Monograph:

Wei, H.L., Lang, Z.Q. and Billings, S.A. (2006) An algorithm for determining the output frequency range of volterra models with multiple inputs. Research Report. ACSE Research Report no. 900 . Automatic Control and Systems Engineering, University of Sheffield

An Algorithm for Determining the Output Frequency Range of Volterra Models with Multiple Inputs

Hua-Liang Wei, Zi-Qiang Lang and Stephen A. Billings

Research Report No. 900

Department of Automatic Control and Systems Engineering
The University of Sheffield
Mappin Street, Sheffield,
S1 3JD, UK

6 September 2006

An Algorithm for Determining the Output Frequency Range of Volterra Models with Multiple Inputs

Hua-Liang Wei, Zi-Qiang Lang, and Stephen A. Billings

***Abstract*—A new algorithm for determining the output frequency range and the frequency components of Volterra models under multiple inputs is introduced for nonlinear system analysis. For a given Volterra model, the output frequency components corresponding to a multi-tone input can easily be calculated using the new algorithm.**

***Index Terms*—generalized frequency response functions, nonlinear systems, output spectrum, Volterra models.**

I. Introduction

One important aspect of system analysis in the frequency domain is the requirement to investigate the relationship between the system input frequencies and the output frequency behaviour. For linear systems, the output frequency function $Y(j\omega)$ is related to the input frequency spectrum $U(j\omega)$ by the system frequency response function $H(j\omega)$ via the simple linear relationship $Y(j\omega) = H(j\omega)U(j\omega)$. This simple basic result provides the foundation for all linear system analysis and design in the frequency domain. In this case, the input frequencies pass independently through the system, that is, an input at a given frequency ω produces at steady state an output at the same frequency and no energy is transferred to or from any other frequency components. The system frequency response function $H(j\omega)$ itself alone can totally characterise a given linear system. For nonlinear systems, however, this is not true. It has been observed that the output frequency components of nonlinear systems are much richer compared to the corresponding input frequencies. The input frequencies pass in a coupled way through a nonlinear system, that is, an input at given frequencies may produce quite different output frequencies. This is quite different from the case for linear systems where the output frequency range is identical in steady state to that of the inputs. This makes it difficult to give a general explicit expression connecting the input and output frequencies for most nonlinear systems. However, for some specified inputs, explicit algorithms are available to determine the output frequency range [1].

This study presents a new and much simpler algorithm for the determination of the output frequency range and the frequency components for Volterra models under multitone inputs. This is very useful for the analysis of nonlinear systems in the frequency domain.

II. Generalized Frequency Response Functions for Nonlinear Systems

It is well known that the input-output relationship of a wide class of nonlinear systems can be approximated in the time domain by the Volterra functional series [4].

$$y(t) = \sum_{n=1}^{L} y_n(t) \tag{1}$$

where the system output $y(t)$ is expressed as a sum of the response of L parallel subsystems, each of which is related to both the system input $u(t)$ and an nth-order kernel. The output of the nth-order nonlinear

Manuscript received September 6, 2006 .This work was supported by the Engineering and Physical Sciences Research Council, U.K. (EPSRC-UK).

The authors are with the Department of Automatic Control and Systems Engineering, University of Sheffield, Sheffield, S1 3JD, UK. E-mail: {s.billings@shef.ac.uk, z.lang@sheffield.ac.uk,w.hualiang@shef.ac.uk }.

subsystem, $y_n(t)$, is characterised by an extension of the familiar convolution integral of linear systems theory to higher dimensions

$$y_n(t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)u(t-\tau_1)\cdots u(t-\tau_n)d\tau_1\cdots d\tau_n=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}[u(t-\tau_i)d\tau_i] \tag{2}$$

where the *nth-order kernel* or *nth-order impulse response* $h_n(\tau_1,\cdots,\tau_n)$ is so called because this reduces to the linear impulse response function for the simplest case n=1. By introducing the concept of the *nth-order associated function* [4] and then taking the multidimensional Fourier transform of the associated function, yields from (2)

$$\hat{Y}_n(j\omega_1,\cdots,j\omega_n)=H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i) \tag{3}$$

where $U(\cdot)$ is the input spectrum defined as the Fourier transform operator. $H_n(j\omega_1,\cdots,j\omega_n)$ is the *nth-order transfer function* or *nth-order generalised frequency response function* (GFRF) defined as

