Monograph:

Jing, X.J., Lang, Z.Q., Billings, S.A. et al. (1 more author) (2006) A new approach to nonlinear feedback control for suppressing periodic disturbances: Part 1. Fundamental Theory. Research Report. ACSE Research Report no. 933 . Automatic Control and Systems Engineering, University of Sheffield

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Part 1. Fundamental Theory

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Research Report No. 933
August 2006

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Part 1. Fundamental Theory

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Abstract: A new nonlinear feedback control approach is proposed in the present study to suppress periodic exogenous disturbances based on a frequency domain theory of nonlinear systems. In Part 1 of this paper, a series of fundamental theoretical results and techniques are established. It is shown that a low order nonlinear feedback may be sufficient for some control problems. A general procedure is then proposed for controller design. The new approach is demonstrated by a case study on the design of an active vibration control system in Part 2. Theoretical analysis and simulation results verify the effectiveness of the new results.

1. Introduction

Suppression of periodic disturbances covers a wide range of applications, for example, active control and isolation of vibrations in engineering systems. Traditionally, an increase in damping can reduce the response at the resonance. However, this is often at the expense of degradation of isolation at high frequencies (Graham and Mcruer 1961). Many methods have been proposed to deal with this problem, such as optimal control, H-infinity control, "skyhook" damper and repetitive learning control *etc* (Hrovat 1997, Graham and Mcruer 1961, Karnopp 1995, Lee and Smith 2000). Nonlinear feedback is an approach that has been noted by several researchers. Lee and Smith (2000) pointed out that, it is not possible to use linear time-invariant controllers to robustly stabilize a controlled plant and to achieve asymptotic rejection of a periodic disturbance. However, the problem is solvable using a nonlinear controller for a linear plant subjected to a triangular wave disturbance. It has also been reported many times that existing nonlinearities or deliberately introduced nonlinearities may bring benefits to control system design (Graham and Mcruer 1961). Hence, the design of a nonlinear feedback controller to suppress periodic disturbances has great potential to achieve a considerably improved control performance.

Recently, significant progress has been achieved in the analysis of nonlinear systems in the frequency domain (Lang and Billings 1996, Billings and Peyton-Jones 1990, Lang and Billings 2005, Lang, Billings, Yue and Li 2006, Lang, Billings, Tomlinson and Yue 2006). The effective determination and analysis of the Generalised Frequency Response Functions (GFRFs) and Output Frequency Response Functions have been achieved for nonlinear systems which can be described by the NARX (Nonlinear Auto-Regressive model with eXogenous input) model. Based on these results, energy transfer characteristics of nonlinear systems have been analysed and discussed in Billings and Lang (2002) and Lang and Billings (2005).

The frequency domain theories for nonlinear systems indicate that under certain conditions, the output spectrum of a nonlinear system is determined by a combination of the contributions of the system nonlinearities of different orders and the input. This implies that for a linear controlled plant subjected to a periodic disturbance, if a nonlinear feedback control is introduced to produce a nonlinear closed loop system, the output frequency response of the closed loop system at the frequency of the disturbance can be reduced, when the contributions of different system nonlinear orders at this frequency interact with each other. Motivated by this basic idea of exploiting nonlinear effects to suppress periodic disturbances, in the present study, a novel frequency domain analysis based nonlinear feedback control approach is proposed to suppress sinusoidal exogenous disturbances for a general linear controlled plant.

This paper is divided into two parts. Part 1 is concerned with a series of fundamental results for the development of this new approach. The general nonlinear feedback controller design problem is divided into several basic problems which can be addressed separately. Then a series of theoretical results and techniques needed to solve these basic problems are established and developed. Finally a general procedure for the design and analysis of the nonlinear feedback controller is proposed. Part 2 is concerned with a case study, where the implementation of the new approach is demonstrated using an active vibration control problem (Daley, Johnson, Pearson and Dixon 2005). Simulation results are provided to illustrate the effectiveness and advantages of the new approach.

2. Problem Formulation

Consider an SISO system described by the following differential equation:

$$\sum_{l=0}^{L} C_x(l) D^l x + b \cdot u + e \cdot \eta = 0 \tag{1}$$

$$y = \sum_{l=0}^{L-1} C_y(l) D^l x + d \cdot u \tag{2}$$

where, x, y, u, $\eta \in \Re^1$ represent the system state, output, control input, and an exogenous disturbance input respectively; η stands for a known, external, bounded and periodical vibration, which can be described by a summation of multiple sinusoidal functions; L is a positive integer; D^l is an operator defined by $D^l x = d^l x / dt^l$. The model of system (1,2) can also be written into a state-space form:

$$\dot{X} = AX + Bu + E\eta \quad (3)$$

$$y = CX + Du \quad (4)$$

where, $X=[x, D^1x, \ldots, D^{L-1}x]^{\mathrm{T}} \in \Re^L$ is the system state variable, *A, B, E, C, D* are matrixes with appropriate dimensions. The problem to be addressed in the present study is:

Given a frequency interval $I(\omega)$ and a desired magnitude level of the output frequency response Y^* over this frequency interval, find a nonlinear state feedback control law

$$u = -\varphi(x, D^1 x, \ldots, D^{L-1} x) \quad (5)$$

such that

$$\int_{I(\omega)} Y(j\omega)Y(-j\omega)d\omega \leq Y^* \quad (6)$$

where the feedback control law $-\varphi(x, D^1 x, \ldots, D^{L-1} x)$ is generally a nonlinear function of x, D^1x, …, $D^{L-1}x$, with the linear state/output feedback as a special case; $Y(j\omega)$ is the spectrum of the system output. Note that a dynamic control law can also be considered here. This will be addressed in a future study.

For simplicity, the present study assumes $I(\omega) = \omega_0$, that is only the output response at a specific frequency is considered. Let $Y = [Y(j\omega)Y(-j\omega)]_{(\omega_0,u)}$, then $Y_0 = [Y(j\omega)Y(-j\omega)]_{(\omega_0,0)}$ shows the magnitude of the system output frequency response at frequency ω_0 under zero control input. Obviously, it is required that

$$[Y(j\omega)Y(-j\omega)]_{(\omega_0,u)} \leq Y^* < Y_0 = [Y(j\omega)Y(-j\omega)]_{(\omega_0,0)} \quad (7)$$

To obtain a nonlinear feedback controller, $\varphi(x, D^1 x, \ldots, D^{L-1} x)$ is written into the general form as

$$\varphi(x, D^1 x, \ldots, D^{L-1} x) = \sum_{p=1}^{M} \sum_{l_1 \cdots l_p = 0}^{L-1} C_{p0}(l_1, \cdots, l_p) \prod_{i=1}^{p} D^{l_i} x \quad (8)$$

where M is a positive integer representing the maximum degree of nonlinearity in the terms $D^i x(t)$ (i=0…L-1); $\sum_{l_1 \cdots l_p=0}^{L-1}(\cdot) = \sum_{l_1=0}^{L-1} \cdots \sum_{l_p=0}^{L-1}(\cdot)$. The nonlinear function in (8) includes a general class of possible linear and nonlinear functions of $D^i x$ (i=0…L-1). Since $D^i x = e(i+1)^{\mathrm{T}} X$, where e(i+1) is a L-dimensional column vector whose (i+1)th element is 1 with all other terms zero, $\varphi(x, D^1 x, \ldots, D^{L-1} x)$ can also be written as a function of X, *i.e.*, $\varphi(X)$. For the parameters $C_{p0}(.)$ (p=1,…,M), when p =1 the parameters will be referred to as the linear parameters corresponding to the linear terms in (8), *e.g.*, $C_{1,0}(2)\frac{d^2 x(t)}{dt^2}$. All the other parameters in (8) will be referred to as the nonlinear parameters corresponding to the nonlinear terms $\prod_{i=1}^{p} D^{l_i} x(t)$. p is the nonlinear order of the nonlinear parameter $C_{p0}(\cdot)$.

