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RECOLLEMENT OF DEFORMED PREPROJECTIVE ALGEBRAS AND THE CALOGERO-MOSER CORRESPONDENCE

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To Vitya Ginzburg for his 50th birthday

1. INTRODUCTION

The aim of this paper is to clarify the relation between the following objects: (*a*) rank 1 projective modules (ideals) over the first Weyl algebra $A_1(\mathbb{C})$; (*b*) simple modules over deformed preprojective algebras $\Pi_\lambda(Q)$ introduced by Crawley-Boevey and Holland [CBH]; and (*c*) simple modules over the rational Cherednik algebras $H_{0,c}(S_n)$ associated to symmetric groups [EG]. The isomorphism classes of each type of these objects can be parametrized geometrically by the same space (namely, the Calogero-Moser algebraic varieties); however, no natural functors between the corresponding module categories seem to be known. We construct such functors by translating our earlier results on A_∞-modules over A_1 (see [BC]) to the more familiar language of quiver representations. We should mention that the question of explaining the "mysterious bijection" between ideal classes of A_1 and simple modules of Cherednik algebras was first raised in [EG] and emphasized further in [BGK2].

2. THE CALOGERO-MOSER CORRESPONDENCE

2.1. **The Calogero-Moser spaces.** For an integer $n \geq 0$, let $\tilde{\mathcal{C}}_n$ be the space of linear maps

$$\{(\bar{X}, \bar{Y}, \bar{v}, \bar{w}) \,:\, \bar{X}, \bar{Y} \in \mathtt{End}(\mathbb{C}^n)\,,\, \bar{v} \in \mathtt{Hom}(\mathbb{C}, \mathbb{C}^n)\,,\, \bar{w} \in \mathtt{Hom}(\mathbb{C}^n, \mathbb{C})\}\,,$$

satisfying the equation $[\bar{X}, \bar{Y}] + \mathtt{Id} + \bar{v}\,\bar{w} = 0$. The group $\mathtt{GL}(n, \mathbb{C})$ acts on $\tilde{\mathcal{C}}_n$ in the natural way:

$$(\bar{X}, \bar{Y}, \bar{v}, \bar{w}) \mapsto (g\bar{X}g^{-1},\, g\bar{Y}g^{-1},\, g\bar{v},\, \bar{w}g^{-1}),\ g \in \mathtt{GL}(n, \mathbb{C})\,,$$

and we define the *n*-th *Calogero-Moser space* $\mathcal{C}_n$ to be the the quotient variety $\tilde{\mathcal{C}}_n /\!/ \mathtt{GL}(n, \mathbb{C})$ modulo this action. It turns out that $\mathtt{GL}(n, \mathbb{C})$ acts freely on $\tilde{\mathcal{C}}_n$, and $\mathcal{C}_n$ is a smooth affine variety of dimension $2n$ (see [W]).

2.2. **Deformed preprojective algebras.** Let $Q = (I, Q)$ be a finite quiver with vertex set I and arrow set Q, and let $\bar{Q}$ be its double (i.e. the quiver obtained from Q by adding a reverse arrow a^* to each arrow $a \in Q$). Following [CBH], for each $\lambda = (\lambda_i) \in \mathbb{C}^I$, we define the *deformed preprojective algebra of weight* λ by

$$\Pi_\lambda(Q) := \mathbb{C}\bar{Q} \left/ \left\langle \sum_{a \in Q} [a, a^*] - \sum_{i \in I} \lambda_i e_i \right\rangle \right. .$$

Here $\mathbb{C}\bar{Q}$ denotes the path algebra of the double quiver $\bar{Q}$ and $e_i \in \mathbb{C}\bar{Q}$ stand for the orthogonal idempotents corresponding to the trivial paths (vertices) in $\bar{Q}$.

In this section we will be concerned with the following example. Let Q_∞ be the quiver consisting of two vertices $\{\infty, 0\}$ and two arrows $v : 0 \to \infty$ and $X : 0 \to 0$. Write $w := v^*$ and $Y := X^*$ for the reverse arrows in $\bar{Q}_\infty$. The algebra $\Pi_\lambda := \Pi_\lambda(Q_\infty)$ is then generated by X, Y, v, w and the idempotents e_0 and e_∞, which, apart from the standard path algebra relations, satisfy

$$(1) \qquad\qquad [X, Y] - wv = \lambda_0 e_0 , \quad vw = \lambda_\infty e_\infty .$$

Thus, right Π_λ-modules can be identified with representations $V = V_\infty \oplus V_0$ of $\bar{Q}_\infty$, in which linear maps $\bar{X}, \bar{Y} \in \mathtt{End}(V_0)$, $\bar{v} \in \mathtt{Hom}(V_\infty, V_0)$, $\bar{w} \in \mathtt{Hom}(V_0, V_\infty)$, corresponding to the (right) action of X, Y, v, w, satisfy the equations

$$(2) \qquad\qquad [\bar{X}, \bar{Y}] + \bar{v}\bar{w} = -\lambda_0 \mathtt{Id}_{V_0} , \quad \bar{w}\bar{v} = \lambda_\infty \mathtt{Id}_{V_\infty} .$$

2.3. **Representation varieties.** Let $\lambda_0 = 1$ and $\lambda_\infty = -n$ in the above example. Comparing (2) to the definition of the Calogero-Moser spaces in Section 2.1, we see that each point of $\tilde{\mathcal{C}}_n$ corresponds naturally to a right Π_λ-module of dimension vector $\alpha = (1, n)$. All such modules are simple and, as it was originally observed by Crawley-Boevey (see [CB2] or Proposition 3 below), every semisimple module of Π_λ of dimension α has this form. Thus, we can identify the varieties $\mathcal{C}_n$ with representation spaces $\mathtt{Rep}(\Pi_\lambda^{\mathrm{opp}}, \alpha)//\mathtt{G}(\alpha)$, parametrizing the isomorphism classes of simple modules of dimension vector α.

On the other other hand, according to [BW, BW1], the Calogero-Moser varieties $\mathcal{C}_n$ also parametrize the isomorphism classes of right ideals of the first Weyl algebra $A_1(\mathbb{C})$. Our aim is to relate simple modules of Π_λ to ideals of A_1 in a natural (functorial) way. To this end we will use a version of "recollement" formalism, due to Beilinson, Bernstein and Deligne [BBD] (see also [CPS]).

2.4. **Recollement.** Originating from geometry, this formalism abstracts functorial relations between the categories of abelian sheaves on a topological space X, its open subspace U and the closed complement of U in X. Unlike [BBD], we will work with abelian (rather than triangulated) categories. For reader's convenience, we review the definition of recollement in this special case.

Let $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ be three abelian categories. We say that $\mathcal{A}$ is a *recollement* of $\mathcal{A}'$ and $\mathcal{A}''$ if there are six additive functors

$$(3) \qquad\qquad \mathcal{A}' \underset{\substack{\longleftarrow \\ i^* \\ \xrightarrow{\;\;i_*\;\;} \\ \longleftarrow \\ i^!}}{} \mathcal{A} \underset{\substack{\xleftarrow{\;\;j_!\;\;} \\ \xrightarrow{\;\;j^*\;\;} \\ \xleftarrow{\;\;j_*\;\;}}}{} \mathcal{A}''$$

satisfying the following conditions:

(R1) i^* and $i^!$ are adjoint to i_* on the left and on the right respectively, the adjunction morphisms $\mathtt{Id}_{\mathcal{A}'} \to i^! i_*$ and $i^* i_* \to \mathtt{Id}_{\mathcal{A}'}$ being isomorphisms;

(R2) $j_!$ and j_* are adjoint to j^* on the left and on the right respectively, the adjunction morphisms $\mathtt{Id}_{\mathcal{A}''} \to j^* j_!$ and $j^* j_* \to \mathtt{Id}_{\mathcal{A}''}$ being isomorphisms;

(R3) i_* is an embedding of $\mathcal{A}'$ onto the full subcategory of $\mathcal{A}$ consisting of objects annihilated by j^*; thus j^* induces an equivalence of abelian categories $\mathcal{A}/\mathcal{A}' \simeq \mathcal{A}''$.

These conditions imply

(R4) $i^* j_! = 0$ and $i^! j_* = 0$ (by adjunction with $j^* i_* = 0$);

(R5) the canonical morphisms $\sigma : j_! j^* \to \mathtt{Id}_{\mathcal{A}}$ and $\eta : \mathtt{Id}_{\mathcal{A}} \to j_* j^*$ give rise to the exact sequences of natural transformations

$$\text{(4)} \qquad j_! j^* \to \mathtt{Id}_{\mathcal{A}} \to i_* i^* \to 0 \quad , \quad 0 \to i_* i^! \to \mathtt{Id}_{\mathcal{A}} \to j_* j^* \ .$$

We will need some simple consequences of the above definition, which we record in the next two lemmas.

