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Nonlinear Output Frequency Response Functions for Multi-Input Nonlinear Volterra Systems

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Abstract: The concept of Nonlinear Output Frequency Response Functions (NOFRFs) is extended to the nonlinear systems that can be described by a multi-input Volterra series model. A new algorithm is also developed to determine the output frequency range of nonlinear systems from the frequency range of the inputs. These results allow the concept of NOFRFs to be applied to a wide range of engineering systems. The phenomenon of the energy transfer in a two degree of freedom nonlinear system is studied using the new concepts to demonstrate the significance of the new results.

1 Introduction

Linear systems, which have been widely studied by practitioners in many different fields, have provided a basis for the development of the majority of control system synthesis, mechanical system analysis and design, and signal processing methods. However, there are certain types of qualitative behaviour encountered in engineering, which cannot be produced by linear models [1], for example, the generation of harmonics and inter-modulation behaviour. In cases where these effects are dominant or significant nonlinear behaviours exist, nonlinear models are required to describe the system, and nonlinear system analysis methods have to be applied to investigate the system dynamics.

The Volterra series approach [2] is a powerful tool for the analysis of nonlinear systems, which extends the familiar concept of the convolution integral for linear systems to a series of multi-dimensional convolution integrals. The Fourier transforms of the Volterra kernels are known as the kernel transforms, Higher-order Frequency Response Functions (HFRFs) [3], or Generalised Frequency Response Functions (GFRFs), and these provide a convenient tool for analyzing nonlinear systems in the frequency domain. If a differential equation or discrete-time model is available for a system, the GFRFs can be determined using the algorithm in [4]~[6]. The GFRFs can be regarded as the extension of the classical frequency response function (FRF) of linear systems to the nonlinear case.

However, the GFRFs are much more complicated than the FRF. GFRFs are multidimensional functions [7][8], which can be difficult to measure, display and interpret in practice. Recently, the novel concept of Nonlinear Output Frequency Response Functions (NOFRFs) was proposed by the authors [9]. The concept can be considered to be an alternative extension of the FRF to the nonlinear case. NOFRFs are one dimensional functions of frequency, which allow the analysis of nonlinear systems to be implemented in a manner similar to the analysis of linear systems and which provides great insight into the mechanisms which dominate many nonlinear behaviours. The NOFRF concept was recently used to investigate the energy transfer properties of bilinear oscillators in the frequency domain [10]. The results revealed the existence of resonances at frequencies different from the frequencies at the input excitation in this class of oscillators.

The objective of this paper is to extend the concept of NOFRFs to multi-input nonlinear Volterra systems so that the concept of NOFRFs can be applied to a much wider range of engineering systems. The phenomenon of energy transfer in a 2DOF nonlinear system is also investigated using the extended concept of NOFRFs to demonstrate the effectiveness and significance of the results obtained in the present study.

2 The Concept of Nonlinear Output Frequency Response Functions

NOFRFs were recently proposed and used to investigate the behaviour of structures with polynomial-type non-linearities. The definition of NOFRFs is based on the Volterra series theory of nonlinear systems.

Consider the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\ldots,\tau_n)\prod_{i=1}^{n}x(t-\tau_i)d\tau_i \tag{1.1}$$

where $y(t)$ and $x(t)$ are the output and input of the system, $h_n(\tau_1,\ldots,\tau_n)$ is the nth order Volterra kernel, and N denotes the maximum order of the system nonlinearity. Lang and Billings [3] have derived an expression for the output frequency response of this class of nonlinear systems to a general input. The result is

$$\begin{cases} Y(j\omega)=\sum_{n=1}^{N}Y_n(j\omega) \quad \text{for } \forall\omega \\ Y_n(j\omega)=\dfrac{1/\sqrt{n}}{(2\pi)^{n-1}}\displaystyle\int_{\omega_1+,\ldots,+\omega_n=\omega}H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n}X(j\omega_i)d\sigma_{n\omega} \end{cases} \tag{1.2}$$

This expression reveals how nonlinear mechanisms operate on the input spectra to produce the system output frequency response. In (1.2), $Y(j\omega)$ and $X(j\omega)$ are the spectra of the system output and input respectively, $Y_n(j\omega)$ represents the nth order output frequency response of the system,

$$H_n(j\omega_1,...,j\omega_n) = \int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} h_n(\tau_1,...,\tau_n) e^{-(\omega_1\tau_1+,...,+\omega_n\tau_n)j} d\tau_1...d\tau_n \qquad (1.3)$$

is the definition of the Generalised Frequency Response Function (GFRF), and

$$\int_{\omega_1+,...,+\omega_n=\omega} H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n} X(j\omega_i)d\sigma_{n\omega}$$

denotes the integration of $H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n} X(j\omega_i)$ over the n-dimensional hyper-plane $\omega_1+\cdots+\omega_n=\omega$. Equation (1.2) is a natural extension of the well-known linear relationship $Y(j\omega)=H_1(j\omega)X(j\omega)$ to the nonlinear case.

For linear systems, the possible output frequencies are the same as the frequencies in the input. For nonlinear systems described by Equation (1.1), however, the relationship between the input and output frequencies is more complicated. Given the frequency range of the input, the output frequencies of system (1.1) can be determined using an explicit expression derived by Lang and Billings in [3].

Based on the above results for output frequency responses of nonlinear systems, a new concept known as the Nonlinear Output Frequency Response Function (NOFRF) was recently introduced by Lang and Billings [9]. The concept was defined as

$$G_n(j\omega) = \frac{\int_{\omega_1+,...,+\omega_n=\omega} H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n} X(j\omega_i)d\sigma_{n\omega}}{\int_{\omega_1+,...,+\omega_n=\omega} \prod_{i=1}^{n} X(j\omega_i)d\sigma_{n\omega}} \qquad (1.4)$$

under the condition that

$$U_n(j\omega) = \int_{\omega_1+,...,+\omega_n=\omega} \prod_{i=1}^{n} X(j\omega_i)d\sigma_{n\omega} \neq 0 \qquad (1.5)$$

Notice that $G_n(j\omega)$ is valid over the frequency range of $U_n(j\omega)$, which can be determined using the algorithm in [3].

By introducing the NOFRFs $G_n(j\omega)$, $n=1,\cdots N$, Equation (1.4) can be written as

$$Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) = \sum_{n=1}^{N} G_n(j\omega)U_n(j\omega) \qquad (1.6)$$

which is similar to the description of the output frequency response of linear systems. For a linear system, the relationship between $Y(j\omega)$ and $X(j\omega)$ can be illustrated as in

Figure 1. Similarly, the nonlinear system input and output relationship of Equation (1.1) can be illustrated in Figure 2.

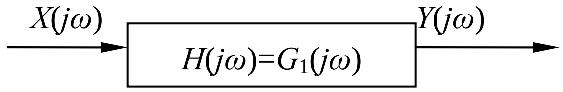


Figure 1. The output frequency response of a linear system

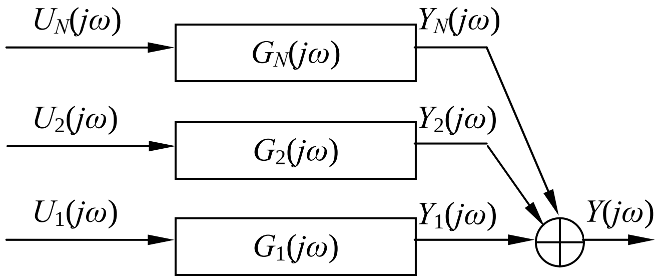


Figure 2. The output frequency response of a nonlinear system

The NOFRFs reflect a combined contribution of the system and the input to the frequency domain output behaviour. It can be seen from Equation (1.4) that $G_n(j\omega)$ depends not only on H_n (i=1,…,N) but also on the input $X(j\omega)$. For a nonlinear system, the dynamical properties are determined by the GFRFs H_n (i= 1,…,N). However, from Equation (1.3) it can be seen that a GFRF is multidimensional [7][8], and may become difficult to measure, display and interpret in practice. Feijoo, Worden and Stanway [11]-[12] demonstrated that the Volterra series can be described by a series of associated linear equations (ALEs) whose corresponding associated frequency response functions (AFRFs) are easier to analyze and interpret than the GFRFs. According to Equation (1.4), the NOFRF $G_n(j\omega)$ is a weighted sum of $H_n(j\omega_1,...,j\omega_n)$ over $\omega_1+\cdots+\omega_n=\omega$ with the weights depending on the input. Therefore $G_n(j\omega)$ can be used as an alternative representation of the structural dynamical characteristics described by H_n. The most important property of the NOFRF $G_n(j\omega)$ is that it is one dimensional, and thus allows the analysis of nonlinear systems to be implemented in a convenient manner similar to the analysis of linear systems. Moreover, there is an effective algorithm [9] available which allows the evaluation of the NOFRFs to be implemented directly using system input output data.