Let

$$C(M)=\left(C_{p0}(l_1,\cdots,l_p)\left|\begin{array}{l}p=1\cdots M\\ l_i=0\cdots L-1\\ i=1\cdots p\end{array}\right.\right) \quad (9)$$

The task for the nonlinear controller design is to determine M and the controller parameters in (9) to make the closed loop system (1,2,5) asymptotically stable within a ball around the zero equilibrium point while satisfying the steady state performance (6). In this paper, without loss of generality it will be assumed that the linear parameters $C_{10}(\cdot)$ =0. The focus of this study is to design the nonlinear parameters in (9) and investigate the advantage of nonlinear state feedback control for system (1-2). The principle of the design is to determine the nonlinear parameters in the controller with the linear part of the system set according to other design requirements (*e.g.*, stability). Moreover, considering that controller (5) should only involve terms which are needed to stabilise the closed loop system and to achieve the steady state performance (6), thus the controller parameters in (9) can be further written as

$$C(M)=\left(C_{p0}(l_1,\cdots,l_p)\left|\begin{array}{l}p=2\cdots M\\ l_i=0\cdots L-1, i=1\cdots p\\ \left[Y(j\omega)Y(-j\omega)\right]_{(\omega_0,u)}<Y_0\end{array}\right.\right) \quad (10)$$

Determine the controller parameters in (10) is the main problem to be addressed.

Substituting (8) into (1) and (2) yields a description for the closed loop system as

$$\sum_{p=1}^{M}\sum_{l_1\cdots l_p=0}^{L}\overline{C}_{p0}(l_1,\cdots,l_p)\prod_{i=1}^{p}D^{l_i}x+e\cdot\eta=0 \quad (11a)$$

$$\sum_{p=1}^{M}\sum_{l_1\cdots l_p=0}^{L-1}\widetilde{C}_{p0}(l_1,\cdots,l_p)\prod_{i=1}^{p}D^{l_i}x=y \quad (11b)$$

where,

$$\overline{C}_{10}(l_1)=C_x(l_1)-bC_{10}(l_1),\ \ \widetilde{C}_{10}(l_1)=C_y(l_1)-dC_{10}(l_1)$$

$$\overline{C}_{p0}(l_1,\cdots,l_p)=-bC_{p0}(l_1,\cdots,l_p),\ \widetilde{C}_{p0}(l_1,\cdots,l_p)=-dC_{p0}(l_1,\cdots,l_p),$$

for $p=2\cdots M, l_i=0\cdots L,$ and $i=1\cdots p$. For simplicity, (11) can also be written into a concise state space form as follows

$$\dot{X}=AX-B\varphi(X)+E\eta:=f(X)+E\eta \quad (12a)$$

$$y=CX-D\varphi(X):=h(X) \quad (12b)$$

Equation (11) is a special case of the continuous NARX model with a single input $\eta(t)$ and two outputs x(t) and y(t). The Generalized Frequency Response Functions (GFRFs) and Output Frequency Response Functions of model (11) can be obtained by using a harmonic probing method introduced in Rugh (1981) and some results developed recently by the authors (Lang and Billings 1996, Billings and Peyton-Jones 1990, Lang, Billings, Yue and Li 2006).

3. Some Fundamental Theoretical Results

The nonlinear frequency domain approach is based on the Volterra series theory of nonlinear systems. It has been shown that, any time invariant, causal, nonlinear system with fading memory can be approximated by a finite Volterra series (*i.e.*, by Volterra polynomials) (Boyd and Chua 1985). This implies that there exists a convergent Volterra series which can approximate a nonlinear system under certain assumptions. In order to apply the nonlinear frequency domain approach to design the parameters of controller (5) and achieve the control objective (6), first the stability of system (11) around its zero equilibrium should be guaranteed. This is to ensure the nonlinear Volterra series theory and hence the corresponding nonlinear frequency analysis approaches are valid and can therefore be used for the system analysis and design. To achieve this objective, a range needs to be determined for the values of the parameters of controller (5) over which the closed loop system (11) is asymptotically stable when $\eta(t)=0$. Second, an analytical relationship between the controller parameters and the closed loop system output spectrum needs to be determined. Lang, Billings, Tomlinson, and Yue (2005) have shown that this relationship is a polynomial function of the controller parameters and have therefore provided a necessary basis for the implementation of this step. Finally, from the analytical relationship, the controller parameters can be determined to achieve the design objective (6). Therefore, there are generally four fundamental issues to be addressed for the nonlinear feedback control problem. These are:

(a) Determination of the analytical relationship between the system output spectrum and the nonlinear controller parameters.

(b) Determination of an appropriate structure for the nonlinear feedback controller. Only significant nonlinear terms are needed in the controller to achieve the control objective.

(c) Derivation of a range for the values of the control parameters over which the stability of the closed loop nonlinear system is guaranteed.

(d) Development of an effective numerical method for the practical implementation of the feedback controller design.

In this two part paper, the focus of Part 1 is to investigate the fundamental issues. In Part 2, a case study will be presented to show how to implement the general results achieved in Part 1.

3.1. The Output Frequency Response of the Closed-Loop System

In this section, the output frequency response of the closed loop nonlinear system (11) is derived. The relationship between the system output spectrum and the controller parameters are investigated to produce a series of important results which are useful to facilitate the controller design.

3.1.1. The Output Spectrum

Consider x(t) and y(t) of the closed loop nonlinear system (11) as two outputs, *i.e.*, $y_1(t)=x(t)$, $y_2(t)=y(t)$, and assume that the relationship between the output x(t) and y(t) and the input $\eta(t)$ of system (11) can be approximated by Volterra functional polynomials up to Nth order as

$$y_j(t)=\sum_{n=1}^{N} y_j^{(n)}(t)\,,\ \ y_j^{(n)}=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n^j(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}\eta(t-\tau_i)d\tau_i\ \text{, j=1,2.} \tag{13}$$

where $h_n^j(\tau_1,\cdots,\tau_n)$ is the nth order Volterra kernel of system (11) corresponding to the jth output. When the input in (13) is a multi-tone function of time t described by

$$\eta(t)=\sum_{i=1}^{K}\left|F_i\right|\cos(\omega_i t+\angle F_i) \tag{14}$$

the frequency domain input output description of the system can be obtained by extending the result in Lang and Billings (1996):

$$Y_j(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n^j(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \qquad j=1,2 \tag{15}$$

where,

$$F(\omega)=\begin{cases}\left|F_i\right|e^{j\angle F_i} & \text{if } \omega\in\{\omega_k, k=\pm1,\cdots,\pm K\}\\ 0 & \text{else}\end{cases} \tag{16a}$$

$$H_n^j(j\omega_{k_1},\cdots,j\omega_{k_n})=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n^j(\tau_1,\cdots,\tau_n)e^{-j(\omega_1\tau_1+\cdots+\omega_n\tau_n)}d\tau_1\cdots d\tau_n \tag{16b}$$

(16b) is the nth order generalised frequency response function (GFRF) of system (11) corresponding to the jth output.

Following the idea of mapping a time domain description of nonlinear systems into the frequency domain introduced in Rugh (1981), Peyton Jones and Billings (1989), and Swain and Billings (2001), the output spectrum $Y(j\omega)$ of nonlinear system (11) is derived and the results are summarized in Proposition 1.

Proposition 1. Suppose the relationship between input $\eta(t)$ and output y(t) of the nonlinear system (11) can be approximated by a convergent Volterra series, then the output frequency response function $Y(j\omega)$ of the system output $y(t)$ under the multi-tone input disturbance (14) can be determined as

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n^2(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{17a}$$

where,

$$H_n^2(j\omega_1,\cdots,j\omega_n)=\sum_{p=1}^{n}\sum_{l_1\cdots l_p=0}^{L-1}\widetilde{C}_{p0}(l_1\cdots l_p)H_{np}^1(j\omega_1,\cdots,j\omega_n) \tag{17b}$$

$$H_{np}^1(j\omega_1,\cdots,j\omega_n)=\sum_{i=1}^{n-p+1}H_i^1(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}^1(j\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)^{l_p} \tag{17c}$$

$$H_{n1}^1(j\omega_1,\cdots,j\omega_n)=H_n^1(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{l_1}\,,\ \ H_1^1(j\omega_1)=e\Big/\sum_{l_1=0}^{L}\overline{C}_{10}(l_1)(j\omega_1)^{l_1} \tag{17d}$$

$$H_n^1(j\omega_1,\cdots,j\omega_n)=-\frac{1}{e}H_1^1(j\omega_1+\cdots+j\omega_n)\left(\sum_{p=2}^{n}\sum_{l_1\cdots l_p=0}^{L-1}\overline{C}_{p0}(l_1\cdots l_p)H_{np}^1(j\omega_1,\cdots,j\omega_n)-e\delta(n-1)\right) \tag{17e}$$

and $\delta(n)=\begin{cases}1 & n=0\\ 0 & \text{otherwise}\end{cases}$.