Lemma 1. *Let L be an object of $\mathcal{A}$ such that $i^* L = i^! L = 0$. Then*

(a) $\sigma_L : j_! j^* L \to L$ is an epimorphism in $\mathcal{A}$ with $\mathtt{Ker}(\sigma_L) \cong i_* i^! j_! j^* L$.

(b) If $\mathcal{A}$ has enough projectives, then we also have $\mathtt{Ker}(\sigma_L) \cong i_*(\mathbf{L}_1 i^*) L$, where $\mathbf{L}_1 i^* : \mathcal{A} \to \mathcal{A}'$ is the left derived functor of i^*.

Proof.   (a) σ_L being epimorphism follows immediately from the first exact sequence of (4). To compute its kernel we apply j^* to the exact sequence $0 \to \mathtt{Ker}(\sigma_L) \to j_! j^* L \to L \to 0$. By (R2), j^* maps σ_L to an isomorphism in $\mathcal{A}''$; hence $j^*(\mathtt{Ker}\,\sigma_L) = 0$ and therefore $\mathtt{Ker}(\sigma_L) = i_* K$ for some $K \in \mathrm{Ob}(\mathcal{A}')$. Now, to see that $K \cong i^! j_! j^* L$ we apply the (left exact) functor $i^!$ to $0 \to i_* K \to j_! j^* L \to L \to 0$ and use the first isomorphism of (R1).

(b) If the category $\mathcal{A}$ has enough projectives, so does its quotient $\mathcal{A}''$. Moreover, the functor $j_! : \mathcal{A}'' \to \mathcal{A}$ is left adjoint to an exact functor, and thus it maps projectives to projectives. Under these conditions, there is a Grothendieck spectral sequence with $E^2_{p,q} = (\mathbf{L}_p i^*)(\mathbf{L}_q j_!)$, converging to $H_n = \mathbf{L}_n(i^* j_!)$. In view of (R4), the limit terms H_n are all zero, and hence so is the edge map $E^2_{1,0} \cong E^\infty_{1,0} \hookrightarrow H_1$. It follows that $(\mathbf{L}_1 i^*) j_! = 0$. Now, applying i^* to the exact sequence $0 \to i_* K \to j_! j^* L \to L \to 0$ and using again (R4), we get $\mathtt{Ker}(\sigma_L) = i_* K \cong i_*(\mathbf{L}_1 i^*) L$. $\square$

Next, in addition to the six basic functors (3), we introduce one more additive functor $j_{!*} : \mathcal{A}'' \to \mathcal{A}$ (cf. [BBD], Section 1.4.6). By axiom (R2), for any $X, Y \in \mathrm{Ob}(\mathcal{A}'')$, there are natural isomorphisms

$$\text{(5)} \qquad \mathtt{Hom}_{\mathcal{A}}(j_! X, \, j_* Y) \cong \mathtt{Hom}_{\mathcal{A}''}(X, \, j^* j_* Y) \cong \mathtt{Hom}_{\mathcal{A}''}(X, \, Y) \ .$$

Letting $X = Y$ in (5), we get a family of morphisms $\mathcal{N}_X : j_! X \to j_* X$ in $\mathcal{A}$, corresponding to the identity maps of objects in $\mathcal{A}''$. This defines a natural transformation $\mathcal{N} : j_! \to j_*$ (see [BBD], 1.4.6.2), satisfying

$$\text{(6)} \qquad\qquad\qquad \mathcal{N} j^* = \eta \circ \sigma \ .$$

The functor $j_{!*} : \mathcal{A}'' \to AB$ is now defined by

$$\text{(7)} \qquad\qquad j_{!*} X := \mathtt{Im}(\mathcal{N}_X) \subseteq j_* X \ , \quad X \in \mathrm{Ob}(\mathcal{A}'') \ .$$

The meaning of this functor, which is referred to as the 'prolongement intermédiare' in [BBD], becomes clear from the following lemma.

Lemma 2.

(*a*) $j_{!*}$ *gives an equivalence between* $\mathcal{A}''$ *and the full subcategory of* $\mathcal{A}$ *consisting of objects* L *with* $i^*L = i^!L = 0$. *The inverse of* $j_{!*}$ *is given by* j^*.

(*b*) j^* *and* $j_{!*}$ *induce the mutually inverse bijections*

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{simple objects } L \in \mathcal{A} \text{ with } j^*L \neq 0 \end{array} \right\} \underset{j_{!*}}{\overset{j^*}{\rightleftarrows}} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{nonzero simple objects of } \mathcal{A}'' \end{array} \right\}$$

Proof.   (*a*) It follows from (R2) that $j^*j_{!*} \simeq \mathrm{Id}_{\mathcal{A}''}$, so $j_{!*}$ is a fully faithful functor. On the other hand, in view of (6), we have $j_{!*}j^*L = \mathrm{Im}(\eta_L \circ \sigma_L)$ for any $L \in \mathrm{Ob}(\mathcal{A})$. If $i^*L = i^!L = 0$, the map σ_L is epi and η_L is mono, see (4), so $j_{!*}j^*L = \mathrm{Im}(\eta_L \circ \sigma_L) \cong L$. Conversely, (R4) forces $i^*j_{!*} = i^!j_{!*} = 0$; hence, if $j_{!*}j^*L \cong L$ for some $L \in \mathrm{Ob}(\mathcal{A})$, then $i^*L = i^!L = 0$. The (essential) image of $j_{!*}$ consists thus of objects $L \in \mathcal{A}$ with $i^*L = i^!L = 0$.

Now, part (*b*) follows from part (*a*) if we observe that the (nonzero) simple objects $L \in \mathcal{A}$ with $i^*L = i^!L = 0$ are exactly the ones, for which $j^*L \neq 0$. $\qquad\square$

2.5. **Main theorem.** From now on, we fix $n \geq 0$ and let $\lambda = (-n, 1)$ as in Section 2.3. The following observation is then immediate from the definition of Π_λ.

Lemma 3. A_1 *is isomorphic to the quotient of* Π_λ *by the ideal generated by* e_∞.

In fact, combined with the canonical projection $\Pi_\lambda \twoheadrightarrow \Pi_\lambda/\langle e_\infty \rangle$, the algebra map $\mathbb{C}\langle x, y \rangle \to \Pi_\lambda$, $x \mapsto X$, $y \mapsto Y$, is an epimorphism with kernel containing $xy - yx - 1$. The induced map $A_1 := \mathbb{C}\langle x, y \rangle/\langle xy - yx - 1 \rangle \to \Pi_\lambda/\langle e_\infty \rangle$ is then an isomorphism of algebras, since A_1 is simple.

Simplifying the notation, we write $\Pi = \Pi_\lambda$, $A = A_1$, and let $B := e_\infty \Pi e_\infty$. Clearly, B is an associative subalgebra of Π with identity element e_∞; by analogy with representation theory of semisimple complex Lie algebras (see Section 2.6 below), we call it the *parabolic subalgebra* of Π.

Now, we set $\mathcal{A} := \mathrm{Mod}(\Pi)$, $\mathcal{A}' := \mathrm{Mod}(A)$ and $\mathcal{A}'' := \mathrm{Mod}(B)$ and define the six functors (3) between these categories as follows. Using the isomorphism of Lemma 3, we first identify $A = \Pi/\langle e_\infty \rangle$, and let i_* be the restriction functor for the canonical epimorphism $i : \Pi \to A$. This functor is fully faithful and has both the right adjoint $i^! := \mathrm{Hom}_\Pi(A, -)$ and the left adjoint $i^* := - \otimes_\Pi A$ with adjunction maps satisfying $i^*i_* \simeq \mathrm{Id}_{\mathcal{A}'} \simeq i^!i_*$. The functor $j^* : \mathcal{A} \to \mathcal{A}''$ is then defined by $j^* := - \otimes_\Pi \Pi e_\infty$. As $e_\infty \in \Pi$ is an idempotent, j^* is exact and has also the right adjoint $j_* = \mathrm{Hom}_B(\Pi e_\infty, -)$ and the left adjoint $j_! = - \otimes_B e_\infty \Pi$ satisfying $j^*j_* \simeq \mathrm{Id}_{\mathcal{A}''} \simeq j^*j_!$. Now, if $j^*M = 0$ in $\mathcal{A}''$, we have $M \otimes_\Pi \Pi e_\infty = M e_\infty = 0$, so $\langle e_\infty \rangle \subseteq \mathrm{Ann}_\Pi(M)$, and therefore $M \cong i_*N$ for some $N \in \mathrm{Ob}(\mathcal{A}')$. Summing up, we have the following

Proposition 1. *The functors* $(i^*, i_*, i^!)$ *and* $(j_!, j^*, j_*)$ *defined above satisfy the recollement conditions of Section 2.4.*

Our main result can now be stated as follows.

