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Mapping from Parametric Characteristics to Generalized Frequency Response Functions of Nonlinear Systems

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Abstract: Based on the parametric characteristic of the *n*th-order GFRF (Generalised Frequency Response Function) for nonlinear systems described by an NDE (nonlinear differential equation) model, a mapping function from the parametric characteristics to the GFRFs is established, by which the *n*th-order GFRF can directly be written into a more straightforward and meaningful form in terms of the first order GFRF, i.e., an *n*-degree polynomial function of the first order GFRF. The new expression has no recursive relationship between different order GFRFs, and demonstrates some new properties of the GFRFs which can explicitly unveil the linear and nonlinear factors included in the GFRFs, and reveal clearly the relationship between the *n*th-order GFRF and its parametric characteristic, and also the relationship between the *n*th-order GFRF and the first order GFRF. The new results provide a novel and useful insight into the frequency domain analysis and design of nonlinear systems based on the GFRFs. Several examples are given to illustrate the theoretical results.

Keywords: Generalised Frequency Response Function (GFRF), Nonlinear systems, Parametric characteristics, Nonlinear differential equation (NDE), Volterra series

1 Introduction

The frequency domain analysis of nonlinear systems has been studied for many years (Taylor 1999, Solomou 2002, Pavlov 2007). Nonlinear systems can also be studied in the frequency domain based on Volterra series theory (Bedrosian and Rice 1971, Rugh 1981, Brilliant 1958, Kotsios 1997, Volterra 1959). It is noted in Boyd and Chua (1985) that nonlinear systems, which are causal and have fading memory, can be approximated by the Volterra series of finite orders. The existence of a Volterra series expansion for a nonlinear system was also studied in Sandberg (1982, 1983). For a Volterra series expansion of a nonlinear system, the *n*th-order Generalized Frequency Response Function (GFRF) of the system is defined as the multi-dimensional Fourier transform of the *n*th order Volterra kernel (George 1959). This concept provides a significant basis for the analysis of nonlinear systems in the frequency domain. Many significant results relating to the estimation and computation of the GFRFs and analysis of output frequency response for a practical nonlinear system have been developed based on this concept (Bendat 1990, Billings and Lang 1996, Chua and Ng 1979, Jing et al 2007).

To compute the GFRFs of nonlinear systems, Bedrosian and Rice (1971) introduced the "harmonic probing" method, by which the higher order GFRFs of the harmonic expansion of the nonlinear system under study can be derived. By applying the probing method (Rugh 1981), algorithms to compute the GFRFs for nonlinear Volterra systems described by NDE model and NARX (Nonlinear Auto-Regressive model with eXogenous input) model were derived, which enable the *n*th-order GFRF to be recursively obtained in terms of the coefficients of the governing NARX or NDE model (Peyton-Jones and Billings 1989, Billings and Peyton-Jones 1990, Chen and Billings 1989). Based on the GFRFs, frequency response characteristics of nonlinear systems can therefore be investigated (Peyton Jones and Billings 1990, Yue et al 2005). These results are important extensions of the well known frequency domain methods for linear systems such as transfer function or Bode diagram, and provide a method to the analysis of nonlinear systems in the frequency domain. Although these progresses have been made and the GFRFs of nonlinear systems described by NARX model and NDE model can be determined effectively, it can be seen that the GFRF is in fact a multivariate complex valued function series in terms of model parameters defined in high dimensional frequency space, and consequently the existing recursive algorithms for the computation of the GFRFs can not explicitly and simply reveal the analytical relationship between system time domain model parameters and system frequency response functions in a clear and straightforward manner such that many problems remain unsolved regarding the characteristics of the GFRFs and the system output frequency response, including how the frequency response functions are influenced by the parameters of the underlying system, and the connection to complex non-linear behaviours. These inhibit the practical application and understanding of the existing theoretical results to a certain extent. In order to solve these problems, the parametric characteristics of the GFRFs were studied in Jing et al (2006), which effectively build up a mapping from the GFRF to its parametric characteristic and thus provides an explicit expression for the analytical relationship between the GFRFs and system time-domain model parameters. The significance of the parametric characteristic analysis of the *n*th-order GFRF is that it can clearly reveal what model parameters contribute to and how these parameters affect system frequency response functions including the GFRFs and output frequency response function. This provides an effective approach to the analysis of the frequency domain characteristics of nonlinear systems in terms of system time domain model parameters.

This study is based on our previous results in Jing et al (2006). It is shown in Jing et al (2006) that the *n*th-order GFRF and output spectrum of a nonlinear Volterra system can both be written as an explicit and straightforward polynomial function in terms of nonlinear model parameters, and this polynomial function is characterized by its parametric characteristic and some related complex valued functions which are dependent on the frequency variables, system's linear factors and even system input (for output spectrum). The parametric characteristics can be analytically determined by the results in Jing et al (2006). In this study, the focus is to analytically determine the complex valued functions related to the parametric characteristics. An inverse mapping function from the parametric characteristics of the GFRFs to the GFRFs is studied. By using this new mapping function, the *n*th-order GFRF can directly be recovered from its parametric characteristic as an *n*-degree polynomial function of the first order GFRF, keeping the explicit analytical relationship between the GFRF and system time-domain model parameters. Compared with the existing recursive algorithm for the computation of the GFRFs, the new mapping function enables the *n*th-order GFRF to be determined in a much more straightforward and meaningful structure. Note from the previous results that the higher order GFRFs are recursively dependent on the lower order GFRFs. This crossing relationship sometimes complicates the qualitative analysis and understanding of system frequency characteristics by using the nth-order GFRF. The new results can effectively overcome this problem, and unveil the system's linear and nonlinear factors included in the *n*th-order GFRF more clearly. This provides a novel and useful insight into the frequency domain analysis and design of nonlinear systems based on the GFRFs, and can be regarded an important extension of the parametric characteristic theory established previously. Several examples are given to illustrate these results.

Nomenclature

$c_{p,q}(k_1,\cdots,k_{p+q})$	A model parameter in the NDE model, k_i is the order of the derivative, *p* represents the order of the involved output nonlinearity, *q* is the order of the involved input nonlinearity, and *p*+*q* is the nonlinear degree of the parameter.
$H_n(j\omega_1,\cdots,j\omega_n)$	The nth-order GFRF
$C_{p,q}=[c_{p,q}(0,\cdots,0),c_{p,q}(0,\cdots,1),\cdots,c_{p,q}(\underbrace{K,\cdots,K}_{p+q=m})]$	A parameter vector consisting of all the nonlinear parameters of the form $c_{p,q}(k_1,\cdots,k_{p+q})$
$CE(.)$	The coefficient extraction operator
$CE(H_n(j\omega_1,\cdots,j\omega_n))$	The parametric characteristics of the nth-order GFRF
$f_n(j\omega_1,\cdots,j\omega_n)$	The correlative function of $CE(H_n(j\omega_1,\cdots,j\omega_n))$
$\otimes$	The reduced Kronecker product defined in the CE operator
$\oplus$	The reduced vectorized summation defined in the CE operator
$c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$	A monomial consisting of nonlinear parameters
$s_{x_1}s_{x_2}\cdots s_{x_p}$	A *p*-partition of a monomial $c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$
s_{x_i}	A monomial of x_i parameters of $\{c_{p_0,q_0}(\cdot),\cdots,c_{p_k,q_k}(\cdot)\}$ of the involved monomial, $0\le x_i\le k$, and $s_0=1$
$\varphi_n: S_C(n)\to S_f(n)$	A new mapping function from the parametric characteristics to the correlative functions, $S_C(n)$ is the set of all the monomials in the parametric characteristics and $S_f(n)$ is the set of all the involved correlative functions in the nth order GFRF.
$n(s_x(\bar{s}))$	The order of the GFRF where the monomial $s_x(\bar{s})$ is generated
$\bar{\lambda}_n(\omega_1,\cdots,\omega_n)$	The maximum eigenvalue of the frequency characteristic matrix Θ_n of the nth-order GFRF

2 The *n*th-order GFRF for nonlinear systems and its parametric characteristic

A large amount of nonlinear systems can be described by the following nonlinear differential equation (NDE) model

$$\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}(k_1,\cdots,k_{p+q})\prod_{i=1}^{p}\frac{d^{k_i}y(t)}{dt^{k_i}}\prod_{i=p+1}^{p+q}\frac{d^{k_i}u(t)}{dt^{k_i}}=0 \tag{1}$$

where $\left.\frac{d^k x(t)}{dt^k}\right|_{k=0}=x(t)$, $p+q=m$, $\sum_{k_1,k_{p+q}=0}^{K}(\cdot)=\sum_{k_1=0}^{K}(\cdot)\cdots\sum_{k_{p+q}=0}^{K}(\cdot)$, M is the maximum degree of nonlinearity in terms of $y(t)$ and $u(t)$, and K is the maximum order of the derivative. In this model, the parameters such as $c_{0,1}(.)$ and $c_{1,0}(.)$ are linear parameters corresponding to the linear terms in the model, *i.e.*, $\frac{d^k y(t)}{dt^k}$ and $\frac{d^k u(t)}{dt^k}$ for k=0,1,…,L, and $c_{p,q}(\cdot)$ for $p+q>1$ are referred to as the nonlinear parameters corresponding to nonlinear terms in the model of the form $\prod_{i=1}^{p}\frac{d^{k_i}y(t)}{dt^{k_i}}\prod_{i=p+1}^{p+q}\frac{d^{k_i}u(t)}{dt^{k_i}}$, *e.g.*, $y(t)^p u(t)^q$. $p+q$ is referred to as the nonlinear degree of parameter $c_{p,q}(\cdot)$.

Consider nonlinear systems which can be approximated by a Volterra series up to maximum order N (Boyd and Chua 1985) as

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \tag{2}$$

where $h_n(\tau_1,\cdots,\tau_n)$ is a real valued function of $\tau_1,\cdots,\tau_n$ called the *n*th-order Volterra kernel. The *n*th-order GFRF of system (2) is defined as (George 1959)

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \tag{3}$$

The concept of GFRF provides a basis for the study of nonlinear systems in the frequency domain. The GFRF for system (2) described by NDE model (1) can be obtained by the probing method (Rugh 1981). An algorithm to compute the *n*th-order GFRF for NDE model (1) was provided in Billings and Peyton-Jone (1990):

$$\begin{aligned}L_n(j\omega_1+\cdots+j\omega_n)\cdot H_n(j\omega_1,\cdots,j\omega_n)&=\sum_{k_1,k_n=1}^{K}c_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n}\\&+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}c_{p,q}(k_1,\cdots,k_{p+q})(j\omega_{n-q+1})^{k_{n-q+1}}\cdots(j\omega_{p+q})^{k_{p+q}}H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})\\&+\sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K}c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n)\end{aligned} \tag{4}$$

$$H_{n,p}(\cdot)=\sum_{i=1}^{n-p+1}H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)^{k_p} \tag{5}$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n)=H_n(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{k_1} \tag{6}$$

where $L_n(j\omega_1+\cdots+j\omega_n)=-\sum_{k_1=0}^{K}c_{1,0}(k_1)(j\omega_1+\cdots+j\omega_n)^{k_1}$. Moreover, $H_{n,p}(j\omega_1,\cdots,j\omega_n)$ in (6) can also be written as

$$H_{n,p}(j\omega_1,\cdots,j\omega_n)=\sum_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\prod_{i=1}^{p}H_{r_i}(j\omega_{X+1},\cdots,j\omega_{X+r_i})(j\omega_{X+1}+\cdots+j\omega_{X+r_i})^{k_i} \tag{7}$$

where $X=\sum_{x=1}^{i-1}r_x$.