2 NOFRFS for Multi-Input Nonlinear Volterra Systems

2.1 Multi-Input Nonlinear Volterra Systems

Multi-input multi-output nonlinear Volterra systems can be expressed so that each output can to be modeled as a multi-input Volterra series. The extension of the single input single output Volterra series representation (1.1) to this more general case is as follows

$$y_i(t)=\sum_{n=1}^{N} y_i^{(n)}(t) \tag{2.1}$$

where

$$\begin{aligned} y_i^{(n)}(t) = &\sum_{N_1+\cdots+N_m=n} \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(\tau_1,\cdots,\tau_n)x_1(t-\tau_1)\cdots x_1(t-\tau_{N_1}) \\ & x_2(t-\tau_{N_1+1})\cdots x_2(t-\tau_{N_1+N_2})\cdots x_m(t-\tau_{N_1+\cdots+N_{m-1}+1})\cdots x_m(t-\tau_n)d\tau_1\cdots d\tau_n \end{aligned} \tag{2.2}$$

and $h_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}$ represents the *n*th order kernel associated with the *i*th output and N_1th input $x_1(t)$, N_2th input $x_2(t)$,…, N_mth input $x_m(t)$.

Equation (2.2) can be rewritten as

$$y_i^{(n)} = \sum_{N_1+\cdots+N_m=n} \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(\tau_1,\cdots,\tau_n)x_{(N_1,\cdots,N_m)}(\tau_1,\cdots,\tau_n)d\tau_1\cdots d\tau_n \tag{2.3}$$

where

$$\begin{aligned} & x_{(N_1,\cdots,N_m)}(\tau_1,\cdots,\tau_n) = \\ & x_1(t-\tau_1)\cdots x_1(t-\tau_{N_1})x_2(t-\tau_{N_1+1})\cdots x_2(t-\tau_{N_1+N_2})\cdots x_m(t-\tau_{N_1+\cdots+N_{m-1}+1})\cdots x_m(t-\tau_n) \end{aligned} \tag{2.4}$$

In the single-input case, the Volterra series has only one kernel for each order of nonlinearity, for example, $h_2(\tau_1,\tau_2)$ is the second order kernel. It can be seen, however, that in the multi-input case more kernels are involved for each order of nonlinearity. For example, for a two input system there are three 2nd order kernels for the *i*th output which are $h_{(i,P_1=2,P_2=0)}^{(2)}(\tau_1,\tau_2)$, $h_{(i,P_1=0,P_2=2)}^{(2)}(\tau_1,\tau_2)$, and $h_{(i,P_1=1,P_2=1)}^{(2)}(\tau_1,\tau_2)$.

The frequency domain description of (2.1), (2.2) can be expressed as

$$Y_i(j\omega)=\sum_{n=1}^{N} y_i^{(n)}(j\omega) \tag{2.5}$$

$$\begin{aligned} Y_i^{(n)}(j\omega) = &\left(\frac{1}{2\pi}\right)^{n-1} \sum_{N_1+\cdots+N_m=n} \int_{\omega_1+\cdots+\omega_n=\omega} H_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega_1,\cdots,j\omega_n) \\ & \times\prod_{i=1}^{N_1} X_1(j\omega_i)\prod_{i=N_1+1}^{N_1+N_2} X_2(j\omega_i)\cdots \prod_{i=N_1+\cdots+N_{m-1}+1}^{n} X_m(j\omega_i)d\sigma_{n\omega} \end{aligned} \tag{2.6}$$

This is an extension of Equation (1.2) for the single-input case to the multi-input case. Define $N_0=0$, then Equation (2.6) can be written as

$$Y_i^{(n)}(j\omega) = \left(\frac{1}{2\pi}\right)^{n-1} \frac{1}{\sqrt{n}} \sum_{N_1+\cdots+N_m=n} \int_{\omega_1+\cdots+\omega_n=\omega} H_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega_1,\cdots,j\omega_n)\prod_{j=1}^{m}\prod_{i=N_0+\cdots+N_{j-1}+1}^{N_0+\cdots+N_j} X_j(j\omega_i)d\sigma_{n\omega} \tag{2.7}$$

For a given set of $N_1,N_2,\cdots N_m$, define

$$\begin{aligned} & Y_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) \\ & = \left(\frac{1}{2\pi}\right)^{n-1} \frac{1}{\sqrt{n}} \int_{\omega_1+\cdots+\omega_n=\omega} H_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega_1,\cdots,j\omega_n)\times\prod_{j=1}^{m}\prod_{i=N_0+\cdots+N_{j-1}+1}^{N_0+\cdots+N_j} X_j(j\omega_i)d\sigma_{n\omega} \end{aligned} \tag{2.8}$$

then Equation (2.7) can be written in a compact form

$$Y_i^{(n)}(j\omega) = \sum_{N_1+\cdots+N_m=n} Y_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) \tag{2.9}$$

2.2 Definition of the NOFRFs for Multi-Input Nonlinear Volterra Systems

Define

$$U_{(P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) = \left(\frac{1}{2\pi}\right)^{n-1} \frac{1}{\sqrt{n}} \int_{\omega_1+\cdots+\omega_n=\omega} \prod_{j=1}^{m} \prod_{i=N_0+\cdots+N_{j-1}+1}^{N_0+\cdots+N_j} X_j(j\omega_i) d\sigma_{n\omega} \tag{2.10}$$

then (2.8) can be rewritten as

$$Y_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) = \frac{\int_{\omega_1+\cdots+\omega_n=\omega} \begin{pmatrix} H_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega_1,\cdots,j\omega_n) \\ \times \prod_{j=1}^{m} \prod_{i=N_0+\cdots+N_{j-1}+1}^{N_0+\cdots+N_j} X_j(j\omega_i) \end{pmatrix} d\sigma_{n\omega}}{\int_{\omega_1+\cdots+\omega_n=\omega} \prod_{j=1}^{m} \prod_{i=N_0+\cdots+N_{j-1}+1}^{N_0+\cdots+N_j} X_j(j\omega_i) d\sigma_{n\omega}} \left(\frac{1}{2\pi}\right)^{n-1} \frac{1}{\sqrt{n}} \int_{\omega_1+\cdots+\omega_n=\omega} \prod_{j=1}^{m} \prod_{i=N_0+\cdots+N_{j-1}+1}^{N_0+\cdots+N_j} X_j(j\omega_i) d\sigma_{n\omega}$$

$$= G_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) U_{(P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) \tag{2.11}$$

where

$$G_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) = \frac{\int_{\omega_1+\cdots+\omega_n=\omega} \begin{pmatrix} H_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega_1,\cdots,j\omega_n) \\ \times \prod_{j=1}^{m} \prod_{i=N_0+\cdots+N_{j-1}+1}^{N_0+\cdots+N_j} X_j(j\omega_i) \end{pmatrix} d\sigma_{n\omega}}{\int_{\omega_1+\cdots+\omega_n=\omega} \prod_{j=1}^{m} \prod_{i=N_0+\cdots+N_{j-1}+1}^{N_0+\cdots+N_j} X_j(j\omega_i) d\sigma_{n\omega}} \tag{2.12}$$

will be referred to as the Nonlinear Output Frequency Response Function for multi-input nonlinear Volterra systems, and is a natural extension of Equation (1.4) to more general case. Substituting (2.12) into (2.9) yields

$$Y_i^{(n)}(j\omega) = \sum_{N_1+\cdots+N_m=n} G_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) U_{(P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) \tag{2.13}$$

It is easy to verify that $G_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega)$ has the following important properties.