Theorem 1. *For each* $n \geq 0$, *the composition of functors*

$$(8) \qquad\qquad \Omega: \mathrm{Mod}(\Pi) \xrightarrow{j^*} \mathrm{Mod}(B) \xrightarrow{j_!} \mathrm{Mod}(\Pi) \xrightarrow{i^!} \mathrm{Mod}(A)$$

maps injectively *the set* $\mathrm{Irr}(\Pi, \alpha)$ *of isomorphism classes of simple* Π-*modules of dimension* $\alpha = (1, n)$ *into the set* $\mathcal{R}$ *of isomorphism classes of right ideals of* A. *If*

we identify $\mathcal{C}_n = \mathtt{Rep}(\Pi^{\mathrm{opp}}, \alpha)//\mathtt{G}(\alpha)$ *as in Section 2.3, then the map induced by* Ω *agrees with the Calogero-Moser map* $\omega : \mathcal{C}_n \to \mathcal{R}$ *constructed in* [BW]. *Collecting such maps for all* $n \geq 0$, *we get thus the bijective correspondence* $\bigsqcup_{n \geq 0} \mathcal{C}_n \xrightarrow{\sim} \mathcal{R}$.

There are several different constructions of the map ω in the literature, the most explicit one being given in [BC]. To prove the theorem, we will compute Ω on simple Π-modules and show that the induced map agrees with [BC]. For this we will need another observation regarding the structure of Π_λ.

Lemma 4. *If* $\lambda = (-n, 1)$ *with* $n \neq 0$, *the algebra* Π_λ *is Morita equivalent to* $e_0 \Pi_\lambda e_0$, *while* $e_0 \Pi_\lambda e_0$ *is isomorphic to a quotient of the free algebra* $R = \mathbb{C}\langle x, y \rangle$.

Proof. As e_0 is an idempotent in $\Pi = \Pi_\lambda$, the natural functor $\mathtt{Mod}(\Pi) \to \mathtt{Mod}(e_0 \Pi e_0)$, $M \mapsto M e_0$, is an equivalence of categories if and only if $\Pi e_0 \Pi = \Pi$ (cf. [MR], Prop. 3.5.6). This last identity holds in Π, since

$$1 = e_0 + e_\infty = e_0 - \frac{1}{n} vw = e_0 - \frac{1}{n} v e_0 w \in \Pi e_0 \Pi$$

in view of the defining relations (1). Now, we have $X, Y \in e_0 \Pi e_0$, and it is again easily seen from (1) that the algebra map

$$(9) \qquad \pi : R \to e_0 \Pi e_0 , \quad 1 \mapsto e_0 , \quad x \mapsto X , \quad y \mapsto Y ,$$

is surjective, with $\mathtt{Ker}(\pi)$ generated by $([x, y] - 1)([x, y] + n - 1) \in R$. $\qquad \square$

Proof of Theorem 1. By definition, the functors i^* and $i^!$ assign to a Π-module L its largest quotient and largest submodule over A respectively. In particular, both i^* and $i^!$ map finite-dimensional modules to finite-dimensional ones. Since 0 is the only such module over A, we have $i^* L = i^! L = 0$ for any $L \in \mathrm{Ob}(\mathcal{A})$ with $\dim(L) < \infty$. By Lemma 1(a), $\Omega(L)$ is then isomorphic to the kernel of $\sigma_L : j_! j^* L = L e_\infty \otimes_B e_\infty \Pi \to L$, which is the obvious multiplication-action map.

Now, let L have dimension vector $\alpha = (1, n)$. Set $M := L e_\infty \otimes_B e_\infty \Pi$. Since $\dim(j^* L) = \dim(L e_\infty) = 1$, M is a cyclic Π-module generated by $\xi \otimes_B e_\infty$, where ξ is a(ny) nonzero vector of $j^* L = L e_\infty$. To describe M explicitly, we will compute the annihilator of $\xi \otimes_B e_\infty$ in Π. First, we observe that, as a subspace of Π, the algebra $B = e_\infty \Pi e_\infty$ is spanned by the elements $v\, a(X, Y)\, w \in \Pi$, where $a(x, y) \in R$. So we can define on R the linear form $\varepsilon : R \to \mathbb{C}$ by

$$(10) \qquad \xi.(v\, a(X, Y)\, w) = \varepsilon(a)\, \xi .$$

(Note that ε is independent of the choice of the basis vector $\xi \in j^* L$, and it determines the B-module $j^* L$ uniquely, up to isomorphism.)

It is easy to see now that $\mathtt{Ann}_\Pi(\xi \otimes_B e_\infty)$ is generated by e_0 and the elements $\{v\, a(X, Y)\, w + \varepsilon(a) : a \in R\}$; thus, with natural splitting $\Pi = e_0 \Pi \oplus e_\infty \Pi$, we have the isomorphism of Π-modules

$$(11) \qquad e_\infty \Pi \bigg/ \sum_{a \in R} [\, v\, a(X, Y)\, w - \varepsilon(a)\,]\, e_\infty \Pi \ \xrightarrow{\sim}\ M , \quad [p] \mapsto \xi \otimes_B p ,$$

where $[\, p\,]$ denotes the residue class of $p \in e_\infty \Pi$.

On the other hand, the left multiplication by $v \in \Pi$ induces the map

$$(12) \quad e_0 \Pi \bigg/ \sum_{a \in R} [\, a(X, Y)\, wv - \varepsilon(a)\,]\, e_0 \Pi \to v\, \Pi \bigg/ \sum_{a \in R} [\, v\, a(X, Y)\, w - \varepsilon(a)\,]\, v\, \Pi ,$$

which is also easily seen to be an isomorphism of Π-modules.

If $n \neq 0$, we have $v\,\Pi = e_\infty\,v\,\Pi \subseteq e_\infty\Pi = \frac{1}{n}vw\,\Pi \subseteq v\,\Pi$. Whence $v\,\Pi = e_\infty\Pi$. Combining now (11) and (12), we get

$$(13) \qquad e_0\Pi \left/ \sum_{a \in R} \left[\, a(X,Y)\,wv - \varepsilon(a)\, \right] e_0\Pi \right. \ \xrightarrow{\sim}\ M \ ,$$

with $[\,e_0\,]$ corresponding to the (cyclic) vector $\xi \otimes_B v \in M$. Under the Morita equivalence of Lemma 4, the module M transforms to $Me_0 = Le_\infty \otimes_B e_\infty\Pi\,e_0$ and the isomorphism (13) becomes

$$(14) \qquad e_0\Pi e_0 \left/ \sum_{a \in R} \left[\, a(X,Y)\,wv - \varepsilon(a)\, \right] e_0\Pi e_0 \right. \ \xrightarrow{\sim}\ Me_0 \ .$$

Through the algebra extension $\pi : R \twoheadrightarrow e_0\Pi e_0$, see (9), we can regard Me_0 as a right R-module and, using (14), we can then identify it with a quotient of R: more precisely, we have

$$(15) \qquad R/J \ \xrightarrow{\sim}\ Me_0 \ , \quad [\,1\,]_J \mapsto \xi \otimes_B v \ ,$$

where J is the right ideal of R generated by $\{a\,([x,y] - 1) - \varepsilon(a) : a \in R\}$.