2.1 A correction for the computation of the *n*th-order GFRF

In the recursive algorithm for the computation of the GFRFs above, the second term in the right side of equation (4), *i.e.*,

$$\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}c_{p,q}(k_1,\cdots,k_{p+q})(j\omega_{n-q+1})^{k_{n-q+1}}\cdots(j\omega_{p+q})^{k_{p+q}}H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})$$

should be

$$\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}c_{p,q}(k_1,\cdots,k_{p+q})(\prod_{i=1}^{q}(j\omega_{n-q+i})^{k_{p+i}})H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \tag{8}$$

That is, equation (4) is corrected as

$$\begin{aligned}L_n(j\omega_1+\cdots+j\omega_n)\cdot H_n(j\omega_1,\cdots,j\omega_n)&=\sum_{k_1,k_n=1}^{K}c_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n}\\&+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}c_{p,q}(k_1,\cdots,k_{p+q})(\prod_{i=1}^{q}(j\omega_{n-q+i})^{k_{p+i}})H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})\\&+\sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K}c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n)\end{aligned} \tag{9}$$

This result can be shown by applying the probing method for the cross input-output nonlinear terms labelled by nonlinear parameter $c_{pq}(.)$ for $p\ge 1,q\ge 1$ in NDE model (1) as demonstrated in Billings and Peyton Jones (1990).

For clarity, consider a simple cross nonlinear term $c_{1,2}(k_1,k_2,k_3)\frac{d^{k_1}y(t)}{dt^{k_1}}\frac{d^{k_2}u(t)}{dt^{k_2}}\frac{d^{k_3}u(t)}{dt^{k_3}}$. The contribution to the asymmetric *n*th-order GFRF from this specific term is

$$\begin{aligned}&C_n\left[\sum_{r_1=1}^{n}H_{r_1}(j\omega_1\cdots j\omega_{r_1})(j\omega_1+\cdots+j\omega_{r_1})^{k_1}e^{j(\omega_1+\cdots+\omega_{r_1})t}\cdot\sum_{r=1}^{n}(j\omega_r)^{k_2}e^{j\omega_r t}\cdot\sum_{r=1}^{n}(j\omega_r)^{k_3}e^{j\omega_r t}\right]\\&=H_{n-2}(j\omega_1\cdots j\omega_{n-2})(j\omega_1+\cdots+j\omega_{n-2})^{k_1}e^{j(\omega_1+\cdots+\omega_{n-2})t}\cdot(j\omega_{n-1})^{k_2}e^{j\omega_{n-1}t}\cdot(j\omega_n)^{k_3}e^{j\omega_n t}\\&=H_{n-2}(j\omega_1\cdots j\omega_{n-2})(j\omega_1+\cdots+j\omega_{n-2})^{k_1}\cdot(j\omega_{n-1})^{k_2}\cdot(j\omega_n)^{k_3}\cdot e^{j(\omega_1+\cdots+\omega_n)t}\end{aligned} \tag{10}$$

where $C_n[.]$ denote the operation of extracting the coefficient of $e^{j(\omega_1+\cdots+\omega_n)t}$ (Billings and Peyton Jones 1990). By using (5) and (7), (10) is equal to

$$(\prod_{i=1}^{2}(j\omega_{n-2+i})^{k_{1+i}})H_{n-2,1}(j\omega_1,\cdots,j\omega_{n-2})$$

This result is consistent with (8). Following the same method and extending to the more general case, (8) and (9) can be achieved. Moreover, for convenience in further derivation, let

$$H_{0,0}(\cdot)=1,\ H_{n,0}(\cdot)=0 \text{ for n>0},\ H_{n,p}(\cdot)=0 \text{ for } n<p, \text{ and } \prod_{i=1}^{q}(\cdot)=\begin{cases}1 & q=0,p>1\\0 & q=0,p\le 1\end{cases} \tag{11}$$

Then (9) can be written for more simplicity as

$$H_n(j\omega_1,\cdots,j\omega_n)=\frac{1}{L_n(j\sum_{i=1}^{n}\omega_i)}\sum_{q=0}^{n}\sum_{p=0}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}c_{p,q}(k_1,\cdots,k_{p+q})(\prod_{i=1}^{q}(j\omega_{n-q+i})^{k_{p+i}})H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \tag{12}$$

Therefore, the corrected recursive algorithm for the computation of GFRFs is (9 or 12, 11, 5-7). This will be used in the following sections. Note that the GFRFs here are asymmetric and the symmetric GFRFs can be obtained as

$$H_n^{sym}(j\omega_1,\cdots,j\omega_n)=\tfrac{1}{n!}\sum_{\substack{\text{all the permutations}\\ \text{of }\{1,2,\ldots,n\}}}H_n(j\omega_1,\cdots,j\omega_n)$$

2.2 The parametric characteristics of the GFRFs

The parametric characteristics were studied in Jing et al (2006) to reveal what model parameters contribute to and how these parameters affect system frequency response functions. By using the parametric characteristic analysis, some frequency domain characteristics of the GFRFs can be obtained, and the explicit relationship between the GFRFs and system time domain model parameters can be unveiled. Let

$$C_{p,q}=[c_{p,q}(0,\cdots,0),c_{p,q}(0,\cdots,1),\cdots,c_{p,q}(\underbrace{K,\cdots,K}_{p+q=m})]$$

From the results in Jing et al (2006), the parametric characteristic of the *n*th-order GFRF in (4) can be computed as

$$CE(H_n(j\omega_1,\cdots,j\omega_n))=C_{0,n}\oplus\left(\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}C_{p,q}\otimes CE(H_{n-q-p+1}(\cdot))\right)\oplus\left(\mathop{\oplus}_{p=2}^{n}C_{p,0}\otimes CE(H_{n-p+1}(\cdot))\right) \tag{13}$$

where *CE*(.) is a novel coefficient extraction operator which has two basic operations "$\oplus$" and "$\otimes$". For the detailed definition and operation rules for *CE*(.), the readers can refer to Appendix A. Based on the parametric characteristic analysis (Jing et al 2006), the *n*th-order GFRF can be expressed as

$$H_n(j\omega_1,\cdots,j\omega_n)=CE(H_n(j\omega_1,\cdots,j\omega_n))\cdot f_n(j\omega_1,\cdots,j\omega_n) \tag{14}$$

where $f_n(j\omega_1,\cdots,j\omega_n)$ is a complex valued function vector with an appropriate dimension, which is referred to as the correlative function of the parametric characteristic $CE(H_n(j\omega_1,\cdots,j\omega_n))$ in this paper.

Equation (14) provides an explicit expression for the analytical relationship between the GFRFs and the system time-domain model parameters. Based on these results, system nonlinear characteristics can be studied in the frequency domain from a novel perspective such as frequency characteristics of system output frequency response, nonlinear effect from some specific nonlinear parameters, parametric sensitivity analysis and so on, as demonstrated in Jing et al (2006, 2007b). In the following sections of this study, an algorithm is provided to explicitly determine the correlative function $f_n(j\omega_1,\cdots,j\omega_n)$ in (14) directly in terms of the first order GFRF $H_1(j\omega_1)$ based on the parametric characteristic vector $CE(H_n(j\omega_1,\cdots,j\omega_n))$. To this objective, a mapping from $CE(H_n(j\omega_1,\cdots,j\omega_n))$ to $H_n(j\omega_1,\cdots,j\omega_n)$ is established such that the *n*th-order GFRF can directly be written into the parametric characteristic function (14) in its detailed and analytical form by using this mapping function, and some new properties of the GFRFs are developed. These results effectively extend the previous established parametric characteristic theory. The GFRFs can directly be determined in a much more straightforward and meaningful structure in terms of model parameters and the first order GFRF without recursive and crossing relationship between different order GFRFs, and the system's linear and nonlinear factors included in the *n*th-order GFRF can be unveiled more clearly. By using the new results, the analytical OFRF can now be determined explicitly. The new results of this study should provide a fundamental basis for the frequency domain analysis of nonlinear Volterra systems.

3 Mapping from the parametric characteristic to the *n*th-order GFRF

The parametric characteristic vector $CE(H_n(j\omega_1,\cdots,j\omega_n))$ of the *n*th-order GFRF can be recursively determined by equation (13), which has elements of the form $C_{p,q}\otimes C_{p_1,q_1}\otimes C_{p_2,q_2}\otimes\cdots\otimes C_{p_k,q_k}$ (n-2$\geq k\geq 0$), and each element of which has a corresponding complex valued correlative function in vector $f_n(j\omega_1,\cdots,j\omega_n)$. For example, $c_{0,n}(k_1,\cdots,k_n)$ corresponds to the complex valued function $(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n}$ in the *n*th-order GFRF. For further development, $CE(H_n(j\omega_1,\cdots,j\omega_n))$ can also be determined by the following result, which allows the direct determination of the parameter characteristic vector of the *n*th-order GFRF without recursive computations and provides a sufficient and necessary condition for which nonlinear parameters and how these parameters are included in $CE(H_n(j\omega_1,\cdots,j\omega_n))$.

Lemma 1 (Jing et al 2006). The elements of $CE(H_n(j\omega_1,\cdots,j\omega_n))$ include and only include the nonlinear parameters in C_{0n} and all the nonlinear parameter monomials in $C_{p,q}\otimes C_{p_1,q_1}\otimes C_{p_2,q_2}\otimes\cdots\otimes C_{p_k,q_k}$ for $0\leq k\leq n-2$, where the subscripts satisfy

$$\left.\begin{array}{c} p+q+\sum_{i=1}^{k}(p_i+q_i)=n+k \\ 1\leq p\leq n-k,\ 2\leq p+q\leq n-k,\ 2\leq p_i+q_i\leq n-k \end{array}\right\} \tag{15}$$

From Lemma 1, an element in $CE(H_n(j\omega_1,\cdots,j\omega_n))$ is either a single parameter coming from pure input nonlinearity such as $c_{0n}(.)$, or a nonlinear parameter monomial function of the form $C_{p,q}\otimes C_{p_1,q_1}\otimes C_{p_2,q_2}\otimes\cdots\otimes C_{p_k,q_k}$ satisfying (15), and the first parameter of $C_{p,q}\otimes C_{p_1,q_1}\otimes C_{p_2,q_2}\otimes\cdots\otimes C_{p_k,q_k}$ must come from pure output nonlinearity or input-output cross nonlinearity, *i.e.*, $c_{pq}(.)$ with $p\geq 1$ and $p+q>1$. For this reason, the following definition is given.

