ORTHOGONALITY RELATIONS AND CHEREDNIK IDENTITIES FOR MULTIVARIABLE BAKER–AKHIEZER FUNCTIONS

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Abstract. We establish orthogonality relations for the Baker–Akhiezer (BA) eigenfunctions of the Macdonald difference operators. We also obtain a version of Cherednik–Macdonald–Mehta integral for these functions. As a corollary, we give a simple derivation of the norm identity and Cherednik–Macdonald–Mehta integral for Macdonald polynomials. In the appendix written by the first author, we prove a summation formula for BA functions. We also introduce more general twisted BA functions and obtain for them identities of Cherednik type. This leads to an implicit construction of new quantum integrable models of Macdonald–Ruijsenaars type. Our approach does not require Hecke algebras and therefore is applicable to deformed root systems. As an example, we consider the deformed root system $R = A_n(m)$, which leads us to an explicit evaluation of a new integral and a sum of Cherednik–Macdonald–Mehta type.

1. Introduction

Around 1988, Macdonald introduced a remarkable family of multivariate orthogonal polynomials related to root systems [M1]. Apart from a root system R, these polynomials depend on two additional (sets of) parameters q, t and specialize to various families of symmetric functions, among which are the characters of simple complex Lie groups, Hall–Littlewood functions, zonal spherical functions, Jack polynomials, and multivariate Jacobi polynomials of Heckman and Opdam [HO]. The Macdonald polynomials have since become a subject of numerous works revealing their links to many different areas of mathematics and mathematical physics.

The Macdonald polynomials are customarily defined as symmetric polynomial eigenfunctions of some rather remarkable partial difference operators, called Macdonald operators. These operators can be viewed as commuting quantum Hamiltonians, and the corresponding quantum model in case $R = A_n$ is equivalent to the trigonometric limit of the Ruijsenaars model [R1], a relativistic version of the Calogero–Moser model. The Macdonald polynomials play the role of eigenstates for these Macdonald–Ruijsenaars models and only exist on certain discrete energy levels. Their orthogonality follows from the fact that the Macdonald operators are self-adjoint with respect to a certain scalar product (Macdonald's product) defined as an integral over n-dimensional torus, with an explicit analytic measure.

For other values of energy, the solutions to the eigenvalue problem are non-elementary functions which can be expressed in terms of q-Harish–Chandra series [LS]. Rather remarkably, in the case $t \in q^{\mathbb{Z}}$ these series reduce to elementary (but still highly nontrivial) functions. These non-polynomial eigenfunctions $\psi(\lambda, x)$ depend on continuous (rather than discrete) spectral parameter λ and can be viewed as the Bloch–Floquet (i.e.quasi-periodic) solutions to the eigenvalue problem. Such solutions were constructed and studied in [Ch2]; in the case $R = A_n$ they were known from the earlier work [FVa, ES]. As shown in [Ch2], the functions $\psi(\lambda, x)$ are uniquely characterized by certain analytic properties, which makes them similar to the Baker–Akhiezer functions from the finite-gap theory [N, DMN, Kr1, Kr2]. For that reason we will refer to $\psi(\lambda, x)$ as multivariable Baker–Akhiezer (BA) functions. The idea that eigenfunctions of the quantum Calogero–Moser model for integral coupling parameters should be given by certain multivariable Baker–Akhiezer functions goes back to the work of the first author and Veselov [CV], see [Ch3] for the survey of known results in that direction.

According to [Ch2], the BA functions $\psi(\lambda, x)$ are related to Macdonald polynomials by a formula that generalizes the Weyl character formula. Using this, some important properties of Macdonald polynomials were derived in *loc. cit.* from analogous properties of ψ. In particular, the duality and evaluation identities for Macdonald polynomials are simple corollaries of the bispectral duality for ψ. The approach of *loc. cit.* led to an elementary proof of Macdonald's conjectures, different from Cherednik's proof that uses double affine Hecke algebras [C1, C2].

Our first main result concerns a question which was not addressed in [Ch2], namely, the orthogonality properties of $\psi(\lambda, x)$. Since these are eigenfunctions for the Macdonald operators (which are self-adjoint with respect to Macdonald's scalar product), one would expect ψ to form an orthogonal family. However, there is a subtlety here due to the fact that the definition of the Macdonald's product requires that $t = q^m$ with positive m, so it does not work for $m \in \mathbb{Z}_-$. In the latter case the Macdonald's product becomes degenerate and the action of Macdonald operators on symmetric polynomials becomes non-semisimple. (There is no such problem for $m \in \mathbb{Z}_+$, however in that case the functions $\psi(\lambda, x)$ have poles on the contour of integration, so the Macdonald's product again is not well-defined.) The way around that problem is suggested by the work of the second author and Varchenko [EV2]. Namely, as we show in Theorem 4.1 below, the correct scalar product can be defined by shifting the contour of integration suitably, after which the integral can be easily evaluated by moving the contour to infinity. Morally, this is the same argument as the one used by Grinevich and Novikov in [GN], where they derive orthogonality relations for BA functions on Riemann surfaces. Similarly to *loc. cit.*, our scalar product is indefinite.

However, to compare with their situation, our $\psi(\lambda, x)$ represents a section of a line bundle on a (singular) n-dimensional algebraic variety, so even the existence of ψ is a non-trivial fact (proved in [Ch2]). Also, our situation is rather special because our BA functions are self-dual unlike those in [GN]. As an application of our result, we present a simple derivation of the norm formula for Macdonald polynomials.

Our second main result is a version of the Cherednik–Macdonald–Mehta integral identity for BA functions (Theorem 5.1). It is a generalization of the self-duality of the Gaussian e^{-x^2}, a basic fact about the Fourier transforms. Again, the proof is quite simple, and it easily implies the integral identity originally proved by Cherednik [C3], in particular, it gives a new proof for the q-analogue of the Macdonald–Mehta integral [M3, C3].

The paper finishes with an appendix written by the first author. In it we prove a version of the summation formula for ψ that involves the Gaussian (Theorem 6.1); this implies the result of [C3, Theorem 1.3]. Part of the motivation behind this work was to find analogues of Cherednik's results for the deformed root systems, discovered in [CFV1, CFV2]. Since our proofs do not require double affine Hecke algebras, they can be adapted for the deformed cases. We illustrate this on one particular example, the deformed system $A_n(m)$ from [CFV1]. The BA function $\psi(\lambda, x)$ in that case was constructed in [Ch2]. We prove the orthogonality relations and Cherednik–Macdonald–Mehta identities for that ψ. In particular, we explicitly evaluate deformed q-Macdonald–Mehta integral and sum for $R = A_n(m)$. In the final section of the appendix, we introduce twisted BA functions $\psi_\ell(\lambda, x)$, $\ell \in \mathbb{Z}_+$, in relation to more general integrals and sums of Cherednik–Macdonald–Mehta type. We show that the functions $\psi_\ell(\lambda, x)$ serve as common eigenfunctions for quantum integrable models of Macdonald–Ruijsenaars type, which we call twisted Macdonald–Ruijsenaars models. To the best of our knowledge, they are new. The commuting quantum Hamiltonians for these models look as lower order perturbations of the Macdonald operators raised to power ℓ. Our construction of these models is implicit and is based on the construction and properties of the twisted BA functions ψ_ℓ. It would be interesting to find an explicit construction for these twisted models.

The structure of the paper is as follows. In Section 2 we introduce notations and recall the definitions of Macdonald scalar product, Macdonald polynomials and Macdonald operators. In Section 3 we collect definitions and main properties of the Baker–Akhiezer functions in Macdonald theory. The material is based on [Ch2] and is not new, apart from the evaluation results in Section 3.3. Section 4 proves orthogonality relations for the BA functions (Theorem 4.1). In Section 4.2 we explain how one can use them to compute the norms of Macdonald polynomials. We also prove orthogonality relations in the case when q is a root of unity (Theorem 4.6). Section 5 establishes a version of Cherednik–Macdonald–Mehta integral for the BA functions. Following [EV2], we also discuss briefly the related integral transforms and use them to rewrite the Cherdnik–Macdonald–Mehta integral with the integration over a real cycle. We finish the section by re-deriving Cherednik identities for Macdonald polynomials (Theorem 5.7) and discuss some special cases, including q-Macdonald–Mehta integral. The paper concludes with an Appendix consisting of three sections. Section 6 is devoted to the proof of the summation formula analogous to [C3, Theorem 1.3]. In Section 7 we apply a similar approach to the deformed root system $A_n(m)$, extending the previous results to that case. In particular, we explicitly evaluate an integral and a sum of Macdonald–Mehta type for $R = A_n(m)$ (Propositions 7.6 and 7.7). Finally, in Section 8 we introduce twisted BA functions, and show that they serve as eigenfunctions of a twisted version of Macdonald–Ruijsenaars models.

Let us finish by mentioning that in the case $R = A_n$ the results of Theorems 4.1 and 5.1 were obtained previously in [EV2] by using representation theory of quantum groups. The strategy of *loc. cit.* was in a sense opposite to the one employed in the present paper. Namely, the results in *loc. cit.* were first derived in the symmetric setting, by representation-theoretic methods from [EK1, EK2, ES, EV1], and then they were extended to statements about ψ by analytic arguments. In contrast, we prove our results directly for ψ, and then use them to derive analogous results for Macdonald polynomials. In both approaches, Proposition 4.2 below plays the crucial role.

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2. Macdonald polynomials and Macdonald operators

2.1. **Notations.** Let $V_{\mathbb{R}}$ be a finite-dimensional real Euclidean vector space with the scalar product $\langle \cdot , \cdot \rangle$. Let $R = \{\alpha\} \subset V_{\mathbb{R}}$ be a reduced irreducible root system and W be the Weyl group of R, generated by orthogonal reflections s_α for $\alpha \in R$. The dual system is $R^\vee = \{\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle} \,|\, \alpha \in R\}$. We choose a basis of simple roots $\{\alpha_1, \dots, \alpha_n\} \subset R$ and denote by R_+ the positive half with respect to that choice. We use the standard notation of [B], so $Q = Q(R)$ and $P = P(R)$ denote the root and weight lattices of R, Q_+ is the positive part of Q and P_+ is the set of dominant weights. Their counterparts for $R^\vee$ are denoted as $Q^\vee$, $P^\vee$, etc. Let $<$ denote the dominant partial ordering on P.