Proof: See Appendix. □

Proposition 1 shows how the GFRFs can be recursively computed from the system time domain model (11) and how the system output spectrum is related to the GFRFs.

3.1.2. Important Properties of the Output Spectrum

The output frequency response function $Y(j\omega)$ given by (17) has several useful properties which provide an important basis for the analysis and design of the nonlinear closed loop system (11).

Property 1 (Lang, Billings, Yue and Li 2006). The output frequency response function $Y(j\omega)$ of the closed loop nonlinear system (11) can be expressed as a polynomial function of the nonlinear controller parameters in (10), *i.e.*,

$$Y(j\omega) = P_0(j\omega) + a_1 P_1(j\omega) + a_2 P_2(j\omega) + \cdots \tag{18a}$$

where $P_0(j\omega)$ is the linear part of the system output frequency response, $P_i(j\omega)$, $i \geq 1$, represent the effects of higher order output frequency responses, and a_i $(i = 1,2,\cdots)$ are functions of the nonlinear controller parameters in (10).□

The detailed form of the coefficients a_i $(i = 1,2,\cdots)$ in (18) will be discussed later. It should be noted that, if the parameters a_i are confined to a small range around zero, then (18) can usually be approximated by a finite number of terms. From Property 1, the following property follows.

Property 2. The output frequency response function $Y(j\omega)$ of the nonlinear closed loop system (11) can be written as a polynomial function of any individual nonlinear controller parameter in (10). That is, for a nonlinear controller parameter *c* in (10), there exists a series of functions of frequency ω $\{\overline{P}_i(j\omega)$, i=0,1,2,3,…} such that

$$Y(j\omega) = \overline{P}_0(j\omega) + c\overline{P}_1(j\omega) + c^2\overline{P}_2(j\omega) + \cdots \tag{18b}$$

□

The properties above show that the nonlinear controller parameters are separable from the system frequency functions, and the output spectrum of the nonlinear closed loop system can be described as a polynomial function of the nonlinear controller parameters. Equation (18) was referred to as the output frequency response function of nonlinear Volterra systems by Lang et al (2006). This concept and associated results developed in the following section reveal an important relationship between the output spectrum and the system parameters. This can considerably facilitate the analysis and design of the nonlinear feedback control system (11). In order to study equation (18) in more detail and reveal the contribution of the nonlinear controller parameters of different orders to the output spectrum more clearly, some useful results regarding the parametric characteristics of the output spectrum of the closed loop system (11) are developed in the following section, which can describe explicitly the detailed form of the coefficients a_i $(i = 1,2,\cdots)$ in (18).

3.1.3. The Parametric Characteristics of the Output Spectrum

The objective of the study of the parametric characteristics of the output spectrum of system (11) is to investigate the polynomial expression (18) in detail, and to reveal how the frequency response functions in (17a-e) of the nonlinear system (11) depend on the nonlinear controller parameters (*i.e.*, $C_{p0}(.)$ for $p>1$ in (10)). For this purpose, a useful operator will be defined below.

Consider a series $H_{cf} = c(1)f(1) + c(2)f(2) + \cdots + c(n)f(n) \in C$, where the coefficients $c(i)$ for $i=1,\ldots,n$ are real or complex numbers, $f(i)$ for $i=1,\ldots,n$ are real or complex valued functions, C denotes all the complex numbers.

Define a **Coefficient Extractor** operator $CE: C \to C^n$ such that for any

$$H_{cf} = c(1)f(1) + c(2)f(2) + \cdots + c(n)f(n) \in C$$

$CE(H_{cf}) = [c(1), c(2), \cdots, c(n)] := C \in C^n$, where C^n is the n-dimensional complex space. This operator has the following properties:

(1) Vectorized sum $\oplus$: $CE(H_{c_1 f_1} + H_{c_2 f_2}) = CE(H_{c_1 f_1}) \oplus CE(H_{c_2 f_2}) = C_1 \oplus C_2 = [C_1, C_2']$, where $C_2'(i)$ for i=1…m are non-repetitive elements in C_2 with respect to C_1, *i.e.*, $\forall i, j, C_2'(i) \neq C_1(j)$

(2) Reduced Kronecker product $\otimes$: $CE(H_{c_1 f_1} \cdot H_{c_2 f_2}) = CE(H_{c_1 f_1}) \otimes CE(H_{c_2 f_2}) = C_1 \otimes C_2$, and "reduced" here means that there are no repetitive components in $C_1 \otimes C_2$.

(3) Invariant: (a) $CE(\alpha \cdot H) = CE(H)$; (b) $CE(H_{cf_1} + H_{cf_2}) = CE(H_{c(f_1+f_2)}) = C$

(4) Unitary: if H is not a function of $c(i)$ for $i=1\ldots n$, $CE(H) = 1$ for operator $\otimes$ and $CE(H) = [\,]$ for operator $\oplus$

(5) Inverse: $CE^{-1}(C) = H_{cf}$.

For simplicity, let $\underset{(*)}{\otimes}(\cdot)$ and $\underset{(*)}{\oplus}(\cdot)$ denote the multiplication and addition by the reduced Kronecker product" $\otimes$ " and vectorized sum" $\oplus$ "of the series (.) satisfying (*), respectively. Moreover, define the pth order parameter vector $C_{p0} = [C_{p0}(0,\cdots,0), C_{p0}(0,\cdots,1), \cdots, C_{p0}(\underbrace{L,\cdots,L}_{p})]$, which includes all the nonlinear parameters of nonlinearity degree p in (10).

For further derivation, $H_{np}^1(j\omega_1,\cdots,j\omega_n)$ in (17c) can be rewritten as (Lang *et al* 2006),

$$H_{np}^1(j\omega_1,\cdots,j\omega_n) = \sum_{\substack{r_1\cdots r_p=1 \\ \sum r_i = n}}^{n-p+1} \prod_{i=1}^{p} H_{r_i}^1(j\omega_{i1},\cdots,j\omega_{ir_{i}})(j\omega_{i1}+\cdots+j\omega_{ir_i})^{l_{p+1-i}} \tag{19}$$

Utilizing (19), it can be shown from equations (17b,e) that

$$H_2^1(j\omega_1, j\omega_2) = -\frac{1}{e} H_1^1(j\omega_1 + j\omega_2) \sum_{l_1,l_2=0}^{L-1} \overline{C}_{20}(l_1,l_2) H_{22}^1(j\omega_1, j\omega_2)$$

$$= -\frac{1}{e} H_1^1(j\omega_1 + j\omega_2) \sum_{l_1,l_2=0}^{L-1} (-b) C_{20}(l_1,l_2) H_1^1(j\omega_1) H_1^1(j\omega_2)(j\omega_1)^{l_1}(j\omega_2)^{l_2}$$

$$H_2^2(j\omega_1, j\omega_2) = \sum_{l_1=0}^{L-1} \widetilde{C}_{10}(l_1) H_{21}^1(j\omega_1, j\omega_2) + \sum_{l_1,l_p=0}^{L-1} \widetilde{C}_{20}(l_1,l_2) H_{22}^1(j\omega_1, j\omega_2)$$

$$= \sum_{l_1=0}^{L-1} \widetilde{C}_{10}(l_1) H_2^1(j\omega_1, j\omega_2)(j\omega_1 + j\omega_2)^{l_1} + \sum_{l_1,l_p=0}^{L-1} (-d) C_{20}(l_1,l_2) H_1^1(j\omega_1) H_1^1(j\omega_2)(j\omega_1)^{l_1}(j\omega_2)^{l_2}$$

$$H_3^1(j\omega_1, \cdots, j\omega_3) = -\frac{1}{e} H_1^1(j\omega_1 + \cdots + j\omega_3) \left(\sum_{l_1,l_2=0}^{L-1} \overline{C}_{20}(l_1,l_2) H_{32}^1(j\omega_1, \cdots, j\omega_3) + \sum_{l_1 \cdots l_3=0}^{L-1} \overline{C}_{30}(l_1 \cdots l_3) H_{33}^1(j\omega_1, \cdots, j\omega_3) \right)$$