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)e^{-j(\omega_1\tau_1+\cdots+\omega_n\tau_n)}d\tau_1\cdots d\tau_n \tag{4}$$

Following [1] and [3], it can easily be shown that

$$y_n(t)=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\hat{Y}_n(j\omega_1,\cdots,j\omega_n)\times e^{j(\omega_1+\cdots+\omega_n)t}d\omega_1\cdots d\omega_n \tag{5}$$

By making a change of variables

$$\begin{cases}\sigma_i=\omega_i,\ 1\le i\le n-1\\ \sigma_n=\sum_{i=1}^{n}\omega_i\end{cases} \tag{6}$$

Eq. (5) becomes

$$\begin{aligned}y_n(t)&=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\hat{Y}_n(j\sigma_1,\cdots,j\sigma_{n-1},j(\sigma_n-\sigma_1-\cdots-\sigma_{n-1}))\\ &\quad\times e^{j\sigma_n t}d\sigma_1\cdots d\sigma_n=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{(2\pi)^{n-1}}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n-1}\hat{Y}_n(j\omega_1,\cdots,j\omega_{n-1},j(\omega-\textstyle\sum_{i=1}^{n-1}\omega_i))d\omega_1\cdots d\omega_{n-1}\right]\\ &\quad\times e^{j\omega t}d\omega\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}Y_n(j\omega)e^{j\omega t}d\omega\end{aligned} \tag{7}$$

where

$$Y_n(j\omega)=\frac{1}{(2\pi)^{n-1}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\hat{Y}_n(j\omega_1,\cdots,j\omega_{n-1},j(\omega-\omega_1-\cdots-\omega_{n-1}))\times d\omega_1\cdots d\omega_{n-1} \tag{8}$$

From (1) and (7)

$$y(t)=\sum_{n=1}^{L}\frac{1}{2\pi}\int_{-\infty}^{\infty}Y_n(j\omega)e^{j\omega t}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_{n=1}^{L}Y_n(j\omega)\right]e^{j\omega t}d\omega \tag{9}$$

Therefore, the system output frequency response or output spectrum to a given general input $u(t)$ is

$$Y(j\omega)=\sum_{n=1}^{L}Y_n(j\omega),\ \omega\in\bigcup_{n=1}^{L}\Omega_n \tag{10}$$

where Ω_n is the effective frequency domain of the nth-order output frequency function $Y_n(j\omega)$. The family $\{\omega_1,\cdots,\omega_n;\omega\}$ in Eq (8) was referred to as the *input-output frequency domain* in [3]. The output spectrum $Y_n(j\omega)$ can therefore be referred to as the *nth-order output frequency (response) function* or *output spectrum*. For a physical interpretation of (5) and (8), see [1][3]. Note from the variable transform (6) that the input-output frequency domain is restricted to $\omega_1+\cdots+\omega_n=\omega$. The valid frequency range of the output spectrum can therefore be determined provided that the input frequencies are known.

III. Determining Output Frequencies Under Multiple Inputs

This section presents a useful result on calculating the output frequencies of nonlinear systems which can be described by the Voterra series.

A. *Description of Output Frequencies*

As a simple example, consider a simple case, where a nonlinear system is driven by a sinusoidal signal

$$u(t)=A\cos(\omega_0 t)=\frac{A}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t}) \tag{11}$$

Substituting (11) into (2), yields [2]

$$y_n(t)=\left(\frac{A}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}H_n(\underbrace{j\omega_0,\cdots,j\omega_0}_{n-k},\underbrace{-j\omega_0,\cdots,-j\omega_0}_{k})e^{j(n-2k)\omega_0 t} \tag{12}$$

From (12), the input to the nth-order submodel $y_n(t)$ contains only one single principal frequency component ω_0, the output of the nth-order submodel $y_n(t)$, however, contains many frequency components distributed at $\pm n\omega_0$, $\pm(n-2)\omega_0$, $\pm(n-4)\omega_0,\cdots$. For example, for the linear submodel of the nonlinear system (1), the output frequencies include $\pm\omega_0$; for the 2nd-order nonlinear subsystem, the output frequencies will appear at 0 and $\pm 2\omega_0$.