Definition 1. A parameter monomial of the form $C_{p,q}\otimes C_{p_1,q_1}\otimes C_{p_2,q_2}\otimes\cdots\otimes C_{p_k,q_k}$ with $k\geq 0$ and $p+q>1$ is said to be effective or an effective combination of the involved nonlinear parameters for $CE(H_n(j\omega_1,\cdots,j\omega_n))$ if $p+q=n(>1)$ for k=0, or (15) is satisfied for k>0. □

From Definition 1, it is obvious that all the monomials in $CE\left(H_n(j\omega_1,\cdots,j\omega_n)\right)$ are effective combinations. The following lemma shows further that what an effective monomial should be for certain order GFRF and how it is generated in the GFRF.

Lemma 2. For a monomial $c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ with k>0, the following statements hold:

(1) it is effective for the Z^{th}-order GFRF if and only if there is at least one parameter $c_{p_iq_i}(.)$ with p_i>0, where $Z=\sum_{i=0}^{k}(p_i+q_i)-k$.

(2) if there are l different parameters with p_i>0, then there are l different cases in which this monomial is produced by the recursive computation of the Z^{th}-order GFRF.

Proof. (1) This is directly from Definition 1. Z can be computed according to Lemma 1, *i.e.*, $p_0+q_0+\sum_{i=1}^{k}(p_i+q_i)=Z+k$, which yields $Z=\sum_{i=0}^{k}(p_i+q_i)-k$. (2) From the second and third terms in the recursive algorithm of Equation (9), *i.e.*,

$$\begin{aligned}&\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}c_{p,q}(k_1,\cdots,k_{p+q})(\prod_{i=1}^{q}(j\omega_{n-q+i})^{k_{p+i}})H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})\\&+\sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K}c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n)\end{aligned}\tag{16}$$

it can be seen that all the nonlinear parameters with p>0 and $p+q\le n$ are involved in the nth-order GFRF, and each of these parameters must correspond to the initial parameter in an effective monomial of $CE\left(H_n(j\omega_1,\cdots,j\omega_n)\right)$. Hence, if there are l different parameters with p_i>0 in the monomial $c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$, then there will be l different cases in which this monomial is produced in the Zth order GFRF. This completes the proof. □

Definition 2. A (p,q)-partition of $H_n(j\omega_1,\cdots,j\omega_n)$ is a combination $H_{r_1}(w_{r_1})H_{r_2}(w_{r_2})\cdots H_{r_p}(w_{r_p})$ satisfying $\sum_{i=1}^{p}r_i=n-q$, where $1\le r_i\le n-p-q+1$, and w_{r_i} is a set consisting of r_i different frequency variables such that $\bigcup_{i=1}^{p}w_{r_i}=\{\omega_1,\omega_2,\cdots,\omega_n\}$ and $w_{r_i}\cap w_{r_j}=\phi$ for $i\neq j$. □

For example, $H_1(\omega_1)H_1(\omega_2)H_3(\omega_3\cdots\omega_5)$ and $H_1(\omega_1)H_2(\omega_2,\omega_3)H_2(\omega_4,\omega_5)$ are two (3,0)-partitions of $H_5(j\omega_1,\cdots,j\omega_5)$.

Definition 3. A p-partition of an effective monomial $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ is a combination $s_{x_1}s_{x_2}\cdots s_{x_p}$, where s_{x_i} is a monomial of x_i parameters in $\{c_{p_1,q_1}(\cdot),\cdots,c_{p_k,q_k}(\cdot)\}$, $0\le x_i\le k$, s_0=1, and each non-unitary s_{x_i} is an effective monomial satisfying $\sum_{i=1}^{p}x_i=k$ and $s_{x_1}s_{x_2}\cdots s_{x_p}=c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$. □

The sub-monomial s_{x_i} in a p-partition of an effective monomial $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ is denoted by $s_{x_i}(c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot))$. Suppose that a p-partition for 1 is still 1, i.e., $\underbrace{1\cdot 1\cdots 1}_{p}=1$. Obviously $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot) = s_{x_1}s_{x_2}\cdots s_{x_p}(c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)) = s_k(c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot))$. For example, $s_1(c_{1,1}(\cdot))s_2(c_{2,1}(\cdot)c_{3,0}(\cdot))$ and $s_2(c_{1,1}(\cdot)c_{2,1}(\cdot))s_1(c_{3,0}(\cdot))$ are two 2-partitions of $c_{1,1}(\cdot)c_{2,1}(\cdot)c_{3,0}(\cdot)$. Moreover, note that when s_0 appear in a p-partition of a monomial, it means that there is a $H_1(.)$ appearing the corresponding (p,q)-partition for $H_n(.)$.

For an effective monomial $c_{p,q}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ in $CE(H_n(j\omega_1,\cdots,j\omega_n))$, without speciality, suppose the first parameter $c_{p,q}(\cdot)$ is directly generated in the recursive computation of $H_n(j\omega_1,\cdots,j\omega_n)$, then the other parameters must be generated from the lower order GFRFs that are involved in the recursive computation of $H_n(j\omega_1,\cdots,j\omega_n)$. From Equations (4-7)) it can be seen that each parameter in a monomial corresponds to a certain order GFRF from which it is generated. The following lemma shows how a monomial is generated in $H_n(j\omega_1,\cdots,j\omega_n)$ by using the new concepts defined above. This provides an important insight into the mapping from a monomial to its correlative function.

Lemma 3. If a monomial $c_{p,q}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ is effective, and $c_{p,q}(\cdot)$ is the initial parameter directly generated in the xth-order GFRF and p>0, then

(1) $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ comes from (p,q)-partitions of the xth-order GFRF, where x= $p+q+\sum_{i=1}^{k}(p_i+q_i)-k$;

(2) if additionally s_0 is supposed to be generated from H_1(.), then each p-partition of $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ corresponds to a (p,q)-partition of the xth-order GFRF, and each (p,q)-partition of the xth-order GFRF produces at least one p-partition for $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$;

(3) the correlative function of $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ is the summation of the correlative functions from all the (p,q)-partitions of the xth-order GFRF which produces $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$, and therefore is the summation of the correlative functions corresponding to all the p-partition of $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$. □

Remark 1. From Lemma 3, it can be seen that all the (p,q)-partitions of the xth-order GFRF which produce $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ are all the (p,q)-partitions corresponding to all the p-partitions for $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$. Therefore, to obtain all the (p,q)-partitions of interest, all the p-partitions for $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ is needed to be determined. □

Based on the results above, in order to determine the mapping between a parameter monomial $c_{p,q}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ and its correlative function in $f_n(j\omega_1,\cdots,j\omega_n)$, the following operator is defined.

Definition 4. Let $S_C(n)$ be a set composed of all the elements in $CE(H_n(j\omega_1,\cdots,j\omega_n))$, and let $S_f(n)$ be a set of the complex-valued functions of the frequency variables $j\omega_1,\cdots,j\omega_n$. Then define a mapping

$$\varphi_n : S_C(n) \to S_f(n) \tag{17a}$$

such that in $\omega_1,\cdots,\omega_n$

$$H_n^{sym}(j\omega_1,\cdots,j\omega_n) = \tfrac{1}{n!}\sum_{\substack{\text{all the permutations}\\ \text{of } \{1,2,\ldots,n\}}} CE(H_n(j\omega_1,\cdots,j\omega_n))\cdot\varphi_n(CE(H_n(j\omega_1,\cdots,j\omega_n))) \tag{17b}$$

□

The existence of this mapping function is obvious. For example, $\varphi_n(c_{0,n}(k_1,\cdots,k_n)) = (j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n}$. The task is to determine the complex valued correlative function $\varphi_n(c_{p,q}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot))$ for any nonlinear parameter monomial $c_{p,q}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ $(0\le k\le n\text{-}2)$ in $CE(H_n(j\omega_1,\cdots,j\omega_n))$.

Based on Lemma 2-3, the following result can be obtained.

Proposition 1. For an effective nonlinear parameter monomial $c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$, let $\bar{s} = c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$, $n(s_x(\bar{s})) = \sum_{i=1}^{x}(p_i+q_i) - x + 1$, where x is the number of the parameters in s_x, $\sum_{i=1}^{x}(p_i+q_i)$ is the summation of the subscripts of all the parameters in s_x, $\sum_{i=1}^{x}(.) = 0$ if x<1 and n(1)=1. Then for $0\le k\le n(\bar{s})$ -2

$$\begin{aligned}
&\varphi_{n(\bar{s})}(c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) \\
&= \sum_{\substack{\text{all the 2-partitions}\\ \text{for } \bar{s} \text{ satisfying}\\ s_1(\bar{s})=c_{p,q}(\cdot) \text{ and } p>0}} \Big\{ f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\cdot \sum_{\substack{\text{all the p-partitions}\\ \text{for } \bar{s}/c_{pq}(\cdot)}} \sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } \{s_{x_1},\cdots,s_{x_p}\}}} \Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{p,q}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)}) \\
&\qquad \cdot\prod_{i=1}^{p}\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot)))}(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot))))}) \Big] \Big\}
\end{aligned} \tag{18a}$$

or simplified as

$$\begin{aligned}
&\varphi_{n(\bar{s})}(c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) \\
&= \sum_{\substack{\text{all the 2-partitions}\\ \text{for } \bar{s} \text{ satisfying}\\ s_1(\bar{s})=c_{p,q}(\cdot) \text{ and } p>0}} \Big\{ f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\cdot \sum_{\substack{\text{all the p-partitions}\\ \text{for } \bar{s}/c_{p,q}(\cdot)}} \Big[f_{2b}(s_{x_1}\cdots s_{x_p}(\bar{s}/c_{p,q}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)}) \\
&\qquad \cdot\prod_{i=1}^{p}\varphi_{n(s_{x_i}(\bar{s}/c_{p,q}(\cdot)))}(s_{x_i}(\bar{s}/c_{p,q}(\cdot));\omega_{l(X(i)+1)}\cdots\omega_{l(X(i)+n(s_{x_i}(\bar{s}/c_{p,q}(\cdot))))}) \Big] \Big\}
\end{aligned} \tag{18b}$$

the terminating condition is k=0 and $\varphi_1(1;\omega_i) = H_1(j\omega_i)$, where,

$$\bar{X}(i) = \sum_{j=1}^{i-1} n(s_{\bar{x}_j}(\bar{s}/c_{pq}(\cdot))) \text{ or } X(i) = \sum_{j=1}^{i-1} n(s_{x_j}(\bar{s}/c_{pq}(\cdot))) \tag{19a}$$

$$f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) = \Big(\prod_{i=1}^{q}(j\omega_{l(n(\bar{s})-q+i)})^{k_{p+i}}\Big/ L_{n(\bar{s})}(j\sum_{i=1}^{n(\bar{s})}\omega_{l(i)}) \tag{19b}$$

$$f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{p,q}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)})=\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot))))})^{k_i} \quad (19c)$$

$$f_{2b}(s_{x_1}\cdots s_{x_p}(\bar{s}/c_{p,q}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)})=\frac{n_x^*}{n_k^*}\sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of }\{k_1,\cdots,k_p\}}}\prod_{i=1}^{p}(j\omega_{l(X(i)+1)}+\cdots+j\omega_{l(X(i)+n(s_{x_i}(\bar{s}/c_{pq}(\cdot))))})^{k_i} \quad (19d)$$