(i) $G_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega)$ allows $Y_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega)$ to be described in a manner similar to the description for the output frequency response of linear systems.

(ii) $G_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega)$ is valid over a frequency range where $U_{(P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) \neq 0$.

(iii) $G_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega)$ is insensitive to the change of the input spectra by a constant gain, that is,

$$G^{(n)}_{(i,P_1=N_1,\cdots,P_m=N_m)}(j\omega)\Big|_{\substack{X_1(j\omega)=\alpha\bar{X}_1(j\omega)\\ \vdots \\ X_m(j\omega)=\alpha\bar{X}_m(j\omega)}} = G^{(n)}_{(i,P_1=N_1,\cdots,P_m=N_m)}(j\omega)\Big|_{\substack{X_1(j\omega)=\bar{X}_1(j\omega)\\ \vdots \\ X_m(j\omega)=\bar{X}_m(j\omega)}} \quad (2.14)$$

2.3 Determination of the Output Frequency Range of Multi-Input nonlinear Volterra Systems

For a nonlinear system that can be modeled as a single-input Volterra series, given the frequency range of the input, Lang and Billings [3] derived an explicit expression for the output frequency range. In the following, a method will be derived to determine the output frequency range of multi-input nonlinear Volterra systems.

Obviously, the frequency range of $U^{(n)}_{(P_1=N_1,\cdots,P_m=N_m)}(j\omega)$ is given as the range of

$$\omega = \sum_{k=1}^{m} \omega^{(k)} \quad (2.15)$$

where $\omega^{(k)}$ is associated to the *k*th input $x_k(t)$ of order N_k, and

$$\omega^{(k)} = \omega_{N_0+\cdots+N_{k-1}+1} + \cdots + \omega_{N_0+\cdots+N_k} \quad (2.16)$$

Define the frequency range of the *k*th input $x_k(t)$ ($k = 1,\ldots,m$) as

$$[-b_k \ -a_k] \cup [a_k \ b_k]$$

Now, assume L_k of N_k components are located in $[-b_k \ -a_k]$, and the remainder are located $[a_k \ b_k]$, in this case, the frequency range is

$$[(N_k - L_k)a_k - L_k b_k \ \ (N_k - L_k)b_k - L_k a_k] \quad (2.17)$$

that is

$$[N_k a_k - L_k(b_k + a_k) \ \ N_k b_k - L_k(a_k + b_k)] \quad (2.18)$$

Therefore the range of $\omega^{(k)}$ is

$$\bigcup_{L_k=0}^{N_k} [N_k a_k - L_k(b_k + a_k) \ \ N_k b_k - L_k(a_k + b_k)] \quad (2.19)$$

that is

$$\omega^{(k)} \in \bigcup_{L_k=0}^{N_k} [N_k a_k - L_k(b_k + a_k) \ \ N_k b_k - L_k(a_k + b_k)] \quad (2.20)$$

This can easily be extended to case of $\omega^{(1)} + \cdots + \omega^{(m)}$

$$\omega = (\omega^{(1)} + \cdots + \omega^{(m)}) \in \bigcup_{L_1=0}^{N_1} \cdots \bigcup_{L_m=0}^{N_m} \left[\sum_{i=1}^{m} N_i a_i - \sum_{i=1}^{m} L_i(a_i + b_i) \quad \sum_{i=1}^{m} N_i b_i - \sum_{i=1}^{m} L_i(a_i + b_i) \right] \quad (2.21)$$

Therefore, the frequency range of $U^{(n)}_{(P_1=N_1,\cdots,P_m=N_m)}(j\omega)$ can be expressed

$$f^{(n)}_{(P_1=N_1,\cdots,P_m=N_m)} = \bigcup_{L_1=0}^{N_1} \cdots \bigcup_{L_m=0}^{N_m} \left[\sum_{i=1}^{m} N_i a_i - \sum_{i=1}^{m} L_i(a_i + b_i) \quad \sum_{i=1}^{m} N_i b_i - \sum_{i=1}^{m} L_i(a_i + b_i) \right] \quad (2.22)$$

From Equation (2.11), it can be shown that $Y^{(n)}_{(i,P_1=N_1,\cdots,P_m=N_m)}(j\omega)$ and $U^{(n)}_{(P_1=N_1,\cdots,P_m=N_m)}(j\omega)$ have the same frequency range. Furthermore, according to Equation (2.9), it can easily be shown that the frequency range of $Y_i^{(n)}(j\omega)$ is

$$f_i^{(n)} = \bigcup_{N_1+\cdots N_m=n} f_{(P_1=N_1,\cdots,P_m=N_m)}^{(n)} \tag{2.23}$$

Finally, according to Equation (2.5), the frequency range of $Y_i(j\omega)$ is given by

$$f_{y_i} = \bigcup_{n=1}^{N} f_i^{(n)} \tag{2.24}$$

Therefore, given the frequency ranges of the inputs $x_k(t)$ (k = 1 ,…, m) as

$$[-b_k \; -a_k] \cup [a_k \; b_k],$$

the output frequency range can be determined by Equations (2.22), (2.23) and (2.24). The validity of this method will be verified by numerical studies in Section 3.

2.4 Evaluation of the NOFRFs for Multi-Input Systems

For single-input nonlinear systems, Lang and Billings [9] derived an effective algorithm for the estimation of the NOFRFs, which can be implemented directly using system input output data. To estimate the NOFRFs up to *N*th order, the algorithm generally requires experimental or simulation results for the system under *N* different input signal excitations, which have the same waveforms but different intensities. This algorithm can be extended to estimate the NOFRFs in the multi-input case. As a multi-input system of nonlinearity up to *N*th order involves more than *N* NOFRFs, more than *N* experiments or simulations under different signal excitations are needed to estimate the NOFRFs.

Combining Equation (2.5) and (2.13) yields

$$y_i(j\omega) = \sum_{n=1}^{N} \sum_{N_1+\cdots+N_m=n} Y_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) \tag{2.25}$$

Equation (2.25) can be further written in the polynomial form

$$\begin{aligned} y_i(j\omega) = &\sum_{k_1=1}^{m} G_{k_1}^{(1)}(j\omega)\left[U_{k_1}(j\omega)\right] + \sum_{k_1=1}^{m}\sum_{k_2=k_1}^{m} G_{k_1k_2}^{2}(j\omega)\left[U_{k_1}(j\omega)U_{k_2}(j\omega)\right] + \cdots \\ &+ \sum_{k_1=1}^{m}\cdots\sum_{k_N=k_{N-1}}^{m} G_{k_1\cdots k_N}^{(N)}(j\omega)\left[U_{k_1}(j\omega)\cdots U_{k_N}(j\omega)\right] \end{aligned} \tag{2.26}$$

where $G_{k_1\cdots k_N}^{(N)}(j\omega)$ represents a specific

$$G_{(i,P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) = G_{\underbrace{1\cdots 1}_{N_1}\underbrace{2\cdots 2}_{N_2}\cdots\underbrace{m\cdots m}_{N_m}}(j\omega) \tag{2.27}$$

with $N_1 + \cdots + N_m = n$, and $n = 1, \cdots, N$, and

$$U_{(P_1=N_1,\cdots,P_m=N_m)}^{(n)}(j\omega) = \left[\underbrace{U_1(j\omega)\cdots U_1(j\omega)}_{N_1}\underbrace{U_2(j\omega)\cdots U_2(j\omega)}_{N_2}\cdots\underbrace{U_m(j\omega)\cdots U_m(j\omega)}_{N_m}\right] \tag{2.28}$$

