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Abstract: The analysis of multi-degree-of-freedom (MDOF) nonlinear systems is studied using the concept of Nonlinear Output Frequency Response Functions (NOFRFs). The results reveal very important properties of MDOF systems, which are of significant importance for the analysis of nonlinear structures. One important application of the results obtained in this study is the detection and location of faults in engineering structures which make the structures behave nonlinearly.

1 Introduction

Linear system methods, which have been widely studied by practitioners in many different fields, provide a basis for the development of the majority of control system synthesis, mechanical system analysis and design, and signal processing algorithms. However, there are many qualitative behaviours in engineering, such as the generation of harmonics and inter-modulations, which cannot be produced by linear models [1]. In these cases, nonlinear models are needed to describe the system, and nonlinear system analysis methods have to be applied to investigate the system dynamics.

The Volterra series approach [2] is a powerful tool for the analysis of nonlinear systems, which extends the familiar concept of the convolution integral for linear systems to a series of multi-dimensional convolution integrals. The Fourier transforms of the Volterra kernels, called Generalised Frequency Response Functions (GFRFs) [3], are an extension of the linear Frequency Response Function (FRF) to the nonlinear case. If a differential equation or discrete-time model is available for a nonlinear system, the GFRFs can be determined using the algorithm in [4]~[6]. However, the GFRFs are multidimensional functions [7][8], and are much more complicated than the linear FRF and can be difficult to measure, display and interpret in practice. Recently, the novel concept known as

Nonlinear Output Frequency Response Functions (NOFRFs) was proposed by the authors [9]. Thus concept can be considered to be an alternative extension of the FRF to the nonlinear case. The NOFRFs are one dimensional functions of frequency. This allows the analysis of nonlinear systems in the frequency domain to be implemented in a manner similar to the analysis of linear systems and provides great insight into mechanisms which dominate important nonlinear behaviours.

In practice, many mechanical and structural systems can be described by MDOF models. In addition, these systems may also behave nonlinearly due to nonlinear characteristics of some components within the systems. For example, a beam with breathing cracks behaves nonlinearly because of the cracked elements inside the beam [10]. These nonlinear MDOF systems can be regarded as locally nonlinear MDOF systems. This paper is concerned with the study the properties of locally nonlinear MDOF systems using the concept of NOFRFs. The results reveal, for the first time, very important properties of these systems and are of significant importance for the analysis of nonlinear structural systems. One important application of the results obtained in the study is the detection and location of faults in engineering structures which make the structures behave nonlinearly.

2. Nonlinear Output Frequency Response Functions

2.1 Nonlinear Output Frequency Response Functions under General Inputs

The definition of the NOFRFs is based on the Volterra series theory of nonlinear systems. The Volterra series extends the well-known convolution integral description for linear systems to a series of multi-dimensional convolution integrals, which can be used to represent a wide class of nonlinear systems [3].

Consider the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,...,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \tag{1}$$

where $y(t)$ and $u(t)$ are the output and input of the system, $h_n(\tau_1,...,\tau_n)$ is the nth order Volterra kernel, and N denotes the maximum order of the system nonlinearity. Lang and Billings [3] derived an expression for the output frequency response of this class of nonlinear systems to a general input. The result is

$$\begin{cases} Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) \quad \text{for } \forall\,\omega \\ Y_n(j\omega) = \dfrac{1/\sqrt{n}}{(2\pi)^{n-1}} \displaystyle\int_{\omega_1+,\ldots,+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega} \end{cases} \tag{2}$$

This expression reveals how nonlinear mechanisms operate on the input spectra to produce the system output frequency response. In (2), $Y(j\omega)$ is the spectrum of the system output, $Y_n(j\omega)$ represents the *n*th order output frequency response of the system,

$$H_n(j\omega_1,\ldots,j\omega_n) = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)e^{-(\omega_1\tau_1+,\ldots,+\omega_n\tau_n)j}d\tau_1\ldots d\tau_n \tag{3}$$

is the *n*th order Generalised Frequency Response Function (GFRF) [3], and

$$\int_{\omega_1+,\ldots,+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega}$$

denotes the integration of $H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)$ over the n-dimensional hyper-plane $\omega_1+\cdots+\omega_n=\omega$. Equation (2) is a natural extension of the well-known linear relationship $Y(j\omega)=H(j\omega)U(j\omega)$, where $H(j\omega)$ is the frequency response function, to the nonlinear case.

For linear systems, the possible output frequencies are the same as the frequencies in the input. For nonlinear systems described by equation (1), however, the relationship between the input and output frequencies is more complicated. Given the frequency range of an input, the output frequencies of system (1) can be determined using the explicit expression derived by Lang and Billings in [3].

Based on the above results for the output frequency response of nonlinear systems, a new concept known as the Nonlinear Output Frequency Response Function (NOFRF) was recently introduced by Lang and Billings [9]. The NOFRF is defined as

$$G_n(j\omega) = \frac{\displaystyle\int_{\omega_1+,\ldots,+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega}}{\displaystyle\int_{\omega_1+,\ldots,+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega}} \tag{4}$$

under the condition that

$$U_n(j\omega) = \int_{\omega_1+,\ldots,+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega} \neq 0 \tag{5}$$

Notice that $G_n(j\omega)$ is valid over the frequency range of $U_n(j\omega)$, which can be determined using the algorithm in [3].

By introducing the NOFRFs $G_n(j\omega)$, $n=1,\cdots N$, equation (2) can be written as

$$Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) = \sum_{n=1}^{N} G_n(j\omega) U_n(j\omega) \tag{6}$$

which is similar to the description of the output frequency response for linear systems. For a linear system, the relationship between $Y(j\omega)$ and $U(j\omega)$ can be illustrated as shown in Figure 1. Similarly, the nonlinear system input and output relationship of Equation (6) can be illustrated as shown in Figure 2.

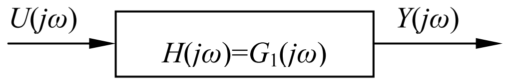

Figure 1. The output frequency response of a linear system

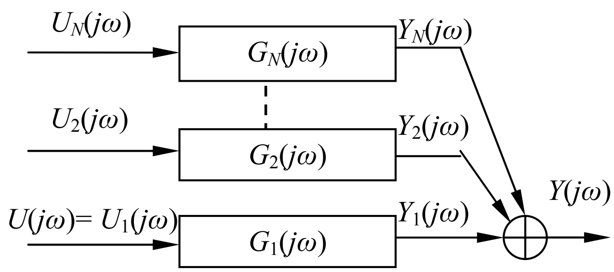

Figure 2. The output frequency response of a nonlinear system

The NOFRFs reflect a combined contribution of the system and the input to the system output frequency response behaviour. It can be seen from equation (4) that $G_n(j\omega)$ depends not only on H_n (n=1,…,N) but also on the input $U(j\omega)$. For a nonlinear system, the dynamical properties are determined by the GFRFs H_n (n= 1,…,N). However, from equation (3) it can be seen that the GFRFs are multidimensional [7][8], which can make the GFRFs difficult to measure, display and interpret in practice. Feijoo, Worden and Stanway [11][12] demonstrated that the Volterra series can be described by a series of associated linear equations (ALEs) whose corresponding associated frequency response functions (AFRFs) are easier to analyze and interpret than the GFRFs. According to equation (4), the NOFRF $G_n(j\omega)$ is a weighted sum of $H_n(j\omega_1,...,j\omega_n)$ over $\omega_1+\cdots+\omega_n=\omega$ with the weights depending on the test input. Therefore $G_n(j\omega)$ can be used as an alternative representation of the dynamical properties described by H_n. The most important property of the NOFRF $G_n(j\omega)$ is that it is one dimensional, and thus allows the analysis of nonlinear systems to be implemented in a convenient manner similar to the analysis of linear systems. Moreover, there is an effective algorithm [9] available which allows the estimation of the NOFRFs to be implemented directly using system input output data.