Moreover, $\{s_{\bar{x}_1},\cdots s_{\bar{x}_p}\}$ is a permutation of $\{s_{x_1},\cdots s_{x_p}\}$, $\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}$ represents the frequency variables involved in the corresponding functions, $l(i)$ for i=1… $n(\bar{s})$ is a positive integer representing the index of the frequency variables, $n_k^*=\dfrac{p!}{n_1!n_2!\cdots n_c!}$, $n_1+\ldots+n_c=p$, c is the number of distinct differentials k_i appearing in the combination, n_i is the number of repetitions of the ith distinct differential k_i, and a similar definition holds for n_x^*.□

Remark 2. Equation (18) is recursive. The terminating condition is k=0, which is also included in (18). For k=0, it can be derived from (18b) that

$$\begin{aligned}
&\varphi_{n(\bar{s})}(c_{p,q}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})=\varphi_{p+q}(c_{p,q}(\cdot);\omega_{l(1)}\cdots\omega_{l(p+q)})\\
&=f_1(c_{p,q}(\cdot),p+q;\omega_{l(1)}\cdots\omega_{l(p+q)})\\
&\quad\cdot\sum_{\substack{\text{all the } p\text{-partitions}\\ \text{for }1}}f_{2b}(s_{x_1}\cdots s_{x_p}(1);\omega_{l(1)}\cdots\omega_{l(p+q-q)})\prod_{i=1}^{p}\varphi_{n(s_{x_i}(1))}(s_{x_i}(1);\omega_{l(X(i)+1)}\cdots\omega_{l(X(i)+n(s_{x_i}(1)))})\\
&=f_1(c_{p,q}(\cdot),p+q;\omega_{l(1)}\cdots\omega_{l(p+q)})\cdot f_{2b}(\underbrace{11\cdots1}_{p};\omega_{l(1)}\cdots\omega_{l(p)})\cdot\prod_{i=1}^{p}\varphi_1(1;\omega_i)\\
&=\frac{1}{L_{p+q}(j\sum_{i=1}^{p+q}\omega_{l(i)})}(\prod_{i=1}^{q}(j\omega_{l(p+i)})^{k_{p+i}}\cdot\prod_{i=1}^{p}(j\omega_{l(i)})^{k_i}\cdot\prod_{i=1}^{p}H_1(j\omega_{l(i)}) \qquad (20)
\end{aligned}$$

Note that in this case, p+q= $n(\bar{s})$ from (15), and $\bar{s}=c_{p,q}(\cdot)$ corresponding to a specific recursive terminal. Hence, (20) can be written as

$$\varphi_{n(\bar{s})}(c_{p,q}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})=\frac{1}{L_{n(\bar{s})}(j\sum_{i=1}^{n(\bar{s})}\omega_{l(i)})}(\prod_{i=1}^{q}(j\omega_{l(p+i)})^{k_{p+i}}\cdot\prod_{i=1}^{p}(j\omega_{l(i)})^{k_i}\cdot\prod_{i=1}^{p}H_1(j\omega_{l(i)}) \quad (21)$$

In order to verify this result, let n=$n(\bar{s})$= p+q, it can be obtained from (12) that for a parameter $c_{p,q}(\cdot)$, its correlative function is

$$\frac{1}{L_{n(\bar{s})}(j\sum_{i=1}^{n(\bar{s})}\omega_i)}\prod_{i=1}^{q}(j\omega_{p+i})^{k_{p+i}})H_{p,p}(j\omega_1,\cdots,j\omega_p)$$

From (7), $H_{p,p}(j\omega_1,\cdots,j\omega_p)=\prod_{i=1}^{p}(j\omega_i)^{k_i}\cdot\prod_{i=1}^{p}H_1(j\omega_i)$. This is consistent with (21). To further understand the results in Proposition 1, the following figure can be referred, which demonstrates the recursive process in the new mapping function and the structure of the theoretical results above (See Figure 1). □

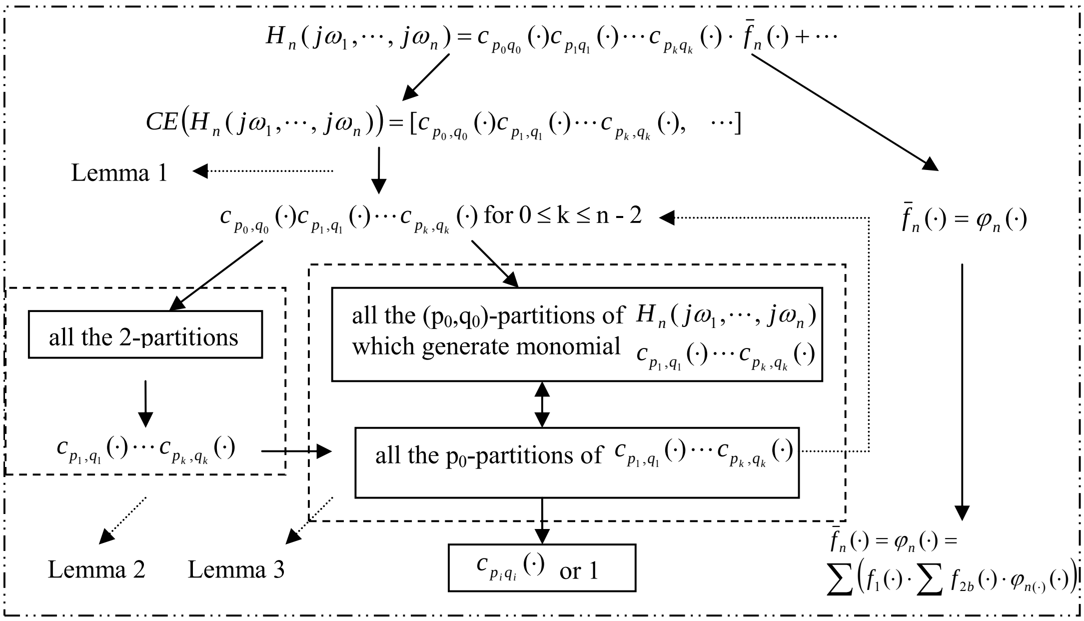

Figure 1. An illustration of the relationships in Proposition 1

To further demonstrate the result in Proposition 1, the following example is given.

Example 1. Consider the 4th-order GFRF. The parametric characteristic of the 4th-order GFRF can be obtained from Lemma 1 that

$$
\begin{aligned}
CE(H_4(j\omega_1,\cdots,j\omega_4)) = {} & \mathrm{C}_{0,4} \oplus \mathrm{C}_{1,3} \oplus \mathrm{C}_{3,1} \oplus \mathrm{C}_{2,2} \oplus \mathrm{C}_{4,0} \oplus \mathrm{C}_{1,1} \otimes \mathrm{C}_{0,3} \oplus \mathrm{C}_{1,1} \otimes \mathrm{C}_{1,2} \\
& \oplus \mathrm{C}_{1,1} \otimes \mathrm{C}_{2,1} \oplus \mathrm{C}_{1,1} \otimes \mathrm{C}_{3,0} \oplus \mathrm{C}_{1,2} \otimes \mathrm{C}_{0,2} \oplus \mathrm{C}_{1,2} \otimes \mathrm{C}_{2,0} \oplus \mathrm{C}_{2,0} \otimes \mathrm{C}_{0,3} \\
& \oplus \mathrm{C}_{2,0} \otimes \mathrm{C}_{2,1} \oplus \mathrm{C}_{2,0} \otimes \mathrm{C}_{3,0} \oplus \mathrm{C}_{2,1} \otimes \mathrm{C}_{0,2} \oplus \mathrm{C}_{3,0} \otimes \mathrm{C}_{0,2} \\
& \oplus \mathrm{C}_{1,1} \otimes \mathrm{C}_{0,2}{}^{2} \oplus \mathrm{C}_{1,1}{}^{2} \otimes \mathrm{C}_{0,2} \oplus \mathrm{C}_{1,1} \otimes \mathrm{C}_{0,2} \otimes \mathrm{C}_{2,0} \oplus \mathrm{C}_{1,1}{}^{3} \oplus \mathrm{C}_{1,1}{}^{2} \otimes \mathrm{C}_{2,0} \\
& \oplus \mathrm{C}_{1,1} \otimes \mathrm{C}_{2,0}{}^{2} \oplus \mathrm{C}_{2,0} \otimes \mathrm{C}_{0,2}{}^{2} \oplus \mathrm{C}_{2,0}{}^{2} \otimes \mathrm{C}_{0,2} \oplus \mathrm{C}_{2,0}{}^{3}
\end{aligned}
$$

By using Proposition 1, the correlative function of each term in $CE(H_4(j\omega_1,\cdots,j\omega_4))$ can all be obtained. For example, for the term $c_{1,1}(.)c_{0,2}(.)c_{2,0}(.)$, it can be derived that

$$
\begin{aligned}
& \varphi_{n(\bar{s})}(c_{1,1}(\cdot)c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) = \varphi_4(c_{1,1}(\cdot)c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_4) \\
& = f_1(c_{1,1}(\cdot),4;\omega_1\cdots\omega_4) \\
& \cdot\left[f_{2b}(s_2(c_{0,2}(\cdot)c_{2,0}(\cdot));\omega_1\cdots\omega_3)\cdot\varphi_{n(s_2(c_{0,2}(\cdot)c_{2,0}(\cdot)))}(s_2(c_{0,2}(\cdot)c_{2,0}(\cdot));\omega_{X(1)+1}\cdots\omega_{X(1)+n(s_2(c_{0,2}(\cdot)c_{2,0}(\cdot)))})\right] \\
& + f_1(c_{2,0}(\cdot),4;\omega_1\cdots\omega_4) \\
& \quad \cdot\left[f_{2b}(s_0s_2(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_{n(s_0(c_{1,1}(\cdot)c_{0,2}(\cdot)))}(s_0(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_{X(1)+1}\cdots\omega_{X(1)+n(s_0(c_{1,1}(\cdot)c_{0,2}(\cdot)))})\right. \\
& \qquad \cdot\varphi_{n(s_2(c_{1,1}(\cdot)c_{0,2}(\cdot)))}(s_2(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_{X(2)+1}\cdots\omega_{X(2)+n(s_2(c_{1,1}(\cdot)c_{0,2}(\cdot)))}) \\
& \quad + f_{2b}(s_1s_1(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_{n(s_1(c_{1,1}(\cdot)))}(s_1(c_{1,1}(\cdot));\omega_{X(1)+1}\cdots\omega_{X(1)+n(s_1(c_{1,1}(\cdot)))}) \\
& \qquad \left.\cdot\varphi_{n(s_1(c_{0,2}(\cdot)))}(s_1(c_{0,2}(\cdot));\omega_{X(2)+1}\cdots\omega_{X(2)+n(s_1(c_{0,2}(\cdot)))})\right]
\end{aligned}
$$