Applying the CE operator to $H_2^1(j\omega_1, j\omega_2)$, $H_2^2(j\omega_1, j\omega_2)$ and $H_3^1(j\omega_1, \cdots, j\omega_3)$ for the nonlinear controller parameters, yields

$$CE(H_2^1(j\omega_1, j\omega_2)) = CE\left(-\frac{1}{e} H_1^1(j\omega_1 + j\omega_2) \sum_{l_1,l_2=0}^{L-1} (-b) C_{20}(l_1,l_2) H_1^1(j\omega_1) H_1^1(j\omega_2)(j\omega_1)^{l_1}(j\omega_2)^{l_2} \right) = C_{20}$$

$$CE(H_2^2(j\omega_1, j\omega_2)) = CE\left(\sum_{l_1=0}^{L-1} \widetilde{C}_{10}(l_1) H_2^1(j\omega_1, j\omega_2)(j\omega_1 + j\omega_2)^{l_1} + \sum_{l_1,l_p=0}^{L-1} (-d) C_{20}(l_1,l_2) H_1^1(j\omega_1) H_1^1(j\omega_2)(j\omega_1)^{l_1}(j\omega_2)^{l_2} \right)$$

$$= CE(H_2^1) \oplus C_{20} = C_{20}$$

Similarly,

$$CE(H_3^1(j\omega_1, \cdots, j\omega_3)) = C_{20} \otimes C_{20} \oplus C_{30}$$

The calculation process above demonstrates how the CE operator is applied. Using the invariant property of the CE operator, it follows that

$$CE(\overline{C}_{p0}(l_1, \cdots, l_p)) = C_{p0}(l_1, \cdots, l_{p+q}),\ CE(\widetilde{C}_{p0}(l_1, \cdots, l_p)) = C_{p0}(l_1, \cdots, l_p),$$

for the nonlinear controller parameters in (10). Hence, following the same procedure it can be shown that

$$CE(H_n^1(j\omega_1, \cdots, j\omega_n)) = CE\left(-\frac{1}{e} H_1^1(j\omega_1 + \cdots + j\omega_n) \left(\sum_{p=2}^{n} \sum_{l_1 \cdots l_p=0}^{L-1} \overline{C}_{p0}(l_1 \cdots l_p) H_{np}^1(j\omega_1, \cdots, j\omega_n) - e\delta(n-1) \right) \right)$$

$$= CE\left(\sum_{p=2}^{n} \sum_{l_1 \cdots l_p=0}^{L-1} \overline{C}_{p0}(l_1 \cdots l_p) H_{np}^1(j\omega_1, \cdots, j\omega_n) \right) = \mathop{\oplus}_{p=2}^{n} CE\left(\sum_{l_1 \cdots l_p=0}^{L-1} \overline{C}_{p0}(l_1 \cdots l_p) H_{np}^1(j\omega_1, \cdots, j\omega_n) \right) \tag{20}$$

$$= \mathop{\oplus}_{p=2}^{n} \left(C_{p0} \otimes CE\left(H_{np}^1(j\omega_1, \cdots, j\omega_n) \right) \right)$$

and

$$CE(H_{np}^1(j\omega_1, \cdots, j\omega_n)) = CE\left(\sum_{\substack{r_1 \cdots r_p=1 \\ \sum r_i=n}}^{n-p+1} \prod_{i=1}^{p} H_{r_i}^1(j\omega_{i1}, \cdots, j\omega_{ir_{i_i}})(j\omega_{i1} + \cdots + j\omega_{ir_i})^{l_{p+1-i}} \right)$$

$$= \mathop{\oplus}_{\substack{r_1 \cdots r_p=1 \\ \sum r_i=n}}^{n-p+1} CE\left(\prod_{i=1}^{p} H_{r_i}^1(j\omega_{i1}, \cdots, j\omega_{ir_{i_i}})(j\omega_{i1} + \cdots + j\omega_{ir_i})^{l_{p+1-i}} \right) \tag{21}$$

$$= \mathop{\oplus}_{\substack{r_1 \cdots r_p=1 \\ \sum r_i=n}}^{n-p+1} \left(CE(H_{r_1}^1(j\omega_{11}, \cdots, j\omega_{1r_{1_i}})) \otimes CE(H_{r_2}^1(j\omega_{21}, \cdots, j\omega_{2r_2}) \otimes \cdots \otimes CE(H_{r_p}^1(j\omega_{p1}, \cdots, j\omega_{pr_p})) \right)$$

$$= \mathop{\oplus}_{\substack{r_1 \cdots r_p=1 \\ \sum r_i=n}}^{n-p+1} \left(\mathop{\otimes}_{i=1}^{p} CE\left(H_{r_i}^1(j\omega_{i1}, \cdots, j\omega_{ir_i}) \right) \right)$$

Substituting (21) into (20) gives

$$CE(H_n^1(j\omega_1,\cdots,j\omega_n)) = \mathop{\oplus}_{p=2}^{n}\left(C_{p0}\otimes CE\left(H_{np}^1(j\omega_1,\cdots,j\omega_n)\right)\right) = \mathop{\oplus}_{p=2}^{n}\left(C_{p0}\otimes\left(\mathop{\oplus}_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\left(\mathop{\otimes}_{i=1}^{p} CE\left(H_{r_i}^1(j\omega_{i1},\cdots,j\omega_{ir_i})\right)\right)\right)\right) \quad (22)$$

Equation (22) provides an explicit expression for the parametric characteristics of the GFRFs of the closed loop system (11), and reveals how the GFRFs depend on the nonlinear controller parameters in (10).

In order to simplify the general result in (22), several further results are summarized in the following lemmas 1 and 2, and two important conclusions are described in Propositions 2 and 3.

Lemma 1. The terms in CE($H_n^1(j\omega_1,\cdots,j\omega_n)$) include all the possible combinations of the nonlinear control parameters in (10) of the form $C_{p0}\otimes C_{r_1 0}\otimes C_{r_2 0}\otimes\cdots\otimes C_{r_q 0}$, satisfying $p+\sum_{i=1}^{q} r_i = n+q$, $2\le r_i\le n$-1, $0\le q\le n-2$ and $2\le p\le n$.

Proof: See Appendix. □

According to Lemma 1, when *p*=*n*, $p+\sum_{i=1}^{q} r_i = n+\sum_{i=1}^{q} r_i = n+q$, which further yields $\sum_{i=1}^{q} r_i = q$. Since $2\le r_i\le n$-1, then *q* equals zero. Thus the nonlinear parameters of the highest nonlinearity order in $H_n^1(j\omega_1,\cdots,j\omega_n)$ are the elements in C_{n0}. Lemma 1 can be used to check whether and how a nonlinear parameter is included in $H_n^1(j\omega_1,\cdots,j\omega_n)$.

Lemma 2. (1) In equation (21), $CE(H_{np}^1(j\omega_1,\cdots,j\omega_n)) = CE\left(H_{n-p+1}^1(j\omega_p,\cdots,j\omega_n)\right)$. (2) $C_{i0}\otimes C_{j0}\subseteq C_{i+j-1,0}$, $CE(H_i(\cdot))\otimes CE(H_j(\cdot))\subseteq CE(H_{i+j-1}(\cdot))$. Where, $a \subseteq b$ is that all the elements in *a* are elements in *b*.

Proof: See Appendix. □

Lemma 2 shows that the parametric characteristics of $H_{np}^1(j\omega_1,\cdots,j\omega_n)$ are the same as those of $H_{n-p+1}^1(j\omega_p,\cdots,j\omega_n)$. This can be used to simplify the parametric characteristics in (22).

From Lemmas 1 and 2, the following conclusions for the parametric characteristics of the GFRFs in (17b,e) follow.

Proposition 2. For the GFRFs of the nonlinear closed loop system (11),

(1) $CE(H_n^2(\cdot)) = CE(H_n^1(\cdot))$ for n>0

(2) $CE(H_n^1(\cdot)) = C_{n0}\oplus\mathop{\oplus}_{p=2}^{[n+1/2]}\left(C_{p0}\otimes CE(H_{n-p+1}^1(\cdot))\right)$ for n>1

where, [n/2] means to get the integer part of [.].

Proof: See Appendix. □

Proposition 3. For the GFRFs of the nonlinear system (11), (1) $(C_{n0})^i$ appears in $H_m(.)$ from the *mth* order, where *m*=1+(*n*-1)*i*. (2) If the 2nd and 3rd order nonlinear control parameters are all zero, *i.e.*, C_{20}=0 and C_{30}=0, then $H_2(.)$=0, and $H_3(.)$=0. However, even if C_{n0}=0 (for *n*>3), the *n*th order GFRF $H_n(.)$ is not zero, providing there are nonzero terms in C_{20} or C_{30}.
Proof: See Appendix. □

Proposition 2 provides a concise and more explicit insight into the parametric characteristic of $H_n^1(\cdot)$. Proposition 3 demonstrates that the nonlinear controller parameters in C_{20} and C_{30} have a more important role in the determination of the GFRFs than any other nonlinear controller parameters, and the nonlinear controller parameters of nonlinearity order higher than 3 only take a role in the higher order GFRFs. This implies that a lower order nonlinear feedback control may be sufficient for many control problems.