Let $\mathbb{R}[P]$ be the group algebra of the weight lattice P. We choose $0 < q < 1$ and think of the elements of $\mathbb{R}[P]$ as functions on $V_{\mathbb{R}}$ of the form

$$f(x) = \sum_{\nu \in P} f_\nu q^{\langle \nu, x\rangle} \quad \text{with } f_\nu \in \mathbb{R}\,.$$

We can view such f as an analytic function on the complexified space $V_{\mathbb{C}} = V_{\mathbb{R}} \oplus iV_{\mathbb{R}}$ by defining $q^{\langle\nu,x\rangle} := e^{\log q \langle \nu, x\rangle}$. Clearly, f is periodic with the lattice of periods $\kappa Q^\vee$, where

$$\kappa = \frac{2\pi i}{\log q}\,. \tag{2.1}$$

Note that $\kappa \in i\mathbb{R}_-$. Later we will allow complex $q \neq 0$; in that case one needs to fix a value of $\log q$ so κ might no longer be purely imaginary. Whenever we allow q to vary, we do it by choosing a local branch of the logarithm.

There are three types of Macdonald's theory; they correspond to [M2], (1.4.1)–(1.4.3). The first two types are associated to any reduced root system R and one or two additional parameters. The third type corresponds to the non-reduced affine root system $(C_n^\vee, C_n)$; this case involves 5 parameters and is related to Koornwinder polynomials [Ko1]. Following [LS], we will refer to these as cases **a**, **b** and **c**, respectively. Each case depends on a data (R, m) consisting of a root system R and certain labels m playing a role of parameters.

2.1.1. *Cases* **a**, **b**. Given an arbitrary reduced irreducible root system R, let us choose W-invariant multiplicity labels $m_\alpha \in \mathbb{R}$ for all $\alpha \in R$. These labels must be the equal for the roots of the same length, so m_α take at most two values, depending on whether R consists of one or two W-orbits.

Let us introduce quantities q_α for $\alpha \in R$ as follows:

$$q_\alpha = \begin{cases} q & \text{in case } \mathbf{a}\,, \\ q^{\frac{\langle\alpha,\alpha\rangle}{2}} & \text{in case } \mathbf{b}\,. \end{cases} \tag{2.2}$$

(By default, we also assume that $q_\alpha = q$ in case **c**.) We will also write t_α for $t_\alpha = q_\alpha^{-m_\alpha}$.

2.1.2. *Case* **c**. Consider $V_\mathbb{R} = \mathbb{R}^n$ with the standard Euclidean product and let $R \subset V_\mathbb{R}$ be the root system of type C_n, that is $R = 2R^1 \cup R^2$ where

$$R^1 = \{\pm e_i \,|\, i = 1, \dots, n\}\,, \quad R^2 = \{\pm e_i \pm e_j \,|\, 1 \le i < j \le n\}\,. \tag{2.3}$$

Choose real parameters m_i, $i = 1, \dots, 5$ and set

$$m_\alpha = \begin{cases} \frac12 + \frac12 \sum_{i=1}^4 m_i & \text{for } \alpha \in 2R^1\,, \\ m_5 \quad \text{for } \alpha \in R^2\,. \end{cases} \tag{2.4}$$

Below we will need the dual parameters m_i' defined as follows:

$$\begin{aligned} m_1' &= \frac12 + \frac12(m_1 + m_2 + m_3 + m_4)\,, \\ m_2' &= -\frac12 + \frac12(m_1 + m_2 - m_3 - m_4)\,, \\ m_3' &= -\frac12 + \frac12(m_1 - m_2 + m_3 - m_4)\,, \\ m_4' &= -\frac12 + \frac12(m_1 - m_2 - m_3 + m_4)\,, \\ m_5' &= m_5\,. \end{aligned} \tag{2.5}$$

Write t for

$$(t_1, t_2, t_3, t_4, t_5) := (q^{-m_1}, q^{-m_2}, -q^{-m_3}, -q^{-m_4}, q^{-m_5})\,. \tag{2.6}$$

In all three cases m will denote the set of m_α or m_i, respectively, and we will use the abbreviation $t = q^{-m}$ to denote the above t_α or t_i.

It will be convenient to use the notation α' for α in cases **a**, **c** and $\alpha^\vee$ in case **b**, and put $R' = \{\alpha' \,|\, \alpha \in R\}$. To have uniform notation, let us also put $m'_\alpha = m_\alpha$ in cases **a**, **b**, while in case **c** we put, according to (2.5), (2.4) that

$$m'_\alpha = \begin{cases} m_1 & \text{for } \alpha \in R^1\,, \\ m_\alpha & \text{for } \alpha \in R^2\,. \end{cases} \tag{2.7}$$

Let us now introduce the **Macdonald weight function** ∇. In cases **a**, **b** it is defined as follows ([M2, (5.1.28)]):

$$\nabla = \nabla(x; q, t) = \prod_{\alpha \in R} \frac{\left(q^{\langle \alpha, x\rangle}; q_\alpha\right)_\infty}{\left(t_\alpha q^{\langle \alpha, x\rangle}; q_\alpha\right)_\infty}\,, \tag{2.8}$$

where we used the standard notation

$$(a; q)_\infty := \prod_{i=0}^{\infty} (1 - a q^i)\,.$$

In case **c** we put ([M2, (5.1.28)])

$$\nabla = \nabla(x; q, t) = \nabla^{(1)} \nabla^{(2)}\,, \tag{2.9}$$

where

$$\nabla^{(1)} = \prod_{\alpha \in R^1} \frac{\left(q^{2\langle \alpha, x\rangle}; q\right)_\infty}{\prod_{i=1}^4 \left(t_i q^{\langle \alpha, x\rangle}; q\right)_\infty}$$

and

$$\nabla^{(2)} = \prod_{\alpha \in R^2} \frac{\left(q^{\langle \alpha, x\rangle}; q\right)_\infty}{\left(t_5 q^{\langle \alpha, x\rangle}; q\right)_\infty}\,.$$

Finally, let ρ, ρ' be the following vectors:

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} m_\alpha \alpha\,, \quad \rho' = \frac{1}{2} \sum_{\alpha \in R_+} m'_\alpha \alpha'\,. \tag{2.10}$$

Remark 2.1. Our notation slightly differs from Macdonald's [M2] who uses parameters

$$k_\alpha = -m_\alpha\,. \tag{2.11}$$

In case **c** the relation between k_i used in [M2, Section 5] and our m_i is as follows:

$$m_1 = -k_1\,, m_2 = -k_3 - \frac{1}{2}\,, m_3 = -k_2\,, m_4 = -k_4 - \frac{1}{2}\,, m_5 = -k_5\,. \tag{2.12}$$

Let us also remark on the notation used in [Ch2]. In case **c** the above t_i were denoted a, b, c, d, t in [Ch2, Section 6], while m_i relate to (k, l, l', m, m') used in *loc. cit.* by

$$(m_1, m_2, m_3, m_4, m_5) = (l, l', m, m', k)\,.$$

More importantly, [Ch2] uses q^2 in place of q.

2.1.3. *Integrality assumptions.* Below we will mostly deal with the case when the parameters m are (half-)integers, so let us introduce some additional notation for that case. Our running assumption will be that

$$m_\alpha \in \mathbb{Z}_+ = \{0, 1, 2, \dots\} \quad \forall\, \alpha \in R \quad (\text{case } \mathbf{a} \text{ and } \mathbf{b}) \tag{2.13}$$

and

$$m_i, m_i' \in \frac{1}{2}\mathbb{Z}_+ \quad \text{for } i = 1, \dots, 4\,, \quad m_5 \in \mathbb{Z}_+\,, \tag{2.14}$$

$$m_1 + m_2 \in \frac{1}{2} + \mathbb{Z}\,, \quad m_3 + m_4 \in \frac{1}{2} + \mathbb{Z}\,. \tag{2.15}$$

The latter assumption means that each pair (m_1, m_2) and (m_3, m_4) consists of an integer and a half-integer. For brevity, we will refer to m as in (2.13)–(2.15) as *integral* parameters.

The following notation will be used below for $a, b, c \in \mathbb{R}$:

$$a \preccurlyeq (b, c) \quad \Leftrightarrow \quad a \in \{b - \mathbb{Z}_+\} \cup \{c - \mathbb{Z}_+\}\,. \tag{2.16}$$

For example, $0 < s \preccurlyeq (3/2, 2)$ means that $s \in \{1/2, 3/2, 1, 2\}$.

2.1.4. *Weight function for $t = q^{-m}$.* Let us write explicitly the Macdonald weight function ∇ for integral parameters m as specified above. It will be convenient to introduce another function Δ as follows. In case **a** and **b** we put

$$\Delta(x) = \prod_{\alpha \in R_+} \prod_{j=1}^{m_\alpha} \left(q_\alpha^{-j/2} q^{\langle \alpha, x \rangle/2} - q_\alpha^{j/2} q^{-\langle \alpha, x \rangle/2} \right)\,. \tag{2.17}$$

In case **c**, we put $\Delta = \Delta_+^{(1)} \Delta_-^{(1)} \Delta^{(2)}$ where

$$\Delta_+^{(1)}(x) = \prod_{\alpha \in R_+^1} \prod_{0 < s \preccurlyeq (m_1, m_2)} \left(q^{-s/2} q^{\langle \alpha, x \rangle/2} - q^{s/2} q^{-\langle \alpha, x \rangle/2} \right)\,, \tag{2.18}$$

$$\Delta_-^{(1)}(x) = \prod_{\alpha \in R_+^1} \prod_{0 < s \preccurlyeq (m_3, m_4)} \left(q^{-s/2} q^{\langle \alpha, x \rangle/2} + q^{s/2} q^{-\langle \alpha, x \rangle/2} \right)\,, \tag{2.19}$$

and

$$\Delta^{(2)}(x) = \prod_{\alpha \in R_+^2} \prod_{j=1}^{m_\alpha} \left(q_\alpha^{-j/2} q^{\langle \alpha, x \rangle/2} - q_\alpha^{j/2} q^{-\langle \alpha, x \rangle/2} \right)\,. \tag{2.20}$$

This is related to ∇ (2.8), (2.9) by

$$\nabla(x; q, q^{-m}) = C\,(\Delta(x)\Delta(-x))^{-1}\,, \tag{2.21}$$

where

$$C = \prod_{\alpha \in R_+} q_\alpha^{m_\alpha(m_\alpha+1)/2} \qquad \text{(cases } \mathbf{a}, \mathbf{b}) \tag{2.22}$$

or

$$C = \prod_{\substack{0<r\preccurlyeq(m_1,m_2)\\ 0<s\preccurlyeq(m_3,m_4)}} q^{n(s+r)} \prod_{\alpha \in R_+^2} q^{m_\alpha(m_\alpha+1)/2}. \qquad \text{(case } \mathbf{c}) \tag{2.23}$$

Finally, if $\Delta = \Delta_{R,m}$ is as above then Δ' will denote the dual function $\Delta' = \Delta_{R',m'}$.