$$
\begin{aligned}
&= f_1(c_{1,1}(\cdot),4;\omega_1\cdots\omega_4) \\
&\cdot\left[f_{2b}(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_3)\cdot\varphi_{n(c_{0,2}(\cdot)c_{2,0}(\cdot))}(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_{0+1}\cdots\omega_{0+n(c_{0,2}(\cdot)c_{2,0}(\cdot))})\right] \\
&+ f_1(c_{2,0}(\cdot),4;\omega_1\cdots\omega_4) \\
&\cdot\left[f_{2b}(s_0s_2(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_{n(1)}(1;\omega_1\cdots\omega_{n(1)})\varphi_{n(c_{1,1}(\cdot)c_{0,2}(\cdot))}(c_{1,1}(\cdot)c_{0,2}(\cdot);\omega_{n(1)+1}\cdots\omega_{n(1)+n(c_{1,1}(\cdot)c_{0,2}(\cdot))})\right. \\
&+ f_{2b}(s_1s_1(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_2(c_{1,1}(\cdot);\omega_{X(1)+1}\cdots\omega_{X(1)+n(s_1(c_{1,1}(\cdot)))}) \\
&\qquad\left.\cdot\varphi_2(c_{0,2}(\cdot);\omega_{X(2)+1}\cdots\omega_{X(2)+n(s_1(c_{0,2}(\cdot)))})\right] \\
&= f_1(c_{1,1}(\cdot),4;\omega_1\cdots\omega_4)\cdot\left[f_{2b}(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_3)\cdot\varphi_3(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_3)\right] \\
&+ f_1(c_{2,0}(\cdot),4;\omega_1\cdots\omega_4)\cdot\left[f_{2b}(s_0s_2(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_1(1;\omega_1)\varphi_3(c_{1,1}(\cdot)c_{0,2}(\cdot);\omega_2\cdots\omega_4)\right. \\
&\qquad\qquad\left.+ f_{2b}(s_1s_1(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_2(c_{1,1}(\cdot);\omega_1,\omega_2)\cdot\varphi_2(c_{0,2}(\cdot);\omega_3,\omega_4)\right] \qquad (22)
\end{aligned}
$$

To proceed with the recursive computation, it can be derived that

$$f_1(c_{1,1}(\cdot),4;\omega_1\cdots\omega_4) = \prod_{i=1}^{1}(j\omega_{3+i})^{k_{1+i}}\Big/ L_4(j\sum_{i=1}^{4}\omega_i) = (j\omega_4)^{k_2}\Big/ L_4(j\sum_{i=1}^{4}\omega_i) \qquad (23a)$$

$$f_1(c_{2,0}(\cdot),4;\omega_1\cdots\omega_4) = 1\Big/ L_4(j\sum_{i=1}^{4}\omega_i) \qquad (23b)$$

$$f_{2b}(s_{x_1}(c_{2,0}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_3) = (j\omega_1+\cdots+j\omega_3)^{k_1} \qquad (23c)$$

$$
\begin{aligned}
f_{2b}(s_0s_2(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4) &= \sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } \{k_1,\cdots,k_p\}}}\prod_{i=1}^{2}(j\omega_{X(i)+1}+\cdots+j\omega_{X(i)+n(s_{x_i}(\bar{s}/c_{pq}(\cdot)))})^{k_i} \qquad (23d) \\
&= (j\omega_1)^{k_1}(j\omega_2+\cdots+j\omega_4)^{k_2} + (j\omega_2+\cdots+j\omega_4)^{k_1}(j\omega_1)^{k_2}
\end{aligned}
$$

$$
\begin{aligned}
&\varphi_3(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_3) \\
&= f_1(c_{2,0}(\cdot),3;\omega_1\cdots\omega_3)\cdot f_{2b}(s_{x_1}s_{x_2}(c_{0,2}(\cdot));\omega_1\cdots\omega_3)\prod_{i=1}^{2}\varphi_{n(s_{x_i}(\bar{s}/c_{pq}(\cdot)))}(s_{x_i}(c_{0,2}(\cdot));\omega_{X(i)+1}\cdots\omega_{X(i)+n(s_{x_i}(c_{0,2}(\cdot)))}) \\
&= f_1(c_{2,0}(\cdot),3;\omega_1\cdots\omega_3)\cdot f_{2b}(s_{x_1}s_{x_2}(c_{0,2}(\cdot));\omega_1\cdots\omega_3)\varphi_1(1;\omega_1)\varphi_2(c_{0,2}(\cdot);\omega_2,\omega_3) \\
&= \frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\left((j\omega_1)^{k_1}(j\omega_2+j\omega_3)^{k_2}+(j\omega_3+j\omega_2)^{k_1}(j\omega_1)^{k_2}\right)\cdot H_1(j\omega_1) \\
&\qquad\cdot\frac{1}{L_2(j\omega_2+j\omega_3)}(j\omega_2)^{k_1}(j\omega_3)^{k_2} \qquad (23e)
\end{aligned}
$$

$$
\begin{aligned}
&\varphi_3(c_{1,1}(\cdot)c_{0,2}(\cdot);\omega_2\cdots\omega_4) \\
&= f_1(c_{1,1}(\cdot),3;\omega_2\cdots\omega_4)\cdot f_{2b}(s_{x_1}(c_{0,2}(\cdot));\omega_2,\omega_3)\cdot\varphi_{n(s_{x_1}(c_{0,2}(\cdot)))}(s_{x_1}(c_{0,2}(\cdot));\omega_2,\omega_3) \\
&= f_1(c_{1,1}(\cdot),3;\omega_2\cdots\omega_4)\cdot f_{2b}(c_{0,2}(\cdot);\omega_2,\omega_3)\cdot\varphi_2(c_{0,2}(\cdot);\omega_2,\omega_3) \qquad (23f) \\
&= \frac{(j\omega_4)^{k_2}}{L_3(j\omega_2+\cdots+j\omega_4)}\cdot(j\omega_2+j\omega_3)^{k_1}\cdot\frac{1}{L_2(j\omega_2+j\omega_3)}(j\omega_2)^{k_1}(j\omega_3)^{k_2}
\end{aligned}
$$

Using equations (23a-f) in (22) yields

$$\begin{aligned}
&\varphi_4(c_{1,1}(\cdot)c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_4)\\
&= f_1(c_{1,1}(\cdot),4;\omega_1\cdots\omega_4)\cdot\left[f_{2b}(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_3)\cdot\varphi_3(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_3)\right]\\
&+ f_1(c_{2,0}(\cdot),4;\omega_1\cdots\omega_4)\cdot\left[f_{2b}(s_0s_2(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_1(1;\omega_1)\varphi_3(c_{1,1}(\cdot)c_{0,2}(\cdot);\omega_2\cdots\omega_4)\right.\\
&\qquad\left.+ f_{2b}(s_1s_1(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_2(c_{1,1}(\cdot);\omega_1,\omega_2)\cdot\varphi_2(c_{0,2}(\cdot);\omega_3,\omega_4)\right]\\
&= \frac{(j\omega_4)^{k_2}(j\omega_1+\cdots+j\omega_3)^{k_1}\left((j\omega_1)^{k_1}(j\omega_2+j\omega_3)^{k_2}+(j\omega_3+j\omega_2)^{k_1}(j\omega_1)^{k_2}\right)(j\omega_2)^{k_1}(j\omega_3)^{k_2}}{L_4(j\omega_1+\cdots+j\omega_4)L_3(j\omega_1+j\omega_2+j\omega_3)L_2(j\omega_2+j\omega_3)}\cdot H_1(j\omega_1)\\
&+ \frac{\left((j\omega_1)^{k_1}(j\omega_2+\cdots+j\omega_4)^{k_2}+(j\omega_2+\cdots+j\omega_4)^{k_1}(j\omega_1)^{k_2}\right)(j\omega_4)^{k_2}(j\omega_2+j\omega_3)^{k_1}(j\omega_2)^{k_1}(j\omega_3)^{k_2}}{L_4(j\omega_1+\cdots+j\omega_4)L_3(j\omega_2+\cdots+j\omega_4)L_2(j\omega_2+j\omega_3)}H_1(j\omega_1)\\
&+ \frac{\left((j\omega_1+j\omega_2)^{k_1}(j\omega_3+j\omega_4)^{k_2}+(j\omega_3+j\omega_4)^{k_1}(j\omega_1+j\omega_2)^{k_2}\right)(j\omega_4)^{k_2}(j\omega_3)^{k_1}(j\omega_1)^{k_1}(j\omega_2)^{k_2}}{L_4(j\omega_1+\cdots+j\omega_4)L_2(j\omega_3+j\omega_4)L_2(j\omega_2+j\omega_1)}H_1(j\omega_1)
\end{aligned} \tag{24}$$

Therefore, the correlative function of the parameter monomial $c_{1,1}(\cdot)c_{0,2}(\cdot)c_{2,0}(\cdot)$ is obtained. It can be verified that the same result can be obtained by using the recursive algorithm in (12, 5-7, 11). For the sake of brevity, this is omitted. By following the same method, the whole correlative function vector $\varphi_4(CE(H_4(j\omega_1,\cdots,j\omega_4)))$ can be determined. Thus the 4$^{\text{th}}$-order GFRF $H_4(j\omega_1,\cdots,j\omega_4)$ can directly be written into a parametric characteristic form which can provide a straightforward and meaningful insight into the relationship between $H_4(j\omega_1,\cdots,j\omega_4)$ and nonlinear parameters, and also between $H_4(j\omega_1,\cdots,j\omega_4)$ and $H_1(j\omega_1)$. □

Remark 3. From Example 1, it can be seen that Proposition 1 provides an effective method to determine the correlative function for an effective monomial $c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$, and the computation process should be able to be carried out automatically without manual intervention. Therefore, Proposition 1 provides a simplified method to determine the *n*th-order GFRF directly into a more meaningful form as (14) which can demonstrate the parametric characteristic clearly and describe the *n*th-order GFRF in terms of the first order GFRF $H_1(j\omega)$ and nonlinear parameters without crossing effect with the lower order GFRFs. This reveals a more straightforward insight into the relationships between $H_n(j\omega_1,\cdots,j\omega_n)$ and nonlinear parameters, and between $H_n(j\omega_1,\cdots,j\omega_n)$ and $H_1(j\omega)$. Note that the high order GFRFs can represent system frequency response characteristics (Peyton Jones and Billings 1990, Yue et al 2005) and $H_1(j\omega)$ represents the linear part of the system model. Hence, the results in Proposition 1 not only facilitate the analysis of the connection between system frequency response characteristics and model linear and nonlinear parameters, but also provide a new perspective on the understanding of the GFRFs and on the analysis of nonlinear systems based on the GFRFs. □

4 Some new properties

Based on the mapping function φ_n established in the last section, some new properties of the *n*th-order GFRF are discussed in this section.