The number of terms contained in Equation (2.26) can easily be calculated using the method given in [13], as

$$C(N,m) = m + (m+1)m/2! + \cdots + (m+N-1)\cdots(m+1)m/N! \tag{2.29}$$

It can be seen that there are $(m+n-1)\cdots(m+1)m/n!$ terms for the nth order NOFRFs. Sorting all $G^{(N)}_{k_1\cdots k_N}(j\omega)$, $k_i = k_{i-1},\cdots,m$, $i=1,\cdots,n$, $k_0 = 1$ as a series, and labeling them as

$$G^{(n)}_{(i,k)}(j\omega), \qquad k = 1:C_{(n,m)} \tag{2.30}$$

where

$$C_{(n,m)} = (m+n-1)\cdots(m+1)m/n! \tag{2.31}$$

Denoting the corresponding $[U_{k_1}(j\omega)\cdots U_{k_n}(j\omega)]$ as

$$U^{(n)}_{(k)}(j\omega), \qquad k = 1:C_{(n,m)} \tag{2.32}$$

then Equation (2.25) can be rewritten as

$$y_i(j\omega) = \left[U^{(1)}_{(1)}\cdots U^{(1)}_{(C_{(1,m)})}\cdots U^{(N)}_{(1)}\cdots U^{(N)}_{(C_{(N.m)})}\right][G_i] \tag{2.33}$$

where

$$[G_i] = \left[G^{(1)}_{(i,1)}\cdots G^{(1)}_{(i,C_{(1,m)})}\cdots G^{(N)}_{(i,1)}\cdots G^{(N)}_{(i,C_{(N,m)})}\right]^T \tag{2.34}$$

when, $x_i(t) = \alpha x_i^*(t), i = 1\cdots m.$, where α is a constant and $x_i^*(t), i = 1\cdots m.$ are the input signals under which the NOFRFs of the system are to be evaluated,

$$U^{(n)}_{(P_1=N_1,\cdots,P_m=N_m)}(j\omega) = \alpha^n\left(\frac{1}{2\pi}\right)^{n-1}\frac{1}{\sqrt{n}}\int_{\omega_1+\cdots+\omega_n=\omega}\prod_{j=1}^{m}\prod_{i=N_0+\cdots+N_{j-1}+1}^{N_0+\cdots+N_j}X_j^*(j\omega_i)d\sigma_{n\omega}$$
$$= \alpha^n U^{*(n)}_{(P_1=N_1,\cdots,P_m=N_m)}(j\omega) \tag{2.35}$$

where

$$U^{*(n)}_{(P_1=N_1,\cdots,P_m=N_m)}(j\omega) = \left(\frac{1}{2\pi}\right)^{n-1}\frac{1}{\sqrt{n}}\int_{\omega_1+\cdots+\omega_n=\omega}\prod_{j=1}^{m}\prod_{i=N_0+\cdots+N_{j-1}+1}^{N_0+\cdots+N_j}X_j^*(j\omega_i)d\sigma_{n\omega} \tag{2.36}$$

In this case, from (2.33), Equation (2.25) can be written as

$$y_i(j\omega) = \left[\alpha U^{*(1)}_{(1)}\cdots\alpha U^{*(1)}_{(C_{(1,m)})}\cdots\alpha^N U^{*(N)}_{(1)}\cdots\alpha^N U^{*(N)}_{(C_{(N.m)})}\right][G_i^*] \tag{2.37}$$

where $[G_i^*] = \left[G^{*(1)}_{(i,1)}\cdots G^{*(1)}_{(i,C_{(1,m)})}\cdots G^{*(N)}_{(i,1)}\cdots G^{*(N)}_{(i,C_{(N,m)})}\right]^T$ are the NOFRFs to be evaluated.

Excite the system $C(N,m) = C_{(1,m)} + \cdots + C_{(N,m)}$ times by the input signals

$$x_i(t) = \alpha_j x_i^*(t), \quad i = 1:m, \text{ and, } \quad j = 1:C(N,m)$$

$$\alpha_{C^{N,m}} > \alpha_{C^{N,m}-1} > \cdots > \alpha_1 > 0$$

to generate $C(N,m)$ output frequency responses $Y_i^k(j\omega)$, $k = 1:C(N,m)$. From (2.37), these output frequency responses can be written as

$$Y_i(j\omega) = \begin{bmatrix} \alpha_1 U^{*(1)}_{(1)} & \cdots & \alpha_1 U^{*(1)}_{(C_{(1,m)})} & \cdots & \alpha_1^N U^{*(N)}_{(1)} & \cdots & \alpha_1^N U^{*(N)}_{(C_{(N,m)})} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{C(N,m)} U^{*(1)}_{(1)} & \cdots & \alpha_{C(N,m)} U^{*(1)}_{(C_{(1,m)})} & \cdots & \alpha^N_{C(N,m)} U^{*(N)}_{(1)} & \cdots & \alpha^N_{C(N,m)} U^{*(N)}_{(C_{(N,m)})} \end{bmatrix}[G_i^*] \tag{2.38}$$

where

$$Y_i(j\omega) = \left[Y_i^1(j\omega)\ \cdots\ Y_i^{C(N,m)}(j\omega)\right]^T \tag{2.39}$$

Moreover, defining

$$AU^{1,\cdots,C(N,m)}(j\omega) = \begin{bmatrix} \alpha_1 U^{*(1)}_{(1)} & \cdots & \alpha_1 U^{*(1)}_{(C_{(1,m)})} & \cdots & \alpha_1^N U^{*(N)}_{(1)} & \cdots & \alpha_1^N U^{*(N)}_{(C_{(N,m)})} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{C(N,m)} U^{*(1)}_{(1)} & \cdots & \alpha_{C(N,m)} U^{*(1)}_{(C_{(1,m)})} & \cdots & \alpha^N_{C(N,m)} U^{*(N)}_{(1)} & \cdots & \alpha^N_{C(N,m)} U^{*(N)}_{(C_{(N,m)})} \end{bmatrix} \tag{2.40}$$

yields

$$Y_i(j\omega) = AU^{1,\cdots,C(N,m)}(j\omega)[G_i] \tag{2.41}$$

From Equation (2.41), $[G_i^*] = \left[G^{*(1)}_{(i,1)} \cdots G^{*(1)}_{(i,C_{(1,m)})} \cdots G^{*(N)}_{(i,1)} \cdots G^{*(N)}_{(i,C_{(N,m)})}\right]^T$ can then be determined using an Least Square based approach to yield

$$[G_i^*] = \left[\left(AU^{1,\cdots,C(N,m)}(j\omega)\right)^T \left(AU^{1,\cdots,C(N,m)}(j\omega)\right)\right]^{-1} \left(AU^{1,\cdots,C(N,m)}(j\omega)\right)^T Y_i(j\omega) \tag{2.42}$$

From Equation (2.29), it is known that the number of NOFRF terms will increase with the number of system inputs. For instance, a single input nonlinear system (m=1) with up to 4th order nonlinearity (N=4) has only 4 NOFRF terms; however, a nonlinear system of N=4 and m=2 will have 14 NOFRF terms. This implies that 14 different signal excitations are needed to generate the data of the output spectra to estimate these NOFRFs.