2.2. **Macdonald scalar product.** Let $\nabla(x; q, t)$ be the Macdonald weight function (2.8)–(2.9) associated to (R, m). We are going to define a scalar product on $\mathbb{R}[P]$, where $P = P(R)$ is the weight lattice of R. Let us first assume that the parameters m are of the form (2.11) or (2.12), respectively, with all k_α or k_i positive integers. In that case it is easy to check that $\nabla \in \mathbb{R}[P]$. For instance, in cases **a** and **b**,

$$\nabla = \prod_{\alpha \in R} \prod_{i=0}^{k_\alpha - 1} (1 - q_\alpha^i q^{\langle \alpha, x\rangle}). \tag{2.24}$$

Then the Macdonald scalar product on $\mathbb{R}[P]$ is defined by

$$\langle f, g\rangle = \mathtt{CT}\left[f(x) g(-x) \nabla(x)\right] \quad \forall f, g \in \mathbb{R}[P], \tag{2.25}$$

where $\mathtt{CT}$ is the linear functional on $\mathbb{R}[P]$ computing the constant term:

$$\mathtt{CT}\left[q^{\langle \nu, x\rangle}\right] = \delta_{\nu,0}.$$

We can rewrite $\langle f, g\rangle$ as an integral over a torus. Namely, if κ is as in (2.1) then

$$\int_{iV_\mathbb{R}/\kappa Q^\vee} q^{\langle \nu, x\rangle}\, dx = \delta_{0,\nu} \quad \text{and} \quad \int_{iV_\mathbb{R}/\kappa Q^\vee} f\, d\, x = \mathtt{CT}[f] \quad \forall f \in \mathbb{R}[P],$$

where dx is the normalized Haar measure on the torus $T = iV_\mathbb{R}/\kappa Q^\vee$. The scalar product (2.25) can therefore be written as

$$\langle f, g\rangle = \int_{iV_\mathbb{R}/\kappa Q^\vee} f(x) g(-x) \nabla(x)\, dx. \tag{2.26}$$

Note that $\nabla(x)$ is real on $iV_\mathbb{R}$, and also for any $f \in \mathbb{R}[P]$ we have that $f(-x) = \overline{f(x)}$. This implies that the scalar product (2.25) is positive definite.

For other values of the parameters, the usual convention is to define $\langle f, g\rangle$ by analytic continuation in t from the above values $t = q^k$. It is easy to see that the restriction of ∇ on $iV_\mathbb{R} \subset V_\mathbb{C}$ depends analytically on t provided that t_α (or t_i in case **c**) belong to $(0, 1)$. Therefore, for such parameters the scalar product is still given by the integral (2.26). However, for other values of parameters the integral (2.25) no longer gives the correct scalar product. Indeed, in the process of analytic continuation one might need to deform the contour of integration when the poles of the weight function ∇ cross through $iV_{\mathbb{R}}$. It is far from obvious how to define the correct scalar product by an analytic formula similar to (2.26), that would remain valid for all t.

The present paper provides a (partial) solution to that problem in the case $t = q^{-m}$ with integral m. As we will see below, a simple recipe in that case is to shift the integration cycle $iV_{\mathbb{R}}$ by a suitable $\xi \in V_{\mathbb{R}}$ (we borrowed that idea from [EV2]). Note that on the shifted cycle $f(-x)$ is no longer equal to the complex conjugate of $f(x)$, therefore we cannot expect the scalar product to remain positive. This has obvious parallels with the work [GN], where indefinite scalar products were associated with the Baker-Akhiezer functions on Riemann surfaces. This is not surprising, since the Baker–Akhiezer functions considered in the present paper can be viewed as multivariable analogues of some of the Baker-Akhiezer functions appearing in the finite-gap theory [N, DMN, Kr1, Kr2].

2.3. **Macdonald polynomials.** We write $\mathbb{R}[P]^W$ for the W-invariant part of $\mathbb{R}[P]$. As a vector space, $\mathbb{R}[P]^W$ is generated by the orbitsums

$$\mathfrak{m}_\lambda = \sum_{\tau \in W\lambda} q^{\langle \tau, x\rangle}\,, \qquad \lambda \in P_+\,. \tag{2.27}$$

Definition. Define polynomials $p_\lambda = p_\lambda(x; q, t)$ as the (unique) elements of $\mathbb{R}[P]^W$ of the form

$$p_\lambda = \mathfrak{m}_\lambda + \sum_{\nu < \lambda} a_{\lambda\nu}\mathfrak{m}_\nu\,, \qquad \lambda \in P_+\,, \tag{2.28}$$

which are orthogonal with respect to the scalar product (2.26):

$$\langle p_\lambda\,,\, p_\mu\rangle = 0 \quad \text{for } \lambda \neq \mu\,.$$

Here $a_{\lambda\nu}$ depend on q, t and $\nu < \lambda$ denotes the dominant partial ordering on P_+.

These polynomials were introduced by Macdonald in [M1] in cases **a** and **b** (and some subcases of **c**). In case **c** they are due to Koornwinder [Ko1]. We will call p_λ Macdonald polynomials in all three cases. The existence of such p_λ is a non-trivial fact. Originally, p_λ were constructed in [M1, Ko1] as eigenfunctions of the form (2.28) for certain remarkable difference operators, see the next section. Later, Cherednik developed his celebrated DAHA theory which, among many other things, led to an alternative construction of D^π and p_λ [C1, C2, C3]. Cherednik's approach was extended to Koornwinder polynomials in [No, Sa, S].

Remark 2.2. One should keep in mind that the coefficients $a_{\lambda\nu}$ in (2.28) are certain rational functions of q_α and t_α. They may have poles and, as a result, some of p_λ do not exist for certain values of q, t. This happens, for instance, in the case when $t = q^{-m}$ with integral m, and it is this case which will be of our main interest below.

2.4. **Macdonald difference operators.** For any $\tau \in V_{\mathbb{C}}$, T^τ will denote the shift operator, which acts on a function of $x \in V_{\mathbb{C}}$ by $(T^\tau f)(x) = f(x+\tau)$. A difference operator D (on a lattice $L \subset V_{\mathbb{C}}$) is a finite sum of $a_\tau(x)T^\tau$ with $\tau \in L$. The Macdonald operators (and Koornwinder operator in case **c**) are certain explicitly written difference operators whose eigenfunctions are the polynomials p_λ. Each $D = D^\pi$ is of the form

$$D^\pi = \sum_{\tau \in W\pi} a_\tau(x)(T^\tau - 1) + a_0\,, \tag{2.29}$$

for certain $\pi \in P(R)$, some $a_\tau(x)$ and a constant coefficient a_0. They were introduced in [M1] for cases **a**, **b** and in [Ko1] in case **c** (in case $R = A_n$ they also appeared in [R1], see Example below). We have [M1, Ko1]:

$$D^\pi p_\lambda = \mathfrak{m}_\pi(\lambda) p_\lambda \quad \forall \lambda \in P_+\,. \tag{2.30}$$

We will write down D^π explicitly for case **b** below; explicit formulas for other cases can be found in the original papers [M1], [Ko1], as well as [LS], [Ch2]. Note that some further explicit difference operators with the property (2.30) are known, see [D, DE].

To write down the Macdonald operators in case **b**, recall that a nonzero weight $\pi \in P(R)$ is called minuscule if $\langle \pi, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for all $\alpha \in R$. It is known that minuscule dominant weights are in one-to-one correspondence with nonzero elements of P/Q, which means that they do not exist for $R = E_8, F_4, G_2$, see [B]. A weaker notion is that of a quasi-minuscule weight. By definition, $\pi \in P(R)$ is called quasi-minuscule if $\pi \in R$ and $\langle \pi, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for all $\alpha \in R \backslash \{\pm\pi\}$. (Note that for for $\alpha = \pm\pi$ we have $\langle \pi, \alpha^\vee \rangle = \pm 2$.) Quasi-minuscule weights exist for all R and they are all of the form $\pi = w\theta$, $w \in W$, where $\theta^\vee$ is the maximal coroot in $R^\vee_+$.

Given a (quasi-)minuscule $\pi \in P(R)$, the Macdonald difference operator D^π of type **b** associated to R has the form (2.29) with

$$a_\tau = \prod_{\substack{\alpha \in R:\\ \langle \alpha, \tau \rangle > 0}} \frac{1 - t_\alpha q^{\langle \alpha, x \rangle}}{t_\alpha^{1/2}(1 - q^{\langle \alpha, x \rangle})} \prod_{\substack{\alpha \in R:\\ \langle \alpha^\vee, \tau \rangle = 2}} \frac{1 - t_\alpha q_\alpha q^{\langle \alpha, x \rangle}}{t_\alpha^{1/2}(1 - q_\alpha q^{\langle \alpha, x \rangle})} \tag{2.31}$$

and with $a_0 = \sum_{\tau \in W\pi} q^{-\langle \rho^\vee, \tau \rangle}$, where $\rho^\vee := \sum_{\alpha \in R_+} m_\alpha \alpha^\vee$.

Remark 2.3. When π is minuscule, the second product in (2.31) is trivial. In that case the expression $a_0 - \sum_{\tau \in W\pi} a_\tau$ cancels and the formula for D^π reduces to

$$D^\pi = \sum_{\tau \in W\pi} a_\tau T^\tau, \quad a_\tau = \prod_{\substack{\alpha \in R:\\ \langle \alpha, \tau \rangle > 0}} \frac{1 - t_\alpha q^{\langle \alpha, x \rangle}}{t_\alpha^{1/2}(1 - q^{\langle \alpha, x \rangle})}\,. \tag{2.32}$$

Example. In case $R = A_{n-1} = \{\pm(e_i - e_j)\,|\, i < j\} \subset \mathbb{R}^n$ with $m_\alpha \equiv m$, each fundamental weight $\pi_s = e_1 + \cdots + e_s$ $(s = 1, \ldots, n)$ is minuscule and the corresponding operators $D_s = D^{\pi_s}$ have the form

$$D_s = \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=s}} \prod_{\substack{i\in I\\ j\notin I}} \frac{q^{-m+x_i-x_j} - q^{m-x_i+x_j}}{q^{x_i-x_j} - q^{-x_i+x_j}}\, T^I\,, \tag{2.33}$$

where T^I stands for $\prod_{i\in I} T^{e_i}$. The (commuting) operators $D_1,\dots,D_n$ are conserved quantities ('integrals') of the quantum trigonometric *Ruijsenaars model* [R1], see also [RS] for its classical counterpart.

3. Baker–Akhiezer function for Macdonald operators

Throughout this section we assume $q\in\mathbb{C}^\times$ is not a root of unity (unless specified otherwise). From now on we will work under the integrality assumptions as specified in 2.1.3. For a given (R,m), the Baker–Akhiezer functions (BA functions for short) are eigenfunctions of special form for the Macdonald operators D^π with $t=q^{-m}$. In cases **a** and **c** they were introduced and studied in [Ch2]; case **b** is entirely similar.