2.2 Nonlinear Output Frequency Response Functions under Harmonic Input

Harmonic inputs are pure sinusoidal signals which have been widely used for the dynamic testing of many engineering structures. Therefore, it is necessary to extend the NOFRF concept to the harmonic input case.

When system (1) is subject to a harmonic input

$$u(t) = A\cos(\omega_F t + \beta) \tag{7}$$

Lang and Billings [3] showed that equation (2) can be expressed as

$$Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) = \sum_{n=1}^{N}\left(\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega} H_n(j\omega_{k_1},\cdots,j\omega_{k_n})A(j\omega_{k_1})\cdots A(j\omega_{k_n})\right) \tag{8}$$

where

$$A(j\omega_{k_i}) = \begin{cases} |A|e^{j\mathrm{sign}(k)\beta} & if \quad \omega_{k_i} \in \{k\omega_F, k=\pm 1\}, i=1,\cdots,n \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Define the frequency components of the *n*th order output of the system as Ω_n, then according to equation (8), the frequency components in the system output can be expressed as

$$\Omega = \bigcup_{n=1}^{N} \Omega_n \tag{10}$$

where Ω_n is determined by the set of frequencies

$$\{\omega = \omega_{k_1} + \cdots + \omega_{k_n} \mid \omega_{k_i} = \pm\omega_F, i=1,\cdots,n\} \tag{11}$$

From equation (11), it is known that if all $\omega_{k_1},\cdots,\omega_{k_n}$ are taken as $-\omega_F$, then $\omega = -n\omega_F$. If *k* of these are taken as ω_F, then $\omega = (-n+2k)\omega_F$. The maximal *k* is *n*. Therefore the possible frequency components of $Y_n(j\omega)$ are

$$\Omega_n = \{(-n+2k)\omega_F, k=0,1,\cdots,n\} \tag{12}$$

Moreover, it is easy to deduce that

$$\Omega = \bigcup_{n=1}^{N} \Omega_n = \{k\omega_F, k=-N,\cdots,-1,0,1,\cdots,N\} \tag{13}$$

Equation (13) explains why superharmonic components are generated when a nonlinear system is subjected to a harmonic excitation. In the following, only those components with positive frequencies will be considered.

The NOFRFs defined in equation (4) can be extended to the case of harmonic inputs as

$$G_n^H(j\omega) = \frac{\dfrac{1}{2^n}\displaystyle\sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega} H_n(j\omega_{k_1},\cdots,j\omega_{k_n})A(j\omega_{k_1})\cdots A(j\omega_{k_n})}{\dfrac{1}{2^n}\displaystyle\sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega} A(j\omega_{k_1})\cdots A(j\omega_{k_n})} \qquad n = 1,\ldots,N \tag{14}$$

under the condition that

$$A_n(j\omega) = \frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega} A(j\omega_{k_1})\cdots A(j\omega_{k_n}) \neq 0 \tag{15}$$

Obviously, $G_n^H(j\omega)$ is only valid over Ω_n defined by equation (12). Consequently, the output spectrum $Y(j\omega)$ of nonlinear systems under a harmonic input can be expressed as

$$Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) = \sum_{n=1}^{N} G_n^H(j\omega)\, A_n(j\omega) \tag{16}$$

When k of the n frequencies of $\omega_{k_1},\cdots,\omega_{k_n}$ are taken as ω_F and the remainders are as $-\omega_F$, substituting equation (9) into equation (15) yields,

$$A_n(j(-n+2k)\omega_F) = \frac{1}{2^n} |A|^n\ e^{j(-n+2k)\beta} \tag{17}$$

Thus $G_n^H(j\omega)$ becomes

$$\begin{aligned} G_n^H(j(-n+2k)\omega_F) &= \frac{\frac{1}{2^n} H_n(\overbrace{j\omega_F,\cdots,j\omega_F}^{k},\overbrace{-j\omega_F,\cdots,-j\omega_F}^{n-k})\,|A|^n\ e^{j(-n+2k)\beta}}{\frac{1}{2^n}|A|^n\ e^{j(-n+2k)\beta}} \\ &= H_n(\overbrace{j\omega_F,\cdots,j\omega_F}^{k},\overbrace{-j\omega_F,\cdots,-j\omega_F}^{n-k}) \end{aligned} \tag{18}$$

where $H_n(j\omega_1,...,j\omega_n)$ is assumed to be a symmetric function. Therefore, in this case, $G_n^H(j\omega)$ over the nth order output frequency range $\Omega_n = \{(-n+2k)\omega_F, k=0,1,\cdots,n\}$ is equal to the GFRF $H_n(j\omega_1,...,j\omega_n)$ evaluated at $\omega_1=\cdots=\omega_k=\omega_F$, $\omega_{k+1}=\cdots=\omega_n=-\omega_F$, $k=0,\cdots,n$.

3. Analysis of Locally Nonlinear MDOF Systems Using NOFRFs

3.1 Locally Nonlinear MDOF Systems

Without loss of generality and for convenience of analysis, consider an undamped multi-degree-of-freedom oscillator as shown in Figure 3.

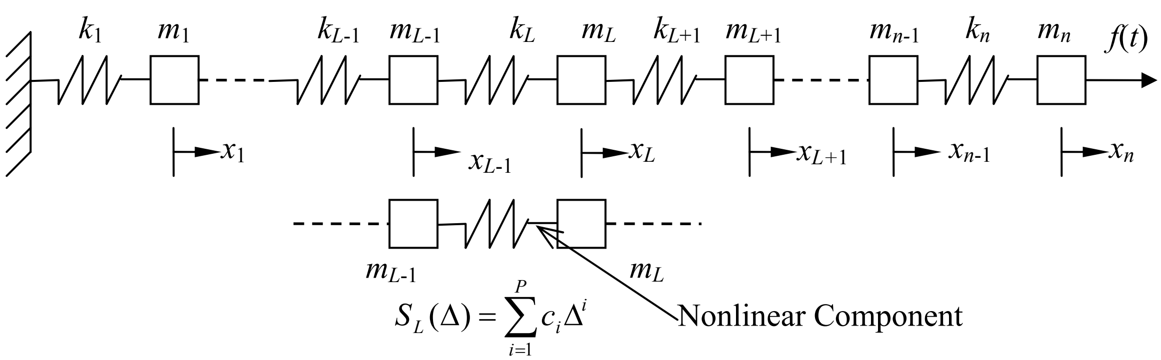

Figure 3, a multi-degree freedom oscillator

If all the springs have linear stiffness, then the governing motion equation of the MDOF oscillator in Figure 3 can be written as

$$M\ddot{x} + Kx = F(t) \tag{19}$$

where

$$M = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix} \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -k_{n-1} & k_{n-1} + k_n & -k_n \\ 0 & \cdots & 0 & -k_n & k_n \end{bmatrix}$$

are the system mass matrix and stiffness matrix respectively. $x = (x_1, \cdots, x_n)'$ is the displacement vector, and $F(t) = (0, \cdots, 0, f(t))'$ is the external force vector acting on the oscillator.