Based on Proposition 2, the parametric characteristics of the output spectrum can be obtained as follows

$$\begin{aligned} CE(Y(j\omega)) &= CE\left(\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n^2(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n})\right) \\ &= CE\left(\sum_{n=1}^{N}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n^2(j\omega_{k_1},\cdots,j\omega_{k_n})\right) = CE\left(\sum_{n=1}^{N} H_n^2(j\omega_{k_1},\cdots,j\omega_{k_n})\right) \qquad (23a) \\ &= CE(H_1^2(\cdot))\oplus CE(H_2^2(\cdot))\oplus\cdots\oplus CE(H_N^2(\cdot)) \\ &= CE(H_1^1(\cdot))\oplus CE(H_2^1(\cdot))\oplus\cdots\oplus CE(H_N^1(\cdot)) \end{aligned}$$

Note that the inverse of $CE(Y(j\omega))$ is obviously a polynomial from the definition of the CE, which is equivalent to equation (18). Equation (23) states clearly that how the system output spectrum of the closed loop nonlinear system (11) are determined by the nonlinear controller parameters. Now the coefficients of the polynomial function (18a) can be described in more detail as

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_K \end{bmatrix} = CE(Y(j\omega)) = CE(H_1^1(\cdot))\oplus CE(H_2^1(\cdot))\oplus\cdots\oplus CE(H_N^1(\cdot)) \qquad (23b)$$

where K is the dimension of the vector $CE(H_1^1(\cdot))\oplus CE(H_2^1(\cdot))\oplus\cdots\oplus CE(H_N^1(\cdot))$. The application of these results is demonstrated in the section 3.3 of Part 2 of this paper and partly in the following section.

3.2. The Structure Issue of the Nonlinear Feedback Controller

The determination of the structure of the nonlinear state feedback function (8) is an important task yet to be tackled. Based on the results developed above, some fundamental results for testing the effectiveness of each nonlinear term can be achieved.

Firstly, as stated in the last section, the structure parameter *M* in (8) or (10) may be chosen as small as possible. Secondly, after *M* is determined, we can test whether a term in C_{p0} is effective or not. It has been mentioned in Section 2 that an effective controller must satisfy the inequality (7). For the effectiveness of a specific nonlinear control parameter *c*, this requirement can be written as

$$\frac{\partial |Y(j\omega_0)|}{\partial c} < 0 \text{ for some c} \tag{24}$$

Consider a specific nonlinear controller parameter *c* in C_{p0} and let all the other nonlinear controller parameters be zero or assumed to be constants. In this case, only the nonlinear coefficient c^i appears in $CE(H^1_{1+(p-1)i}(.))$ according to Proposition 3 (1). Therefore, only the GFRFs for the orders 1+(*p*-1)*i* (for *i*=1,2,3,…) need to be computed to obtain the system output spectrum in (17a). According to (23a,b), we can finally obtain the output spectrum in polynomial form as (18b),

$$Y(j\omega) = \overline{P}_0(j\omega) + c\overline{P}_1(j\omega) + c^2\overline{P}_2(j\omega) + \cdots \tag{25}$$

Regarding the effectiveness of a specific controller parameter, we have the following result.

Proposition 4. Consider equation (25).

(1) $\frac{\partial |Y(j\omega)|}{\partial c} < 0$ for some c $\Rightarrow \exists$some $n > 0, \Re(\sum_{\substack{0\le i\le n-1 \\ i\le j\le n, i+j=n}} < \overline{P}_i(j\omega), sign(c^{n-1})\overline{P}_j(j\omega) >) < 0$

(2) $\Re(< \overline{P}_0(j\omega), \overline{P}_1(j\omega) >) < 0 \Rightarrow$ there exists ε >0 such that $\frac{\partial |Y(j\omega)|}{\partial c} < 0$ for $0 < c < \varepsilon$ or $-\varepsilon < c < 0$

where, $\Re(\cdot)$ is to get the real part of (.), <*x*,*y*> is the inner product of *x* and *y*, $sign(x) = \begin{cases} 1 & x \ge 1 \\ -1 & \text{else} \end{cases}$.

Proof: See Appendix. □

If a nonlinear controller parameter satisfies (24) provided that all or part of the other nonlinear controller parameters being zero, then the corresponding nonlinear term will be said to be conditionally effective; if a nonlinear controller parameter satisfies (24) for any cases, then the corresponding nonlinear term will be said to be absolutely effective. Obviously, a nonlinear parameter satisfying Proposition 4(1) is conditionally effective and one satisfying Proposition 4(2) is absolutely effective. And clearly, only an effective nonlinear term can be utilized in the nonlinear feedback control in order to realize the control objective. How to find the conditionally or even absolutely effective nonlinear terms and make full use of them is yet to be developed.

3.3. Stability of the Closed-loop System

The Volterra series model can be used to describe a rather general class of nonlinear systems (Rugh 1981). Boyd and Chua (1985) showed that any nonlinear system which is time invariant, causal, and has fading memory can be approximated by a Volterra polynomial of sufficient order. This also implies that the stability of a nonlinear system should be guaranteed before the nonlinear system can be represented by a convergent

Volterra series approximation and then analyzed in the frequency domain. For this purpose, a range for the nonlinear controller parameters which can ensure the stability of the closed loop nonlinear system (11) should first be determined.

Noting that the exogenous disturbance in (12) is a periodic bounded signal, and the objective in vibration control is often to suppress the output vibration below a desired level, a concept of asymptotic stability to a ball is introduced in this section. This concept implies that the magnitude of the output for a system is asymptotically controlled to a satisfactory predefined level. Based on this concept, a general result is then derived which can provide some conditions to ensure the stability of the closed loop nonlinear system (12).

A Ball $B_\rho(X)$ is defined as: $B_\rho(X) = \{X \| \|X\| \le \rho, \rho > 0\}$. A K-function γ(s) is an increasing function of s, and a KL-function β(s,t) is an increasing function of s, but a decreasing function of *t*. For detailed definitions of K /*KL*-functions can refer to Alberto (1999).

Asymptotic Stability to a Ball. Given an initial state $X_0 \in \Re^n$ and disturbance input η of a nonlinear system, if there exists a KL-function β such that the solution $X(t, X_0, \eta)$ (for $t \ge 0$) of the system satisfies $\|X(t, X_0, \eta)\| \le \beta(\|X_0\|, t) + \rho$, $\forall t > 0$, then the nonlinear system is said to be asymptotically stable to a ball $B_\rho(X)$, where ρ can be expressed as an upper bound function of η, *i.e.*, there exist a K-function γ such that $\rho = \gamma(\|\eta\|_\infty)$.

Lemma 3. Consider two positive, scalar and continuous process in time *t*, x(t) and y(t) satisfying y(t)$\le \alpha(x(t))$ (for $t \ge 0$), where α is a *K*-function. If x(t) is asymptotically stable to a ball $B_\rho(x)$, then y(t) is asymptotically stable to a ball $B_{\alpha(2\rho)}(y)$.

Proof. There exists a KL-function β, such that function x(t) (for $t \ge 0$) satisfies $x(t) \le \beta(x(0), t) + \rho$, $\forall t > 0$. Therefore, y(t) $\le \alpha(x(t)) = \alpha(\beta(x(0), t) + \rho) \le \alpha(\max(2\beta(x(0), t), 2\rho))$ $= \max(\alpha(2\beta(x(0), t)), \alpha(2\rho)) \le \alpha(2\beta(x(0), t)) + \alpha(2\rho)$. Note that $\alpha(2\beta(x(0), t))$ is still a KL-function of x(0) and *t*, thus the lemma is concluded. □

From Lemma 3, if there exists a *K*-function o such that the output function $h(X)$ of a nonlinear system satisfies $\|h(X)\| \le o(\|X\|)$, then the system output is asymptotically stable to a ball if the system is asymptotically stable to a ball.

Assumption 1. There exists a K-function o such that the output function $h(X)$ of the nonlinear system (12) satisfies $\|h(X)\| \le o(\|X\|)$.