Let us denote by $\mathcal{N}\subset V_\mathbb{R}$ the following polytope associated to (R,m):

$$\mathcal{N} = \{\nu = \frac{1}{2}\sum_{\alpha\in R_+} l_\alpha\alpha \mid -m_\alpha\le l_\alpha\le m_\alpha\}\,. \tag{3.1}$$

By $\mathcal{N}'$ we denote the counterpart of $\mathcal{N}$ for (R',m'), i.e.

$$\mathcal{N}' = \{\nu = \frac{1}{2}\sum_{\alpha\in R_+} l_\alpha\alpha' \mid -m'_\alpha\le l_\alpha\le m'_\alpha\}\,. \tag{3.2}$$

Note that the vertices of $\mathcal{N}$ and $\mathcal{N}'$ are of the form $w\rho$ and $w\rho'$, respectively, with $w\in W$.

3.1. **Baker–Akhiezer function.** Let $\psi(\lambda,x)$ be a function of $(\lambda,x)\in V_\mathbb{C}\times V_\mathbb{C}$ of the form

$$\psi = q^{\langle\lambda,x\rangle}\sum_{\nu\in\mathcal{N}\cap P}\psi_\nu(\lambda)q^{\langle\nu,x\rangle}\,. \tag{3.3}$$

Let us assume that ψ has the following properties for each $\alpha\in R$ in cases **a**, **b** or $\alpha\in R^2$ in case **c**, and for every $j=1,\dots,m_\alpha$:

$$\psi\left(\lambda,\, x+\frac{1}{2}j\alpha'\right) = \psi\left(\lambda,\, x-\frac{1}{2}j\alpha'\right)\quad\text{when}\quad q^{\langle\alpha,x\rangle}=1\,. \tag{3.4}$$

In case **c**, we require additionally for each $\alpha=e_i\in R^1$ that

(1) for all $0<s\preccurlyeq(m_1,m_2)$

$$\psi(\lambda,x-se_i)=\psi(\lambda,x+se_i)\qquad\text{for } q^{x_i}=1\,; \tag{3.5}$$

(2) for all $0<s\preccurlyeq(m_3,m_4)$

$$\psi(\lambda,x-se_i)=\psi(\lambda,x+se_i)\qquad\text{for } q^{x_i}=-1\,. \tag{3.6}$$

Notice that for each $\alpha=e_i$ we get $1+\sum_{i=1}^4 m_i = 2m'_1$ different conditions.

Definition. A (nonzero) function $\psi(\lambda, x)$ with the properties (3.3)–(3.6) is called a **Baker–Akhiezer (BA) function** associated to $\{R, m\}$.

Theorem 3.1. *[cf.*[Ch2, Proposition 3.1 and Theorem 3.7]*] A Baker–Akhiezer function $\psi(\lambda, x)$ exists and is unique up to multiplication by a factor depending on λ. As a function of x, ψ is an eigenfunction of the Macdonald operators D^π with $t = q^{-m}$, namely, we have*

$$D^\pi \psi = \mathfrak{m}_\pi(\lambda)\psi\,, \qquad \mathfrak{m}_\pi(\lambda) = \sum_{\tau \in W\pi} q^{\langle \tau, \lambda \rangle}\,.$$

Proof. In cases **a** and **c** this is proved in [Ch2], Sections 3 and 6, respectively (notice that the variables (λ, x) are denoted (x, z) in [Ch2]). The proof in case **b** is the same. The main observation is as follows (cf. [Ch2, Proposition 2.1]). Let $\mathcal{Q} \subset \mathbb{R}[P]$ denote the subspace of all $f(x) \in \mathbb{R}[P]$ that have the same properties (3.4)–(3.6) as ψ. E.g., in case **b** we have that, for every $j = 1, \ldots, m_\alpha$,

$$f\left(x + \frac{1}{2}j\alpha\right) = f\left(x - \frac{1}{2}j\alpha\right) \quad \text{when} \quad q^{\langle \alpha, x \rangle} = 1\,. \tag{3.7}$$

It is easy to see that $\mathcal{Q}$ is in fact a ring. ($\mathcal{Q}$ is a q-analogue of the ring of *quasi-invariants* [CV, FVe].) Then the operators D^π preserve that ring, i.e.

$$D^\pi(\mathcal{Q}) \subseteq \mathcal{Q}\,.$$

Using this fact, one constructs ψ by repeatedly applying D^π to a suitable function, see the proof of Theorem 3.7 in [Ch2]. □

Remark 3.2. For a finite linear combination $f(x) = \sum_{\nu \in V_{\mathbb{R}}} a_\nu q^{\langle \nu, x \rangle}$, we call the **support** of f to be the convex hull of those points $\nu \in V_{\mathbb{R}}$ where $a_\nu \neq 0$. Then the property (3.3) means that, for a fixed λ, the support of ψ is contained in the set $\lambda + \mathcal{N}$. Also it is useful to note that the coefficients ψ_ν in (3.3) are nonzero only if $\nu \in \rho + Q$, cf. [Ch2, Corollary 3.4]. In fact, in the axiomatics for ψ one can replace P by any lattice containing Q and ρ: that would still define the same object.

Let us next impose the following normalization condition on ψ, prescribing its coefficient ψ_ν at one of the vertices of $\mathcal{N}$:

$$\psi_\rho = \Delta'(\lambda)\,, \tag{3.8}$$

where $\Delta' = \Delta_{R', m'}$.

Definition. A **normalized BA function** is the unique function $\psi(\lambda, x)$ with the properties (3.3)–(3.6) and normalization (3.8).

Our choice of normalization is justified by the following result, which in cases **a** and **c** was obtained in [Ch2, Sections 4 and 6].

Theorem 3.3. *The normalized BA function ψ has the following properties:*
(i) for all $w \in W$ the coefficient $\psi_{w\rho}$ in (3.3) *has the form*

$$\psi_{w\rho} = \Delta'(w^{-1}\lambda)\,; \tag{3.9}$$

(ii) $\psi(\lambda, x)$ can be presented in the form

$$\psi = q^{\langle \lambda, x\rangle} \sum_{\nu \in \mathcal{N}, \nu' \in \mathcal{N}'} \psi_{\nu\nu'} q^{\langle \nu, \lambda\rangle} q^{\langle \nu', x\rangle}\,, \tag{3.10}$$

with $\psi_{\nu\nu'} \in \mathbb{Q}(q_\alpha^{1/2})$, where $\mathbb{Q}(q_\alpha^{1/2})$ is the field extension of $\mathbb{Q}$ by all $q_\alpha^{1/2}$;

(iii) We have the following bispectral duality:

$$\psi(\lambda, x) = \psi'(x, \lambda)\,, \tag{3.11}$$

where ψ' is the normalized BA function associated to (R', m').

Proof. In case **a** and **c** this follows from Proposition 4.4 and Theorem 4.7 of [Ch2]. The cases **b** and **c** can be treated similarly (see e.g. Theorem 6.7 of *loc. cit.*). The statement that $\psi_{\nu\nu'} \in \mathbb{Q}(q_\alpha^{1/2})$ is not mentioned in [Ch2], but it follows immediately from the construction of ψ, see e.g. formula (3.16) of *loc. cit.* □

Note that the duality (3.11) implies that $\psi(\lambda, x)$ has the following properties in the λ-variable: for each $\alpha \in R$ (or $\alpha \in R^2$ in case **c**) and $j = 1, \dots, m_\alpha$

$$\psi\left(\lambda + \frac{1}{2}j\alpha, x\right) \equiv \psi\left(\lambda - \frac{1}{2}j\alpha, x\right) \quad \text{for} \quad q^{\langle \alpha', \lambda\rangle} = 1\,, \tag{3.12}$$

and, additionally in case **c**,
(1) for all $0 < s \preccurlyeq (m_1', m_2')$

$$\psi(\lambda - se_i, x) = \psi(\lambda + se_i, x) \qquad \text{for } q^{\lambda_i} = 1\,; \tag{3.13}$$

(2) for all $0 < s \preccurlyeq (m_3', m_4')$

$$\psi(\lambda - se_i, x) = \psi(\lambda + se_i, x) \qquad \text{for } q^{\lambda_i} = -1\,. \tag{3.14}$$

Part (i) of the above theorem, together with uniqueness of ψ, implies the following symmetries of ψ.

Lemma 3.4. The normalized BA function has the following invariance properties:
(i) $\psi(w\lambda, wx) = \psi(\lambda, x)$ for any $w \in W$;
(ii) $\psi(-\lambda, -x) = \psi(\lambda, x)$.

For the proof of the first part, see [Ch2, Lemma 5.4]. Proof of part (ii) is similar. □

Remark 3.5. In the rank-one case $R = A_1$, $\psi(\lambda, x)$ is a very particular case of Baker–Akhiezer functions associated to algebraic curves in the framework of finite-gap theory [N, DMN, Kr1, Kr2]. Namely, to relate it to the setting of [Kr2], one needs to consider a rational singular curve Γ with m double points. Note that in this case one can express ψ in terms of the basic hypergeometric series ${}_2\phi_1(a, b; c; q, z)$, which reduces the properties (3.4) and the statements of Theorem 3.3 to the known identities for ${}_2\phi_1$, see [Ko2]. Other explicit presentation of ψ in the rank-one case also exist [R2, EV1]. In higher rank for $R = A_n$ a function closely related to ψ was constructed in [FVa] via a version of Bethe ansatz, and in [ES] via representation theory of quantum groups.

Remark 3.6. From (3.1), (3.3) and Remark 3.2 it follows that ψ can be presented in the form

$$\psi(\lambda, x) = q^{\langle \lambda+\rho, x\rangle} \sum_{\nu \in Q_-} \Gamma_\nu(\lambda) q^{\langle \nu, x\rangle} . \tag{3.15}$$

Here $Q_- = -Q_+$ and the sum is finite. The leading coefficient Γ_0 can be determined from (3.9) as

$$\Gamma_0 = \Delta'(\lambda) . \tag{3.16}$$

Recall that this ψ is an eigenfunction of Macdonald difference operators with $t = q^{-m}$. For generic t the eigenfunctions are no longer given by finite sums, but rather infinite series of the form (3.15). Such infinite series solutions were studied in [LS], [vMS], [vM]. The fact that for $t = q^{-m}$ with (half-)integer m those series terminate is non-obvious, but it follows from the above results and the uniqueness of the formal series solution, cf. [LS, Proposition 4.13]. Note also that for $t = q^{m+1}$ the series solutions (3.15) are no longer finite, but are in fact still elementary functions. For example, in cases **a** and **b** they are obtained by dividing the BA function ψ by the function δ introduced in (3.21) below.

