolator, the transmissibility of the linear isolator (3) for the two cases of ξ =0.1 and ξ =1.0 is also shown in the figures. All results clearly indicate that the introduction of the nonlinear anti-symmetric damping can not only significantly reduce $T(\overline{\Omega})$ and consequently suppress the vibration at the resonant frequency $\overline{\Omega} \approx 1$, but these designs also keeps $T(\overline{\Omega})$ almost unchanged over the isolation frequency ranges where $\overline{\Omega} << 1$ and $\overline{\Omega} >> 1$. These results confirm the theoretical analysis results proved in Section 3.3. Therefore, the numerical studies have verified the important conclusions revealed in Section 3.3.

The theoretical analysis based on the concept of OFRFs and the numerical studies clearly show the effects of vibration isolators with a nonlinear anti-symmetric damping characteristic are equivalent to that of adaptive passive isolators, which have the ideal dynamic damping response as shown in Fig. 3. Consequently, the nonlinear isolators can be used to overcome the dilemma associated with the design of passive linear vibration isolators.

4.2 Discussion

The validity of the important properties described by Eqs. (41) and (56) are based on the premise that the nonlinear damping characteristic of the vibration isolator is anti-symmetric and the nonlinear characteristic parameters are positive. However, these premises may not always be true in practice. Therefore it is necessary to test the sensitivity and robustness of the designs when, for example, the damping characteristic is not exactly anti-symmetric and some nonlinear damping characteristic parameters are negative.

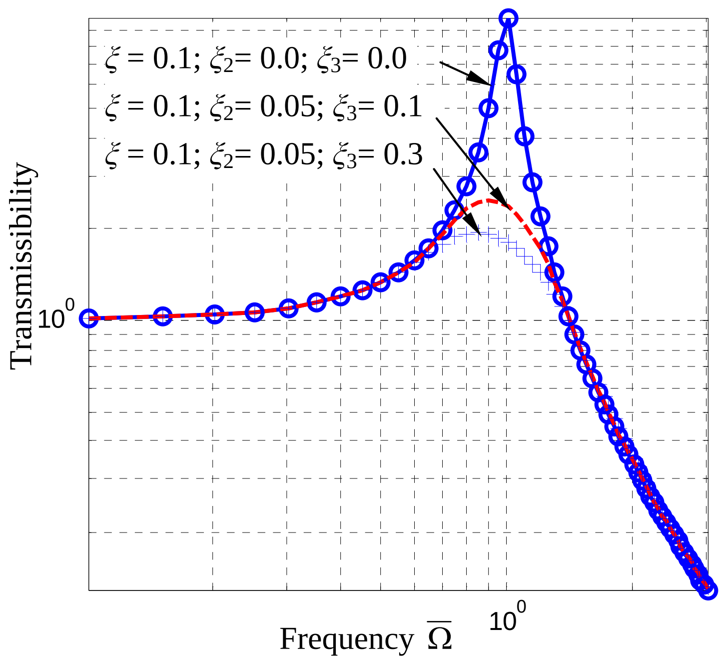

Fig. 7, The transmissibility of the nonlinear isolator with a non-anti-symmetric damping characteristic

In order to test the effects of a small deviation from an anti-symmetry damping characteristic on the performance of the nonlinear isolator, the damping force of the nonlinear vibration isolator is considered to be of the form below

$$F_d = \sum_{p=1}^{3} \xi_p \left[\dot{y}_1(t)\right]^p \quad (57)$$

where the presence of the 2nd power term makes the nonlinear damping characteristic no longer anti-asymmetry. Fig. 7 shows the transmissibility of the nonlinear vibration isolator in this case. Clearly, the increase of ξ_3 can still significantly reduce the transmissibility around the resonant frequency region and there is almost no effect on the transmissibility over the non-resonant frequency region. Therefore, the properties given by Eqs. (41) and (56) are still valid in the case where the anti-symmetry requirement for the nonlinear damping characteristic is not exactly satisfied.