3 Energy Transfer Phenomena of a Multi-Input Nonlinear System

In this section, the concept of NOFRFs for multi-input nonlinear systems is applied to investigate the energy transfer phenomena in a 2-DOF nonlinear system [2]. The differential equation of the considered nonlinear system is given by

$$\begin{aligned} & m_1\ddot{y}_1(t) + (c_{11}+c_{12})\dot{y}_1(t) - c_{12}\dot{y}_2(t) + (k_{11}+k_{12})y_1(t) - k_{12}y_2(t) + c_2(\dot{y}_1(t)-\dot{y}_2(t))^2 \\ & + c_3(\dot{y}_1(t)-\dot{y}_2(t))^3 + k_2 y_1^2(t) + k_3 y_1^3(t) = u_1(t) \\ & m_2\ddot{y}_2(t) + (c_{12}+c_{22})\dot{y}_2(t) - c_{12}\dot{y}_1(t) + (k_{12}+k_{22})y_2(t) - k_{12}y_1(t) - c_2(\dot{y}_1(t)-\dot{y}_2(t))^2 \\ & - c_3(\dot{y}_1(t)-\dot{y}_2(t))^3 = u_2(t) \end{aligned} \tag{3.1}$$

Where $y_1(t), y_2(t)$ are the two outputs of the system, m_1, m_2, c_{11}, c_{12}, c_{22}, c_2, c_3, k_{11}, k_{12}, k_{22}, k_2, k_3 are the system parameters: mass, damping and stiffness respectively. The nonlinear system can be illustrated as a mechanical oscillator shown in Figure 3.

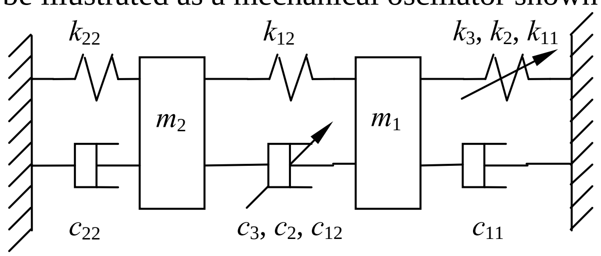


Figure 3, a 2-DOF nonlinear system

In the following study, the values of all the parameters used are $m_1 = m_2 = 1\,\text{kg}$, $c_{11} = c_{12} = c_{22} = 20$ N/m/s , $c_2 = 1 \times 500$ N(m/s)^2 , $c_3 = 1 \times 10^4$ N(m/s)^3 , $k_{11} = k_{12} = k_{22} = 1 \times 10^4$ N/m, $k_2 = 1 \times 10^7$ N/m^2, $k_3 = 5 \times 10^9$ N/m^3, and the two input excitations are

$$u_1(t) = \frac{3}{2\pi}\frac{\sin(2\times 35\times \pi \times t) - \sin(2\times 10\times \pi \times t)}{t} \qquad -10\sec \le t \le 10\sec \qquad (3.2)$$

$$u_2(t) = \frac{3}{2\pi}\frac{\sin(2\times 100\times \pi \times t) - \sin(2\times 85\times \pi \times t)}{t} \qquad -10\sec \le t \le 10\sec \qquad (3.3)$$

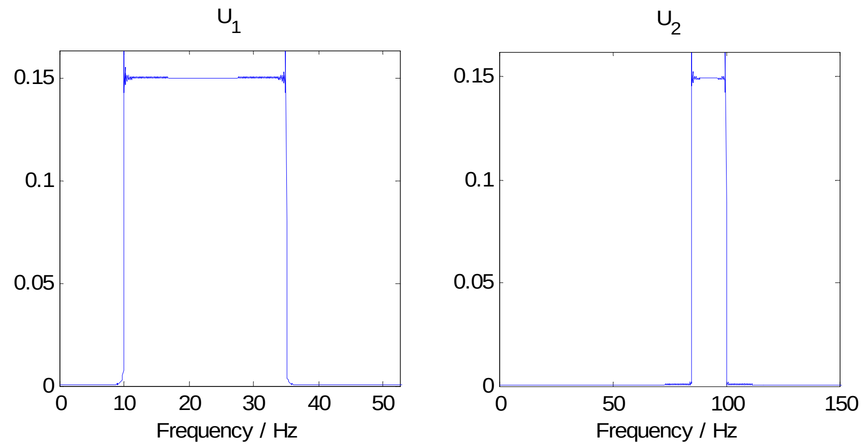


Figure 4, The spectra of the two inputs for the system in Equation (3.1)

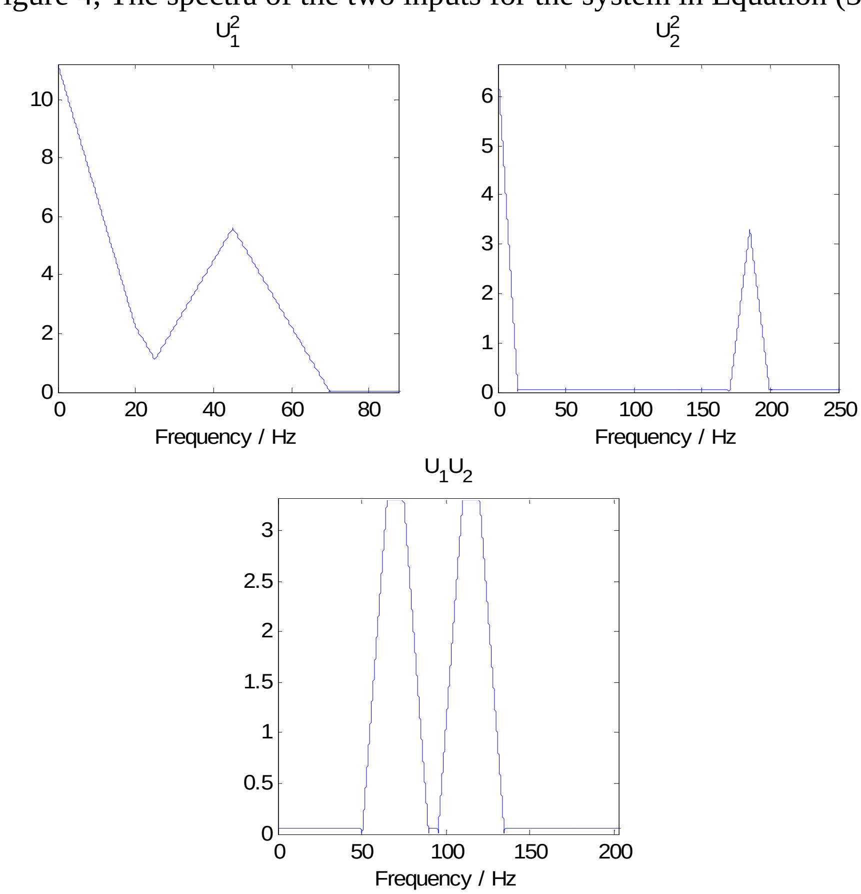


Figure 5 The spectra of $u_1^2(t)$, $u_2^2(t)$ and $u_1(t)u_2(t)$ for the system in Equation (3.1)

The frequency ranges of the first input and the second input are [-10 -35]$\cup$[10 35] Hz and [-85 -100]$\cup$ [85 100] Hz respectively. These spectra are shown in Figure 4. According to Equation (2.21), it can be known that the frequency range of $U^{(2)}_{(P_1=2,P_2=0)}(j\omega)$ is [2$\times$10 2$\times$35]$\cup$[10-35 35-10] $\cup$[-2$\times$35 -2$\times$10] = [-70 70] Hz. Similarly, it can be deduced that the frequency range of $U^{(2)}_{(P_1=0,P_2=2)}(j\omega)$ is [-200 -170]$\cup$[-15 15] $\cup$[170 200] Hz. According to Equation (2.23), it can be known that the frequency range of $U^{(2)}_{(P_1=1,P_2=1)}(j\omega)$ is [-135 -95]$\cup$[-90 -50] $\cup$[50 90] $\cup$[95 135] Hz. These results are verified by the spectra of $u_2^2(t)$ and $u_1(t)u_2(t)$ shown in Figure 5. Furthermore, using Equation (2.25), the possible frequency range of the output can be calculated to be [-200 -170]$\cup$[-135 135]$\cup$[170 200] Hz.