3.1.1. *Roots of unity.* The proofs of the above results in [Ch2] require q not being a root of unity; this is needed for the proof of the crucial Lemma 3.2 of *loc. cit.* In fact, for given multiplicities m one has to avoid only certain roots of unity. Namely, let us assume that the function Δ defined by (2.17)–(2.20) has simple zeroes, i.e. all the factors are distinct. Explicitly, in case **a** and **b** this means that for all $\alpha \in R$

$$q_\alpha^j \neq 1 \quad \text{for } j = 1, \ldots, m_\alpha - 1 . \tag{3.17}$$

In case **c** our assumption is that

$$q^j \neq 1 \quad \text{for } j = 1, \ldots, m_5 - 1 , \tag{3.18}$$

and that the following numbers are pairwise distinct:

$$q^s \text{ with } 0 < s \preccurlyeq (m_1, m_2) \quad \text{and} \quad -q^s \text{ with } 0 < s \preccurlyeq (m_3, m_4) . \tag{3.19}$$

Proposition 3.7. *With the conditions* (3.17)–(3.19) *the statements of Theorems 3.1, 3.3 remain valid.*

Proof. The conditions (3.4)–(3.6) are equivalent to an overdetermined linear system for coefficients ψ_ν, see the proof of [Ch2, Lemma 3.2]. We know that ψ exists for generic q, therefore, it must exist for any q. Its uniqueness is based on [Ch2, Lemma 3.2] and elementary geometric arguments. Looking at the proof of that lemma in *loc. cit.*, given in case **a**, one sees that it only requires an assumption that $q^j \neq 1$ for $j = 1, \dots, m_\alpha - 1$. In case **b** it should be replaced by (3.17). In case **c** everything is analogous for the roots $\alpha \in R^2$, which gives (3.18). Finally, one needs to look at the corresponding linear system for $\alpha = e_i \in R^1$. In that case one can see, similarly to case **a**, that in the limit $q^{\lambda_i} \to \infty$ this system has the matrix of coefficients being the Vandermonde matrix built from the numbers appearing in (3.19). Therefore, the system has only zero solution provided that these numbers are pairwise distinct. This proves that conditions (3.18)–(3.19) are sufficient for the uniqueness of ψ in case **c**. □

Remark 3.8. If one is interested in eigenfunctions of the difference operators D^π, then the assumptions (3.17)–(3.19) are not very restrictive. Indeed, a quick look at the formula (2.31) for the coefficients of the Macdonald operator in case **b** shows that if q_α is a primitive nth root of unity then m_α can be reduced modulo n as this does not change $t_\alpha = q^{-m_\alpha}$. Therefore, we can always assume that $m_\alpha < n$, and in that case (3.17) is automatic. The situation in cases **a** and **c** is similar. Thus, the Macdonald operators with $q = t^{-m}$ with integral m will always have BA functions $\psi(\lambda, x)$ as their eigenfunctions, for any $q \in \mathbb{C}^\times$. For fixed $t = q^{-m}$ these eigenfunctions are analytic in q provided (3.17)–(3.19).

3.2. **Generalized Weyl formula.** Let us explain, following [Ch2], the relationship between $\psi(\lambda, x)$ and Macdonald polynomials p_λ. Given the normalized BA function $\psi(\lambda, x)$, we consider two functions $\Phi_\pm$ obtained by (anti)symmetrization in λ:

$$\Phi_+(\lambda, x) = \sum_{w \in W} \psi(w\lambda, x)\,, \quad \Phi_-(\lambda, x) = \sum_{w \in W} (-1)^w \psi(w\lambda, x)\,. \tag{3.20}$$

Note that (anti)symmetrization in x would give the same result, due to Lemma 3.4; hence, Φ_+ is W-symmetric in x, and Φ_- is antisymmetric.

Introduce the following function:

$$\delta(x) = \Delta(x)\Delta(-x)\delta_0(x)\,, \tag{3.21}$$

where

$$\delta_0(x) = \prod_{\alpha \in R_+} \left(q^{\langle \alpha, x \rangle/2} - q^{-\langle \alpha, x \rangle/2} \right) .$$

Let $\widetilde{m} = m+1$ denote the shifted parameters $\widetilde{m}_\alpha = m_\alpha + 1$ in cases **a** and **b** and $\widetilde{m}_i := m_i + 1$ in case **c**. Recall the vector ρ (2.10) and let

$$\widetilde{\rho} = \frac{1}{2} \sum_{\alpha \in R_+} (m_\alpha + 1)\alpha \,, \tag{3.22}$$

in all three cases.

Theorem 3.9 (cf. [Ch2, Theorem 5.11])**.** *For $\lambda \in \widetilde{\rho} + P_+$ we have*

$$\Phi_+(\lambda, x) = \Delta'(\lambda)\, p_{\lambda+\rho}(x; q, q^{-m}) \tag{3.23}$$

and

$$\Phi_-(\lambda, x) = \Delta'(\lambda)\delta(x)\, p_{\lambda-\widetilde{\rho}}(x; q, q^{m+1})\,. \tag{3.24}$$

Note that the condition $\lambda \in \widetilde{\rho} + P_+$ ensures that $\Delta'(\lambda) \neq 0$.

In case **a** this is [Ch2, Theorem 5.11]. The same proof works in cases **b** and **c**. □

Remark 3.10. For $m = 0$ $\psi(\lambda, x)$ is simply $q^{\langle \lambda, x\rangle}$ while $p_\lambda(x; q, q)$ are the characters of the corresponding Lie algebra of type R. Thus, for $m = 0$ formula (3.24) turns into the classical Weyl character formula. In case $R = A_n$ formula (3.24) was conjectured by Felder and Varchenko [FVa] and proved by Etingof and Styrkas [ES]. We note that the evaluation and duality identities for p_λ are trivial consequences of this formula and the duality (3.11), see [Ch2, Section 5.5] for the details.

3.3. **Evaluation.** Relation 3.23 gives a well-defined expression for p_λ only if $\lambda \in \widetilde{\rho} + \rho + P_+$, i.e. if λ is sufficiently large. This reflects the fact that for $m_\alpha \in \mathbb{Z}_+$, some of p_λ are not well-defined, cf. [Ch2, Corollary 5.13]. This is also related to the fact that while for generic λ the function $\psi(\lambda, x)$ has the support $\lambda + \mathcal{N}$, for special λ the support becomes smaller. In particular, the support can reduce to a single point, as the following proposition shows.

Proposition 3.11. *For $\lambda = w\rho$, $w \in W$, the normalized BA function $\psi(\lambda, x)$ does not depend on x and is equal to $\Delta'(-\rho) \neq 0$.*

Proof. The vectors $w\rho$ point to the vertices of the polytope $\mathcal{N}$. Each vertex corresponds to a choice of a positive half $R_+ \subset R$, and for any two adjacent vertices λ_1, λ_2 we have $\lambda_2 = s_\alpha(\lambda_1)$ and $\lambda_2 = \lambda_1 - m_\alpha \alpha$ for a suitable $\alpha \in R$. Put $\lambda = \frac{1}{2}(\lambda_1 + \lambda_2)$. Then $\langle \alpha, \lambda\rangle = 0$ so that $q^{\langle \alpha', \lambda\rangle} = 1$; also $\lambda_{1,2} = \lambda \pm \frac{1}{2} m_\alpha \alpha$. Therefore, applying (3.12) with $j = m_\alpha$ gives us that $\psi(\lambda_1, x) = \psi(\lambda_2, x)$ in cases **a**, **b**, or **c** with $\alpha \in R^2$. In the remaining case $\alpha = 2e_i \in 2R^1$ this also works, because in that case we can use (3.13) for $s = m_1'$:

$$\psi(\lambda - m_i' e_i, x) = \psi(\lambda + m_i' e_i, x)\,.$$

According to (2.4), we have $m_i' e_i = \frac{1}{2} m_\alpha \alpha$. Therefore, in that case we also obtain that $\psi(\lambda_1, x) = \psi(\lambda_2, x)$.

So, in all cases we obtain that the functions $\psi(w\rho, x)$ with $w \in W$ are all the same. Each of these functions has support within $w\rho + \mathcal{N}$. Since

$$\cap_{w\in W}\{w\rho + \mathcal{N}\} = \{0\}\,,$$

they all are constants. To evaluate this constant, we need to look at the coefficient $\psi_\rho(-\rho)$ in (3.3), which equals $\Delta'(-\rho)$ by (3.8). The fact that this is nonzero is easy to check. ☐

Remark 3.12. By duality (3.11), we also have $\psi(\lambda, w\rho') = \Delta(-\rho')$ for all $w \in W$. In particular, for $\lambda = \rho$ and $x = \rho'$ this gives

$$\psi(\rho, \rho') = \Delta(-\rho') = \Delta'(-\rho)\,.$$

More generally, we have the following result, which reduces to Proposition 3.11 in the case $\mu = 0$.

Proposition 3.13. *Let $\mu \in P_+$ be such that $\langle\alpha^\vee, \mu\rangle \le m_\alpha$ for every simple root $\alpha \in R_+$, and $\lambda := \mu - \rho$. Then $\psi(\lambda, x) \in \mathbb{R}[P]^W$ and we have*

$$\psi(\lambda, x) = \sum_{\nu\le\mu} a_{\mu\nu}\mathfrak{m}_\nu\,, \qquad a_{\mu\mu} = \Delta'(\lambda) \ne 0\,.$$

Here $\nu \le \mu$ denotes the dominant ordering on P_+.

Proof. For any simple root $\alpha \in R_+$ we have $\langle\alpha^\vee, \lambda\rangle = \langle\alpha^\vee, \mu\rangle - m_\alpha$, therefore, $-m_\alpha \le \langle\alpha^\vee, \lambda\rangle \le 0$. Put $\lambda_1 = \lambda$ and $\lambda_2 = s_\alpha\lambda$, where s_α is the corresponding simple reflection. Then the same argument as above shows that $\psi(\lambda_1, x) = \psi(\lambda_2, x)$, i.e. $\psi(\lambda, x) = \psi(s_\alpha\lambda, x)$. By Lemma 3.4($i$), we have $\psi(\lambda, x) = \psi(\lambda, s_\alpha x)$. Since this applies to every simple reflection, we conclude that $\psi(\lambda, x)$ is W-invariant. The support of $\psi(\lambda, x)$ is contained in $\lambda + \mathcal{N}$, thus ψ must be a combination of orbitsums $\mathfrak{m}_\nu$ with $\nu \le \lambda + \rho = \mu$. The leading coefficient $a_{\mu\mu}$ can be found as $\psi_\rho(\lambda)$, which equals $\Delta'(\lambda)$. Since $\langle\alpha', \lambda\rangle \le 0$ for all simple roots, we have $\Delta'(\lambda) \ne 0$. ☐

Because $\psi(\lambda, x)$ is an eigenfunction of the Macdonald difference operators, the polynomials constructed in the proposition will be symmetric eigenfunctions. As mentioned earlier, in the case $t = q^{-m}$ some of the Macdonald polynomials p_λ do not exist. We have seen that there are two types of Macdonald polynomials that exist for $t = q^{-m}$, namely, p_λ with large λ as in (3.23), or p_μ with small μ as in Proposition 3.13. It is interesting to note that, as the formula (4.1) below shows, p_λ's have positive norms, while the norms of p_μ are all zero.