Equation (19) is often used for a simplified modal analysis. Modal analysis is a well-established approach for determining dynamic characteristics of engineering structures [13]. In the linear case, the displacements $x_i(t)$ ($i = 1, \cdots, n$) can be expressed as

$$x_i(t) = \int_{-\infty}^{+\infty} h_{(i)}(t-\tau) f(\tau) d\tau \tag{20}$$

where $h_{(i)}(t)$ ($i = 1, \cdots, n$) are the impulse response functions that are determined by equation (19), and the Fourier transform of $h_{(i)}(t)$ is the well-known FRF.

Consider the case where one of the springs, say the Lth spring, has a nonlinear stiffness, and, as shown in Figure 3, assume the restoring force $S_L(\Delta)$ of the spring is a polynomial function of the deformation Δ, i.e.,

$$S_L(\Delta) = \sum_{i=1}^{P} c_i \Delta^i \tag{21}$$

where P is the degree of the polynomial. Without loss of generality, assume $L \neq 1$ and $L \neq n$ and $k_L = c_1$. Then for the masses which are not connected to the Lth spring, the governing motion equations are

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \tag{22}$$

$$m_i\ddot{x}_i + (k_i + k_{i+1})x_i - k_i x_{i-1} - k_{i+1}x_{i+1} = 0 \quad (i \neq L-1 \text{ and } i \neq L) \tag{23}$$

$$m_n\ddot{x}_n + k_n x_n - k_n x_{n-1} = f(t) \tag{24}$$

For the mass that is connected to the left end of the Lth spring, the governing motion equation is

$$m_{L-1}\ddot{x}_{L-1} + (k_{L-1} + k_L)x_{L-1} - k_{L-1}x_{L-2} - k_L x_L + \sum_{i=2}^{P} c_i (x_{L-1} - x_L)^i = 0 \tag{25}$$

For the mass that connects to the right end of the Lth spring, the governing motion equation is

$$m_L \ddot{x}_L + (k_L + k_{L+1})x_L - k_L x_{L-1} - k_{L+1}x_{L+1} - \sum_{i=2}^{P} c_i (x_{L-1} - x_L)^i = 0 \tag{26}$$

Denote

$$NF = \begin{pmatrix} \overbrace{0 \cdots 0}^{L-2} & \sum_{i=2}^{P} c_i (x_{L-1} - x_L)^i & -\sum_{i=2}^{P} c_i (x_{L-1} - x_L)^i & \overbrace{0 \cdots 0}^{n-L} \end{pmatrix}' \tag{27}$$

Then, equations (22)~(27) can be written in a matrix form as

$$M\ddot{x} + Kx = -NF + F(t) \tag{28}$$

The system described by equations (27)(28) is a typical locally nonlinear MDOF system. The Lth nonlinear spring component can lead the whole system to behave nonlinearly. In this case, the Volterra series can be used to describe the relationships between the displacements $x_i(t)$ ($i = 1, \cdots, n$) and the input force $f(t)$ as below

$$x_i(t) = \sum_{j=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{(i,j)}(\tau_1, ..., \tau_j) \prod_{i=1}^{j} f(t - \tau_i) d\tau_i \tag{29}$$

under quite general conditions [3]. In equation (29), $h_{(i,j)}(\tau_1, ..., \tau_j)$, ($i = 1, \cdots, n$, $j = 1, \cdots, N$), represents the jth order Volterra kernel for the relationship between $f(t)$ and the displacement of m_i. The Fourier transform of $h_{(i,j)}(\tau_1, ..., \tau_j)$ is the corresponding GFRF $H_{(i,j)}(j\omega_1, ..., j\omega_j)$ ($i = 1, \cdots, n$, $j = 1, \cdots, N$).

3.2 GFRFs of Locally Nonlinear MDOF System

From equations (22)~(26), the GFRFs $H_{(i,j)}(j\omega_1, ..., j\omega_j)$, ($i = 1, \cdots, n$, $j = 1, \cdots, N$) can be determined using the harmonic probing method [5][6].

First consider the input $f(t)$ is of a single harmonic

$$f(t) = e^{j\omega t} \tag{30}$$

Substituting (30) and

$$x_i(t) = H_{(i,1)}(j\omega) e^{j\omega t} \qquad (i = 1, \cdots, n) \tag{31}$$

into equation (28) and extracting the coefficients of $e^{j\omega t}$ yields

$$\left(-M\omega^2 + K\right) H_1(j\omega) = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}' \tag{32}$$

where

$$H_1(j\omega) = \begin{pmatrix} H_{(1,1)}(j\omega) & \cdots & H_{(n,1)}(j\omega) \end{pmatrix}' \tag{33}$$

From equation (32), it is known that

$$H_1(j\omega) = \left(-M\omega^2 + K\right)^{-1} \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}' \tag{34}$$

Denote

$$\Theta(j\omega) = -M\omega^2 + K \tag{35}$$

and

$$\Theta^{-1}(j\omega)=\begin{pmatrix} Q_{(1,1)}(j\omega) & \cdots & Q_{(1,n)}(j\omega) \\ \vdots & \ddots & \vdots \\ Q_{(n,1)}(j\omega) & \cdots & Q_{(n,n)}(j\omega) \end{pmatrix} \tag{36}$$

From (34)~(36), it follows that

$$H_{(i,1)}(j\omega)=Q_{(i,n)}(j\omega) \qquad (i=1,\cdots,n) \tag{37}$$

Thus, for any two consecutive masses, the relationship between the first order GFRFs can be expressed as

$$\frac{H_{(i,1)}(j\omega)}{H_{(i+1,1)}(j\omega)}=\frac{Q_{(i,n)}(j\omega)}{Q_{(i+1,n)}(j\omega)} \qquad (i=1,\cdots,n-1) \tag{38}$$

Moreover, from the first row of equation (32), it can be shown that

$$-m_1\omega^2 H_{(1,1)}(j\omega)+(k_1+k_2)H_{(1,1)}(j\omega)-k_2 H_{(2,1)}(j\omega)=0 \tag{39}$$

From equations (38) and (39), the relationship between $H_{(1,1)}(j\omega)$ and $H_{(2,1)}(j\omega)$ can be described as

$$\frac{H_{(1,1)}(j\omega)}{H_{(2,1)}(j\omega)}=\frac{Q_{(1,n)}(j\omega)}{Q_{(2,n)}(j\omega)}=\frac{k_2}{\left(-m_1\omega^2+k_1+k_2\right)}=\lambda_1^{1,2}(j\omega) \tag{40}$$

Similarly, for the other first order GFRFs, the relationship between the GFRFs of any two consecutive masses can be expressed as

$$\frac{H_{(i,1)}(j\omega)}{H_{(i+1,1)}(j\omega)}=\frac{Q_{(i,n)}(j\omega)}{Q_{(i+1,n)}(j\omega)}=\frac{k_{i+1}}{\left[-m_i\omega^2+\left(1-\lambda_1^{i-1,i}(j\omega)\right)k_i+k_{i+1}\right]}=\lambda_1^{i,i+1}(j\omega)$$
$$(i=2,\cdots,n-1) \tag{41}$$

Denote $\lambda_1^{0,1}(j\omega)=0$. Then equations (40) and (41) can be written together as

$$\frac{H_{(i,1)}(j\omega)}{H_{(i+1,1)}(j\omega)}=\frac{Q_{(i,n)}(j\omega)}{Q_{(i+1,n)}(j\omega)}=\frac{k_{i+1}}{\left[-m_i\omega^2+\left(1-\lambda_1^{i-1,i}(j\omega)\right)k_i+k_{i+1}\right]}=\lambda_1^{i,i+1}(j\omega)$$
$$(i=1,\cdots,n-1) \tag{42}$$

Equation (42) gives a comprehensive description of the relationships between the *n* first order GFRFs of the locally nonlinear *n*-DOF systems.