The forced response of the system is obtained through integrating Equation (3.1) using a fourth-order *Runge–Kutta* method, and the of results over ($-3 \le t \le 3$) are shown in Figure 6. Figure 7 shows the spectra of the outputs, which clearly indicate the two outputs have the same frequency range over [0 135] $\cup$[170 200] Hz, and this frequency range is the same as determined using the analysis result by Equation (2.24). From Figure 7, it can be seen that considerable input energy is transferred by the system from the input frequency band [10 35] $\cup$[85 100] Hz to the other frequency ranges [0 10) $\cup$(35 70] Hz.

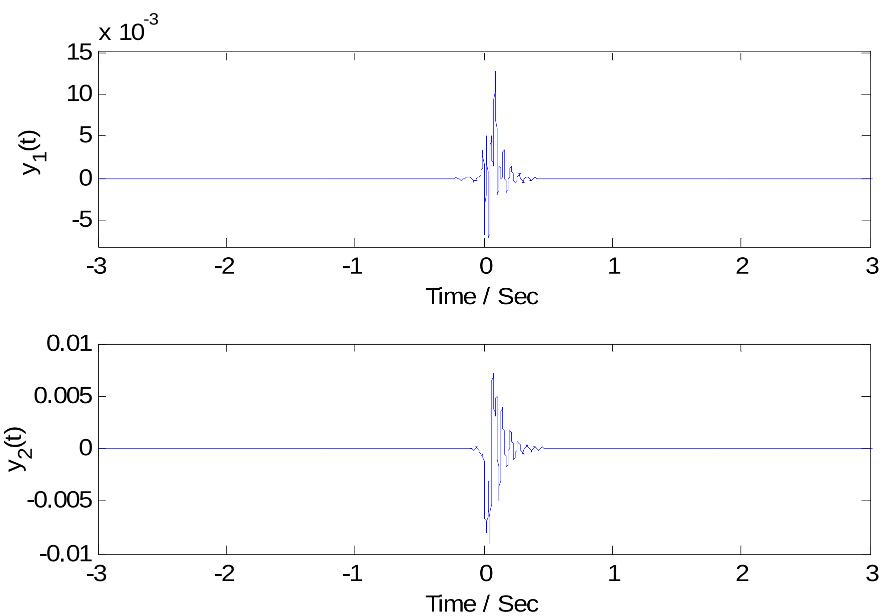


Figure 6 The output response of the system in Equation (3.1)

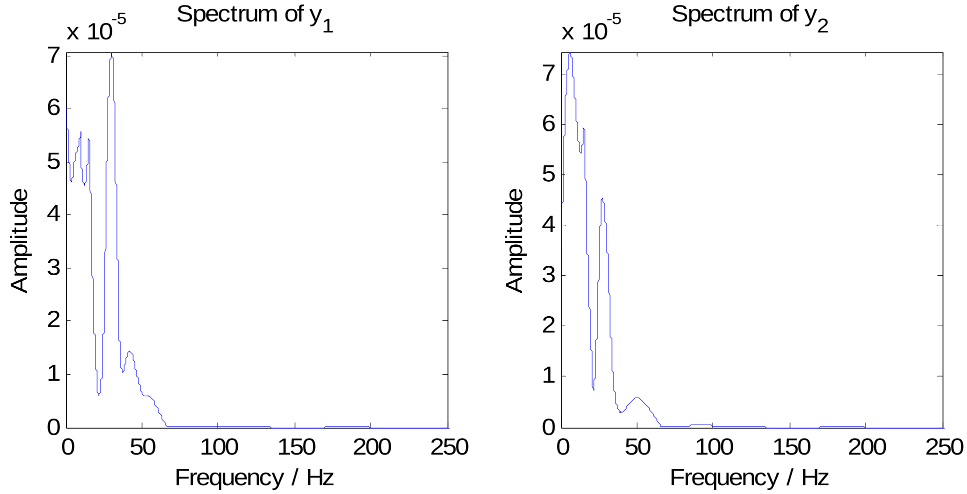


Figure 7 The output spectra of the system in Equation (3.1)

The NOFRFs of system (3.1) under the excitation (3.2) and (3.3) have been evaluated up to second order over the frequency range $[0\ 135]\bigcup[170\ 200]$ Hz. According to Equation (2.30), to evaluate the NOFRFs of a 2-DOF nonlinear system up to the second order, generally, five different signal excitations are needed. However, from the frequency ranges of $U^{(1)}_{(P_1=1)}(j\omega)$, $U^{(1)}_{(P_2=1)}(j\omega)$, $U^{(2)}_{(P_1=2,P_2=0)}(j\omega)$, $U^{(2)}_{(P_1=0,P_2=2)}(j\omega)$ and $U^{(2)}_{(P_1=1,P_2=1)}(j\omega)$, the output frequency responses in Equation (2.26) can be simplified as the below

$$y_i(j\omega)=G^{(2)}_{(i,P_1=2,P_2=0)}(j\omega)U^{(2)}_{(P_1=2,P_2=0)}(j\omega)+G^{(2)}_{(i,P_1=0,P_2=2)}(j\omega)U^{(2)}_{(P_1=0,P_2=2)}(j\omega)\quad \omega\in[0\ \ 10)\,\text{Hz}\quad \text{(a)}$$

$$y_i(j\omega)=G^{(1)}_{(i,P_1=1)}(j\omega)U^{(1)}_{(P_1=1)}(j\omega)+G^{(2)}_{(i,P_1=2,P_2=0)}(j\omega)U^{(2)}_{(P_1=2,P_2=0)}(j\omega)+G^{(2)}_{(i,P_1=0,P_2=2)}(j\omega)U^{(2)}_{(P_1=0,P_2=2)}(j\omega)\quad \omega\in[10\ \ 15]\,\text{Hz}\quad \text{(b)}$$

$$y_i(j\omega)=G^{(1)}_{(i,P_1=1)}(j\omega)U^{(1)}_{(P_1=1)}(j\omega)+G^{(2)}_{(i,P_1=2,P_2=0)}(j\omega)U^{(2)}_{(P_1=2,P_2=0)}(j\omega)\quad \omega\in(15\ \ 35]\,\text{Hz}\quad \text{(c)}$$

$$y_i(j\omega)=G^{(2)}_{(i,P_1=2,P_2=0)}(j\omega)U^{(2)}_{(P_1=2,P_2=0)}(j\omega)\quad \omega\in(35\ \ 50)\,\text{Hz}\quad \text{(d)}$$

$$y_i(j\omega)=G^{(2)}_{(i,P_1=2,P_2=0)}(j\omega)U^{(2)}_{(P_1=2,P_2=0)}(j\omega)+G^{(2)}_{(i,P_1=1,P_2=1)}(j\omega)U^{(2)}_{(P_1=1,P_2=1)}(j\omega)\quad \omega\in[50\ \ 70]\,\text{Hz}\quad \text{(e)}$$

$$y_i(j\omega)=G^{(2)}_{(i,P_1=1,P_2=1)}(j\omega)U^{(2)}_{(P_1=1,P_2=1)}(j\omega)\quad \omega\in(70\ \ 85)\bigcup(100\ \ 135)\,\text{Hz}\quad \text{(f)}$$

$$y_i(j\omega)=G^{(1)}_{(i,P_2=1)}(j\omega)U^{(1)}_{(P_2=1)}(j\omega)+G^{(2)}_{(i,P_1=1,P_2=1)}(j\omega)U^{(2)}_{(P_1=1,P_2=1)}(j\omega)\quad \omega\in[85\ \ 90]\bigcup[95\ \ 100]\ \text{Hz}\quad \text{(g)}$$

$$y_i(j\omega)=G^{(1)}_{(i,P_2=1)}(j\omega)U^{(1)}_{(P_2=1)}(j\omega)\quad \omega\in(90\ \ 95)\ \text{Hz}\quad \text{(h)}$$

$$y_i(j\omega)=G^{(2)}_{(i,P_1=0,P_2=2)}(j\omega)U^{(2)}_{(P_1=0,P_2=2)}(j\omega)\quad \omega\in[170\ \ 200]\ \text{Hz}\quad \text{(i)}$$

for i = 1, 2 (3.4)