For a general case, where the input is a summation of multiple sinusoidal waves

$$u(t)=\sum_{i=1}^{K}A_i\cos(\omega_i t)=\sum_{i=-K}^{K}\frac{A_i}{2}e^{j\omega_i t} \tag{13}$$

with $\omega_0=0$, $\omega_{-i}=-\omega_i$, $A_0=0$, $A_{-i}=A_i$, the output of the nth-order submodel $y_n(t)$ can be calculated to be [1]

$$y_n(t) = \frac{1}{2^n} \sum_{k_1=-K}^{K} \cdots \sum_{k_n=-K}^{K} B(\omega_{k_1}) \cdots B(\omega_{k_n}) H_n(\omega_{k_1}, \cdots, \omega_{k_n}) \times e^{j(\omega_{k_1} + \cdots + \omega_{k_n})t} \tag{14}$$

where

$$B(\omega) = \begin{cases} A_k & \omega \in \{\omega_k : k = \pm 1, \cdots, \pm K\} \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Following [1], the nth-order output frequency function $Y_n(j\omega)$ can be expressed as

$$Y_n j\omega) = \frac{1}{2^{n-1}} \underbrace{\sum_{k_1=-K}^{K} \cdots \sum_{k_n=-K}^{K}}_{\omega_{k_1} + \cdots + \omega_{k_n} = \omega} B(\omega_{k_1}) \cdots B(\omega_{k_n}) H_n(\omega_{k_1}, \cdots, \omega_{k_n}) \tag{16}$$

As will be shown in the next section, the output frequency components of the nth-order submodel $y_n(t)$ will be much richer compared with the input frequency components since each frequency component ω determined by the combination $\omega = \sum_{i=1}^{n} \omega_{k_i}$ with $k_i \in \{\pm 1, \pm 2, \cdots, \pm K\}$ might appear in the output frequency domain. An important point is that these possible output frequency components can be determined beforehand once the frequency components in the multiple input are given.

B. An Algorithm for Determining the Output Frequency Range

It is observed that the output frequency components of nonlinear systems are much richer compared to the corresponding input frequencies. The input frequencies will pass in a coupled way through a nonlinear system, that is, an input at given frequencies may produce quite different output frequencies. Therefore energy may be transferred to or from other frequency components. This is quite different from the case for linear systems where the output frequency range is identical in steady state to that of the input. It would be difficult to give a general explicit expression connecting the input and output frequencies for all nonlinear systems. However, for some specified inputs, explicit algorithms are available to determine the effective frequency range for arbitrary order output frequency response functions. Lang and Billings [1] proposed an algorithm to compute the frequency range of the arbitrary order output frequency function $Y_n(j\omega)$ defined by (7) and (8). In this study, however, a much improved and compact recursive algorithm is proposed for calculating the effective frequency range of arbitrary order output frequency functions.

From (7) and (8), the input and output frequencies for the nth-order subsystem with a multiple input of the form (13) will be constrained by

$$\omega = \sum_{i=1}^{n} \omega_{k_i}, \ k_i \in \{\pm 1, \pm 2, \cdots, \pm K\} \tag{17}$$

This will be used to determine the frequency range of the nth-order output frequency function. For convenience of description, denote

$$\begin{cases} \sigma_1 = -\omega_K \\ \cdots \\ \sigma_K = -\omega_1 \\ \sigma_{K+1} = \omega_1 \\ \cdots \\ \sigma_{2K} = \omega_K \end{cases} \tag{18}$$

For the simplest case of n=1, it is clear that the effective frequency range of the output spectrum is $\omega \in \Omega_1 = \{\sigma_k : k = 1,2,\cdots,2K\} = \{\omega_k : k = \pm 1,\cdots,\pm K\}$.