Now, suppose that L is a simple module representing a point $[(\bar{X}, \bar{Y}, \bar{v}, \bar{w})]$ of $\mathcal{C}_n$, see Section 2.1. Under the equivalence of Lemma 4, it corresponds to Le_0, which we can again regard as an R-module via (9). With identification of Section 2.3, Le_0 is then simply $\mathbb{C}^n$ with (right) action of $x, y \in R$ defined by the matrices $\bar{X}, \bar{Y} \in \mathtt{End}(\mathbb{C}^n)$. The associated functional $\varepsilon : R \to \mathbb{C}$ is given by $a(x,y) \mapsto \bar{w}\,a^\tau(\bar{X}, \bar{Y})\,\bar{v}$, where $a \mapsto a^\tau$ is the natural anti-involution of the algebra R acting identically on x and y. Now, with identification (15), the canonical map $Me_0 \to Le_0$, corresponding to σ_L under the Morita equivalence, becomes the homomorphism of R-modules

$$R/J \to \mathbb{C}^n \ , \quad [\,1\,]_J \mapsto \bar{v} \ .$$

As shown in [BC], Theorem 5, the kernel of this last homomorphism represents an ideal class in $\mathcal{R}$, which is exactly the image of $[(\bar{X}, \bar{Y}, \bar{v}, \bar{w})]$ under the Calogero-Moser map $\omega : \mathcal{C}_n \to \mathcal{R}$ of [BW]. Thus we get $[\Omega(L)] = \omega[(\bar{X}, \bar{Y}, \bar{v}, \bar{w})]$. Now, if Ω maps two simple Π-modules L and L' to isomorphic A-modules, the corresponding functionals ε and ε' on R must coincide (see [BC], Theorem 3). Hence, if $[\Omega(L)] = [\Omega(L')]$, we have $j^*L \cong j^*L'$, and by Lemma 2(b), we then conclude $L \cong L'$. This finishes the proof of the theorem.

Remark. Notice that, thanks to Lemma 1, the functor Ω can be described on finite-dimensional modules $L \in \mathtt{Mod}(\Pi)$ in two equivalent ways: either as the kernel of $\sigma_L : j_!j^*L \to L$ (see above) or as $(\mathbf{L}_1 i^*)L \cong \mathtt{Tor}_1^\Pi(L, A)$. This last formula should be compared to formula (A.5) in the appendix to [BW1].

2.6. **Analogy with highest weight modules.** We would like to draw reader's attention to an interesting analogy with representation theory of semisimple complex Lie algebras. The modules M and L, which appear in the proof of Theorem 1, should be compared to the Verma module $M(\lambda)$ and its simple quotient $L(\lambda)$ in the case when $\lambda \in \mathfrak{h}^*$ is a dominant integral weight (see [D], Chapter 7). By analogy with Verma modules, we can characterize M as a universal cyclic Π-module generated by a "highest weight" vector $\boldsymbol{\xi} \in M$, satisfying $\boldsymbol{\xi}\,e_0 = 0$ and $\boldsymbol{\xi}\,b = \varepsilon(b)\,\boldsymbol{\xi}$ for all $b \in B$ (cf. (10) above). The module L is then the unique simple quotient of M;

the subalgebra B (or rather, its unital extension $\tilde{B} = \mathbb{C}e_0 + B \subset \Pi$) plays a role similar to (the universal enveloping algebra of) the Borel subalgebra $U(\mathfrak{b}) \subset U(\mathfrak{g})$ in Lie theory, and ε is an analogue of the weight functional $\lambda \in \mathfrak{h}^*$.

Now, in the Lie case the assertion of Theorem 1 amounts to injectivity of the map $[L(\lambda)] \mapsto [K(\lambda)]$, assigning to the isomorphism class of a finite-dimensional simple $\mathfrak{g}$-module $L(\lambda)$ the isomorphism class of the unique maximal submodule $K(\lambda)$ of $M(\lambda)$ (i. e., the kernel of the canonical epimorphism $M(\lambda) \to L(\lambda)$). By the classical Cartan-Weyl theory, the isomorphism classes of finite-dimensional simple $\mathfrak{g}$-modules are in bijection with the set $\mathcal{P}(\mathfrak{g})_+$ of dominant integral weights, and the map $\lambda \mapsto [K(\lambda)]$ is indeed injective on this subset of $\mathfrak{h}^*$. To see this we observe that, for any $\lambda \in \mathfrak{h}^*$, the module $K(\lambda)$ admits a (unique) central character χ_λ equal to the central character of the Verma module $M(\lambda)$, of which it is a submodule. Thus, if $K(\lambda) \cong K(\lambda')$ for some $\lambda, \lambda' \in \mathfrak{h}^*$, we have $\chi_\lambda = \chi_{\lambda'}$, and therefore $\lambda + \varrho$ and $\lambda' + \varrho$ lie on the same orbit (in $\mathfrak{h}^*$) of the Weyl group W associated to $\mathfrak{g}$. Now, if λ and λ' are both dominant, then $\lambda + \varrho$ and $\lambda' + \varrho$ are regular dominant: that is, $\lambda + \varrho$ and $\lambda' + \varrho$ belong to the same (positive) Weyl chamber of W and hence coincide. In this way, for $\lambda, \lambda' \in \mathcal{P}(\mathfrak{g})_+$, we have $K(\lambda) \cong K(\lambda') \Rightarrow \lambda = \lambda'$.

2.7. **Simple Π-modules of arbitrary dimension.** Using the results of [CB3], we can easily compute the dimension vectors of all simple modules of Π_λ: if $\lambda = (-n, 1)$, these vectors are given by (k, kn), where $k = 0, 1, 2, \ldots$. Theorem 1 holds, however, only for the modules of dimension $(1, n)$: the functor Ω fails to be injective on simple Π-modules of other dimensions.

3. Simple Modules of Cherednik Algebras

In this section we will deal with rational Cherednik algebras $H_{0,c}(S_n)$ associated to a symmetric group S_n, with deformation parameters $t = 0$ and $c \neq 0$ (see [EG]). By a result of Etingof and Ginzburg, the isomorphism classes of simple modules over $H_{0,c}(S_n)$ are parametrized by points of the Calogero-Moser space $\mathcal{C}_n$. Our goal is to relate these modules to ideals of A_1 functorially. We will do this in two steps: first, we construct a functor $\Xi : \mathtt{Mod}(H) \to \mathtt{Mod}(\Pi)$, inducing a natural bijection between simple H-modules and simple Π-modules of dimension $\alpha = (1, n)$; then we will apply Theorem 1, combining Ξ with the functor Ω.

3.1. **Cherednik algebras and their spherical subalgebras.** Recall that, for a fixed integer $n \geq 1$ and a parameter $c \in \mathbb{C}$, $H_{0,c}(S_n)$ is generated by two polynomial subalgebras $\mathbb{C}[x_1, x_2, \ldots, x_n]$, $\mathbb{C}[y_1, y_2, \ldots, y_n]$ and the group algebra $\mathbb{C}S_n$ subject to the following "deformed crossed product" relations (cf. [EG], (4.1)):

$$(16) \qquad s_{ij} x_i = x_j s_{ij} , \quad s_{ij} y_i = y_j s_{ij} ,$$

$$(17) \qquad [x_i, y_j] = c\, s_{ij} \ (i \neq j) , \quad [x_k, y_k] = -c \sum_{i \neq k} s_{ik} .$$

(Here s_{ij} denote the elementary transpositions $i \leftrightarrow j$ generating S_n.) It is easy to see that the algebras $H_{0,c}(S_n)$ with $c \neq 0$ are all isomorphic to each other, and it will be convenient for us to fix $c = 1$. Thus we write $H = H_{0,1}(S_n)$ and let $U := eHe$, where $e := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$ is the symmetrizing idempotent in $\mathbb{C}S_n \subset H$. By definition, U is an associative subalgebra of H with identity element e; following [EG], we call it the *spherical subalgebra*. Two properties of U, which

play a crucial role in representation theory of Cherednik algebras, are given in the following theorem.

Theorem 2 (see [EG], Theorems 1.23 and 1.24)**.**
  (a) U *is Morita equivalent to* H, *the equivalence* $\mathrm{Mod}(H) \to \mathrm{Mod}(U)$ *being the natural functor* $e : M \mapsto Me$;
  (b) U *is a commutative algebra isomorphic to* $\mathcal{O}(\mathcal{C}_n)$, *the coordinate ring of the Calogero-Moser variety* $\mathcal{C}_n$.

Note that it is a consequence of Theorem 2 that the isomorphism classes of simple H-modules are in bijection with (closed) points of the variety $\mathcal{C}_n$.

3.2. **The functor** Ξ. There seem to be no natural maps relating the algebras Π and H; however, at the level of their subalgebras B and U such a map does exist.

Lemma 5. *The assignment* $v\,a(X,Y)\,w \mapsto -\sum_{i=1}^{n} e\,a(x_i, y_i)\,e$ *defines a homomorphism of (unital) algebras* $\theta : B \to U$.

Proof.   A straightforward calculation using the defining relations of H and Π. $\square$

Let $\theta_* : \mathrm{Mod}(U) \to \mathrm{Mod}(B)$ be the restriction functor corresponding to the map θ of Lemma 5. Recall the functor $j_{!*} : \mathrm{Mod}(B) \to \mathrm{Mod}(\Pi)$ defined in Section 2.4.