The above procedure used to analyze the relationships of the first order GFRFs can be extended to investigate the relationship between the $\overline{N}$ th order GFRFs with $\overline{N}\geq 2$. To achieve this, consider the input

$$f(t)=\sum_{k=1}^{\overline{N}} e^{j\omega_k t} \tag{43}$$

Substituting (43) and

$$\begin{aligned} x_i(t)=&H_{(i,1)}(j\omega_1)e^{j\omega_1 t}+\cdots+H_{(i,1)}(j\omega_{\overline{N}})e^{j\omega_N t}+\cdots \\ &+\overline{N}!H_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})e^{j(\omega_1+\cdots+\omega_{\overline{N}})t}+\cdots \end{aligned} \qquad (i=1,\cdots,n) \tag{44}$$

into (22)~(26) and extracting the coefficients of $e^{j(\omega_1+\cdots+\omega_N)t}$ yields

$$-m_1(\omega_1+\cdots+\omega_{\overline{N}})^2 H_{(1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})+(k_1+k_2)H_{(1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}}) \\ -k_2 H_{(2,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})=0 \tag{45}$$

$$-m_n(\omega_1+\cdots+j\omega_{\overline{N}})^2 H_{(n,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})+k_n H_{(n,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}}) \\ -k_n H_{(n-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})=0 \tag{46}$$

$$-m_i(\omega_1+\cdots+\omega_{\overline{N}})^2 H_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})+(k_i+k_{i+1})H_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}}) \\ -k_i H_{(i-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})-k_{i+1}H_{(i+1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})=0 \\ (i\neq 1,L-1,L,n) \tag{47}$$

$$-m_{L-1}(\omega_1+\cdots+\omega_{\overline{N}})^2 H_{(L-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})+(k_{L-1}+k_L)H_{(L-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}}) \\ -k_{L-1}H_{(L-2,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})-k_L H_{(L,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})+\Lambda_{\overline{N}}^{L-1,L}(j\omega_1,\cdots,j\omega_{\overline{N}})=0 \tag{48}$$

$$-m_L(\omega_1+\cdots+\omega_{\overline{N}})^2 H_{(L,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})+(k_L+k_{L+1})H_{(L,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}}) \\ -k_L H_{(L-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})-k_{L+1}H_{(L+1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})-\Lambda_{\overline{N}}^{L-1,L}(j\omega_1,\cdots,j\omega_{\overline{N}})=0 \tag{49}$$

In equations (48) and (49), $\Lambda_N^{L-1,L}(j\omega_1,\cdots,j\omega_N)$ represents the extra terms introduced by $\sum_{i=2}^{P} c_i(x_{L-1}-x_L)^i$ for the *N*th order GFRFs, for example, for the second order GFRFs,

$$\Lambda_2^{L-1,L}(j\omega_1,j\omega_2)=c_2\Big(H_{(L-1,1)}(j\omega_1)H_{(L-1,1)}(j\omega_2)+H_{(L,1)}(j\omega_1)H_{(L,1)}(j\omega_2) \\ -H_{(L-1,1)}(j\omega_1)H_{(L,1)}(j\omega_2)-H_{(L-1,1)}(j\omega_2)H_{(L,1)}(j\omega_1)\Big) \tag{50}$$

Denote

$$H_{\overline{N}}(j\omega_1,\cdots,j\omega_{\overline{N}})=\Big(H_{(1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}}) \quad \cdots \quad H_{(n,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})\Big)' \tag{51}$$

and

$$\Lambda_{\overline{N}}(j\omega_1,\cdots,j\omega_{\overline{N}})=\begin{bmatrix}\overbrace{0\cdots0}^{L-2} & \Lambda_{\overline{N}}^{L-1,L}(j\omega_1,\cdots,j\omega_{\overline{N}}) & -\Lambda_{\overline{N}}^{L-1,L}(j\omega_1,\cdots,j\omega_{\overline{N}}) & \overbrace{0\cdots0}^{n-L}\end{bmatrix}' \tag{52}$$

Then equations (45)~(49) can be written in a matrix form as

$$\Theta(j(\omega_1+\cdots+\omega_{\overline{N}}))H_{\overline{N}}(j\omega_1,\cdots,j\omega_{\overline{N}})=\Lambda_{\overline{N}}(j\omega_1,\cdots,j\omega_{\overline{N}}) \tag{53}$$

so that

$$H_{\overline{N}}(j\omega_1,\cdots,j\omega_{\overline{N}})=\Theta^{-1}(j(\omega_1+\cdots+\omega_{\overline{N}}))\Lambda_{\overline{N}}(j\omega_1,\cdots,j\omega_{\overline{N}}) \tag{54}$$

Therefore, for each mass, the $\overline{N}$ th order GFRF can be calculated as

$$H_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}}) \\ =\Big(Q_{i,L-1}(j(\omega_1+\cdots+\omega_{\overline{N}})),Q_{i,L}(j(\omega_1+\cdots+\omega_{\overline{N}})\Big)\begin{pmatrix}\Lambda_{\overline{N}}^{L-1,L}(j\omega_1,\cdots,j\omega_{\overline{N}}) \\ -\Lambda_{\overline{N}}^{L-1,L}(j\omega_1,\cdots,j\omega_{\overline{N}})\end{pmatrix} \\ (i=1,\cdots,n) \tag{55}$$

and consequently, for two consecutive masses, the $\overline{N}$ th order GFRFs have the following relationships

$$\frac{H_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(i+1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}=\frac{Q_{i,L-1}(j(\omega_1+\cdots+\omega_{\overline{N}}))-Q_{i,L}(j(\omega_1+\cdots+\omega_{\overline{N}}))}{Q_{i+1,L-1}(j(\omega_1+\cdots+\omega_{\overline{N}}))-Q_{i+1,L}(j(\omega_1+\cdots+\omega_{\overline{N}}))}$$

$$=Q^{i,i+1}(j(\omega_1+\cdots+\omega_{\overline{N}}))=\lambda_{\overline{N}}^{i,i+1}(\omega_1+\cdots+\omega_{\overline{N}})\qquad(i=1,\cdots,n-1)\quad(56)$$

From equation (45), it is known that

$$\frac{H_{(1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(2,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}=\frac{k_2}{\left[-m_1(\omega_1+\cdots+\omega_{\overline{N}})^2+k_1+k_2\right]}$$

$$=\lambda_{\overline{N}}^{1,2}(\omega_1,\cdots,\omega_{\overline{N}})\quad(57)$$

Moreover, from equation (47), it can be deduced that

$$\frac{H_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(i+1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}=\frac{k_{i+1}}{\left[-m_i(\omega_1+\cdots+\omega_{\overline{N}})^2+\left(1-\lambda_{\overline{N}}^{i-1,i}(\omega_1+\cdots+\omega_{\overline{N}})\right)k_i+k_{i+1}\right]}$$

$$=\lambda_{\overline{N}}^{i,i+1}(\omega_1+\cdots+\omega_{\overline{N}})\qquad(1<i<L-1\ \text{and}\ L<i<n)\ (58)$$