4.1 Determination of FRFs based on parametric characteristics

There are several relationships involved in this paper. $H_n(j\omega_1,\cdots,j\omega_n)$ is determined from the NDE model in terms of the model parameters. Thus there is a bijective mapping between $H_n(j\omega_1,\cdots,j\omega_n)$ and the NDE model. The CE operator is a mapping from $H_n(j\omega_1,\cdots,j\omega_n)$ to its parametric characteristic, which can also be regarded as a mapping from the nonlinear parameters of the NDE model to the parametric characteristics of $H_n(j\omega_1,\cdots,j\omega_n)$. The function φ_n can be regarded as an inverse mapping of the CE operator such that the *n*th-order GFRF can be reconstructed from its parametric characteristic, which can also be regarded as a mapping from the nonlinear parameters of the NDE model to $H_n(j\omega_1,\cdots,j\omega_n)$. This can refer to Figure 2, where "•" represents the point multiplication between the parametric monomial and its correlative function.

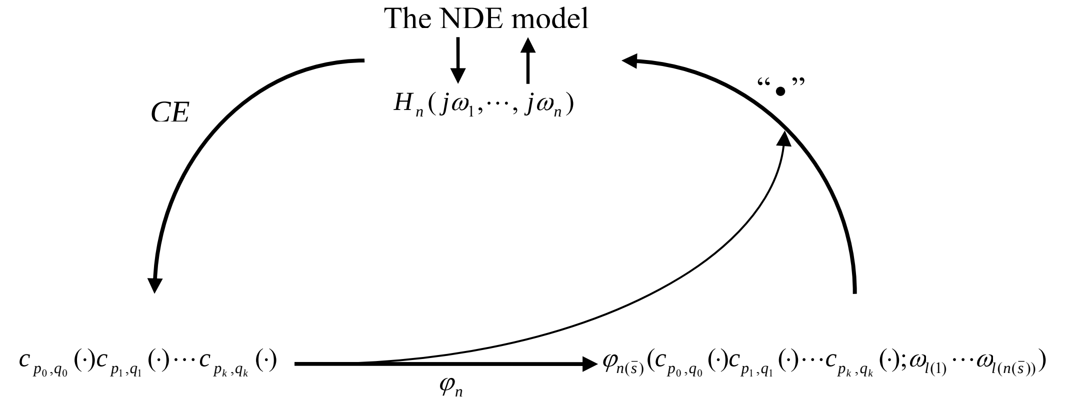

Figure 2. Relationship between φ_n and *CE*

It can be seen from Figure 2 that

$$H_n(j\omega_1,\cdots,j\omega_n) = CE(H_n(\cdot))\cdot\varphi_n(CE(H_n(\cdot))) \tag{25}$$

From (25), the inverse of the operator *CE* can simply be written as (x=$CE(H_n(\cdot))$)

$$CE^{-1}(x) = x\cdot\varphi_n(x)$$

which constructs a mapping directly from the parametric characteristic of the *n*th-order GFRF to the *n*th-order GFRF itself. Note that $CE(H_n(\cdot))$ includes all the nonlinear parameters of degree from 2 to *n* of the nonlinear system of interest, and $\varphi_n(CE(H_n(\cdot)))$ is a complex valued function vector including the effect of the complicated nonlinear behaviour and also the effect of the linear part of the nonlinear system. Hence, Equation (25) reveals a new perspective on the computation and understanding of the GFRFs as discussed in Section 3, and also provides a new insight into the frequency domain analysis of nonlinear systems based on the GFRFs.

From the results in Jing et al (2006), the output spectrum for system (1) can now be determined as

$$Y(j\omega) = \sum_{n=1}^{N} CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)\cdot\hat{F}_n(j\omega) \tag{26a}$$

when the input is a general input $U(j\omega)$,

$$\hat{F}_n(j\omega)=\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}\varphi_n(CE(H_n(j\omega_1,\cdots,j\omega_n)))\cdot\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \tag{26b}$$

when the input is a multi-tone function $u(t)=\sum_{i=1}^{K}|F_i|\cos(\omega_i t+\angle F_i)$,

$$\hat{F}_n(j\omega)=\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}\varphi_n(CE(H_n(j\omega_{k_1},\cdots,j\omega_{k_n})))\cdot F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{26c}$$

It is obvious that Equation (26a) is an explicitly analytical polynomial functions with coefficients in $S_C(1)\cup\cdots\cup S_C(N)$ and the corresponding correlative functions in $S_f(1)\cup\cdots\cup S_f(N)$. This demonstrates a direct analytical relationship between system output spectrum and system time-domain model parameters. The effects on system output spectrum from the linear parameters are included in $S_f(1)\cup\cdots\cup S_f(N)$, and the effects from the nonlinear parameters are included in $S_C(1)\cup\cdots\cup S_C(N)$ and also embodied in $S_f(1)\cup\cdots\cup S_f(N)$. This will facilitate the analysis of output frequency response characteristics of nonlinear systems. For example, for any interested parameters of model (1), which may represent some specific physical characteristics, the output spectrum can therefore directly be written as a polynomial in terms of these parameters. Then how these parameters affect the system output spectrum need only be investigated by studying the frequency characteristics of the new mapping functions involved in the polynomial and simultaneously optimizing the values of these nonlinear parameters. Further study in this topic will be introduced in another publication.

4.2 Magnitude of the *n*th-order GFRF

Based on Equation (25), magnitude of the *n*th-order GFRF can be expressed into a novel form in terms of its parametric characteristic.

Corollary 1. Let $CE_n=CE(H_n(\cdot))$, $\Theta_n=\varphi_n(CE(H_n(\cdot)))\cdot\varphi_n(CE(H_n(\cdot)))^*$, $\varphi_n=\varphi_n(CE(H_n(\cdot)))$, and $\Lambda_n=CE(H_n(\cdot))^T CE(H_n(\cdot))$, then

$$|H_n(j\omega_1,\cdots,j\omega_n)|^2=CE_n\Theta_n CE_n^T \tag{27a}$$

$$|H_n(j\omega_1,\cdots,j\omega_n)|^2=\varphi_n^*\Lambda_n\varphi_n \tag{27b}$$

Proof. It can be derived from (25) that

$$\begin{aligned}|H_n(j\omega_1,\cdots,j\omega_n)|^2&=H_n(j\omega_1,\cdots,j\omega_n)\cdot H_n^*(j\omega_1,\cdots,j\omega_n)\\&=CE(H_n(\cdot))\cdot\varphi_n(CE(H_n(\cdot)))\cdot(CE(H_n(\cdot))\cdot\varphi_n(CE(H_n(\cdot))))^*\\&=CE(H_n(\cdot))\cdot\left(\varphi_n(CE(H_n(\cdot)))\cdot\varphi_n(CE(H_n(\cdot)))^*\right)\cdot CE(H_n(\cdot))^T=CE_n\Theta_n CE_n^T\end{aligned}$$

The result in equation (27b) can also be achieved by following the same method. This completes the proof. □

From Corollary 1, the square of the magnitude of the *n*th-order GFRF is proportional to a quadratic function of the parametric characteristic and also proportional to a quadratic function of the corresponding correlative function. Corollary 1 provides a new property of the *n*th-order GFRF, which reveals the relationship between the magnitude of $H_n(j\omega_1,\cdots,j\omega_n)$ and its nonlinear parametric characteristic, and also the relationship between the magnitude of $H_n(j\omega_1,\cdots,j\omega_n)$ and the correlative functions which include the linear and nonlinear behavior. Given a requirement on $|H_n(j\omega_1,\cdots,j\omega_n)|$, the condition on model parameters can be derived by using equations (27ab). This may provide a new technique for the analysis and design of nonlinear systems based on the *n*th- order GFRF in the frequency domain.

Moreover, it can be seen that the frequency characteristic matrix Θ_n is a Hermitian matrix, whose eigenvalues are positive real valued functions of linear parameters but invariant to the values of the nonlinear parameters in $CE(H_n(\cdot))$. Thus different nonlinearities may result in different frequency characteristic matrix Θ_n, but the same nonlinearities will have an invariant matrix Θ_n. This property of the nth-order GFRF provides a new insight into the nonlinear effect on the high order GFRFs from different nonlinearities. For this purpose, define a new function

$$\bar{\lambda}_n(\omega_1,\cdots,\omega_n)=\lambda_{\max}(\Theta_n) \tag{28}$$

which is the maximum eigenvalue of the frequency characteristic matrix Θ_n. As mentioned, the frequency spectrum of this function can act as a novel insight into the nonlinear effect on the GFRFs from different nonlinearities, since this function is only dependent on different nonlinearities but independent of their values. However, the frequency response spectrum of the GFRFs will change greatly with different values of the involved nonlinear parameters, which can not provide a clear insight into the nonlinear effects between different nonlinearities.

4.3 Relationship between $H_n(j\omega_1,\cdots,j\omega_n)$ and $H_1(j\omega_1)$

As illustrated in Example 1, $H_n(j\omega_1,\cdots,j\omega_n)$ can directly be determined in terms of the first order GFRF $H_1(j\omega)$ based on the novel mapping function φ_n according to its parametric characteristic. The following results can be concluded.

Corollary 2. For an effective parametric monomial $c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$, its correlative function is a ρ-degree function of $H_1(j\omega_{l(1)})$ which can be written as a symmetric form

$$\varphi_{n(\bar{s})}(c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})$$
$$=\frac{(n(\bar{s})-\rho)!\rho!}{n(\bar{s})!}\sum_{\substack{\text{all the combinations of } \rho \text{ integers } \{r_1,r_2,\cdots,r_\rho\} \\ \text{taken from } \{1,2,\cdots,n(\bar{s})\} \text{ without repetition} \\ j \text{ is for different combination}}}\mu_j(\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\prod_{i=1}^{\rho}H_1(j\omega_{\bar{l}(i)})$$

where $\rho=n(\bar{s})-\sum_{i=0}^{k}q_i=\sum_{i=0}^{k}p_i-k$, $\bar{l}=[r_1,r_2,\cdots,r_\rho]$, and $\mu_j(\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})$ can be determined by equations (18-19). Therefore, the *n*th-order GFRF can be regarded as an *n*-degree polynomial function of $H_1(j\omega_{l(1)})$. □

The proof is omitted.

Corollary 2 demonstrates the relationship between $H_n(j\omega_1,\cdots,j\omega_n)$ and $H_1(j\omega)$, and reveals how the first order GFRF, which represents the linear part of system model, affects the higher order GFRFs, together with the nonlinear dynamics. Note that for any specific interested parameters, the polynomial structure of the FRFs is explicitly determined in terms of these parameters, thus the property of this polynomial function is greatly dependent on the "coefficients" of these parameter monomials in the polynomial, which correspond to the correlative functions of the parametric characteristics of the polynomial and are determined by the new mapping function. Hence, Corollary 2 is important for the qualitative analysis of the connection between $H_n(j\omega_1,\cdots,j\omega_n)$ and $H_1(j\omega)$, and also between nonlinear parameters and high order GFRFs .