Theorem 3. *The composition of functors*

$$\Xi \,:\, \mathrm{Mod}(H) \overset{e}{\simeq} \mathrm{Mod}(U) \xrightarrow{\theta_*} \mathrm{Mod}(B) \xrightarrow{j_{!*}} \mathrm{Mod}(\Pi)$$

maps the set $\mathrm{Irr}(H)$ *of isomorphism classes of simple H-modules* bijectively *into the set* $\mathrm{Irr}(\Pi, \alpha)$ *of isomorphism classes of simple Π-modules of dimension* $\alpha = (1, n)$. *If we identify* $\mathcal{C}_n = \mathrm{Rep}(\Pi^{\mathrm{opp}}, \alpha)//\mathrm{G}(\alpha)$ *as in Section 2.3, then the map* $\mathrm{Irr}(H) \to \mathrm{Irr}(\Pi, \alpha)$ *induced by* Ξ *agrees with the Etingof-Ginzburg map in* [EG].

Proof.   We will compute Ξ on simple H-modules and show that the induced map agrees with [EG]. We begin by recalling the Etingof-Ginzburg construction (see *loc. cit.*, Section 11).

Let $S_{n-1} \subset S_n$ be the subgroup of permutations of $\{1, 2, \ldots, n\}$ acting trivially on 1. Let $\bar{e} := \frac{1}{(n-1)!} \sum_{\sigma \in S_{n-1}} \sigma$ be the corresponding idempotent in $\mathbb{C}S_n \subset H$. Given a simple H-module V, set $\bar{V} := V\bar{e} \subseteq V$. It is known (see [EG], Theorem 1.7) that $V \cong \mathbb{C}S_n$ as an S_n-module, so $\dim(V) = n!$ and $\dim(\bar{V}) = n$. On the other hand, in view of the defining relations (16), the elements x_1 and y_1 commute with $\bar{e}$ in H, and the subspace $\bar{V}$ is therefore stable under their action on V. If we define now $\bar{X} \in \mathrm{End}(\bar{V})$ and $\bar{Y} \in \mathrm{End}(\bar{V})$ by restricting the action of x_1 and y_1 to $\bar{V}$, we get $\mathrm{Rk}([\bar{X}, \bar{Y}] + \mathrm{Id}) = 1$. Indeed, using (17), it is easy to see that

$$(18) \qquad\qquad \bar{e}\,([x_1, y_1] - 1) = -n\,e \quad \text{in } H.$$

Whence $\mathrm{Im}([\bar{X}, \bar{Y}] + \mathrm{Id}) \subseteq \bar{V}e = Ve$. On the other hand, by part (a) of Theorem 2, Ve is a simple U-module, and by part (b), it is then 1-dimensional. Thus, we have $[\bar{X}, \bar{Y}] + \mathrm{Id} = -\bar{v}\,\bar{w}$ for some $\bar{v} \in \bar{V}$ and $\bar{w} \in \bar{V}^*$. It follows that $(\bar{V}; \bar{X}, \bar{Y}, \bar{v}, \bar{w})$ represents a point of the variety $\mathcal{C}_n$. Now, let L be a simple Π-module corresponding to this point under the identification of Section 2.3. Restricting L to the parabolic subalgebra $B = e_\infty \Pi e_\infty$, we get the 1-dimensional B-module $j^*L = Le_\infty$. As mentioned in the proof of Theorem 1, the module j^*L is determined, up to isomorphism, by its "weight" functional $\varepsilon : R \to \mathbb{C}$, see (10). The latter can be written

explicitly in terms of the Calogero-Moser matrices: $\varepsilon(a) = \bar{w}\,a^\tau(\bar{X}, \bar{Y})\,\bar{v}$, where τ is the canonical anti-involution on R satisfying $x^\tau = x$ and $y^\tau = y$. Since

$$\bar{w}\,a^\tau(\bar{X}, \bar{Y})\,\bar{v} = \mathtt{Tr}_{\bar{V}}[\,a^\tau(\bar{X}, \bar{Y})\,\bar{v}\bar{w}\,] = -\,\mathtt{Tr}_{\bar{V}}[\,a^\tau(\bar{X}, \bar{Y})\,([\bar{X}, \bar{Y}] + \mathtt{Id})\,]\,,$$

using (18), we get

$$(19) \qquad \varepsilon(a) = -n\,\mathtt{Tr}_{\bar{V}}[\,\cdot\,e\,a(x_1, y_1)] = -n\,\mathtt{Tr}_V[\,\cdot\,e\,a(x_1, y_1)]\,,$$

where "$\cdot$" denotes the action of H on V.

Now, let us regard Ve as a B-module via the algebra map θ. As V is simple, Ve is 1-dimensional, its "weight" functional $\tilde{\varepsilon} \colon R \to \mathbb{C}$ being

$$(20) \qquad \tilde{\varepsilon}(a) = -\sum_{i=1}^{n}\mathtt{Tr}_{Ve}[\,\cdot\,e\,a(x_i, y_i)\,e]\,.$$

By symmetry, we have $e\,a(x_i, y_i)\,e = e\,a(x_1, y_1)\,e$ in H for any $i = 1, 2, \ldots, n$, so the sum in the right-hand side of (20) equals $-n\,\mathtt{Tr}_{Ve}[\,\cdot\,e\,a(x_1, y_1)\,e]$. Comparing this to (19), we see that $\tilde{\varepsilon} = \varepsilon$. Hence $\theta_*(Ve) \cong j^*L$ as B-modules. Now, by Lemma 2(a), we have $\Xi(V) = j_{!*}\theta_*(Ve) \cong j_{!*}j^*L \cong L$, which shows that the Etingof-Ginzburg map is indeed induced by the functor Ξ. $\qquad\square$

Combining Theorems 1 and 3 together, we get our second main result.

Theorem 4. *The composition of functors*

$$(21) \qquad \mathtt{Mod}(H) \overset{e}{\simeq} \mathtt{Mod}(U) \xrightarrow{\theta_*} \mathtt{Mod}(B) \xrightarrow{j_!} \mathtt{Mod}(\Pi) \xrightarrow{i^!} \mathtt{Mod}(A)$$

maps the set $\mathtt{Irr}(H)$ of isomorphism classes of simple H-modules injectively into the set $\mathcal{R}$ of isomorphism classes of right ideals of A. Collecting such maps for all $n \geq 0$, we get a bijective correspondence $\bigsqcup_{n \geq 0}\mathtt{Irr}[H(S_n)] \overset{\sim}{\to} \mathcal{R}$.

4. Quiver Generalization

The above results generalize to an arbitrary affine Dynkin quiver associated (via McKay's construction) to a finite subgroup Γ of $\mathtt{SL}(2, \mathbb{C})$. The Weyl algebra A_1 is replaced in this situation by a "quantized coordinate ring" $O_\tau(\Gamma)$ of the Kleinian singularity $\mathbb{C}^2//\Gamma$ and the rational Cherednik algebra $H_{0,c}(S_n)$ by a symplectic reflection algebra $H_{0,k,c}(\mathbf{\Gamma}_n)$ associated to the n-th wreath product $\mathbf{\Gamma}_n = S_n \ltimes \Gamma^n$.

In what follows we outline this generalization in the special case of cyclic groups. The construction of A_∞-envelopes of ideals of $O_\tau(\Gamma)$ for such Γ's, using the approach of [BC], has been carried out recently in [E]. The results of [E] can be reinterpreted in terms of representation theory of deformed preprojective algebras of Dynkin quivers of type $\tilde{A}_m$, and (the analogues of) Theorems 1, 3 and 4 hold true in this case. The proofs repeat almost verbatim the above arguments for the Weyl algebra, so we omit them for the sake of brevity. The case of general (noncyclic) Dynkin quivers seems to be more instructive and will be treated in detail elsewhere.
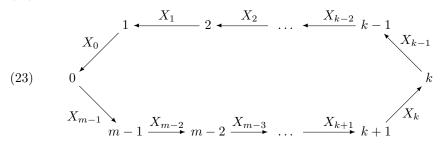
4.1. **Deformations of Kleinian singularities.** Let (L, ω) be a two-dimensional complex symplectic vector space with symplectic form ω, and let Γ be a finite subgroup of $\mathtt{Sp}(L, \omega)$. The natural action of Γ on $L^* = \mathtt{Hom}(L, \mathbb{C})$ extends diagonally to the tensor algebra TL^*, and we write $R := TL^* \# \Gamma$ for the crossed product of TL^* with Γ. Since $\omega \in L^* \otimes L^* \subset TL^* \subset R$, we can form the (two-sided) ideals $\langle \omega - \tau \rangle$ in R, one for each $\tau \in \mathbb{C}\Gamma$, and define $\mathcal{S}_\tau(\Gamma) := R/\langle \omega - \tau \rangle$. Furthermore, taking the idempotent $\mathfrak{e} := \frac{1}{|\Gamma|}\sum_{g \in \Gamma} g \in \mathbb{C}\Gamma \subset \mathcal{S}_\tau(\Gamma)$, we set $O_\tau(\Gamma) := \mathfrak{e}\,\mathcal{S}_\tau(\Gamma)\,\mathfrak{e}$. By

definition, $O_\tau(\Gamma)$ is a unital associative algebra, with $\mathfrak{e}$ being the identity element. For $\tau = 0$, it is a commutative ring isomorphic to $\mathbb{C}[L]^\Gamma$. The family of algebras $\{O_\tau(\Gamma)\}_{\tau \in \mathbb{C}\Gamma}$ can thus be viewed as a (noncommutative) deformation of $\mathbb{C}[L]^\Gamma$, the coordinate ring of the classical Kleinian singularity $L//\Gamma$, see [CBH].