Lemma 4. Consider a scalar differential inequality $\dot{y}(t) \le -\alpha(y(t)) + \gamma$, where α is a K-function and γ is a constant and y(t) satisfies Lipschitz condition. Then there exists KL-function β such that $y(t) \le \beta(|y(t_0) - \alpha^{-1}(\gamma)|, t) + \alpha^{-1}(\gamma)$.

Proof. Consider the differential equation $\dot{y}(t) = -\alpha(y(t))$. From Lemma 10.1.2 in Alberto (1999) it is known that, there is a KL -function β such that $y(t) = \beta(y(t_0), t)$. Similarly, considering the differential equation $\dot{y}(t) = -\alpha(y(t)) + \gamma$, then $y(t) = sign(y(t_0) - \alpha^{-1}(\gamma)) \cdot \beta(\left|y(t_0) - \alpha^{-1}(\gamma)\right|, t) + \alpha^{-1}(\gamma)$. Thus from the comparison principle and the differential inequality $\dot{y}(t) \le -\alpha(y(t)) + \gamma$, the lemma follows. □

For the stability of nonlinear system (12), a general result is given as follows.

Theorem 1. Suppose Assumption 1 holds, then the following statements are equivalent:

(a) There exist a smooth (C^∞) function $V : \Re^L \to \Re_{\ge 0}$ and K_∞ -functions β_1, β_2 and K -functions α , γ such that

$$\beta_1(\|X\|) \le V(X) \le \beta_2(\|X\|) \text{ and } \frac{\partial V(X)}{\partial X}\{f(X) + E\eta\} \le -\alpha(\|X\|) + \gamma(\|\eta\|_\infty) \quad (26)$$

(b) The nonlinear system (12) is asymptotically stable to the ball $B_\rho(X)$ with $\rho = \beta_1(2 \cdot \beta_2^{-1} \cdot \alpha^{-1} \cdot \gamma(\|\eta\|_\infty))$, and the output of the nonlinear system (12) is asymptotically stable to the ball $B_{o(2\rho)}(y)$. □

Proof: See Appendix. □

Though it is not easy to derive a specific stability condition for the general controller (5), there are always various methods which can be used to obtain a stability condition for some specific controllers with a well-chosen Lyapunov function based on Theorem 1. This will be illustrated in Part 2 of this paper, where a case study will be conducted to demonstrate how to apply the general theorem to obtain the stability condition of a specific system.

3.4. A Numerical Method for the Nonlinear Feedback Controller Design

After the structure of the nonlinear feedback controller is determined (for example, the conditionally effective nonlinearity terms and the largest nonlinearity order *M* have been obtained), the nonlinear feedback controller parameters have to be determined to achieve the control objective (6). The values of the nonlinear controller parameters can be evaluated through solving equation (18) to satisfy the performance (6) or (7) under the constraint from the stability condition derived in the last section. However, it can be seen from (18) that the derivation of the output spectrum of the nonlinear closed loop system (11) involves complicated symbolic manipulation and calculation especially when the orders involved are higher. In order to circumvent the symbolic computation complexity, a simulation-based method is provided in this section. The procedure of this method is described as follows:

(1) The system output frequency response function can be expressed as $Y(j\omega)Y(-j\omega) = |Y(j\omega)|^2 = C \cdot \widetilde{P}$ according to (23) with a finite polynomial order, where

$$C = [1 \quad c_1 \quad c_2 \quad c_3 \quad \cdots \quad c_{K!}]$$
$$= \left(CE(H_1^1(\cdot)) \oplus CE(H_2^1(\cdot)) \oplus \cdots \oplus CE(H_N^1(\cdot))\right) \otimes \left(CE(H_1^1(\cdot)) \oplus CE(H_2^1(\cdot)) \oplus \cdots \oplus CE(H_N^1(\cdot))\right) \quad (27)$$

(2) Collect the system steady output in the time domain $y_i(t)$ under different values of the controller parameters C_i=[1 $c_{1i}, c_{2i}, \ldots c_{(K!)i}$] for i=1,2,3,…N_i;

(3) Apply the FFT to $y_i(t)$ to obtain the frequency response function $Y_i(j\omega)$, then obtain the magnitude $|Y_i(j\omega_0)|^2$ at frequency ω_0, and finally form a vector $YY = [|Y_1(j\omega_0)|^2, \cdots, |Y_{N_i}(j\omega_0)|^2]^T$

(4) Obtain the equation $\psi_C \cdot \widetilde{P} = YY$,

$$\begin{bmatrix} 1, & c_{11}, & c_{12}, \cdots, c_{1,K!} \\ 1, & c_{21}, & c_{22}, \cdots, c_{2,K!} \\ \cdots, \cdots, & \cdots, & \cdots, \cdots \\ 1, c_{N_i 1}, & c_{N_i 2}, & \cdots, c_{N_i,K!} \end{bmatrix} \cdot \begin{bmatrix} \widetilde{P}_0 \\ \widetilde{P}_1 \\ \vdots \\ \widetilde{P}_{K!} \end{bmatrix} = \begin{bmatrix} |Y_1(j\omega_0)|^2 \\ |Y_2(j\omega_0)|^2 \\ \vdots \\ |Y_{N_i}(j\omega_0)|^2 \end{bmatrix} \quad (28)$$

(5) Evaluate the function $\widetilde{P}$ by using Least Squares,

$$\widetilde{P}(j\omega_0) = \left(\psi_C^{\ T} \cdot \psi_C\right)^{-1} \cdot \psi_C^{\ T} \cdot YY \quad (29)$$

(6) Finally, the desired nonlinear controller parameters C^* for any given Y^* at a specific frequency ω_0 can be determined according to

$$Y^* == C \cdot \widetilde{P}$$

The numerical method above is very effective for the implementation of the design of the proposed nonlinear feedback controllers, which will be verified by a simulation study in Part 2 of this paper.

4. A General Procedure for the Controller Design

Although there are some existing time domain methods which can address the nonlinear control problems based on Lyapunov stability theory such as the backstepping technique and feedback linearization (Alberto 1999) *etc*, few results have been achieved for the design of a nonlinear feedback controller to achieve a desired frequency domain performance. In this section, a general procedure for the design of the nonlinear feedback controller (5) for the nonlinear closed loop system (11) is described based on the fundamental results obtained in the last section. This procedure gives the general steps by which the nonlinear feedback controller of a plant with a periodical exogenous disturbance can be designed to achieve a predefined control objective as described in (6) or (7). Corresponding to the four basic problems as discussed in the last section, there are mainly five steps in the procedure:

(A) Determination of the structure of the nonlinear feedback function in (8).

The task is to determine the largest nonlinear order M and which of the nonlinear controller parameters $C_{p0}(.)$ (p=2,3,…,M) should be considered for the design. Based on the analysis of the parametric characteristics in Section 3.1, the nonlinear controller parameters included in C_{20} and C_{30} take a dominant role in the determination of GFRFs and output spectrum. Thus a larger M may not be necessary. Hence, M can be chosen as 2 or 3 at the beginning, and increased later if needed.

(B) Derivation of the region for the nonlinear controller parameters in $C_{p0}(.)$ (p=2,3,…,M).

This is to ensure the stability of the nonlinear closed loop system (11) or (12). Based on Theorem 1, a stability condition can be derived for the closed loop system in terms of the nonlinear controller parameters, which will define a region for the nonlinear controller parameters where the specific values are to be determined for those parameters to implement the controller design.

(C) Derivation of the system output spectrum.

This step is to derive a detailed polynomial expression for the output spectrum according to (18) and (23).

(D) Examination of the effectiveness of nonlinear parameters.

Arrange the output spectrum into a polynomial form as (25) with respect to each nonlinear parameter in C_{20} and C_{30}, respectively. For example, with respect to the parameter c

$$Y(j\omega) = \overline{P}_0(j\omega) + c\overline{P}_1(j\omega) + c^2\overline{P}_2(j\omega) + \cdots$$

According to Proposition 6, the effectiveness of the parameter c can be checked. If the parameter c is not effective, it will be discarded.

(E) Determination of the optimal values for the nonlinear controller parameters.

Use the numerical method provided in Section 3.4 to determine the desired value for each nonlinear controller parameter within the stability region obtained in Step (B) to achieve the control objective (6) or (7).

Following the above procedure, a nonlinear feedback controller can be achieved for the frequency domain control objective of the nonlinear closed loop system (11). It should be noted that the procedure just provides some gross guidance for the controller design. More systematic approaches based on this general procedure are under study and will be discussed in later publications.