For the two masses that are connected to the nonlinear spring, denote

$$\frac{k_L}{\left[-m_{L-1}(\omega_1+\cdots+\omega_{\overline{N}})^2+\left(1-\lambda_{\overline{N}}^{L-2,L-1}(\omega_1+\cdots+\omega_{\overline{N}})\right)k_{L-1}+k_L\right]}=\overline{\lambda}_{\overline{N}}^{L-1,L}(\omega_1+\cdots+\omega_{\overline{N}})\quad(59)$$

$$\frac{k_{L+1}}{\left[-m_L(\omega_1+\cdots+\omega_{\overline{N}})^2+\left(1-\lambda_{\overline{N}}^{L-1,L}(\omega_1+\cdots+\omega_{\overline{N}})\right)k_L+k_{L+1}\right]}=\overline{\lambda}_{\overline{N}}^{L,L+1}(\omega_1+\cdots+\omega_{\overline{N}})\quad(60)$$

Then, from equations (48),(49), it can be known that

$$\frac{H_{(L-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(L,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}=\overline{\lambda}_{\overline{N}}^{L-1,L}(\omega_1+\cdots+\omega_{\overline{N}})\left(1-\frac{1}{k_L}\frac{\Lambda_{\overline{N}}^{L-1,L}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(L,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}\right)$$

$$=\lambda_{\overline{N}}^{L-1,L}(\omega_1+\cdots+\omega_{\overline{N}})\quad(61)$$

$$\frac{H_{(L,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(L+1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}=\overline{\lambda}_{\overline{N}}^{L,L+1}(\omega_1+\cdots+\omega_{\overline{N}})\left(1+\frac{1}{k_{L+1}}\frac{\Lambda_{\overline{N}}^{L-1,L}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(L+1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}\right)$$

$$=\lambda_{\overline{N}}^{L,L+1}(\omega_1+\cdots+\omega_{\overline{N}})\quad(62)$$

Equations (56)~(62) give a comprehensive description of the relationships between the n $\overline{N}$ th order GFRFs of the locally nonlinear n-DOF system.

It can be seen that $\lambda_1^{i,i+1}(\omega_1)$ and $\lambda_{\overline{N}}^{i,i+1}(\omega_1+\cdots+\omega_{\overline{N}})$ have the same form. Thus, by denoting

$$\Lambda_1^{L-1,L}(j\omega)=0\ \text{and}\ \lambda_{\overline{N}}^{0,1}(\omega_1+\cdots+\omega_{\overline{N}})=0\qquad(1\le\overline{N}\le N)\ (63)$$

the relationship between GFRFs up to Nth order of two consecutive masses of the locally nonlinear n-DOF system can be summarized as

$$\lambda_{\overline{N}}^{i,i+1}(\omega_1+\cdots+\omega_{\overline{N}})=\frac{k_{i+1}}{\left[-m_i(\omega_1+\cdots+\omega_{\overline{N}})^2+\left(1-\lambda_{\overline{N}}^{i-1,i}(\omega_1+\cdots+\omega_{\overline{N}})\right)k_i+k_{i+1}\right]}$$

$$(1\le \overline{N}\le N,\ 1\le i<n, i\ne L-1,L\ ) \quad (64)$$

and

$$\lambda_{\overline{N}}^{L-1,L}(\omega_1+\cdots+\omega_{\overline{N}})=\overline{\lambda}_{\overline{N}}^{L-1,L}(\omega_1+\cdots+\omega_{\overline{N}})\left(1-\frac{1}{k_L}\frac{\Lambda_{\overline{N}}^{L-1,L}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(L,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}\right)$$

$$(1\le \overline{N}\le N) \quad (65)$$

$$\lambda_{\overline{N}}^{L,L+1}(\omega_1+\cdots+\omega_{\overline{N}})=\overline{\lambda}_{\overline{N}}^{L,L+1}(\omega_1+\cdots+\omega_{\overline{N}})\left(1+\frac{1}{k_{L+1}}\frac{\Lambda_{\overline{N}}^{L-1,L}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(L+1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}\right)$$

$$(1\le \overline{N}\le N) \quad (66)$$

3.3 NOFRFs of Locally Nonlinear MDOF Systems

According to the definition of NOFRFs in equation (4), the $\overline{N}$ th order NOFRF of the *i*th mass can be expressed as

$$G_{(i,\overline{N})}(j\omega)=\frac{\int_{\omega_1+,\ldots,+\omega_{\overline{N}}=\omega}H_{(i,\overline{N})}(j\omega_1,\ldots,j\omega_{\overline{N}})\prod_{q=1}^{\overline{N}}F(j\omega_q)d\sigma_{\overline{N}\omega}}{\int_{\omega_1+,\ldots,+\omega_{\overline{N}}=\omega}\prod_{q=1}^{\overline{N}}F(j\omega_q)d\sigma_{\overline{N}\omega}}$$

$$(1\le \overline{N}\le N,\ 1\le i\le n) \quad (67)$$

where $F(j\omega)$ is the Fourier transform of $f(t)$.

According to equation (56), for any $\overline{N}\ge 2$, equation (67) can be rewritten as

$$G_{(i,\overline{N})}(j\omega)=\frac{\int_{\omega_1+,\ldots,+\omega_{\overline{N}}=\omega}Q^{i,i+1}(j(\omega_1+\cdots+\omega_{\overline{N}}))H_{(i+1,\overline{N})}(j\omega_1,\ldots,j\omega_{\overline{N}})\prod_{q=1}^{\overline{N}}F(j\omega_q)d\sigma_{\overline{N}\omega}}{\int_{\omega_1+,\ldots,+\omega_{\overline{N}}=\omega}\prod_{q=1}^{\overline{N}}F(j\omega_q)d\sigma_{\overline{N}\omega}}$$

$$=Q^{i,i+1}(j\omega)G_{(i+1,\overline{N})}(j\omega)=\lambda_{\overline{N}}^{i,i+1}(\omega)G_{(i+1,\overline{N})}(j\omega)$$

$$(2\le \overline{N}\le N,\ 1\le i\le n) \quad (68)$$

For $\overline{N}$ = 1, it is known from equation (42) that

$$G_{(i,1)}(j\omega)=\lambda_1^{i,i+1}(\omega)G_{(i+1,1)}(j\omega) \qquad (1\le i<n) \quad (69)$$

Moreover, denote $\lambda_{\overline{N}}^{0,1}(\omega)=0$, for the first mass, it can be deduced that

$$\frac{G_{(1,\overline{N})}(j\omega)}{G_{(2,\overline{N})}(j\omega)}=\frac{k_2}{\left(-m_1\omega^2+(1-\lambda_{\overline{N}}^{0,1}(\omega))k_1+k_2\right)}=\lambda_{\overline{N}}^{1,2}(\omega) \qquad \left(1\le \overline{N}\le N\right) \quad (70)$$