Remark 3.14. The result of Proposition 3.11 can be viewed as a counterpart of the evaluation formula for p_λ. Indeed, let us substitute $x = -\rho'$ into (3.23). Using Remark 3.12, we get that $\psi(w\lambda, -\rho') = \Delta'(-\rho)$ for all $w \in W$. As a result, (3.23) gives us that

$$|W|\Delta'(-\rho) = \Delta'(\lambda)\, p_{\lambda+\rho}(-\rho'; q, q^{-m})\,,$$

provided λ sufficiently large so that $p_{\lambda+\rho}$ are well-defined. Denoting $\mu = \lambda+\rho$ and using the notation ρ_m, ρ'_m for vectors (2.10), we get

$$p_\mu(\rho'_{-m}; q, q^{-m}) = |W| \frac{\Delta'(\rho_{-m})}{\Delta'(\mu + \rho_{-m})} \,. \tag{3.25}$$

This should be compared with the evaluation identity for p_λ, see [C2] in cases **a**, **b**, or [M2] for all three cases. In fact, formula (3.25) can be obtained from the formula [M2, (5.3.12)] for $p_\lambda(\rho'_k; q, q^k)$ by analytic continuation in k, assuming the existence of $p_\lambda(x; q, q^{-m})$.

4. Orthogonality relations for BA functions

Let us say that $\xi \in V_{\mathbb{R}}$ is **big** if $|\langle \alpha, \xi\rangle| \gg 1$ for all $\alpha \in R$; more precisely, we will require that

$$|\langle \alpha^\vee, \xi\rangle| > m_\alpha \quad \text{for all } \alpha \in R \,. \tag{4.1}$$

Let $C_\xi = \xi + iV_{\mathbb{R}}$ be the imaginary subspace in $V_{\mathbb{C}}$; it is invariant under translations by $\kappa Q^\vee$. Let dx denote the translation invariant measure on C_ξ normalized by the condition

$$\int_{C_\xi/\kappa Q^\vee} dx = 1 \,.$$

Theorem 4.1. *For any $\lambda, \mu \in V_{\mathbb{R}}$ with $\lambda - \mu \in P$ and any big $\xi \in V_{\mathbb{R}}$ we have*

$$\int_{C_\xi/\kappa Q^\vee} \frac{\psi(\lambda, x)\psi(\mu, -x)}{\Delta(x)\Delta(-x)}\, dx = \delta_{\lambda,\mu} (-1)^M \Delta'(\lambda)\Delta'(-\lambda) \,, \tag{4.2}$$

where $\delta_{\lambda,\mu}$ is the Kronecker delta and $M = \sum_{\alpha\in R_+} m_\alpha$.

Proof. The condition $\lambda - \mu \in P$ guarantees that $\psi(\lambda, x)\psi(\mu, -x)$ is periodic in x with respect to the lattice $\kappa Q^\vee$, thus, the integral is well-defined. The proof of the theorem rests on the following result.

Proposition 4.2 (cf. [EV2, Theorem 5.1])**.** *Let $I(\xi)$ denote the integral in the left-hand side of* (4.2)*. Then $I(\xi)$ does not depend on ξ provided it is big in the sense of* (4.1)*.*

The proof of the proposition occupies the next section. Assuming it, we can evaluate the integral by taking the limit $\xi \to \infty$ in a suitable Weyl chamber. Indeed, let us assume that ξ stays deep inside the negative Weyl chamber, i.e. $\langle \alpha, \xi\rangle \ll 0$ for every $\alpha \in R_+$. In that case

$$\mathrm{Re}\langle \alpha, x\rangle = \langle \alpha, \xi\rangle \ll 0 \quad \text{for any } x \in \xi + iV_{\mathbb{R}} \,,$$

hence $\left|q^{-\langle \alpha, x\rangle}\right| \ll 1$. The properties (3.15)–(3.16) give us the asymptotic behaviour of ψ for $x \in C_\xi$ as $\xi \to \infty$ inside the negative Weyl chamber:

$$\psi(\lambda, x) \sim \Delta'(\lambda) q^{\langle \lambda+\rho, x\rangle} \quad \text{and}$$

$$\psi(\mu, -x) = \psi(-\mu, x) \sim \Delta'(-\mu) q^{\langle -\mu+\rho, x\rangle} \,.$$

For those x we also have

$$\Delta(x)\Delta(-x) \sim (-1)^{\sum_{\alpha\in R_+} m_\alpha} q^{2\langle\rho,x\rangle}\,.$$

As a result, the asymptotic value of the integrand is

$$(-1)^M \Delta'(\lambda)\Delta'(-\mu) q^{\langle\lambda-\mu,x\rangle}\,.$$

In the case $\mu=\lambda$ this immediately leads to (4.2). On the other hand, when $\mu-\lambda$ is dominant the integrand tends to zero as $\xi\to\infty$ in the negative chamber, thus the integral must vanish. Finally, by switching to another Weyl chamber one obtains the same result in the general case. □

4.1. **Proof of Proposition 4.2.** The proof is parallel to the proof of [EV2, Theorem 5.1]. Let us first demonstrate the idea in the rank-one case of $R=A_1=\{\alpha,-\alpha\}\subset\mathbb{R}$, $Q=\mathbb{Z}\alpha$, $P=\frac12 Q$. In that case the integrand in (4.2) is a meromorphic function of a single complex variable $x\in\mathbb{C}$, periodic with the period $\kappa\alpha^\vee$; we denote the integrand as $F(x)$. Thus, we have

$$I(\xi)=\int_\xi^{\xi+\kappa\alpha^\vee} F(x)\,dx\,.$$

To prove that $I(\xi)=I(\xi')$, we need to look at the residues of F between the lines $\mathrm{Re}(x)=\xi$ and $\mathrm{Re}(x)=\xi'$. The integrand has simple poles at points where $q^{\langle\alpha,x\rangle}=q_\alpha^{\pm j}$ with $j=1,2,\dots,m_\alpha$. These poles are naturally organized in groups, with $2m_\alpha$ poles in each group. Namely, for any y such that $q^{\langle\alpha,y\rangle}=1$, we have $2m_\alpha$ poles of F at

$$x=y_{\pm j}:=y\pm\frac12 j\alpha' \quad\text{with}\quad j=1,\dots,m_\alpha\,. \tag{4.3}$$

The requirement that ξ is big is equivalent to saying that these poles lie on one side of the line $\mathrm{Re}(x)=\xi$. We need to check that $I(\xi)=I(-\xi')$ for $\xi,\xi'\gg 0$. For that it is sufficient to check that the sum of the residues of F at the points (4.3) equals zero.

From (3.4) we have

$$\psi(\lambda,y_{-j})=\psi(\lambda,y_j)\qquad\forall\ j=1,2,\dots,m_\alpha\,,$$

and the same for $\psi(\mu,x)$. Also, it is clear that $\Delta(x)\Delta(-x)$ is invariant under the group $\{\pm1\}\ltimes\kappa\mathbb{Z}\alpha^\vee$, which is isomorphic to the affine Weyl group of $R=A_1$. From that it easily follows that

$$\mathrm{res}_{x=y_{-j}} F(x) = -\mathrm{res}_{x=y_j} F(x)\qquad\forall\ j=1,2,\dots,m_\alpha\,.$$

Thus, the sum of the residues is indeed zero, and we are done.

The higher rank case is similar. We will give a proof for cases **a** and **b**; case **c** is similar. Since later we will deal with deformed root systems (see Section 7), we will make most of our arguments independent of the properties of root systems. We will only assume that the lattices $P,Q^\vee\subset V_\mathbb{R}$ have full rank, with $R\subset P$ and $Q^\vee$ contained in the dual to P, i.e. with $\langle P,Q^\vee\rangle\subset\mathbb{Z}$.

The hyperplanes $\langle \alpha, x\rangle = 0$ with $\alpha \in R$ separate $V_{\mathbb{R}}$ into several connected regions (chambers). Clearly, $I(\xi)$ does not change when ξ stays within a particular chamber while remaining big. To show that the value of the integral is the same for every chamber, it is enough to check that $I(\xi) = I(\xi')$ when ξ and ξ' belong to adjacent chambers. Suppose that the two chambers are separated by the hyperplane $\langle \alpha, x\rangle = 0$ for some $\alpha \in R$. Without loss of generality, we may assume that $\xi' = s_\alpha \xi$, with $\langle \alpha^\vee, \xi\rangle > m_\alpha$. Moreover, we can move ξ and ξ' inside the chambers to achieve that

$$|\langle \beta, \xi\rangle| \gg |\langle \alpha, \xi\rangle| \quad \text{for all } \beta \neq \pm\alpha \text{ in } R\,, \tag{4.4}$$

and the same for ξ'.

The integral over $C_\xi/\kappa Q^\vee$ can be computed by integrating over any (bounded, measurable) fundamental region for the action of $\kappa Q^\vee$ on C_ξ. For example, we can choose a basis $\{\epsilon_1, \dots, \epsilon_n\}$ of $Q^\vee$ and integrate over the set of $x \in V_{\mathbb{C}}$ of the form

$$x(t_1, \dots, t_n) = \xi + \kappa \sum_{i=1}^n t_i \epsilon_i\,, \qquad t_i \in (0,1)\,. \tag{4.5}$$

Moreover, one can replace ϵ_i by $\epsilon_i' = \sum a_{ij}\epsilon_j$ where the matrix $A = (a_{ij})$ is upper-triangular with $a_{ii} = 1$: it is easy to see that the set (4.5) for $\{\epsilon_i'\}$ will still be a fundamental region. (Note that the entries of A do not have to be integers, so ϵ_i' may not belong to $Q^\vee$.) Using this, we can change the direction of ϵ_1 arbitrarily; we will assume that ϵ_1 is parallel to the above α.