Equation (3.4) indicates that, to estimate the NOFRFs up to the second order, three different excitations are enough. Equations (3.4-a~3.4-i) also clearly show how the energy transfer happens in the nonlinear system (3.1) when subjected to the inputs (3.2) and (3.3). For example, from Equation (3.4-a), it is clearly that it is the 2nd order NOFRFs

$G^{(2)}_{(1,P_1=2,P_2=0)}(j\omega)$, $G^{(2)}_{(1,P_1=0,P_2=2)}(j\omega)$ which transfer the energy from the frequency bands of the first input ([10 35] Hz) and the second input ([85 100] Hz) respectively to the frequency band [0 10) Hz in the output. The evaluated NOFRFs are shown in Figure 8 and Figure 9.

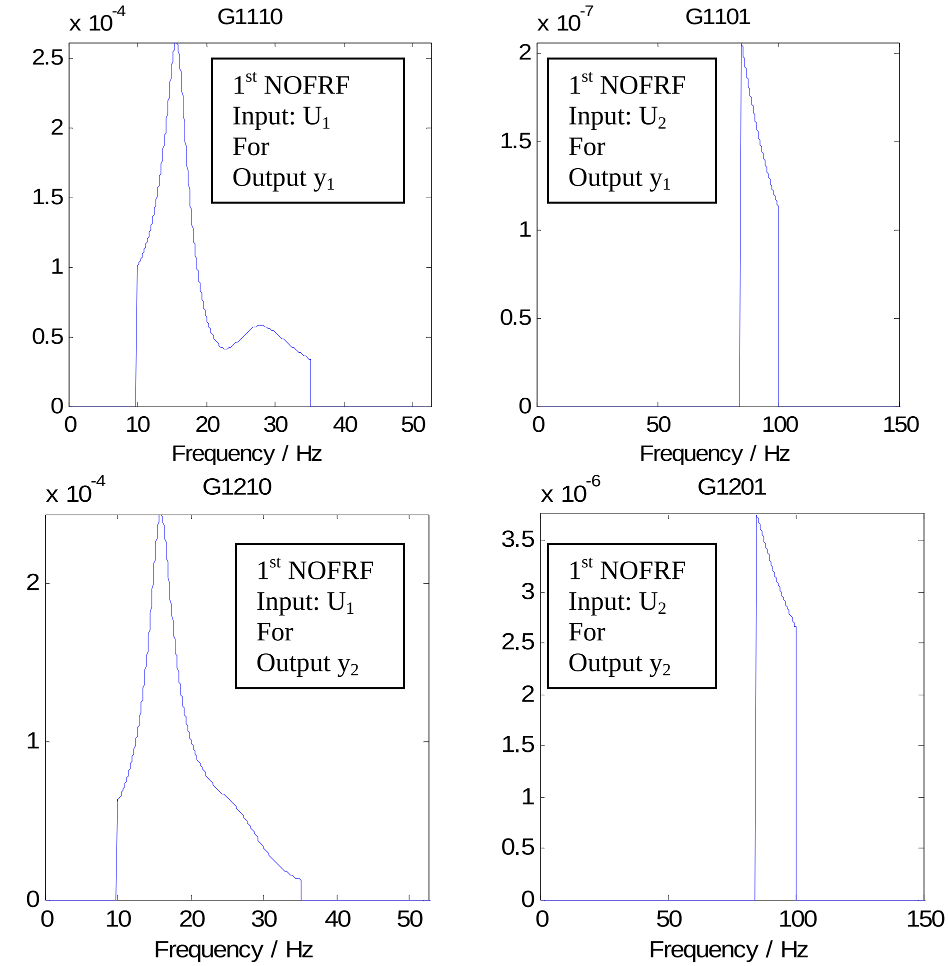
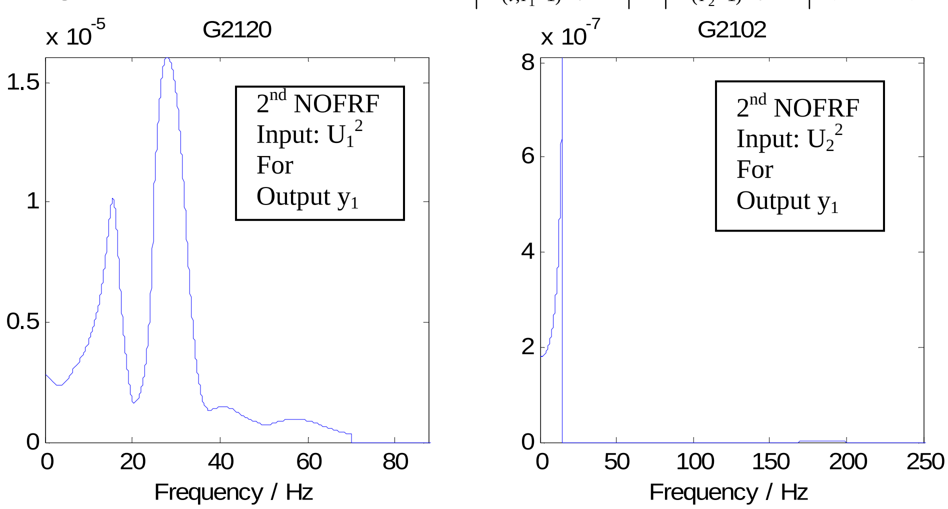


Figure 8, The first order NOFRFs $\left|G^{(1)}_{(i,P_1=1)}(j\omega)\right|$, $\left|G^{(1)}_{(P_2=1)}(j\omega)\right|$ $(i=1,2)$