As mentioned above, we will be dealing here only with cyclic Γ's. Thus we fix $m \geq 1$ and assume $\Gamma \cong \mathbb{Z}/\mathbb{Z}m$. In this case we can choose a basis $\{x, y\}$ in L^*, so that $\omega = x \otimes y - y \otimes x$ and L^* decomposes as $\chi \oplus \chi^{-1}$, with Γ acting on x by χ and on y by χ^{-1}, where χ is a primitive character of Γ. The algebra $\mathcal{S}_\tau(\Gamma)$ can then be identified with a quotient of $\mathbb{C}\langle x, y \rangle \# \Gamma$, with x, y and $g \in \Gamma$ satisfying the relations

$$(22) \qquad g \cdot x = \chi(g)\, x \cdot g \ , \quad g \cdot y = \chi^{-1}(g)\, y \cdot g \ , \quad x \cdot y - y \cdot x = \tau \ .$$

Following [CBH], we can give another construction of $\mathcal{S}_\tau(\Gamma)$. Recall that with each finite subgroup $\Gamma \subset \mathtt{SL}(2, \mathbb{C})$ one can associate a quiver $Q(\Gamma)$, whose underlying graph is an extended Dynkin diagram (see [M]). In case when $\Gamma \cong \mathbb{Z}/\mathbb{Z}m$, the quiver $Q(\Gamma)$ has type $\tilde{A}_m$: it consists of m vertices, indexed by $I = \{0, 1, ..., m-1\}$, and m arrows $\{X_0, X_1, ..., X_{m-1}\}$, forming a cycle, see (23).

$$(23)$$

Given $\tau = (\tau_0, \tau_1, \ldots, \tau_{m-1}) \in \mathbb{C}^m$, we consider the deformed preprojective algebra $\Pi_\tau(Q)$ of weight τ. Explicitly, this algebra is generated by $2m$ arrows X_k, Y_k and m vertices $e_0, e_1, ..., e_{m-1}$, which apart from the standard path algebra relations, satisfy

$$(24) \qquad X_0 Y_0 - Y_{m-1} X_{m-1} = \tau_0\, e_0$$

and

$$(25) \qquad X_k Y_k - Y_{k-1} X_{k-1} = \tau_k\, e_k \ ,$$

where $k = 1, 2, \ldots, m-1$. Now, if we identify $\mathbb{C}\Gamma$ with $\mathbb{C}^m$ by choosing a basis $\{\mathfrak{e}_i\}$ in $\mathbb{C}\Gamma$ with

$$(26) \qquad \mathfrak{e}_i := \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi^i(g)\, g \ , \quad i = 0, 1, \ldots, m-1 \ ,$$

then the mapping

$$(27) \qquad \mathfrak{e}_i \mapsto e_i \ , \quad x \mapsto \sum_{i=0}^{m-1} X_i \ , \quad y \mapsto \sum_{i=0}^{m-1} Y_i \ ,$$

extends to an algebra isomorphism

$$(28) \qquad \mathcal{S}_\tau(\Gamma) \xrightarrow{\sim} \Pi_\tau(Q) \ .$$

Indeed, it is easy to check, using the relations (22) and (24), (25), that (27) yields a well-defined algebra map, and its inverse is given by $X_i \mapsto \mathfrak{e}_i x$, $Y_i \mapsto \mathfrak{e}_i y$.

Remark.   For an arbitrary $\Gamma \subset \mathtt{SL}(2, \mathbb{C})$, the relationship between the algebras $S_\tau(\Gamma)$ and $\Pi_\tau(Q)$ is weaker: one has only a Morita equivalence rather than an isomorphism (see [CBH], Theorem 0.1).

Next, we recall that there is a root system $\Delta(Q) \subset \mathbb{Z}^I$ attached to any finite quiver $Q = (I, Q)$, see [K]. If Q is an (extended) Dynkin quiver of type A, D or E, the corresponding $\Delta(Q)$ is the usual (affine) root system of the same type; in particular, for the cyclic quiver (23), $\Delta(Q)$ is the affine root system of type $\tilde{A}_m$. In this last case, we say that $\tau \in \mathbb{C}^m$ is *regular*, if $\tau \cdot \alpha \neq 0$ for all $\alpha \in \Delta(Q)$.

The following proposition is one of the main results of [CBH] (see also [H, S]).

Proposition 2 ([CBH], Theorem 0.4)**.** *If τ is regular, then $\mathcal{S}_\tau(\Gamma)$ and $O_\tau(\Gamma)$ are simple rings, Morita equivalent to each other.*

4.2. **Nakajima varieties.** In case of nontrivial Γ's, the Calogero-Moser spaces $\mathcal{C}_n$ are replaced by Nakajima quiver varieties [N, N1]. We define these varieties as representation spaces of "framed" Dynkin quivers, following [CB3].

First, as in Section 2.2, we introduce a quiver $Q_\infty(\Gamma)$ by adding to $Q(\Gamma)$ an extra vertex ∞ and an extra arrow $v : 0 \to \infty$. Given $\lambda = (\lambda_\infty, \lambda_0, \ldots, \lambda_{m-1}) \in \mathbb{C}^{m+1}$, we write $\Pi_\lambda := \Pi_\lambda(Q_\infty)$ for the deformed preprojective algebra of weight λ. Explicitly, this algebra is generated by $2(m+1)$ arrows X_k, Y_k, v, w and $m+1$ vertices $e_\infty, e_0, e_1, \ldots, e_{m-1}$, which apart from the standard path algebra relations, satisfy

$$(29) \qquad vw = \lambda_\infty\, e_\infty\ , \quad X_0 Y_0 - Y_{m-1} X_{m-1} - wv = \lambda_0\, e_0$$

and

$$(30) \qquad X_k Y_k - Y_{k-1} X_{k-1} = \lambda_k\, e_k\ ,$$

where $k = 1, 2, \ldots, m - 1$.

The relation between the algebras $\Pi_\lambda(Q_\infty)$ and $\Pi_\tau(Q)$ is given by the following lemma, which is a generalization of Lemma 3.

Lemma 6. *If $\tau = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1}) \in \mathbb{C}^m$, the algebra $\Pi_\tau(Q)$ is isomorphic to the quotient of $\Pi_\lambda(Q_\infty)$ by the idempotent ideal generated by e_∞.*

Proof.   One can show this easily by using the explicit presentations (24)-(25) and (29)-(30). There is, however, a more conceptual way to construct an isomorphism

$$(31) \qquad\qquad \Pi_\tau(Q) \cong \Pi_\lambda(Q_\infty)/\langle e_\infty \rangle\ .$$

Consider the natural projection $\pi : \mathbb{C}Q_\infty \twoheadrightarrow \mathbb{C}Q$, which maps all paths of Q_∞ ending at ∞ to zero. Clearly, $\mathtt{Ker}\,\pi$ is generated by e_∞, and hence it is an idempotent ideal in $\mathbb{C}Q_\infty$. It follows that $\mathtt{Tor}_1^{\mathbb{C}Q_\infty}(\mathbb{C}Q, \mathbb{C}Q) \cong \mathtt{Ker}\,\pi/(\mathtt{Ker}\,\pi)^2 = 0$, so θ is a pseudoflat ring epimorphism. By [CB2], Theorem 0.7, it extends then canonically to an algebra map $\tilde{\pi} : \Pi_\lambda(Q_\infty) \to \Pi_\tau(Q)$, making the commutative diagram

$$(32)$$

$$\begin{array}{ccc} \mathbb{C}Q_\infty & \xrightarrow{\ \pi\ } & \mathbb{C}Q \\ \big\downarrow & & \big\downarrow \\ \Pi_\lambda(Q_\infty) & \xrightarrow{\ \tilde{\pi}\ } & \Pi_\tau(Q) \end{array}$$

This diagram is a pushout in the category of rings, so the surjectivity of π implies the surjectivity of $\tilde{\pi}$, and $\mathtt{Ker}\,\tilde{\pi} = \langle e_\infty \rangle$. $\square$

The next proposition is a special case of [CB2], Theorem 1.2; for reader's convenience, we will sketch a direct proof of it, based on Lemma 6.