Up to an irrelevant constant factor we have $dx = dt_1 \dots dt_n$ and

$$I(\xi) = \int F(x)\, dt_1 \dots dt_n\,, \qquad x = x(t_1, \dots, t_n)\,,$$

with

$$F(x) = \frac{\psi(\lambda, x)\psi(\mu, -x)}{\Delta(x)\Delta(-x)}\,. \tag{4.6}$$

For $I(\xi')$ we have a similar formula

$$I(\xi') = \int F(x')\, dt_1 \dots dt_n\,, \quad x'(t_1, \dots, t_n) = \xi' + \kappa \sum_{i=1}^n t_i \epsilon_i\,.$$

Both integrals can be computed by repeated integration. Therefore, to prove that $I(\xi) = I(\xi')$ it suffices to check that for any $t_2, \dots, t_n \in \mathbb{R}$ we have

$$\int_0^1 F(x)\, dt_1 = \int_0^1 F(x')\, dt_1\,. \tag{4.7}$$

Since ϵ_1 is parallel to α, the variable x in the first integral moves in the direction of $\kappa\alpha^\vee$ through the point

$$y = \xi + \kappa \sum_{i=2}^n t_i \epsilon_i\,.$$

Similarly, x' in the second integral moves in the same direction through the point

$$y' = \xi' + \kappa \sum_{i=2}^{n} t_i \epsilon_i \,.$$

Since $y - y' = \xi - \xi' = \xi - s_\alpha \xi = \langle \alpha^\vee, \xi \rangle \alpha$, the integration takes place along two parallel lines in the complex plane $\{y + z\alpha' \mid z \in \mathbb{C}\}$, which makes the situation similar to the rank-one case above. Namely, if we denote by L and L' the above two lines through y and y' then the relation (4.7) is equivalent to

$$\int_{L/\kappa\mathbb{Z}\alpha^\vee} F(y + z\alpha')\, dz = \int_{L'/\kappa\mathbb{Z}\alpha^\vee} F(y + z\alpha')\, dz \,. \tag{4.8}$$

We therefore need to look at the poles of $F(y + z\alpha')$ as a function of $z \in \mathbb{C}$. The poles between L and L' are those where

$$q^{\langle \alpha, y + z\alpha' \pm \frac{1}{2} j \alpha' \rangle} = 1 \quad \text{with } j = 1, 2, \dots, m_\alpha \,. \tag{4.9}$$

Other factors in $\Delta(x)\Delta(-x)$ will not contribute because of the assumption (4.4) and the fact that $y \in \xi + iV_\mathbb{R}$.

Similarly to the rank-one case, the poles (4.9) are organized into groups with $2m_\alpha$ poles in each group. Namely, by a suitable shift in the z-variable, we can always make $q^{\langle \alpha, y \rangle} = 1$ in such a way that the poles (4.9) will correspond to $z = \pm \frac{1}{2} j$ with $j = 1, \dots, m_\alpha$. Now everything boils down to the following property of the integrand (4.6).

Lemma 4.3. For any $x \in V_\mathbb{C}$ with $q^{\langle \alpha, x \rangle} = 1$ and for all $j = 1, \dots, m_\alpha$ we have

$$\mathrm{res}_{z=-j/2}\, f(z) + \mathrm{res}_{z=j/2} f(z) = 0 \,, \quad \text{where } f(z) := F(x + z\alpha') \,. \tag{4.10}$$

The lemma can be proved in the same manner as in the rank-one case, by using the properties (3.12) and the invariance of $\Delta(x)\Delta(-x)$ under the group $W \ltimes \kappa Q^\vee$. □

Using the lemma, we conclude that the relation (4.8) is valid, and this finishes the proof of Proposition 4.2. □

Remark 4.4. The proof of Lemma 4.3 uses in an essential way the W-invariance of $\Delta(x)\Delta(-x)$, so it has to be modified in the case of deformed root systems, cf. Lemma 7.3 below.

4.2. **Norm identity for Macdonald polynomials.** Let us keep the notation of section 3.2. We can use Theorems 3.9 and 4.1 to easily compute the norms of polynomials $p_\lambda(x; q, t)$. Namely, take $\widetilde{\lambda} = \widetilde{\rho} + \lambda$ with $\lambda \in P_+$, and consider the function $\Phi_-(\widetilde{\lambda}, x)$ as defined in (3.20). Then we can use Theorem 4.2 to compute the integral

$$\int_{C_\xi/\kappa Q^\vee} \frac{\Phi_-(\widetilde{\lambda}, x)\Phi_-(\widetilde{\lambda}, -x)}{\Delta(x)\Delta(-x)}\, dx \,.$$

Indeed, expanding Φ_- in terms of ψ's and using the fact that $w\widetilde{\lambda} = w'\widetilde{\lambda}$ only when $w = w'$, we obtain that the integral equals

$$\sum_{w\in W}(-1)^M\Delta'(w\widetilde{\lambda})\Delta'(-w\widetilde{\lambda}) = |W|(-1)^M\Delta'(\widetilde{\lambda})\Delta'(-\widetilde{\lambda})\,.$$

(Here we used the W-invariance of $\Delta'(\lambda)\Delta'(-\lambda)$.)

According to (3.24), we have

$$\Phi_-(\widetilde{\lambda},x) = (-1)^M\Delta'(\widetilde{\lambda})\delta(x)\,p_\lambda(x)\,,\qquad p_\lambda(x) = p_\lambda(x;q,q^{m+1})\,.$$

Substituting this into the integral gives:

$$\int_{C_\xi/\kappa Q^\vee} p_\lambda(x)p_\lambda(-x)\frac{\delta(x)\delta(-x)}{\Delta(x)\Delta(-x)}\,dx = |W|(-1)^M\frac{\Delta'(-\widetilde{\lambda})}{\Delta'(\widetilde{\lambda})}\,.$$

Now it is easy to check that

$$\frac{\delta(x)\delta(-x)}{\Delta(x)\Delta(-x)} = C^{-1}(-1)^{|R_+|}\nabla(x;q,q^{m+1})\,, \tag{4.11}$$

where δ is as in 3.2 and C is the constant (2.22)–(2.23).

As a result, we obtain that

$$\int_{C_\xi/\kappa Q^\vee} p_\lambda(x)p_\lambda(-x)\nabla(x)\,dx = C(-1)^{\widetilde{M}}|W|\frac{\Delta'(-\lambda-\widetilde{\rho})}{\Delta'(\lambda+\widetilde{\rho})}\,,$$

where we used $\widetilde{M} := \sum_{\alpha\in R_+}(m_\alpha+1)$.

Since now the integrand has no poles, we can shift the cycle C_ξ back to $iV_\mathbb{R}$, so the left-hand side becomes the Macdonald scalar product $\langle p_\lambda, p_\lambda\rangle$. This leads to the formula for the norms of $p_\lambda(x;q,t)$ in the case $t = q^{m+1}$, cf. [C1, M2]. ☐

Remark 4.5. Note that the above proof of the norm identity does not use shift operators or an inductive step from m to $m+1$. In that respect it is very different from other known proofs that use the idea going back to [O].

4.3. **The case of $|q| = 1$.** The relations (4.2) and their proof remain true for $q \in \mathbb{C}^\times$ with $|q| \neq 1$. In that case one still uses $C_\xi = \xi + \kappa V_\mathbb{R}$ with κ given by (2.1). Moreover, a similar result is true for $|q| = 1$ when $\kappa \in \mathbb{R}$. In that case we know that the BA function ψ exists and is analytic in q provided (3.17)–(3.19). Then we have the following analogue of Theorem 4.1.

Theorem 4.6. *Assume that $|q| = 1$ and conditions (3.17)–(3.19) are satisfied. Put $C_\xi = i\xi + V_\mathbb{R}$ with $\xi \in V_\mathbb{R}$, assuming ξ is regular, i.e. $\langle\alpha,\xi\rangle \neq 0$ for all $\alpha \in R$. Then for such C_ξ and $\lambda,\mu \in V_\mathbb{R}$ with $\lambda-\mu \in P$, the relations (4.2) remain valid.*

For generic q on the unit circle this is proved similarly to Theorem 4.1. Namely, due to a cancelation of residues the integral does not depend on ξ (provided it stays regular), after which the integral is evaluated by letting $\xi \to \infty$. For non-generic q such that (3.17)–(3.19) are satisfied, the integrand depends analytically on q, so the result survives when q approaches those values. $\square$

5. Cherednik–Macdonald–Mehta integral

Throughout this section $0 < q < 1$ and $\psi(\lambda, x)$ is the normalized BA function of type **b** associated to (R, m). Recall that in this case we have $(R', m') = (R, m)$, so $\psi(\lambda, x) = \psi(x, \lambda)$ and $\Delta' = \Delta$, where Δ is given by (2.17) with $q_\alpha = q^{\langle\alpha,\alpha\rangle/2}$.

Let dx be the translation invariant measure on $C_\xi = \xi + iV_{\mathbb{R}}$, normalized by the condition

$$\int_{C_\xi} q^{-|x|^2/2} dx = 1\,, \qquad |x|^2 := \langle x, x\rangle\,.$$

(Note that $|x|^2 < 0$ for $x \in iV_{\mathbb{R}}$.)

Our goal is to prove the following integral identity.

Theorem 5.1. *For any $\lambda, \mu \in V_{\mathbb{C}}$ and any big $\xi \in V_{\mathbb{R}}$ we have*

$$\int_{C_\xi} \frac{\psi(\lambda, x)\psi(\mu, x)}{\Delta(x)\Delta(-x)} q^{-|x|^2/2}\, dx = (-1)^M C^{-1/2} q^{(|\lambda|^2+|\mu|^2)/2} \psi(\lambda, \mu)\,, \tag{5.1}$$

where C is the constant (2.22) *and $M = \sum_{\alpha\in R_+} m_\alpha$.*

The proof of the theorem will be based on the following proposition, similar to Proposition 4.2.

Proposition 5.2. *Let $I(\xi)$ denote the integral in the left-hand side of* (5.1)*. Then $I(\xi)$ does not depend on ξ provided ξ remains big in the sense of* (4.1)*.*

Note that in this case we integrate over a non-compact cycle, but the integral converges absolutely due to the rapidly decaying factor $q^{-|x|^2/2}$. The proposition can be proved by looking at the residues of the integrand in (5.1) given by

$$G(x) = \frac{\psi(\lambda, x)\psi(\mu, x)}{\Delta(x)\Delta(-x)} q^{-|x|^2/2}\,.$$

Without the factor $q^{-|x|^2/2}$ we would have a cancelation of the residues as in Lemma 4.3. Now, the crucial fact is that the function $g(x) = q^{-|x|^2/2}$ satisfies conditions (3.7). Indeed, we have for $j \in \mathbb{Z}$ that

$$g(x - \frac{1}{2}j\alpha)/g(x + \frac{1}{2}j\alpha) = q^{j\langle\alpha,x\rangle} = 1 \qquad \text{for } q^{\langle\alpha,x\rangle} = 1\,. \tag{5.2}$$

As a result, the same cancelation of the residues as in Lemma 4.3 also takes place for G, and the rest of the proof remains the same. $\square$

Before proving the theorem, let us mention a 'compact' version of the integral (5.1). Let $\theta(x)$ denote the theta-function associated with the lattice P:

$$\theta(x) = \sum_{\gamma \in P} q^{\langle \gamma, x \rangle} q^{|\gamma|^2/2} . \tag{5.3}$$

We have the following standard fact (see e.g. [EV2, Lemma 4.3]):

Lemma 5.3. If $f(x)$ is a smooth function on C_ξ, which is periodic with respect to the lattice $\kappa Q^\vee$, then

$$\int_{C_\xi} f(x) q^{-|x|^2/2}\, dx = \int_{C_\xi/\kappa Q^\vee} f(x)\theta(x)\, dx .$$

□

When $\lambda + \mu \in P$, the product $\psi(\lambda, x)\psi(\mu, x)$ is $\kappa Q^\vee$-periodic. In that case we can reformulate Theorem 5.1 in the following way.