Thus, for all $(L-1)$th and Lth masses, the following relationship can be established,

$$\frac{G_{(i,\overline{N})}(j\omega)}{G_{(i+1,\overline{N})}(j\omega)}=\frac{k_{i+1}}{\left(-m_i\omega^2+(1-\lambda_{\overline{N}}^{i-1,i}(\omega))k_i+k_{i+1}\right)}=\lambda_{\overline{N}}^{i,i+1}(\omega)$$
$$\left(1\le\overline{N}\le N,\ 1\le i<n, i\ne L-1, L\right)\quad(71)$$

For the (L-1)th and Lth masses

$$\frac{G_{(L-1,\overline{N})}(j\omega)}{G_{(L,\overline{N})}(j\omega)}=\overline{\lambda}_{\overline{N}}^{L-1,L}(\omega)\left(1-\frac{1}{k_L}\frac{\Gamma_{(L-1,\overline{N})}(j\omega)}{G_{(L,\overline{N})}(j\omega)}\right)=\lambda_{\overline{N}}^{L-1,L}(\omega)\qquad\left(1\le\overline{N}\le N\right)\quad(72)$$

$$\frac{G_{(L,\overline{N})}(j\omega)}{G_{(L+1,\overline{N})}(j\omega)}=\overline{\lambda}_{\overline{N}}^{L,L+1}(\omega)\left(1+\frac{1}{k_{L+1}}\frac{\Gamma_{(L-1,\overline{N})}(j\omega)}{G_{(L+1,\overline{N})}(j\omega)}\right)=\lambda_{\overline{N}}^{L,L+1}(\omega)\qquad\left(1\le\overline{N}\le N\right)\quad(73)$$

where

$$\frac{k_L}{\left[-m_{L-1}\omega^2+\left(1-\lambda_{\overline{N}}^{L-2,L-1}(\omega)\right)k_{L-1}+k_L\right]}=\overline{\lambda}_{\overline{N}}^{L-1,L}(\omega)\qquad\left(1\le\overline{N}\le N\right)\quad(74)$$

$$\frac{k_{L+1}}{\left[-m_L\omega^2+\left(1-\lambda_{\overline{N}}^{L-1,L}(\omega)\right)k_L+k_{L+1}\right]}=\overline{\lambda}_{\overline{N}}^{L,L+1}(\omega)\qquad\left(1\le\overline{N}\le N\right)\quad(75)$$

and

$$\Gamma_{(L-1,\overline{N})}(j\omega)=\frac{\int_{\omega_1+,\ldots,+\omega_{\overline{N}}=\omega}\Lambda_{\overline{N}}^{L-1,L}(j\omega_1,\cdots,j\omega_{\overline{N}})\prod_{q=1}^{\overline{N}}F(j\omega_q)d\sigma_{\overline{N}\omega}}{\int_{\omega_1+,\ldots,+\omega_{\overline{N}}=\omega}\prod_{q=1}^{\overline{N}}F(j\omega_q)d\sigma_{\overline{N}\omega}}\qquad\left(1\le\overline{N}\le N\right)\quad(76)$$

Equations (71)~(76) give a comprehensive description of the relationships between the n $\overline{N}$ th order NOFRFs of two consecutive masses of the locally nonlinear n-DOF system.

3.4 The Properties of NOFRFs of Locally Nonlinear MDOF Systems

Important properties of the NOFRFs of locally nonlinear n-DOF systems can be obtained from the equation (69)~(76), as follows

i) For the masses on the left of the nonlinear spring, excluding the (L-1)th mass

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)}=\cdots=\frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)}\qquad(1\le i<L-1)\quad(77)$$

that is

$$\lambda_1^{i,i+1}(\omega)=\cdots=\lambda_N^{i,i+1}(\omega)=\lambda^{i,i+1}(\omega)\qquad(1\le i<L-1)\quad(78)$$

where $\lambda^{0,1}(\omega)=0$, and

$$\lambda^{i,i+1}(\omega)=\frac{k_{i+1}}{\left(-m_i\omega^2+(1-\lambda^{i-1,i}(\omega))k_i+k_{i+1}\right)}\qquad(1\le i<L-1)\quad(79)$$

ii) For the masses on the right of the nonlinear spring, including the (L-1)th mass

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)}\ne\frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)}=\cdots=\frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)}\qquad(L-1\le i<n)\quad(80)$$

that is

$$\lambda_1^{i,i+1}(\omega) \neq \lambda_2^{i,i+1}(\omega) = \cdots = \lambda_N^{i,i+1}(\omega) = \lambda^{i,i+1}(\omega) \qquad (L-1 \leq i < n) \quad (81)$$

iii) For the masses on the left of the nonlinear spring, the following relationships of the output frequency responses hold

$$x_i(j\omega) = \lambda^{i,i+1}(\omega) x_{i+1}(j\omega) \qquad (1 \leq i < L-1) \quad (82)$$

The first property is straightforward. From equation (70), it can be known that

$$\lambda_1^{1,2}(\omega) = \cdots = \lambda_N^{1,2}(\omega) = \frac{k_2}{\left(-m_1\omega^2 + k_1 + k_2\right)} = \lambda^{1,2}(\omega) \quad (83)$$

Consequently, substituting (83) into equation (71) yields

$$\lambda_1^{2,3}(\omega) = \cdots = \lambda_N^{2,3}(\omega) = \frac{k_3}{\left(-m_2\omega^2 + (1-\lambda^{1,2}(\omega))k_2 + k_3\right)} = \lambda^{2,3}(\omega) \quad (84)$$

Following the same procedure until i=L-2, the first property can be proved.

Using equation (68), it is known that, for the masses on the right of the nonlinear spring, including the (L-1)th mass, the relationship

$$\frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} = Q^{i,i+1}(j\omega) \qquad (L-1 \leq i < n) \quad (85)$$

is tenable.

For the (L-1)th mass,

$$\frac{G_{(L-1,1)}(j\omega)}{G_{(L,1)}(j\omega)} = \frac{k_L}{\left(-m_{L-1}\omega^2 + (1-\lambda_1^{L-2,L-1}(\omega))k_{L-1} + k_L\right)} = \lambda_1^{L-1,L}(\omega) \quad (86)$$

and

$$\frac{G_{(L-1,2)}(j\omega)}{G_{(L,2)}(j\omega)} = \frac{k_L}{\left(-m_{L-1}\omega^2 + (1-\lambda_2^{L-2,L-1}(\omega))k_{L-1} + k_L\right)}\left(1 - \frac{1}{k_L}\frac{\Gamma_{(L-1,2)}(j\omega)}{G_{(L,2)}(j\omega)}\right) = \lambda_2^{L-1,L}(\omega) \quad (87)$$

According to the first property, $\lambda_1^{L-2,L-1}(\omega) = \lambda_2^{L-2,L-1}(\omega)$. So, from equations (86) and (87)

$$\lambda_2^{L-1,L}(\omega) = \left(1 - \frac{1}{k_L}\frac{\Gamma_{(L-1,2)}(j\omega)}{G_{(L,2)}(j\omega)}\right)\lambda_1^{L-1,L}(\omega) \quad (88)$$

Obviously, $\lambda_1^{L-1,L}(\omega) \neq \lambda_2^{L-1,L}(\omega)$ since $\Gamma_{(L-1,2)}(j\omega) \neq 0$.