In order to determine the effective frequency range Ω_2 for the case of n=2, consider the following combinations of two frequency components

$$\begin{cases} \sigma_1 + \sigma_1 \\ \vdots \\ \sigma_1 + \sigma_{2K} \\ \sigma_2 + \sigma_1 \\ \vdots \\ \sigma_2 + \sigma_{2K} \\ \vdots \\ \sigma_{2K} + \sigma_1 \\ \vdots \\ \sigma_{2K} + \sigma_{2K} \end{cases} \tag{19}$$

This can be expressed in a vector form as

$$\Gamma_2 = \begin{bmatrix} \sigma_1 I_{2K} + V \\ \vdots \\ \sigma_{2K} I_{2K} + V \end{bmatrix} = V \otimes I_{2K} + I_{2K} \otimes V \tag{20}$$

where $I_{2K} = [\underbrace{1,\cdots,1}_{2K}]^T$, $V = [\sigma_1,\cdots,\sigma_{2K}]^T$. The symbol '$\otimes$' denotes the Kronecker product, which is defined for two vectors $A = [a_1,\cdots,a_p]^T$ and $B = [b_1,\cdots,b_q]^T$ as

$$A_{p\times 1} \otimes B_{q\times 1} = \begin{bmatrix} a_1 B \\ a_2 B \\ \cdots \\ a_p B \end{bmatrix} \tag{21}$$

For a given vector $X = [x_1, x_2, \cdots, x_p]^T$, let X^S denote a set whose elements are formed by the entities of X in the sense that $X^S = \{x_i : 1 \le i \le p\}$. It can easily be proved that all the different entities of the vector Γ_2^S are identical to all the effective frequency components of the second order output frequency function $Y_2(j\omega)$. Note that some entities in the vector Γ_2^S may be the same. Therefore Γ_2^S is redundant for determining the effective frequency components of $Y_2(j\omega)$.

In general, the effective frequency components of the nth order output frequency function $Y_n(j\omega)$ can be calculated using the recursive algorithm below:

Algorithm 1 *Assume that a nonlinear system is excited by a multiple input signal u(t) of the form (13) with K fundamental frequency components ,* $\{\omega_1, \omega_2, \cdots, \omega_K\}$. *The effective frequency components of the nth-order output frequency function can be determined by searching all the different entities of* Ω_n, *which is defined as*

$$\Gamma_1 = V \tag{22}$$

$$\Gamma_n = \Gamma_{n-1} \otimes I_{2K} + I_{<\Gamma_{n-1}^S>} \otimes V \tag{23}$$

$$\Omega_n = \Gamma_n^S, \; n \geq 2 \tag{24}$$

where V is defined as in (20), $<\Gamma_{n-1}^S> = <\Omega_{n-1}>$ *indicates the number of entities in the vector* Γ_{n-1}, *and*

$$I_m = \underbrace{[1, 1, \cdots, 1]}_{m}^T \tag{25}$$

The above recursive algorithm is very simple and quite easy to implement using vector-oriented software tools. As an example, consider the case of K=3, ω_1=2, ω_2=3 and ω_3=7. For n=2 and 3, the frequency components of the output frequency functions were calculated to be Ω_2={0,±1, ±4, ±5, ±6, ±9, ±10, ±14} and Ω_3={±1, ±2, ±3, ±4, ±6, ±7, ±8, ±9, ±11, ±12, ±13, ±16, ±17, ±21}.

Proof of Algorithm 1 Assume that all the different entities of the vector Γ_n are identical to all the effective frequency components of the nth-order output frequency function $Y_n(j\omega)$. Let $\Gamma_n = [\sigma_1^{(n)}, \cdots, \sigma_{\alpha(n)}^{(n)}]^T$, where $\alpha(n) = <\Gamma_n^S>$. All the possible frequency components for the (n+1)th-order output frequency function $Y_{n+1}(j\omega)$ can then be determined by inspecting the following combinations:

$$\begin{cases} \sigma_1^{(n)} + \sigma_1 \\ \quad\vdots \\ \sigma_1^{(n)} + \sigma_{2K} \\ \sigma_2^{(n)} + \sigma_1 \\ \quad\vdots \\ \sigma_2^{(n)} + \sigma_{2K} \\ \quad\vdots \\ \sigma_{\alpha(n)}^{(n)} + \sigma_1 \\ \quad\vdots \\ \sigma_{\alpha(n)}^{(n)} + \sigma_{2K} \end{cases} \tag{26}$$