Example 2. To demonstrate the theoretical results above, consider a simple mechanical system shown in Figure 2.

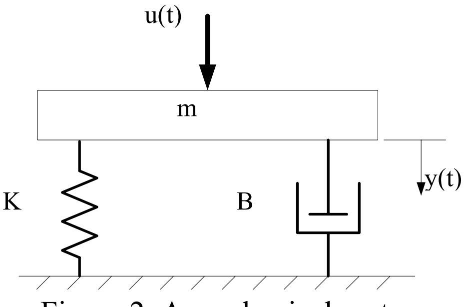

Figure 2. A mechanical system

The output property of the spring satisfies $F = Ky + c_1 y^3$, and the damper $F = B\dot{y} + c_2 \dot{y}^3$. u(t) is the external input force. The system dynamics can be described by

$$m\ddot{y} = -Ky - B\dot{y} - c_1 y^3 - c_2 \dot{y}^3 + u(t) \tag{29}$$

which can be written into the form of NDE model (1) with M=3, K=2, $c_{1,0}(2) = m$, $c_{1,0}(1) = B$, $c_{1,0}(0) = K$, $c_{3,0}(000) = c_1$, $c_{3,0}(111) = c_2$, $c_{0,1}(0) = -1$, and all the other parameters are zero.

There are two nonlinear terms $c_{3,0}(000) = c_1$ and $c_{3,0}(111) = c_2$ in model (29), which are all pure input nonlinearity and can be written as $C_{3,0} = [c_1, c_2]$. The parametric characteristics of the GFRFs of model (29) with respect to nonlinear parameter $C_{3,0}$ can be obtained according to equation (13) or Lemma 1 as

For i=0,1,2,…, $CE(H_{2i+1}(.))=C_{3,0}{}^{i}$, otherwise $CE(H_{2i}(.))=0$ for i=1,2,3,…

Therefore,

$CE(H_1(.))=1$;

$CE(H_3(.))=C_{3,0} = [c_1\ \ c_2]$;

$CE(H_5(.))=C_{3,0}\otimes C_{3,0} = [c_1^2\ \ c_1c_2\ \ c_2^2]$;

$CE(H_7(.))=C_{3,0}\otimes C_{3,0}\otimes C_{3,0}= [c_1^3\ \ c_1^2c_2\ \ c_1c_2^2\ \ c_2^3]$……

By using (18-21), it can be obtained that

$$\varphi_3(c_{3,0}(000);\omega_1,\omega_2,\omega_3) = \frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\prod_{i=1}^{3}(j\omega_i)^0\cdot\prod_{i=1}^{3}H_1(j\omega_i) = \frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\prod_{i=1}^{3}H_1(j\omega_i)$$

$$\varphi_3(c_{3,0}(111);\omega_1,\omega_2,\omega_3)=\frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\prod_{i=1}^{3}(j\omega_i)\cdot\prod_{i=1}^{3}H_1(j\omega_i)=\frac{\prod_{i=1}^{3}(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\prod_{i=1}^{3}H_1(j\omega_i)$$

$$\begin{aligned}
&\varphi_5(c_{3,0}(000)c_{3,0}(000);\omega_1,\cdots,\omega_5)\\
&=f_1(c_{3,0}(000),5;\omega_1,\cdots,\omega_5)\cdot\sum_{\substack{\text{all the 3-partitions}\\ \text{for } c_{3,0}(000)}}\ \sum_{\substack{\text{all the different}\\ \text{permutations of }\{0,0,1\}}}\Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(c_{3,0}(000));\omega_1\cdots\omega_5)\\
&\qquad\cdot\prod_{i=1}^{3}\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot)))}(s_{\bar{x}_i}(c_{3,0}(000));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot))))})\Big]\\
&=f_1(c_{3,0}(000),5;\omega_1,\cdots,\omega_5)\cdot\begin{pmatrix}f_{2a}(s_0s_0s_1(c_{3,0}(000));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_1(1;\omega_2)\varphi_3(c_{3,0}(000);\omega_3\cdots\omega_5)\\
+f_{2a}(s_0s_1s_0(c_{3,0}(000));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_3(c_{3,0}(000);\omega_2\cdots\omega_4)\varphi_1(1;\omega_5)\\
+f_{2a}(s_1s_0s_0(c_{3,0}(000));\omega_1\cdots\omega_5)\varphi_3(c_{3,0}(000);\omega_1\cdots\omega_3)\varphi_1(1;\omega_4)\varphi_1(1;\omega_5)\end{pmatrix}\\
&=\frac{1}{L_5(j\sum_{i=1}^{5}\omega_i)}\cdot\begin{pmatrix}H_1(\omega_1)H_1(\omega_2)\prod_{i=3}^{5}H_1(j\omega_i)\Big/L_3(j\sum_{i=3}^{5}\omega_i)\\
+H_1(\omega_1)\prod_{i=2}^{4}H_1(j\omega_i)H_1(\omega_5)\Big/L_3(j\sum_{i=2}^{4}\omega_i)\\
+\prod_{i=1}^{3}H_1(j\omega_i)H_1(\omega_4)H_1(\omega_5)\Big/L_3(j\sum_{i=1}^{3}\omega_i)\end{pmatrix}\\
&=\frac{1}{L_5(j\sum_{i=1}^{5}\omega_i)}\cdot\Big(\frac{1}{L_3(j\sum_{i=3}^{5}\omega_i)}+\frac{1}{L_3(j\sum_{i=2}^{4}\omega_i)}+\frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\Big)\cdot\prod_{i=1}^{5}H_1(j\omega_i)
\end{aligned}$$

$$\begin{aligned}
&\varphi_5(c_{3,0}(111)c_{3,0}(111);\omega_1,\cdots,\omega_5)\\
&=f_1(c_{3,0}(111),5;\omega_1,\cdots,\omega_5)\cdot\sum_{\substack{\text{all the 3-partitions}\\ \text{for } c_{3,0}(111)}}\ \sum_{\substack{\text{all the different}\\ \text{permutations of }\{0,0,1\}}}\Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(c_{3,0}(111));\omega_1\cdots\omega_5)\\
&\qquad\cdot\prod_{i=1}^{3}\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot)))}(s_{\bar{x}_i}(c_{3,0}(111));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot))))})\Big]\\
&=f_1(c_{3,0}(111),5;\omega_1,\cdots,\omega_5)\cdot\begin{pmatrix}f_{2a}(s_0s_0s_1(c_{3,0}(111));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_1(1;\omega_2)\varphi_3(c_{3,0}(111);\omega_3\cdots\omega_5)\\
+f_{2a}(s_0s_1s_0(c_{3,0}(111));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_3(c_{3,0}(111);\omega_2\cdots\omega_4)\varphi_1(1;\omega_5)\\
+f_{2a}(s_1s_0s_0(c_{30}(111));\omega_1\cdots\omega_5)\varphi_3(c_{30}(111);\omega_1\cdots\omega_3)\varphi_1(1;\omega_4)\varphi_1(1;\omega_5)\end{pmatrix}\\
&=\frac{1}{L_5(j\sum_{i=1}^{5}\omega_i)}\cdot\left(\frac{(j\sum_{i=3}^{5}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=3}^{5}\omega_i)}+\frac{(j\sum_{i=2}^{4}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=2}^{4}\omega_i)}+\frac{(j\sum_{i=1}^{3}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\right)\cdot\prod_{i=1}^{5}H_1(j\omega_i)
\end{aligned}$$

$$
\begin{aligned}
&\varphi_5(c_{3,0}(000)c_{3,0}(111);\omega_1,\cdots,\omega_5)\\
&= f_1(c_{3,0}(000),5;\omega_1,\cdots,\omega_5)\cdot \sum_{\substack{\text{all the 3-partitions}\\ \text{for } c_{3,0}(111)}} \sum_{\substack{\text{all the different}\\ \text{permutations of } \{0,0,1\}}} \Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(c_{3,0}(111));\omega_1\cdots\omega_5)\\
&\qquad\qquad \cdot\prod_{i=1}^{3}\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot)))}(s_{\bar{x}_i}(c_{3,0}(111));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot))))})\Big]\\
&+ f_1(c_{3,0}(111),5;\omega_1,\cdots,\omega_5)\cdot \sum_{\substack{\text{all the 3-partitions}\\ \text{for } c_{3,0}(000)}} \sum_{\substack{\text{all the different}\\ \text{permutations of } \{0,0,1\}}} \Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(c_{3,0}(000));\omega_1\cdots\omega_5)\\
&\qquad\qquad \cdot\prod_{i=1}^{3}\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot)))}(s_{\bar{x}_i}(c_{3,0}(000));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot))))})\Big]\\
&= f_1(c_{3,0}(000),5;\omega_1,\cdots,\omega_5)\cdot\left(\begin{array}{l}
f_{2a}(s_0s_0s_1(c_{3,0}(111));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_1(1;\omega_2)\varphi_3(c_{3,0}(111);\omega_3\cdots\omega_5)\\
+f_{2a}(s_0s_1s_0(c_{3,0}(111));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_3(c_{3,0}(111);\omega_2\cdots\omega_4)\varphi_1(1;\omega_5)\\
+f_{2a}(s_1s_0s_0(c_{3,0}(111));\omega_1\cdots\omega_5)\varphi_3(c_{3,0}(111);\omega_1\cdots\omega_3)\varphi_1(1;\omega_4)\varphi_1(1;\omega_5)
\end{array}\right)\\
&+ f_1(c_{3,0}(111),5;\omega_1,\cdots,\omega_5)\cdot\left(\begin{array}{l}
f_{2a}(s_0s_0s_1(c_{3,0}(000));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_1(1;\omega_2)\varphi_3(c_{3,0}(000);\omega_3\cdots\omega_5)\\
+f_{2a}(s_0s_1s_0(c_{3,0}(000));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_3(c_{3,0}(000);\omega_2\cdots\omega_4)\varphi_1(1;\omega_5)\\
+f_{2a}(s_1s_0s_0(c_{3,0}(000));\omega_1\cdots\omega_5)\varphi_3(c_{3,0}(000);\omega_1\cdots\omega_3)\varphi_1(1;\omega_4)\varphi_1(1;\omega_5)
\end{array}\right)\\
&=\frac{1}{L_5(j\sum_{i=1}^{5}\omega_i)}\cdot\left(\frac{1+(j\sum_{i=3}^{5}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=3}^{5}\omega_i)}+\frac{1+(j\sum_{i=2}^{4}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=2}^{4}\omega_i)}+\frac{1+(j\sum_{i=1}^{3}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\right)\cdot\prod_{i=1}^{5}H_1(j\omega_i)
\end{aligned}
$$