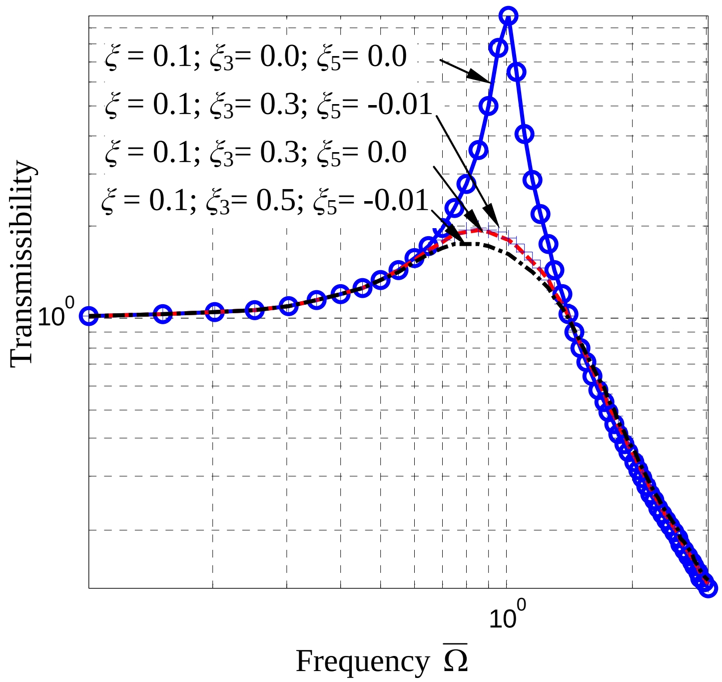

Fig. 8, The transmissibility of the nonlinear isolator with a negative nonlinear damping characteristic parameter

To investigate the effects of the non-positive nonlinear damping characteristic parameters on the vibration isolation performance, the damping force of the nonlinear vibration isolator is considered to be of the following form

$$F_d = \sum_{p=1}^{3} \xi_{(2p-1)} \left[\dot{y}_1(t)\right]^{2p-1} \tag{58}$$

where ξ_5 is negative. The numerical simulation results shown in Fig. 8 clearly indicate that the increase of ξ_3 can also significantly reduce the transmissibility around the resonant region and has no effect on the transmissibility over the non-resonant frequency regions, i.e, the properties given by Eqs. (41) and (56) are still valid.

5. Conclusions

The concept of the OFRF has been used to investigate the effects of a nonlinear anti-symmetric damping characteristic on the transmissibility of nonlinear vibration isolators. The following four important conclusions have been established by theoretical analysis and / or numerical simulation studies:

i) A nonlinear anti-symmetric damping characteristic has almost no effect on the transmissibility of SDOF vibration isolators over both low and high frequency ranges where the frequencies are much lower or much higher than the isolator's resonant frequency.

ii) The introduction of a nonlinear anti-symmetric damping into vibration isolators can significantly reduce the transmissibility over the resonant frequency region.

iii) Properties 1) and 2) are valid even in the case where the anti-symmetry requirement for the nonlinear damping characteristic is not exactly satisfied.

iv) Properties 1) and 2) generally hold when the damping characteristic parameters are positive but are still valid when some of these parameters are relatively small but negative.

The performance of nonlinear vibration isolators with an anti-symmetric damping characteristic imply that the effects of such nonlinear isolators are equivalent to that of adaptive passive isolators having an ideal frequency-dependent damping effect which is significant around the resonant frequency region but less significant over the non-resonant frequency regions. These conclusions are of significant importance for the design of vibration isolators as they reveal that the nonlinear vibration isolator with an anti-symmetric damping characteristic has great potential to overcome the dilemma associated with the design of passive linear vibration isolators.

Results for MDOF systems and other related cases would be reported in forthcoming publications.

Acknowledgements

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Appendix:

The recursive algorithm proposed by the authors [23] can be used to determine how many and what monomials involved in Eq. (21), as follows:

Denote the set of all the monomials involved in Eq. (21) as $M_{(2n+1)}$, and $M_1 = [1]$, then $M_{(2n+1)}$ can be determined as

$$M_{(2n+1)} = \bigcup_{p=1}^{\min(n,Q)} \xi_{(2p+1)} \otimes M_{(2n+1),(2p+1)} \tag{A-1}$$

where $\otimes$ is the Kronecker product , and

$$M_{n,Z} = \bigcup_{i=1}^{\lfloor (n-Z)/2 \rfloor +1} \left(M_{(2i-1)} \otimes M_{(n-2i+1),(Z-1)} \right) \text{ and } M_{n,1} = M_n \tag{A-2}$$