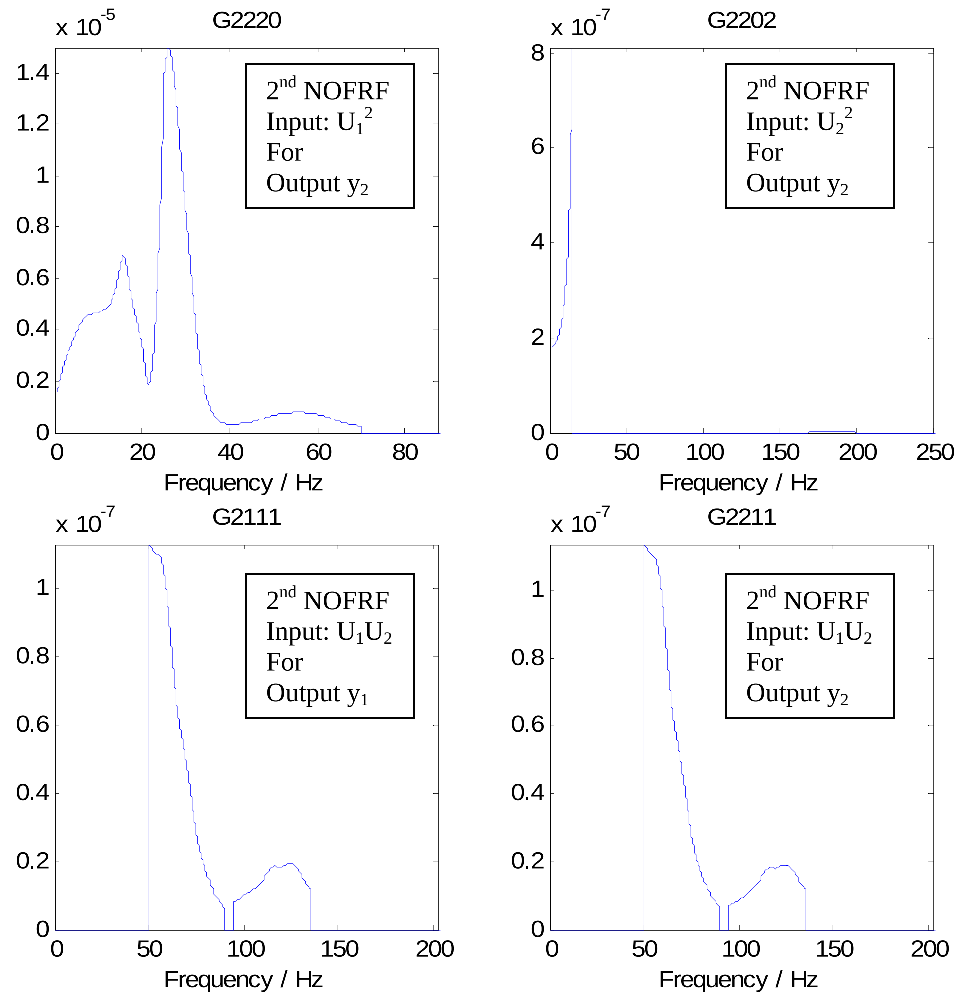


Figure 9, The second order NOFRFs $\left|G^{(2)}_{(i,P_1=2,P_2=0)}(j\omega)\right|$, $\left|G^{(2)}_{(i,P_1=0,P_2=2)}(j\omega)\right|$ and $\left|G^{(2)}_{(i,P_1=1,P_2=1)}(j\omega)\right|$, $(i=1,2)$

As the spectra in Figure 7 show, most of the output energy is located in the frequency range [0 70] Hz. From Figure 8 and Figure 9, it can be seen that, in the frequency range of [0 70] Hz, the first input dependent NOFRFs, such as $G^{(1)}_{(i,P_1=1)}(j\omega)$ and $G^{(2)}_{(i,P_1=2,P_2=0)}(j\omega)$, are much bigger than the other NOFRFs. This implies that the output energy in this frequency range is mainly contributed by the first input. For example, according to Equation (3.4-b), the first output response at 14 Hz is contributed by the three terms $G^{(1)}_{(1,P_1=1)}(j\omega)U^{(1)}_{(P_1=1)}(j\omega)$, $G^{(2)}_{(1,P_1=2,P_2=0)}(j\omega)U^{(2)}_{(P_1=2,P_2=0)}(j\omega)$, and $G^{(2)}_{(1,P_1=0,P_2=2)}(j\omega)U^{(2)}_{(P_1=0,P_2=2)}(j\omega)$. The contributions from these terms to the output are given in Table 1.

Table 1. The contributions of different terms to the output response at frequency of 14 Hz

Terms	Values	Modulus
$G^{(1)}_{(1,P_1=1)}(j\omega)U^{(1)}_{(P_1=1)}(j\omega)$	(0.1496 + 0.0066i)× (1.5899e-004-1.1215e-004i)	2.9135e-005
$G^{(2)}_{(1,P_1=2,P_2=0)}(j\omega)U^{(2)}_{(P_1=2,P_2=0)}(j\omega)$	(4.9330 + 0.2171i)× (-6.0092e-006 +5.1081e-006i)	3.8944e-005
$G^{(2)}_{(1,P_1=0,P_2=2)}(j\omega)U^{(2)}_{(P_1=0,P_2=2)}(j\omega)$	(0.3888 + 0.0171i) × (4.8131e-007 -3.9035e-007i)	2.4117e-007

From Table 1, it can be seen that the contribution of $G^{(2)}_{(1,P_1=0,P_2=2)}(j\omega)U^{(2)}_{(P_1=0,P_2=2)}(j\omega)$ is so small that it can be ignored. The output response at other frequencies can be analyzed in a similar way.

Comparing Equation (1.4) and Equation (2.12), the main difference between the NOFRFs of the single-input and the multi-input nonlinear systems is that the multi-input NOFRFs have more cross-NOFRF terms, for instance, $G^{(2)}_{(i,P_1=1,P_2=1)}(j\omega)$, ($i$ =1,2) for the second order NOFRF. From Equations (3.4-e, f, g), it can be seen that, in this study, $G^{(2)}_{(i,P_1=1,P_2=1)}(j\omega)$, ($i$=1,2) only influence the components at [50 90] $\cup$ [95 135] Hz. Equation (3.4-f) also indicates that the output responses at (70 85) $\cup$ (100 135) Hz are only determined by $G^{(2)}_{(i,P_1=1,P_2=1)}(j\omega)$, ($i$=1,2) and have a very small amplitude. At other frequency ranges, $G^{(2)}_{(i,P_1=1,P_2=1)}(j\omega)$, ($i$=1,2) will influence the output response together with other NOFRF terms, for example with $G^{(2)}_{(i,P_1=2,P_2=0)}(j\omega)$, ($i$=1,2) at [50 70] Hz. For the first output response at 55 Hz, the contributions by $G^{(2)}_{(1,P_1=2,P_2=0)}(j\omega)$ and $G^{(2)}_{(1,P_1=1,P_2=1)}(j\omega)$ are given in below Table 2.

Table 2. The contributions of different terms to the output response at frequency of 55 Hz

Terms	Values	Modulus
$G^{(2)}_{(1,P_1=2,P_2=0)}(j\omega)U^{(2)}_{(P_1=2,P_2=0)}(j\omega)$	(3.3131 + 0.5782i)× (-9.0729e-007 +8.2225e-008i)	3.0639e-006
$G^{(2)}_{(1,P_1=1,P_2=1)}(j\omega)U^{(2)}_{(P_1=1,P_2=1)}(j\omega)$	(1.0498 + 0.1832i) × (8.5484e-008 -6.8922e-008i)	1.1702e-007

The results in Table2 show that, compared with $G^{(2)}_{(1,P_1=2,P_2=0)}(j\omega)U^{(2)}_{(P_1=2,P_2=0)}(j\omega)$, the contribution of $G^{(2)}_{(1,P_1=1,P_2=1)}(j\omega)U^{(2)}_{(P_1=1,P_2=1)}(j\omega)$ to the output response at 55 Hz is very small and can be ignored. Similarly, it can be found that the contribution of the cross-NOFRF to the output responses at other frequencies is also very small. To a certain degree, this implies that the influence of the cross-NOFRFs on the output responses can be ignored in this specific case.

The results shown in Figure 8 and Figure 9 indicate that the maximum gains in the NOFRFs of $\left|G^{(2)}_{(i,P_1=2,P_2=0)}(j\omega)\right|$ appear near 16Hz and 28Hz, (i =1,2). This means that, at these frequencies, the energy transfer through these NOFRFs becomes more efficient, and the frequency components at these frequencies will become significant in the output spectra. This can be confirmed by the output spectra shown in Figure 7 where some significant components can be found at these frequencies.

The above qualitative analysis gives a clear interpretation regarding why and how the generation of new frequencies happens in a multi-input nonlinear system, and extends the procedure for the same analysis for single-input nonlinear systems to the more general multi-input nonlinear system case.

4 Conclusions and Remarks

In the present study, the concept of NOFRFs has been extended from the single-input nonlinear system case to the multi-input nonlinear system case. Given the frequency range of the inputs, a new method was also developed to determine the output frequency range. The phenomenon of energy transfer in a 2DOF nonlinear system subjected to two input excitations was investigated using the concept of NOFRFs for multi-input systems.

Multi-input systems are important in many engineering systems and structures. For example, multi-degree of freedom mechanical structures are a typical example of this category of systems. Therefore, the extension of the NOFRF concept to the more general multi-input case of nonlinear systems is important for the potential applications of the NOFRF concept to a wide range of engineering areas.

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