Similar to (25), the above equation can be expressed in a vector form as

$$\Gamma_{n+1} = \begin{bmatrix} \sigma_1^{(n)} I_{2K} + V \\ \vdots \\ \sigma_{\alpha(n)}^{(n)} I_{2K} + V \end{bmatrix} = \Gamma_n \otimes I_{2K} + I_{\alpha(n)} \otimes V \qquad (27)$$

This is just (23). Therefore, Algorithm 1 can be used to determine the effective frequency range for the arbitrary order output frequency function $Y_n(j\omega)$. Note that some entities in Γ_n are the same and Γ_n is often redundant for determining the effective frequency components of the *n*th-order output frequency function $Y_n(j\omega)$.

It is known that the positive and negative frequencies are symmetrical about the origin, therefore only the non-negative frequencies need to be calculated. It can easily be shown that the non-negative frequency components of the *n*th-order output frequency function $Y_n(j\omega)$ can be calculated using the recursive algorithm below:

Algorithm 2 *Assume that a nonlinear system is excited by a multiple input signal u(t) of the form (13) with K fundamental frequency components ,* $\Omega_1 = \{\omega_1, \omega_2, \cdots, \omega_K\}$. *The non-negative frequency components of the nth order output frequency function can be determined by searching all the different entities of* Ω_n^+, *which is defined as*

$$\Gamma_1 = [\omega_1, \omega_2, \cdots, \omega_K]^T \qquad (28)$$

$$\Gamma_n = \Gamma_{n-1} \otimes I_{2K} + I_{K(2K)^{n-2}} \otimes V \qquad (29)$$

$$\Omega_n^+ = |\Gamma_n|^S, \; n \geq 2 \qquad (30)$$

where *V* is defined as in (20), I_m is defined by (25), and $|\Gamma_n|^S$ is a set whose elements are composed by all the different entities of the vector Γ_n by taking absolute values.

Algorithm 2 can be proved in the same way as Algorithm 1. The recursive algorithm is very simple and quite easy to implement using vector-oriented software tools. For the case of *n*=2, (29) becomes

$$\Gamma_2 = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_K \end{bmatrix} \otimes I_{2K} + I_K \otimes V = \begin{bmatrix} \omega_1 + \sigma_1 \\ \vdots \\ \omega_1 + \sigma_{2K} \\ \vdots \\ \omega_K + \sigma_1 \\ \vdots \\ \omega_K + \sigma_{2K} \end{bmatrix} \qquad (31)$$

Clearly, the absolute values of all the different entities of the vector Γ_2 are identical to all the non-negative frequency components of the second order output frequency function $Y_2(j\omega)$.

As an example, consider the case of *K*=3, $\omega_1 = 2\pi f_1$, $\omega_2 = 2\pi f_2$, $\omega_3 = 2\pi f_3$ with f_1=2, f_2=3 and f_3=7. For *n*=3, the non-negative frequency components of the output frequency functions were calculated to be Ω_3^+={1,2,3,4,6,7,8,9, 11,12,13,16,17,21}.

IV. Conclusion

A new algorithm has been introduced to determine the output frequency range and the frequency components for the Volterra class of nonlinear systems with multitone inputs. The new algorithm is quite simple and easy to implement using vector and matrix-oriented software tools. Thus compared to previous results [1], the new algorithm is more compact in form and much simpler to implement.

REFERENCES

[1] Z. Q. Lang and S. A. Billings, "Output frequency characteristics of nonlinear systems," *Int. J. Control*, vol. 64, no.6, pp.1049-1067, Aug.1996.

[2] G. M. Lee, "Estimation of non-linear system parameters using higher-order frequency response functions," *Mech. Syst. Signal Proc.*, vol. 11, no. 2, pp. 219-228, Mar.1997.

[3] J. C. P. Jones and S. A. Billings, "Interpretation of nonlinear frequency response functions," *Int. J. Control*, vol. 52, no. 2, pp. 319-346, Aug. 1990.

[4] M. Schetzen, *The Volterra and Wiener Theories of Nonlinear Systems*. New York: John Wiley & Sons, 1980.