Similar procedure can be used to determine the corresponding function $\Theta_{(2n+1)}^{(j_3 \cdots j_{(2Q+1)})}$ for the monomial $\xi_3^{j_3} \cdots \xi_{(2Q+1)}^{j_{(2Q+1)}}$. Denote the set of all the functions $\Theta_{(2n+1)}^{(j_3 \cdots j_{(2Q+1)})}\left(j\omega_1, \cdots, j\omega_{(2n+1)}\right)$ involved in Eq. (21) as $\Theta_{(2n+1)}$, then $\Theta_{(2n+1)}$ can be determined as

$$\Theta_{(2n+1)} = \bigcup_{p=1}^{\min(n,Q)} \Theta_{(2n+1),(2p+1)} \tag{A-3}$$

where

$$\Theta_{n,Z} = \bigcup_{i=1}^{\lfloor (n-Z)/2 \rfloor +1} B(j\omega_1 + \cdots + j\omega_{(2i-1)}) \left(\Theta_{(2i-1)} \otimes \Theta_{(n-2i+1),(Z-1)} \right) \quad \text{with } \Theta_{n,1} = \Theta_n \tag{A-4}$$

For example, applying the methods (A-1) and (A-2) to the isolator (9) with Q = 2 up to the 7$^{\text{th}}$ order yields

$M_3 = \left[[\xi_3] \otimes [1] \right] = \left[\xi_3 \right]$

$M_5 = \left[[\xi_3] \otimes M_3 \otimes [1] \otimes [1] \right] \cup \left[[\xi_5] \otimes [1] \right] = \left[\xi_3^2 \quad \xi_5 \right]$

$M_7 = \left[[\xi_3] \otimes M_3 \otimes M_3 \otimes [1] \right] \cup \left[[\xi_3] \otimes M_5 \otimes [1] \otimes [1] \right] \cup \left[[\xi_5] \otimes M_3 \otimes [1] \otimes [1] \otimes [1] \otimes [1] \right]$

$\quad = \left[\xi_3^3 \quad \xi_3 \xi_5 \right]$

The results indicate that, in Eq. (21),

$$J_3 = \{(1,0)\},\ J_5 = \{(2,0),(0,1)\},\ J_7 = \{(3,0),(1,1)\}.$$

When conducting the procedures (A-3) and (A-4) to determine $\Theta_{(2n+1)}$, $B\left(j\omega_{l(1)} + \cdots + j\omega_{l(Z)}\right)$ will be denoted as B_Z for the simplicity and also because the specification of B_Z will not play a curial role in the following analysis. The results of $\Theta_{(2n+1)}$ up to 7$^{\text{th}}$ order are given as follows,

$\Theta_3 = \left[[1] \otimes [1] \otimes [1] \right] = \left[1 \right]$

$\Theta_5 = \left[B_3 \otimes \Theta_3 \otimes [1] \otimes [1] \right] \cup \left[[1] \otimes [1] \otimes [1] \otimes [1] \otimes [1] \right] = \left[B_3 \quad 1 \right]$

$\Theta_7 = \left[B_3 \otimes B_3 \otimes \Theta_3 \otimes \Theta_3 \otimes [1] \right] \cup \left[B_5 \otimes \Theta_5 \otimes [1] \otimes [1] \right] \cup \left[B_3 \otimes \Theta_3 \otimes [1] \otimes [1] \otimes [1] \otimes [1] \right]$

$\quad = \left[B_3 B_3 \quad B_5 B_3 \quad B_5 \quad B_3 \right]$

There are two more terms in Θ_7 than in M_7, and the first two elements in Θ_7 are associated with ξ_3^3 and the other two are associated with $\xi_3\xi_5$, therefore, in Eq. (21)

$$\Theta_3^{(1,0)}\left(j\omega_1,\cdots,j\omega_3\right)=0;\ \Theta_5^{(2,0)}\left(j\omega_1,\cdots,j\omega_5\right)=B_3;\ \Theta_5^{(0,1)}\left(j\omega_1,\cdots,j\omega_5\right)=1;$$

$$\Theta_7^{(3,0)}\left(j\omega_1,\cdots,j\omega_7\right)=B_3B_3+B_5B_3;\ \Theta_7^{(1,1)}\left(j\omega_1,\cdots,j\omega_7\right)=B_5+B_3$$

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