Substituting $\lambda_1^{L-1,L}(\omega)$ and $\lambda_2^{L-1,L}(\omega)$ into equations (73) and (75) for the Lth mass, it can be deduced that

$$\begin{aligned}\lambda_1^{L,L+1}(\omega) &= \frac{k_{L+1}}{\left(-m_L\omega^2 + (1-\lambda_1^{L-1,L}(\omega))k_L + k_{L+1}\right)} \\ &\neq \frac{k_{L+1}}{\left(-m_L\omega^2 + (1-\lambda_2^{L-1,L}(\omega))k_L + k_{L+1}\right)}\left(1 + \frac{1}{k_{L+1}}\frac{\Gamma_{(L-1,2)}(j\omega)}{G_{(L+1,2)}(j\omega)}\right) = \lambda_2^{L,L+1}(\omega)\end{aligned} \quad (89)$$

Iteratively using the above procedure and terminating at the (n-1)th mass, the relationship $\lambda_1^{i,i+1}(\omega) \neq \lambda_2^{i,i+1}(\omega)$ ($L-1 \leq i < n$) can be proved. So far, the whole second property is proved.

The third property is also straightforward since, according to equation (6), the output frequency response of the ith mass can be expressed as

$$x_{i+1}(j\omega) = \sum_{k=1}^{N} G_{(i+1,k)}(j\omega)\, F_k(j\omega) \tag{90}$$

Using the first property, equation (90) can be written as

$$x_{i+1}(j\omega) = \sum_{k=1}^{N} \lambda^{i,i+1}(j\omega) G_{(i,k)}(j\omega)\, F_k(j\omega) \tag{91}$$

Obviously, $x_{i+1}(j\omega) = \lambda^{i,i+1}(j\omega) x_i(j\omega)$, then the third property is proved.

Above three properties can be easily extended to a more general case, as follows.

iv) For any two masses on the left of the nonlinear spring, the following relationship holds.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+k,1)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+k,N)}(j\omega)} = \lambda^{i,i+k}(\omega) \quad (1 \leq i < L-1 \text{ and } i+k < L) \tag{92}$$

and

$$\lambda^{i,i+k}(\omega) = \prod_{d=0}^{k-1} \lambda^{i+d,i+d+1}(\omega) \tag{93}$$

v) For any two masses on the left of the nonlinear spring, the output frequency responses satisfy the following relationship

$$x_i(j\omega) = \lambda^{i,i+k}(\omega) x_{i+k}(j\omega) \qquad (1 \leq i < L-1 \text{ and } i+k < L) \tag{94}$$

vi) For any two masses on the right of the nonlinear spring, including the (L-1)th mass, the following relationships can be deduced from property ii).

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+k,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(i+k,2)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+k,N)}(j\omega)} = \lambda^{i,i+k}(\omega)$$
$$(L-1 \leq i < n \text{ and } i+k \leq n) \tag{95}$$

and

$$\lambda^{i,i+k}(\omega) = \prod_{d=0}^{k-1} \lambda^{i+d,i+d+1}(\omega) \tag{96}$$

vii) For any two masses on different sides of the nonlinear spring, the following relationship holds.

$$\frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(k,2)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(k,N)}(j\omega)}$$
$$(1 \leq i \leq L-1 \text{ and } L \leq k \leq n) \tag{97}$$

The above four properties are straightforward, so the details of the proofs are omitted.

4 Numerical Study

In order to verify the analysis results in the last section, a 6-DOF oscillator was adopted, in which the fourth spring was nonlinear. As widely used in modal analysis [13], a damping characteristic was considered in this numerical study where the damping was assumed to be a proportional damping, e.g., $C=\mu K$. The values of the system parameters used are

$m_1=\cdots=m_6=1$, $c_1=k_1=\cdots=k_6=3.5531\times10^4$, $c_2=0.8\times c_1^2$, $c_1=0.4\times c_1^3$, μ=0.01

and the input is a harmonic force, $f(t)=A\sin(2\pi\times20t)$.

If only the NOFRFs up to the 4th order is considered, according to equations (16) and (17), the frequency components of the outputs of the 6 masses can be written as

$$\begin{aligned}
&x_i(j\omega_F)=G^H_{(i,1)}(j\omega_F)F_1(j\omega_F)+G^H_{(i,3)}(j\omega_F)F_3(j\omega_F)\\
&x_i(j2\omega_F)=G^H_{(i,2)}(j2\omega_F)F_2(j2\omega_F)+G^H_{(i,4)}(j2\omega_F)F_4(j2\omega_F)\\
&x_i(j3\omega_F)=G^H_{(i,3)}(j3\omega_F)F_3(j3\omega_F)\\
&x_i(j4\omega_F)=G^H_{(i,4)}(j4\omega_F)F_4(j4\omega_F) \qquad (i=1,\cdots,6)
\end{aligned} \tag{98}$$

From equation (94), it can be seen that, using the method in [9], two different inputs with the same waveform but different strengths are sufficient to estimate the NOFRFs up to 4th order. Therefore, in this numerical study, two different inputs of amplitudes A_1=0.8 and A_2=1.0 were used. The simulation studies were conducted using a fourth-order *Runge–Kutta* method to obtain the forced response of the oscillator.

The evaluated results of $G_1^H(j\omega_F)$, $G_3^H(j\omega_F)$, $G_2^H(j2\omega_F)$ and $G_4^H(j2\omega_F)$ for all masses are given in Table 1 and Table 2. According to analysis results in the previous section, it can be known that the following relationships should be tenable.

$$\begin{aligned}
&\lambda_1^{i,i+1}(j\omega_F)=\frac{G^H_{(i,1)}(j\omega_F)}{G^H_{(i+1,1)}(j\omega_F)}=\frac{G^H_{(i,3)}(j\omega_F)}{G^H_{(i+1,3)}(j\omega_F)}=\lambda_3^{i,i+1}(j\omega_F) \qquad \text{for } i=1,2\\
&\lambda_1^{i,i+1}(j\omega_F)=\frac{G^H_{(i,1)}(j\omega_F)}{G^H_{(i+1,1)}(j\omega_F)}\neq\frac{G^H_{(i,3)}(j\omega_F)}{G^H_{(i+1,3)}(j\omega_F)}=\lambda_3^{i,i+1}(j\omega_F) \qquad \text{for } i=3,4,5\\
&\lambda_2^{i,i+1}(j2\omega_F)=\frac{G^H_{(i,2)}(j2\omega_F)}{G^H_{(i+1,2)}(j2\omega_F)}=\frac{G^H_{(i,4)}(j2\omega_F)}{G^H_{(i+1,4)}(j2\omega_F)}=\lambda_4^{i,i+1}(j2\omega_F) \qquad \text{for } i=1,2,3,4,5
\end{aligned} \tag{99}$$

From the NOFRFs given in Table 1 and Table 2, $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$ (i=1,2,3,4,5) can be evaluated. Moreover, from equations (36), (65) and (66), the theoretical values of $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$ (i=1,2,3,4,5) can also be calculated. Both the evaluated and theoretical values of $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$ (i=1,2,3,4,5) are given in Tables 3, 4, 6 and 7 respectively. The associated moduli are given in Table 5 and Table 8.