5. Conclusions

Based on the frequency domain theory of nonlinear systems, a new approach to the design of nonlinear feedback controllers to suppress periodic disturbance for SISO linear plants is proposed. Some fundamental theoretical results have been established and developed for the controller design utilizing the frequency response functions of nonlinear systems. It is shown that a lower order nonlinearity feedback may be sufficient for many control problems. A general procedure was also proposed, which can be used as a useful guidance in practical controller design. The approach can be used to design a nonlinear feedback controller to achieve a desired frequency domain performance, and is therefore totally different from existing methods for nonlinear feedback control. Although the results in this paper are developed for the problem of periodic disturbance suppression for SISO linear plants, the idea can be extended to a more general case (*i.e.*, nonlinear controlled plants) and to address more complicated control problems. These will be the focus of our further research in this subject.

Part 2 of this paper will consider the practical application of the theoretical results and techniques developed in this paper through a case study.

APPENDIX: Proofs

PROOF OF PROPOSITION 1:

Consider the disturbance input of system (11) as a multi-tone function of time *t*, *i.e.*, $\eta(t)=\sum_{i=1}^{K}|F_i|\cos(\omega_i t+\angle F_i)$. In this case, $F(\omega_{k_l})\quad l=1,\cdots,n$ in the output frequency response (15) is equation (16a). Now Regarding x(t) and y(t) of the nonlinear system (11) as two outputs, *i.e.*, $y_1(t)=x(t)$, $y_2(t)=y(t)$, and the exogenous disturbance $\eta(t)$ as the input of system (11), we derive the GFRFs $H_n^j(j\omega_{k_1},\cdots,j\omega_{k_n})$ (for j=1,2 and n=1,2,3…) of the nonlinear system (11).

Consider equation (11a). It has only pure output nonlinearities in terms of $D^i x$, and a linear pure input term. Hence, the nth GFRF of (11a), denoted by $H_n^1(j\omega_1,\cdots,j\omega_n)$, involves two terms, respectively. Based on the results and methods in Billings and Peyton-Jones (1990), the nth order of GFRF of (11a) can be obtained as follows

$$H_n^1(j\omega_1,\cdots,j\omega_n)=\frac{-1}{\sum_{l_1=0}^{L}\overline{C}_{10}(l_1)(j\omega_1+\cdots+j\omega_n)^{l_1}}\left(\sum_{p=2}^{n}\sum_{l_1\cdots l_p=0}^{L}\overline{C}_{p0}(l_1\cdots l_p)H_{np}^1(j\omega_1,\cdots,j\omega_n)-\sum_{l_1\cdots l_n=0}^{L}\overline{C}_{0n}(l_1\cdots l_n)(j\omega_1)^{l_1}\cdots(j\omega_n)^{l_n}\right)\tag{A1}$$

where, $\overline{C}_{01}(0)=e$, all other $\overline{C}_{0n}(\cdot)=0$; $H_{np}^1(j\omega_1,\cdots,j\omega_n)$ and $H_{n1}^1(j\omega_1,\cdots,j\omega_n)$ are equation (13c) and (13d), respectively. Note that $\overline{C}_{01}(0)=e$ and all other $\overline{C}_{0n}(\cdot)=0$, the first order GFRF (linear frequency response function) is

$$H_1^1(j\omega_1)=\frac{\sum_{l_1=0}^{L}\overline{C}_{01}(l_1)(j\omega_1)^{l_1}}{\sum_{l_1=0}^{L}\overline{C}_{10}(l_1)(j\omega_1)^{l_1}}=\frac{e}{\sum_{l_1=0}^{L}\overline{C}_{10}(l_1)(j\omega_1)^{l_1}}\tag{A2}$$

Using (A2), equation (A1) can be rewritten into (17e).

Similarly, equation (11b) has one pure output nonlinearities in terms of $D^i x(t)$ and one linear pure output term *y*(*t*). Hence, the nth order of GFRF of the output y(t) has relation with the nth GFRF of the first "output" *x*(*t*), which can be obtained as

$$H_n^2(j\omega_1,\cdots,j\omega_n)=\sum_{p=1}^{n}\sum_{l_1\cdots l_p=0}^{L}\widetilde{C}_{p0}(l_1\cdots l_p)H_{np}^1(j\omega_1,\cdots,j\omega_n)$$

This is equation (17b). Therefore, substituting the GFRF (17b) into (15) for j=2, the nonlinear output frequency response function $Y_2(j\omega)$ of system (11) under the disturbance input (14) can be obtained

$$Y(j\omega)=Y_2(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n^2(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}):=\sum_{n=1}^{N}y_n(j\omega)$$

This completes the proof. □

PROOF OF LEMMA 1:

For convenience in discussion, substituting (19) into (17e), we have

$$
\begin{aligned}
H_n^1(j\omega_1,\cdots,j\omega_n) &= -\frac{1}{e}H_1^1(j\omega_1+\cdots+j\omega_n)\sum_{p=2}^{n}\sum_{l_1\cdots l_p=0}^{L}\overline{C}_{p0}(l_1\cdots l_p)H_{np}^1(j\omega_1,\cdots,j\omega_n) \\
&= -\frac{1}{e}H_1^1(j\omega_1+\cdots+j\omega_n)\sum_{p=2}^{n}\sum_{l_1\cdots l_p=0}^{L}\left(\overline{C}_{p0}(l_1\cdots l_p)\sum_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\prod_{i=1}^{p}H_{r_i}^1(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})(j\omega_{r_{X+1}}+\cdots+j\omega_{r_{X+r_i}})^{l_i}\right)
\end{aligned}
\tag{A3}
$$

It should be noted from (A3) that, $H_n^1(j\omega_1,\cdots,j\omega_n)$ includes all the possible combinations of $(r_1,r_2,\ldots,r_p)$ satisfying $\sum_{i=1}^{p}r_i=n$ and $1\le r_i\le n-p+1$, and so does CE($H_n^1(j\omega_1,\cdots,j\omega_n)$). Moreover, CE($H_1^1(j\omega_{r_x})$)=1 since there are no nonlinear control parameters in it, and any repetitive combinations in (21) have no contribution. Hence, CE($H_{np}^1(j\omega_1,\cdots,j\omega_n)$) should include all the possible non-repetitive combinations of $(r_1,r_2,\ldots,r_q)$ satisfying $\sum_{i=1}^{q}r_i=n-p+q$, $1\le r_i\le n-p+1$ and $1\le q\le p$. Note that each combination corresponds to a combination of the involved nonlinear control parameters. Including the nonlinear control parameter C_{p0}, CE($H_n^1(j\omega_1,\cdots,j\omega_n)$) therefore includes all the possible non-repetitive combination of the nonlinear control parameters $C_{p0}\otimes C_{r_1 0}\otimes C_{r_2 0}\otimes\cdots\otimes C_{r_q 0}$ satisfying $p+\sum_{i=1}^{q}r_i=n+q$, $2\le r_i\le n$, $0\le q\le n-2$ and $2\le p\le n$. □

PROOF OF LEMMA 2:

According to Lemma 1, $CE\left(H_{n-p+1}^1(j\omega_p,\cdots,j\omega_n)\right)$ includes all the terms $C_{r_1 0}\otimes C_{r_2 0}\otimes\cdots\otimes C_{r_{q'} 0}$ satisfying $\sum_{i=1}^{q'}r_i=n-p+1+q'-1=n-p+q'$, and $2\le r_i\le n-p+1$ for some q'. Note that q' is the number of different nonlinear control parameters in $C_{r_1 0}\otimes C_{r_2 0}\otimes\cdots\otimes C_{r_{q'} 0}$ with $r_i>1$. Equation (21) can be rewritten as

$$
\begin{aligned}
&CE(H_{np}^1(j\omega_1,\cdots,j\omega_n)) \\
&=\left(\bigoplus_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p}\left(CE(H_{r_1}^1(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}}))\otimes CE(H_{r_2}^1(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}}))\otimes\cdots\otimes CE(H_{r_p}^1(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}}))\right)\right) \\
&\quad\oplus CE(H_{n-p+1}^1(j\omega_p,\cdots,j\omega_n))
\end{aligned}
$$