Proposition 3. *Let $\tau \in \mathbb{C}^m$ be a* regular *vector for the Dynkin quiver* (23), *and let $\alpha = (1, \boldsymbol{n}) \in \mathbb{Z}^{m+1}$. Then there exists a Π_λ-module of dimension vector α if and only if $\lambda = (-\tau \cdot \boldsymbol{n}, \tau)$ and α is a positive root of Q_∞. Every Π_λ-module of dimension vector α is simple.*

Proof.   We start with the last assertion. Assume that there is a Π_λ-module V of dimension α. Let $V' \subseteq V$ be a submodule of V. Consider the intersection $V' \cap V_\infty$, where $V_\infty := Ve_\infty$. Since $\dim(V_\infty) = 1$, there are two possibilities: either $V' \cap V_\infty = 0$ or $V' \cap V_\infty = V_\infty$. In the first case we have $V'e_\infty = 0$, so V' can be viewed as a module over the quotient algebra $\Pi_\lambda/\langle e_\infty \rangle$. By Lemma 6, this last algebra is isomorphic to $\Pi_\tau(Q)$, which, in view of regularity of τ, is a simple ring, having no non-trivial finite-dimensional modules (see Proposition 2). Thus $V' \cap V_0 = 0$ implies $V' = 0$. In the second case, if $V' \cap V_\infty = V_\infty$, we have $Ve_\infty = V_\infty \subseteq V'$, so e_∞ acts trivially on the quotient V/V'. Applying the above argument to V/V' instead of V' yields $V' = V$. Thus, in any case V' is 0 or V, which means that V is a simple module.

Now, the existence of a Π_λ-module V implies that $\lambda \cdot \alpha = 0$, or equivalently $\lambda = (-\tau \cdot \boldsymbol{n}, \tau)$. The simplicity of V forces it to be indecomposable as a $\mathbb{C}Q_\infty$-module. A well-known theorem of Kac [K] then ensures that α is a positive root of Q_∞. Conversely, if α is a positive root of Q_∞, then (again by Kac's Theorem) there is an indecomposable $\mathbb{C}Q_\infty$-module. If, in addition, $\lambda \cdot \alpha = 0$, this module extends to a Π_λ-module by [CBH], Theorem 4.3. $\square$

Now, for $\tau \in \mathbb{C}^m$ and $\alpha = (1, \boldsymbol{n}) \in \mathbb{Z}^{m+1}$, we set

$$\tilde{\mathcal{C}}_{\boldsymbol{n},\tau}(\Gamma) := \mathtt{Rep}(\Pi_\lambda(Q_\infty)^{\mathrm{opp}}, \alpha)\,, \quad \lambda = (-\tau \cdot \boldsymbol{n}, \tau)\,.$$

By Proposition 3, if τ is regular and α is a positive root of Q_∞, the variety $\tilde{\mathcal{C}}_{\boldsymbol{n},\tau}(\Gamma)$ is non-empty, and the natural action of the group

$$\mathtt{G}(\alpha) := \left(\mathbb{C}^* \times \prod_{i=0}^{m-1} \mathtt{GL}(n_i, \mathbb{C})\right) \Big/ \mathbb{C}^* \cong \prod_{i=0}^{m-1} \mathtt{GL}(n_i, \mathbb{C})$$

on $\tilde{\mathcal{C}}_{\boldsymbol{n},\tau}(\Gamma)$ is free. In that case, we define

(33) $$\mathcal{C}_{\boldsymbol{n},\tau}(\Gamma) := \tilde{\mathcal{C}}_{\boldsymbol{n},\tau}(\Gamma)//\mathtt{G}(\alpha)\,.$$

Thus, by definition, $\mathcal{C}_{\boldsymbol{n},\tau}(\Gamma)$ are affine varieties, whose (closed) points are in bijection with isomorphism classes of simple Π_λ-modules of dimension vector α. If Γ is trivial and $\tau = 1$, these varieties coincide with the usual Calogero-Moser spaces $\mathcal{C}_n$, see Section 2.3. In general, they can be identified with Nakajima's *quiver varieties* $\mathfrak{M}_\Gamma^\tau(V, W)$ with

$$V = W_0^{\oplus n_0} \oplus W_1^{\oplus n_1} \oplus ... \oplus W_{m-1}^{\oplus n_{m-1}} \quad \text{and} \quad W = W_0\,,$$

where $\{W_0, W_1, ..., W_{m-1}\}$ is a complete set of irreducible representations of Γ and W_0 is its trivial representation (see [CB3]).

4.3. **The generalized Calogero-Moser correspondence.** We are now in position to state our generalization of Theorem 1. First, we fix $\tau \in \mathbb{C}^m$ and identify $\mathcal{S}_\tau(\Gamma) = \Pi_\lambda(Q_\infty)/\langle e_\infty \rangle$ by combining the isomorphisms (28) and (31). We denote by $i : \Pi_\lambda \to \mathcal{S}_\tau$ the canonical projection and associate to it the triple of adjoint functors $(i^*, i_*, i^!)$, relating $\mathrm{Mod}(\Pi_\lambda)$ and $\mathrm{Mod}(\mathcal{S}_\tau)$, as in Section 2.5. Next, we set $B_\lambda := e_\infty \Pi_\lambda e_\infty$ and introduce the functors $(j_!, j^*, j_*)$ between the categories $\mathrm{Mod}(\Pi_\lambda)$ and $\mathrm{Mod}(B_\lambda)$. As explained in Section 2.5, $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ satisfy the recollement axioms (R1)–(R5). Finally, we introduce the functor $\mathfrak{e} : \mathrm{Mod}(\mathcal{S}_\tau) \to \mathrm{Mod}(O_\tau)$, $M \mapsto M\mathfrak{e}$, which, in the case of regular τ, is an equivalence of categories (see Proposition 2).

Theorem 5. *Let $\tau \in \mathbb{C}^m$ be a regular vector for the Dynkin quiver (23), let $\alpha = (1, \boldsymbol{n}) \in \mathbb{Z}^{m+1}$ be a positive root for Q_∞, and let $\lambda := (-\tau \cdot \boldsymbol{n}, \tau)$. Then the composition of functors*

$$\Omega_\tau : \mathrm{Mod}(\Pi_\lambda) \xrightarrow{j^*} \mathrm{Mod}(B_\lambda) \xrightarrow{j_!} \mathrm{Mod}(\Pi_\lambda) \xrightarrow{i^!} \mathrm{Mod}(\mathcal{S}_\tau) \xrightarrow{\mathfrak{e}} \mathrm{Mod}(O_\tau)$$

maps injectively the set $\mathrm{Irr}(\Pi_\lambda, \alpha)$ of isomorphism classes of simple Π_λ-modules of dimension vector α into the set $\mathcal{R}_\tau$ of isomorphism classes of (right) ideals of O_τ. If we identify $\mathrm{Irr}(\Pi_\lambda, \alpha) = \mathcal{C}_{\boldsymbol{n}, \tau}(\Gamma)$ as above, then the induced map agrees with the bijections constructed in [BGK1] and [E].

Remark.   If we choose $\boldsymbol{n} = (n, n, \ldots, n)$ with $n \in \mathbb{N}$, then the functors Ω_τ induce a bijection between $\bigsqcup_{n \geq 0} \mathcal{C}_{\boldsymbol{n}, \tau}(\Gamma)$ and the set $\mathcal{R}_\tau^o \subset \mathcal{R}_\tau$ of isomorphism classes of *stably free* ideals of $O_\tau(\Gamma)$, in agreement with the original conjecture of Crawley-Boevey and Holland (see [CB1], p. 45).

4.4. **Relation to symplectic reflection algebras.** In this section we will state the generalizations of Theorems 3 and 4 to the case of cyclic quivers. As mentioned above, the rational Cherednik algebras $H_{0,c}(S_n)$ are replaced in this case by more general symplectic reflection algebras $H_{0,k,c}(\boldsymbol{\Gamma}_n)$. We recall the definition of these algebras, following essentially [EG]; we should warn the reader, however, that our notation slightly differs from the notation of [EG].