Table 1, the evaluated results of $G_1^H(j\omega_F)$ and $G_3^H(j\omega_F)$

	$G_1^H(j\omega_F)$ ($\times 10^{-5}$)	$G_3^H(j\omega_F)$ ($\times 10^{-8}$)
Mass1	0.248755981292+0.105083252913i	-2.640653692313-0.134346651049i
Mass2	-0.431887829736-0.245927063111i	4.797297679103+ 0.817418504289i
Mass 3	-0.984866548167-0.648079124466i	11.23083043936+ 2.666885065817i
Mass 4	-1.227215155076-1.151976267814i	-3.690332450620+4.775081343252i
Mass 5	-1.009037585829-1.722958265454i	-0.167765943288+6.975012211228i
Mass 6	-0.243826521314-2.215514088592i	1.874038990979+7.936955064990i

Table 2, the evaluated results of $G_2^H(j2\omega_F)$ and $G_4^H(j2\omega_F)$

	$G_2^H(j2\omega_F)$ ($\times 10^{-8}$)	$G_4^H(j2\omega_F)$ ($\times 10^{-9}$)
Mass 1	-3.600986560925-0.843396489345i	-1.744362683969+1.791582813072i
Mass 2	5.812267183838+3.680180095536i	4.158928956405-2.083020311251i
Mass 3	11.56344137006+10.86504193730i	10.32469589716-2.908397493893i
Mass 4	-8.362748079065-6.117344659423i	-6.458494748740+2.710328095026i
Mass 5	-6.288845518790+ 1.868823507184i	-1.111106770191+4.294340994126i
Mass 6	-3.826040708263+ 5.554108059052i	1.882654455592+4.153136744233i

Table 3, the evaluated and theoretical values of $\lambda_1^{i,i+1}(j\omega_F)$

	Evaluated	Theoretical
i=1	-0.53956923942555+ 0.06393193691330i	-0.53956812565722+0.06392927976739i
i=2	0.42068003742786-0.02711726504942i	0.42068338945362-0.02712178311738i
i=3	0.69013114133652-0.11973089757392i	0.69003905565205-0.11950391902279i
i=4	0.80845793934758-0.23880480216057i	0.80846919607333-0.23881686890438i
i=5	0.81789573641784-0.36542891683225i	0.81789109869642-0.36541946257437i

Table 4, the evaluated and theoretical values of $\lambda_3^{i,i+1}(j\omega_F)$

	Evaluated	Theoretical
i=1	-0.53955286025570 + 0.06393054204275i	-0.53956812565722 + 0.06392927976739i
i=2	0.42071440540958 - 0.02711985209971i	0.42068338945362 - 0.02712178311738i
i=3	-0.78832894098333 - 1.74272100693032i	-0.78847736178192 - 1.74269593806636i
i=4	0.69692072700836 + 0.51231636293108i	0.69688121980439 + 0.51240060371508i
i=5	0.82766819458198 + 0.21656269910234i	0.82767705033815 + 0.21654740508103i

Table 5, the evaluated and theoretical values of $\left|\lambda_1^{i,i+1}(j\omega_F)\right|$ and $\left|\lambda_3^{i,i+1}(j\omega_F)\right|$

	$\left\|\lambda_1^{i,i+1}(j\omega_F)\right\|$		$\left\|\lambda_3^{i,i+1}(j\omega_F)\right\|$	
	Evaluated	Theoretical	Evaluated	Theoretical
i=1	0.54334358990583	0.54334217122990	0.54332716038864	0.54334217122990
i=2	0.42155312826982	0.42155676400890	0.42158759148846	0.42155676400890
i=3	0.70044020449736	0.70031070603482	1.91273077749801	1.91276911377081
i=4	0.84298990102437	0.84300411497973	0.86496621636963	0.86498428494605
i=5	0.89581902687299	0.89581093594295	0.85553143891097	0.85553613500763

Table 6, the evaluated and theoretical values of $\lambda_2^{i,i+1}(j2\omega_F)$

	Evaluated	Theoretical
i=1	-0.50783201497968 + 0.17644006230670i	-0.50780274212003 + 0.17649386871931i
i=2	0.42577906425913 - 0.08180326802304i	0.42588382195297 - 0.08164496612163i
i=3	-1.51985108140719 - 0.18744903308276i	-1.51994279230261 - 0.18777462404959i
i=4	0.95626870235269 + 1.25689875287077i	0.95679114362216 + 1.25620885732202i
i=5	0.75716690291690 + 0.61070006006679i	0.75701955822647 + 0.61067645667337i

Table 7, the evaluated and theoretical values of $\lambda_4^{i,i+1}(j2\omega_F)$

	Evaluated	Theoretical
i=1	-0.50779971566753 + 0.17644583472841i	-0.50780274212003 + 0.17649386871931i
i=2	0.42585370950162 - 0.08179112084103i	0.42588382195297 - 0.08164496612163i
i=3	-1.51992857059729 - 0.18752165334900i	-1.51994279230261 - 0.18777462404959i
i=4	0.95625249450690 + 1.25653648265525i	0.95679114362216 + 1.25620885732202i
i=5	0.75713972408720 + 0.61075264347404i	0.75701955822647 + 0.61067645667337i

Table 8, the evaluated and theoretical values of $\left|\lambda_2^{i,i+1}(j2\omega_F)\right|$ and $\left|\lambda_4^{i,i+1}(j2\omega_F)\right|$

	$\left\|\lambda_2^{i,i+1}(j2\omega_F)\right\|$		$\left\|\lambda_4^{i,i+1}(j2\omega_F)\right\|$	
	Evaluated	Theoretical	Evaluated	Theoretical
i=1	0.53760994319778	0.53759995405517	0.53758132763800	0.53759995405517
i=2	0.43356612669883	0.43363917061801	0.43363714018142	0.43363917061801
i=3	1.53136685665397	1.53149776405636	1.53145265359157	1.53149776405636
i=4	1.57931767104259	1.57908517367750	1.57901955830012	1.57908517367750
i=5	0.97275705201163	0.97262754755998	0.97276891053393	0.97262754755998

It can be seen that the evaluated results match the theoretical results very well. Moreover, the results shown in Tables 5 and 8 have a strict accordance with the relationships of equations (78) and (81). Therefore, the numerical study verifies the properties of NOFRFs of locally nonlinear MDOF systems described in previous section.

From Table 5, it can be seen that, for i=4, $\left|\lambda_1^{i,i+1}(j\omega_F)\right|$ and $\left|\lambda_3^{i,i+1}(j\omega_F)\right|$ are only slightly different, but they have a significant difference for i=3. This means that, $\left|\lambda_1^{i,i+1}(j\omega_F)\right|$ and $\left|\lambda_3^{i,i+1}(j\omega_F)\right|$, for the two masses connected to the nonlinear spring, have a considerable difference. This result implies that a class of novel approaches can be developed based on the properties of NOFRFs derived in the present study for MDOF nonlinear systems to detect and locate fault elements which make engineering structures behave nonlinearly. This is the focus of our current research studies. The results will be present in a series of later publications.

5 Conclusions and Remarks

In this paper, the properties of locally nonlinear MDOF systems have been investigated using the concept of Nonlinear Output Frequency Response Functions (NOFRFs). Important results regarding the relationships between the NOFRFs of MDOF systems have been derived, and these reveal, for the first time, very significant characteristics of this class of nonlinear systems. A direct application of the derived results is the location of the position of the nonlinear element in a locally nonlinear MDOF system. This idea can be used to develop novel fault diagnosis techniques for a wide range of engineering structures [10][14][15]. Numerical studies verified the theoretical analysis results. It is worthy noting that, although, for convenience, it was assumed that each mass has only one degree of freedom, the obtained results are still tenable for the cases where each mass in the MDOF system has more than one degree of freedom. During the analysis, if $x_i,(i=1,\cdots,n)$ is taken as a vector, the same results will be achieved.

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