Hence, it can be obtained that

$$
\varphi_3(CE(H_3(\cdot)))=\frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\left[\begin{array}{c}1\\ \prod_{i=1}^{3}(j\omega_i)\end{array}\right]\cdot\prod_{i=1}^{3}H_1(j\omega_i)
$$

$$
\varphi_5(CE(H_5(\omega_1,\cdots,\omega_5)))=
$$

$$
\frac{1}{L_5(j\sum_{i=1}^{5}\omega_i)}\cdot\left[\begin{array}{c}
\frac{1}{L_3(j\sum_{i=3}^{5}\omega_i)}+\frac{1}{L_3(j\sum_{i=2}^{4}\omega_i)}+\frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\\
\frac{1+(j\sum_{i=3}^{5}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=3}^{5}\omega_i)}+\frac{1+(j\sum_{i=2}^{4}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=2}^{4}\omega_i)}+\frac{1+(j\sum_{i=1}^{3}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\\
\frac{(j\sum_{i=3}^{5}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=3}^{5}\omega_i)}+\frac{(j\sum_{i=2}^{4}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=2}^{4}\omega_i)}+\frac{(j\sum_{i=1}^{3}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}
\end{array}\right]\cdot\prod_{i=1}^{5}H_1(j\omega_i)
$$

By using equation (25), the GFRFs for n=3 and 5 of system (29) can be obtained. Proceeding with the computation process above, any higher order of the GFRFs of system (29) can be derived and written in a much more meaningful form. It can be seen that, the correlative function of a monomial in the parametric characteristic of the *n*th-order GFRF is an *n*-degree polynomial of the first order GFRF as stated in Corollary 2, and so does the *n*th-order GFRF. Based on equation (25), the first order parametric sensitivity of the GFRFs with respect to any nonlinear parameter can be studied as

$$\frac{\partial H_n(j\omega_1,\cdots,j\omega_n)}{\partial c} = \frac{\partial CE(H_n(\cdot))}{\partial c}\cdot \varphi_n(CE(H_n(\cdot)))$$

For example,

$$\frac{\partial H_3(j\omega_1,\cdots,j\omega_3)}{\partial c_1} = \frac{\partial CE(H_3(\cdot))}{\partial c_1}\cdot \varphi_3(CE(H_3(\cdot))) = [1,0]\cdot \varphi_3(CE(H_3(\cdot))) = \prod_{i=1}^{3} H_1(j\omega_i) \Big/ L_3(j\sum_{i=1}^{3}\omega_i)$$

similarly,

$$\frac{\partial H_5(j\omega_1,\cdots,j\omega_5)}{\partial c_1} = \frac{\partial CE(H_5(\cdot))}{\partial c_1}\cdot \varphi_5(CE(H_5(\cdot))) = [2c_1,c_2,0]\cdot \varphi_5(CE(H_5(\cdot))).$$

Similar results can also be obtained for parameter c_2. It can be seen that the sensitivity of the third order GFRF with respect to the nonlinear spring c_1 and nonlinear damping c_2 is constant which is dependent on linear parameters, but the sensitivity of the higher order GFRFs will be a function of these nonlinearities and the linear parameters. Note that for a Volterra system, the system output is usually dominated by its first several order GFRFs (Boyd and Chua 1985). Hence, in order to make the system less sensitive to these nonlinearities, the linear parameters should properly be designed.

Moreover, the magnitude of $H_n(j\omega_1,\cdots,j\omega_n)$ can also be evaluated readily according to Corollary 1. For example, for *n*= 3

$$\left|H_3(j\omega_1,\cdots,j\omega_3)\right|^2 = CE_3\Theta_3 CE_3^T = \left|\frac{\prod_{i=1}^{3} H_1(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\right|^2 \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^T \begin{bmatrix} 1 & \prod_{i=1}^{3}(j\omega_i) \\ -\prod_{i=1}^{3}(j\omega_i) & \prod_{i=1}^{3}(\omega_i^2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

As mentioned above, instead of studying the Bode diagram of $H_3(j\omega_1,\cdots,j\omega_3)$, the frequency response spectrum of the maximum eigenvalue of the third order frequency characteristic matrix defined in Corollary 1 can be investigated. See Figures 3-4. Different values of the linear parameters will result in a different view. An increase of the linear damping enables the magnitude to increase for higher $\omega_1+\omega_2+\omega_3$ along the line $\omega_1+\omega_3=0$. Note that the system output spectrum (26a-c) involves the computation of the GFRFs along a super-plane $\omega_1+\cdots+\omega_n=\omega$. The frequency response spectra of the maximum eigenvalue on the plane $\omega_1+\cdots+\omega_3=\omega$ with different output frequency ω are given in Figures 5-6. The peak and valley in the figures can represent special properties of the system. Understanding of these diagrams can follow the method in Yue et al (2005), and further results are under study.

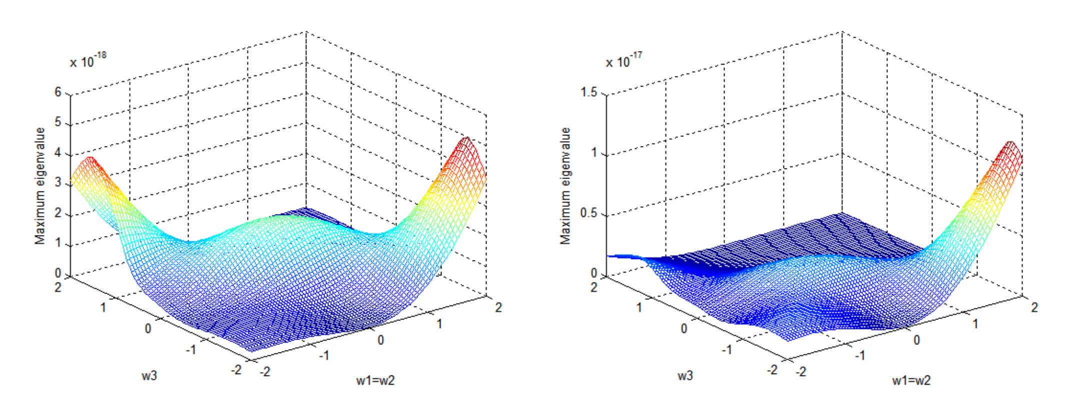

Figure 3. Frequency response spectrum of the maximum eigenvalue
when m=24, B=2.96(left) or 29.6(right), K=160

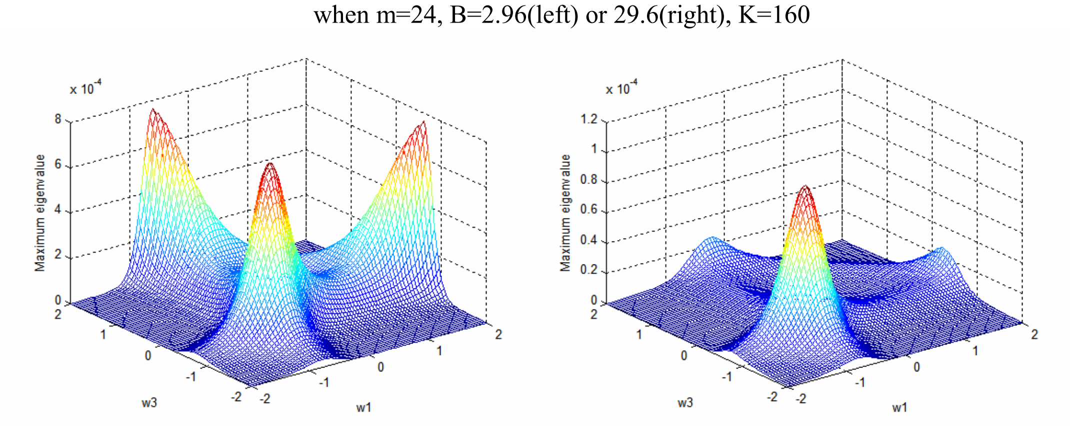

Figure 4. Frequency response spectrum of the maximum eigenvalue
when m=2.4, B=2.96, K=1.6 and $\omega_1+\omega_2+\omega_3=0.8$ (left) or 1.5(right)

The system output spectrum can also be studied. For example, suppose the system is subject to a harmonic input $u(t)=F_d\sin(\omega_0 t)\quad(F_d>0)$, then the magnitude of the third order output spectrum can be evaluated as (Jing et al 2007a)

$$|Y_3(j\omega)|\le\frac{1}{2^3}\sum_{\omega_{k_1}+\cdots+\omega_{k_3}=\omega}|H_3(j\omega_{k_1},\cdots,j\omega_{k_3})||F(\omega_{k_1})\cdots F(\omega_{k_3})|\le\frac{F_d^3}{2^3}\sum_{\omega_{k_1}+\cdots+\omega_{k_3}=\omega}|H_3(j\omega_{k_1},\cdots,j\omega_{k_3})|$$

From corollary 1, $|H_3(j\omega_1,\cdots,j\omega_3)|\le\sqrt{\lambda_3(j\omega_1,\cdots,j\omega_n)}\|CE_3^T\|$. Therefore,

$$|Y_3(j\omega)|\le\frac{F_d^3}{2^3}\sum_{\omega_{k_1}+\cdots+\omega_{k_3}=\omega}\sqrt{\lambda_3(j\omega_1,\cdots,j\omega_n)}\|CE_3^T\|=\frac{F_d^3}{2^3}\sqrt{c_1^2+c_2^2}\sum_{\omega_{k_1}+\cdots+\omega_{k_3}=\omega}\sqrt{\lambda_3(j\omega_1,\cdots,j\omega_n)}$$

For ω =0.8 and m=2.4, B=29.6, K=1.6, it can be obtained that $\sqrt{\lambda_3(j\omega_1,\cdots,j\omega_n)}\le 0.006055896$. Hence, in this case

$$|Y_3(j\omega)|\le 0.00227096F_d^3\sqrt{c_1^2+c_2^2}$$

Obviously, given a requirement on the bound of $|Y_3(j\omega)|$, the design restriction on the nonlinear parameters c_1 and c_2 can further be derived. □

5 Conclusions

A mapping function from the parametric characteristics of the GFRFs to the GFRFs is established, such that the *n*th-order GFRF can directly be written into a more straightforward and meaningful form in terms of the first order GFRF and model parameters based on the parametric characteristic, which explicitly unveils the linear and nonlinear factors included in the GFRFs and can be regarded as an n-degree polynomial function of the first order GFRF. The new results demonstrate some new properties of the GFRFs, which can reveal clearly the relationship between the nth-order GFRF and its parametric characteristic, and also the relationship between the nth-order GFRF and the first order GFRF. These provide a novel and useful insight into the frequency domain analysis and design of nonlinear systems based on the GFRFs. Note that the results of this study are established for nonlinear systems described by the NDE model, further study will extend these results to discrete time nonlinear systems described by NARX model. The frequency characteristics of system output frequency response of nonlinear systems will also be studied by using these new results. Moreover, further study will also focus on some detailed issues relating to the application of the theoretical results developed in the present study.

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