Let Γ be a cyclic subgroup of $\mathrm{SL}(2, \mathbb{C})$ of order $m > 1$ with a fixed generator $\alpha \in \Gamma$ and a primitive character χ. For an integer $n \geq 1$, let S_n be the permutation group of $\{1, ..., n\}$ with elementary transpositions $s_{ij} : i \leftrightarrow j$, and let $\boldsymbol{\Gamma}_n$ be the wreath product of Γ and S_n, i. e. the semidirect product $S_n \ltimes \Gamma^n$ relative to the obvious action of S_n on $\Gamma^n = \Gamma \times \ldots \times \Gamma$.

Let $L = \mathbb{C}^2$ be the natural representation of Γ with a basis $\{x, y\}$, such that $\alpha(x) = \epsilon x$ and $\alpha(y) = \epsilon^{-1}y$, where $\epsilon := \chi(\alpha)$. The group $\boldsymbol{\Gamma}_n$ acts naturally on L^n. Given $i \in \{1, ..., n\}$ and $\gamma \in \Gamma$, resp. $u \in L$, we write $\gamma_i \in \boldsymbol{\Gamma}_n$ for γ placed in i-th factor of Γ^n, resp. u_i for u placed in the i-th factor of L^n.

Now, fix parameters $k, c_1, \ldots, c_{m-1} \in \mathbb{C}$, and let $c := \sum_{l=1}^{m-1} c_l \alpha^l \in \mathbb{C}\Gamma$. The *symplectic reflection algebra* $H_{k,c} := H_{0,k,c}(\boldsymbol{\Gamma}_n)$ is then defined as the quotient of

$T(L^n)\#\mathbf{\Gamma}_n$ by the following relations

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0 , \quad 1 \le i, j \le n ,$$

(34)
$$[y_i, x_i] = k \sum_{j \ne i} \sum_{l=0}^{m-1} s_{ij} \alpha_i^l \alpha_j^{-l} + \sum_{l=1}^{m-1} c_l \alpha_i^l , \quad 1 \le i \le n ,$$

$$[y_i, x_j] = -k \sum_{l=0}^{m-1} s_{ij} \epsilon^l \alpha_i^l \alpha_j^{-l}, \quad 1 \le i \ne j \le n .$$

Write $\mathbf{e} := \frac{1}{|\mathbf{\Gamma}_n|} \sum_{g \in \mathbf{\Gamma}_n} g$ for the symmetrizing idempotent in $\mathbb{C}\mathbf{\Gamma}_n \subset H_{k,c}$ and denote by $U_{k,c} = \mathbf{e}H_{k,c}\mathbf{e}$ the corresponding spherical subalgebra of $H_{k,c}$. Next, put $\sigma_n := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$ and $\sigma_{n-1} := \frac{1}{(n-1)!} \sum_{\sigma \in S_{n-1}} \sigma$, where S_{n-1} is the subgroup of S_n permuting $\{2, ..., n\}$. It is easy to see that $\mathbf{e} = \sigma_n(\mathfrak{e}_0 \otimes \mathfrak{e}_0 \otimes \ldots \otimes \mathfrak{e}_0)$, where $\mathfrak{e}_0 = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g$. Now, recall the idempotents (26) and, for each $i = 0, 1, \ldots, m-1$, define

$$\nu_i := \sigma_{n-1}(\mathfrak{e}_i \otimes \mathfrak{e}_0 \otimes \ldots \otimes \mathfrak{e}_0) .$$

One can check easily that ν_i are idempotents in $\mathbb{C}\mathbf{\Gamma}_n$ satisfying the relations

$$x_1 \cdot \nu_{i+1} = \nu_i \cdot x_1 , \quad y_1 \cdot \nu_i = \nu_{i+1} \cdot y_1 .$$

The following lemma is a generalization of Lemma 5: it gives an explicit homomorphism from B_λ to the spherical subalgebra of $H_{k,c}$.

Lemma 7. *Let $\Pi_\lambda = \Pi_\lambda(Q_\infty)$ be the deformed preprojective algebra of weight $\lambda = (\lambda_\infty, \lambda_0, \ldots \lambda_{m-1}) \in \mathbb{C}^{m+1}$ with $\lambda_\infty = -n \sum_{i=0}^{m-1} \lambda_i$, and let $B_\lambda = e_\infty \Pi_\lambda e_\infty$. Then there is a natural homomorphism of unital algebras $\theta : B_\lambda \to U_{k,c}$, where $k = \frac{\lambda_\infty}{mn}$ and $c = (\lambda_0 + \frac{\lambda_\infty}{n})\mathfrak{e}_0 + \sum_{i=1}^{m-1} \lambda_i \mathfrak{e}_i$.*

Proof. The assignment

$$X_i \mapsto \nu_i x_1 , \quad Y_i \mapsto -y_1 \nu_i , \quad e_i \mapsto \nu_i ,$$

$$v \mapsto \frac{1}{n}\lambda_\infty (1 + s_{12} + \ldots + s_{1n}), \quad w \mapsto \nu_0 , \quad e_\infty \mapsto \mathbf{e} ,$$

extends by multiplicativity to a linear map $\theta : \mathbb{C}\bar{Q}_\infty \to H_{k,c}$. While this map is *not* a homomorphism of unital rings (e.g., $0 = \theta(e_\infty e_0) \ne \theta(e_\infty) \theta(e_0) = \mathbf{e}$), it becomes so when restricted to the subalgebra $e_\infty(\mathbb{C}\bar{Q}_\infty)e_\infty$. Under this restriction, the image of θ gets in $U_{k,c}$. Thus, to prove the theorem it suffices to check the defining relations (29)–(30), and this boils down to a routine calculation. $\square$

Remark. It appears that θ is a special case of the map Θ^{quiver} defined (somewhat implicitly) in [EGGO], see (1.6.3). We thank Vitya Ginzburg for this remark.

Write $\theta_* : \text{Mod}(U_{k,c}) \to \text{Mod}(B_\lambda)$ for the restriction functor corresponding to θ. Then our next result can be formulated as follows (cf. Theorem 3).

Theorem 6. *Let $\tau \in \mathbb{C}^m$ be a regular vector for the Dynkin quiver (23), let $\alpha = (1, \boldsymbol{n}) \in \mathbb{Z}^{m+1}$ with $\boldsymbol{n} = (n, n, \ldots, n)$, and let $\lambda := (-\tau \cdot \boldsymbol{n}, \tau)$. Then the composition of functors*

$$\Xi_\tau : \text{Mod}(H_{k,c}) \xrightarrow{\ \mathbf{e}\ } \text{Mod}(U_{k,c}) \xrightarrow{\ \theta_*\ } \text{Mod}(B_\lambda) \xrightarrow{\ j_{!*}\ } \text{Mod}(\Pi_\lambda)$$

maps the set $\mathtt{Irr}(H_{k,c})$ of isomorphism classes of simple $H_{k,c}$-modules bijectively onto the set $\mathtt{Irr}(\Pi_\lambda, \alpha)$ of isomorphism classes of simple Π_λ-modules of dimension vector α. This map agrees with Etingof-Ginzburg's construction in [EG].

As we pointed out earlier, there is a natural bijection between $\bigsqcup_{n\geq 0} \mathtt{Irr}(\Pi_\lambda, \alpha)$ and the set $\mathcal{R}_\tau^o$ of isomorphism classes of stably free ideals of O_τ (see Remark after Theorem 5). Thus, combining Theorems 5 and 6 together, we get

Theorem 7. *The composition of functors*

$$\mathtt{Mod}(H_{k,c}) \xrightarrow{\ \mathbf{e}\ } \mathtt{Mod}(U_{k,c}) \xrightarrow{\ \theta_*\ } \mathtt{Mod}(B_\lambda) \xrightarrow{\ j_!\ } \mathtt{Mod}(\Pi_\lambda) \xrightarrow{\ i^!\ } \mathtt{Mod}(\mathcal{S}_\tau) \xrightarrow{\ \mathfrak{e}\ } \mathtt{Mod}(O_\tau)$$

maps the set $\mathtt{Irr}(H_{k,c})$ of isomorphism classes of simple $H_{k,c}$-modules injectively into the set of isomorphism classes of stably free ideals of O_τ. Collecting such maps for all $n \geq 0$, we get a bijective correspondence $\bigsqcup_{n\geq 0} \mathtt{Irr}[H_{k,c}(\mathbf{\Gamma}_n)] \xrightarrow{\sim} \mathcal{R}_\tau^o$.

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