From the proof of Lemma 1, the term before $CE(H_{n-p+1}^1(j\omega_p,\cdots,j\omega_n))$ can be written as

$$
\bigoplus_{\substack{r_1\cdots r_p=1\\ \sum r_i=n-p+q}}^{n-p}\left(CE(H_{r_1}^1(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}}))\otimes CE(H_{r_2}^1(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}}))\otimes\cdots\otimes CE(H_{r_q}^1(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}}))\right) \tag{A4}
$$

That is, all the terms in (A4) satisfy $\sum_{i=1}^{q'}r_i=n-p+1+q'-1=n-p+q'$, $2\le r_i\le n-p+1$ and $0\le q'\le p$. Hence, the terms in (A4) are included in $CE\left(H_{n-p+1}^1(j\omega_p,\cdots,j\omega_n)\right)$. The second point of this lemma directly follows from the discussion above and Lemma 1. This completes the proof. □

PROOF OF PROPOSITION 2:

(1). Applying CE operator to equation (13b),

$$
\begin{aligned}
CE(H_n^2(j\omega_1,\cdots,j\omega_n)) &= CE\left(\sum_{l_1\cdots l_p=0}^{L}\widetilde{C}_{p0}(l_1\cdots l_p)H_{np}^1(j\omega_1,\cdots,j\omega_n)+\sum_{p=2}^{n}\sum_{l_1\cdots l_p=0}^{L}\widetilde{C}_{p0}(l_1\cdots l_p)H_{np}^1(j\omega_1,\cdots,j\omega_n)\right)\\
&= CE\left(\sum_{l_1=0}^{L}(C_y(l_1)-C_{10}(l_1))H_{n1}^1(j\omega_1,\cdots,j\omega_n)+\sum_{p=2}^{n}\sum_{l_1\cdots l_p=0}^{L}(-d)C_{p0}(l_1\cdots l_p)H_{np}^1(j\omega_1,\cdots,j\omega_n)\right) \qquad \text{(A5)}\\
&= \begin{cases} 1 \text{ or } [\,] & n=1\\ \displaystyle\mathop{\oplus}_{p=2}^{n}\left(C_{p0}\otimes CE(H_{np}^1(j\omega_1,\cdots,j\omega_n))\right) & n>1\end{cases}
\end{aligned}
$$

Similarly, we can have the same result for $CE(H_n^1(j\omega_1,\cdots,j\omega_n))$. This completes the proof for (1).

(2). From (1), we have for n>1,

$$CE(H_n^1(j\omega_1,\cdots,j\omega_n)) = \mathop{\oplus}_{p=2}^{n}\left(C_{p0}\otimes CE(H_{np}^1(j\omega_1,\cdots,j\omega_n))\right)$$

Use Lemma 2, *i.e.*, $CE(H_{np}^1(j\omega_1,\cdots,j\omega_n)) = CE\left(H_{n-p+1}^1(j\omega_p,\cdots,j\omega_n)\right)$, giving

$$CE(H_n^1(j\omega_1,\cdots,j\omega_n)) = \mathop{\oplus}_{p=2}^{n}\left(C_{p0}\otimes CE(H_{n-p+1}^1(j\omega_p,\cdots,j\omega_n))\right)$$

Considering the symmetry of this equation, only half of the sum is enough to include all the possible combinations except the new term C_{n0}. Hence, we have $CE(H_n^1(\cdot)) = C_{n0}\oplus\mathop{\oplus}_{p=2}^{\left[n+1/2\right]}\left(C_{p0}\otimes CE(H_{n-p+1}^1(\cdot))\right)$. This completes the proof for (2). □

PROOF OF PROPOSITION 3:

(1) According to Lemma 1, the term $(C_{n0})^i$ should be included in the GFRF $H_m(.)$, where m is computed as $m+q=m+i-1=ni$. Hence we have $m= ni - i+1=1+(n-1)i$.

(2) These results can be directly obtained from Proposition 2, *i.e.*,

$$CE(H_2^1(\cdot)) = C_{20}\oplus\mathop{\oplus}_{p=2}^{\left[2+1/2\right]}\left(C_{20}\otimes CE(H_{2-p+1}^1(\cdot))\right) = C_{20} = 0$$

$$CE(H_3^1(\cdot)) = C_{30}\oplus\mathop{\oplus}_{p=2}^{\left[3+1/2\right]}\left(C_{20}\otimes CE(H_{3-p+1}^1(\cdot))\right) = C_{20}\otimes CE(H_{3-2+1}^1(\cdot))\oplus C_{30} = 0$$

For n>3, if C_{n0}=0, then

$$
\begin{aligned}
CE(H_n^1(\cdot)) &= C_{n0}\oplus\mathop{\oplus}_{p=2}^{\left[n+1/2\right]}\left(C_{p0}\otimes CE(H_{n-p+1}^1(\cdot))\right) = \mathop{\oplus}_{p=2}^{\left[n+1/2\right]}\left(C_{p0}\otimes CE(H_{n-p+1}^1(\cdot))\right)\\
&= C_{20}\otimes CE(H_{n-1}^1(\cdot))\oplus C_{30}\otimes CE(H_{n-2}^1(\cdot))\oplus\cdots
\end{aligned}
$$

It can be known from (1) that the nonzero terms in $(C_{n0})^i$ for n=2,3 and i>1 must be included in all higher order GFRFs than 2 or 3 respectively. Hence, $CE(H_n^1(\cdot))$ is not zero. □

PROOF OF PROPOSITION 4:

From (25), we have

$$\left|Y(j\omega)\right|^2 = Y(j\omega)Y(-j\omega) = (P_0(j\omega)+cP_1(j\omega)+c^2P_2(j\omega)+\cdots)(P_0(-j\omega)+cP_1(-j\omega)+c^2P_2(-j\omega)+\cdots)$$

$$= \langle P_0, P_0 \rangle + \sum_{n=1}^{\infty} \left(2c^n \Re(\sum_{\substack{0 \le i \le n-1 \\ i \le j \le n, i+j=n}} \langle P_i, P_j \rangle) \right)$$

From this equation, we have for $|Y| \neq 0$.

$$\frac{\partial |Y|}{\partial c} = \frac{1}{2|Y|} \frac{\partial |Y|^2}{\partial c} = \frac{1}{2|Y|} \sum_{n=1}^{\infty} \left(2nc^{n-1} \Re(\sum_{\substack{0 \le i \le n-1 \\ i \le j \le n, i+j=n}} \langle P_i, P_j \rangle) \right)$$

It can be seen that, if $\frac{\partial |Y(j\omega)|}{\partial c} < 0$ for some *c*, there must be some terms on the right side of the inequality is negative. This follows the first point of the proposition. For the second point of the proposition, it is obvious from the above equation. This completes the proof. □

PROOF OF THEOREM 1:

It follows from (26) that

$$\dot{V}(X(t)) \le -\alpha(\|X\|) + \gamma(\|\eta\|_\infty) \tag{A6}$$

Noting $V(X) \le \beta_2(\|X\|)$, we have $\|X\| \ge \beta_2^{-1}(V(X))$. Substituting this inequality into (A6), we have

$$\dot{V}(X(t)) \le -\alpha(\beta_2^{-1}(V(X))) + \gamma(\|\eta\|_\infty)$$

From lemma 4, it follows that, there exist a KL -function β, such that

$$V(X(t)) \le \beta(V_0, t) + \beta_2^{-1} \cdot \alpha^{-1} \cdot \gamma(\|\eta\|_\infty) \tag{A7}$$

where, $V_0 = \left| V(X(t_0)) - \beta_2^{-1} \cdot \alpha^{-1} \cdot \gamma(\|\eta\|_\infty) \right|$. From (A7), *V*(*X*(*t*)) is asymptotically stable to the ball $B_{\beta_2^{-1} \cdot \alpha^{-1} \cdot \gamma(\|\eta\|_\infty)}(V)$. Noting $\beta_1(\|X\|) \le V(X)$, we have $\|X\| \le \beta_1(V(X))$. From lemma 3, *X*(*t*) is asymptotically stable to the ball $B_\rho(X)$. Furthermore, since assumption 1 holds, from lemma 3, *y*(*t*) is asymptotically stable to the ball $B_{o(2\rho)}(y)$. This completes the proof of sufficiency. The proof of the necessity of the theorem can follow a similar method as demonstrated in the appendix of Hu, Teel and Lin (2005). The proof completes. □

Acknowledgement

The authors gratefully acknowledge the support of the Engineering and Physical Science Research Council, UK and the EPSRC-Hutchison Whampoa Dorothy Hodgkin Postgraduate Award, for this work.

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