Theorem 5.4. *If $\xi \in V_{\mathbb{R}}$ is big and $\lambda + \mu \in P$, then*

$$\int_{C_\xi/\kappa Q^\vee} \frac{\psi(\lambda, x)\psi(\mu, x)}{\Delta(x)\Delta(-x)}\, \theta(x)\, dx = (-1)^M C^{1/2} q^{(|\lambda|^2 + |\mu|^2)/2} \psi(\lambda, \mu) .$$

5.1. **Proof of Theorem 5.1.** Let us first assume that ξ belongs to the negative Weyl chamber, i.e. $\langle \alpha, \xi \rangle \ll 0$ for $\alpha \in R_+$. The denominator in (5) can be presented as

$$\Delta(x)\Delta(-x) = (-1)^M q^{2\langle \rho, x \rangle} \prod_{\alpha \in R_+} \prod_{j=\pm 1}^{\pm m_\alpha} \left(1 - q_\alpha^j q^{-\langle \alpha, x \rangle}\right) .$$

For $x \in \xi + iV_{\mathbb{R}}$ we have $\mathrm{Re}\langle \alpha, x \rangle = \langle \alpha, \xi \rangle \ll 0$ and $\left| q^{-\langle \alpha, x \rangle} \right| \ll 1$ for $\alpha \in R_+$. Therefore, we can expand each of the factors $(1 - q_\alpha^j q^{-\langle \alpha, x \rangle})^{-1}$ into a geometric series and obtain that

$$[\Delta(x)\Delta(-x)]^{-1} = q^{-2\langle \rho, x \rangle} \sum_{\gamma \in Q_-} a_\gamma q^{\langle \gamma, x \rangle} , \quad a_0 = (-1)^M . \tag{5.4}$$

The series converges uniformly and absolutely on C_ξ provided that ξ lies deep inside the negative Weyl chamber. Using Remark (3.6), we can expand the function

$$F(x) = \frac{\psi(\lambda, x)\psi(\mu, x)}{\Delta(x)\Delta(-x)}$$

into a similar convergent series:

$$F(x) = q^{\langle \lambda + \mu, x \rangle} \sum_{\gamma \in Q_-} f_\gamma q^{\langle \gamma, x \rangle} , \qquad f_0 = (-1)^M \Delta(\lambda)\Delta(\mu) . \tag{5.5}$$

All the coefficients f_γ in the series are functions of λ and μ of the form:

$$f_\gamma = \sum_{\nu, \nu' \in \mathcal{N}} a_{\gamma; \nu, \nu'} q^{\langle \nu, \lambda \rangle} q^{\langle \nu', \mu \rangle} \tag{5.6}$$

with suitable coefficients $a_{\gamma;\nu,\nu'}$, and with the summation taken over all $\nu, \nu' \in P$ lying inside the polytope (3.1) (this is immediate from (3.3)).

Note that the coefficients a_γ in (5.4) and, as a consequence, $a_{\gamma;\nu,\nu'}$ in (5.6) have moderate ('exponentially linear') growth, namely,

$$|a_\gamma| < Aq^{\langle u,\gamma\rangle} \quad \text{and} \quad |a_{\gamma;\nu,\nu'}| < A'q^{\langle u',\gamma\rangle} \quad \text{for all } \gamma, \nu, \nu'\,, \tag{5.7}$$

for suitable constants A, A' and vectors $u, u' \in V_{\mathbb{R}}$.

Substituting the series (5.5) into (5.1) and integrating termwise, we obtain a series expansion for the integral (5.1) as follows:

$$I(\xi) = \sum_{\gamma\in Q_-} f_\gamma \int_{C_\xi} q^{\langle\lambda+\mu+\gamma,x\rangle} q^{-|x|^2/2}\,dx = \sum_{\gamma\in Q_-} f_\gamma q^{|\lambda+\mu+\gamma|^2/2} = q^{|\lambda+\mu|^2/2} \sum_{\gamma\in Q_-} f_\gamma q^{\langle\gamma,\lambda+\mu\rangle} q^{|\gamma|^2/2}\,. \tag{5.8}$$

Let us view now this expression as a function of λ. Since each of the coefficients f_γ, as a function of λ, is a polynomial in $\mathbb{R}[P]$ whose exponents spread over the polytope $\mathcal{N}$, we have that

$$I(\xi) = q^{|\lambda+\mu|^2/2} \sum_{\gamma\in\rho+Q_-} g_\gamma q^{\langle\gamma,\lambda\rangle}\,, \tag{5.9}$$

with some coefficients g_γ that depend on μ. It follows from (5.5) that

$$g_\rho = (-1)^M \Delta(\mu) \prod_{\alpha\in R_+} q_\alpha^{-m_\alpha(m_\alpha+1)/4}\,. \tag{5.10}$$

From the way the expression (5.9) was obtained, it is clear that each g_γ is a finite combination of the terms $a_{\gamma';\nu,\nu'} q^{\langle\gamma'+\nu',\mu\rangle} q^{|\gamma'|^2/2}$ with $\gamma' \in \gamma + \mathcal{N}$. Since we are keeping μ fixed, we can use (5.7) to obtain an estimate for g_γ:

$$|g_\gamma| < Bq^{\langle v,\gamma\rangle} q^{|\gamma|^2/2} \quad \text{for all } \gamma\,, \tag{5.11}$$

with a suitable constant B and $v \in V_{\mathbb{R}}$.

It follows that the coefficients g_γ are fast decreasing as $|\gamma| \to \infty$, therefore, the series (5.9) defines an analytic function of λ of the form

$$I(\xi) = q^{|\lambda+\mu|^2/2} \sum_{\gamma\in P} g_\gamma q^{\langle\gamma,\lambda\rangle}\,, \tag{5.12}$$

where $g_\gamma = 0$ unless $\gamma \in \rho + Q_-$. Note that presentation of $I(\xi)$ in the form (5.12) is unique, as it comes from the Fourier series of $g(\lambda) = I(\xi)q^{-|\lambda+\mu|^2/2}$ on the torus $T = iV_{\mathbb{R}}/\kappa Q^\vee$.

We arrive at the conclusion that for ξ deep in the negative Weyl chamber, $I(\xi)$ is given by the series (5.12), where $g_\gamma = 0$ unless $\gamma \in \rho + Q_-$. If we apply the same arguments for, say, ξ' in the positive Weyl chamber, we would get a similar series for $I(\xi')$, but with nonzero Fourier coefficients only for $\gamma \in -\rho + Q_+$. Since $I(\xi) = I(\xi')$, we conclude that the two series coincide and, therefore, have only a finite number of terms. Moreover, by moving ξ to various Weyl chambers, we conclude that

$$I(\xi) = q^{|\lambda+\mu|^2/2} \sum_{\gamma \in P \cap \mathcal{N}} g_\gamma q^{\langle \gamma, \lambda \rangle} ,$$

where $\mathcal{N}$ is the polytope (3.1).

As a function of λ, $I(\xi)$ inherits from $\psi(\lambda, x)$ the properties (3.12). The multiplication by $q^{|\lambda|^2/2}$ does not affect these properties (see (5.2)). Thus, the function $I(\xi)q^{-(|\lambda|^2+|\mu|^2)/2}$ satisfies (3.3) and (3.4) (with (λ, μ) taking place of (x, λ)). By Theorem 3.1, these properties characterize ψ uniquely up to a factor depending on the second variable. Hence,

$$I(\xi)q^{-(|\lambda|^2+|\mu|^2)/2} = C(\mu)\psi(\lambda, \mu) , \quad \text{for some } C(\mu) .$$

Comparing (3.8) and (5.10), we conclude that $C(\mu) = (-1)^M C^{-1/2}$, as needed. This finishes the proof of the theorem. □

5.2. **Integral transforms.** In this section $\psi(\lambda, x)$ is the normalized BA function in any of the cases **a**, **b** or **c**. Let us introduce

$$F(\lambda, x) = \frac{\psi(\lambda, -x)}{\Delta'(\lambda)\Delta(x)} . \tag{5.13}$$

Note that $F(x, \lambda) = F'(\lambda, x)$, where F' is the counterpart of F for the dual data (R', m'). In particular, in case **b** we have

$$F(\lambda, x) = \frac{\psi(\lambda, -x)}{\Delta(\lambda)\Delta(x)} , \qquad F(\lambda, x) = F(x, \lambda) .$$

The relations (4.2) can be rewritten as

$$\int_{C_\xi/\kappa Q^\vee} F(\lambda, -x)F(\mu, x)\, dx = \delta_{\lambda,\mu} Q^{-1}(\lambda) , \qquad \text{where}$$

$$Q(\lambda) = (-1)^M \frac{\Delta'(\lambda)}{\Delta'(-\lambda)} . \tag{5.14}$$

This makes them look similar to [EV2, Theorem 2.2].

The formula (5.1) in case **b**, when written in terms of $F(\lambda, x)$, is equivalent to

$$\int_{C_\xi} F(\lambda, -x)F(\mu, x)q^{-|x|^2/2}\, dx = (-1)^M C^{-1/2} q^{(|\lambda|^2+|\mu|^2)/2} F(\lambda, \mu) , \tag{5.15}$$

where C is the constant (2.22) (cf. [EV2, Theorem 2.3]).

We can use functions (5.13) to define Fourier transforms, following the approach of [EV2]. Since the proofs repeat verbatim those in *loc. cit.*, we will only formulate the results, referring the reader to the above paper for the details.

For $\xi, \eta \in V_\mathbb{R}$ consider the imaginary subspace $C_\xi = \xi + iV_\mathbb{R}$ and the real subspace $D_\eta = i\eta + V_\mathbb{R}$. Let $\mathcal{S}(C_\xi)$ and $\mathcal{S}(D_\eta)$ be the Schwartz spaces of functions on C_ξ and D_η respectively. Introduce the spaces $\mathcal{S}_\eta(C_\xi) = \{\phi :$

$C_\xi \to \mathbb{C} \,|\, q^{2i\langle \eta, x\rangle}\phi(x) \in \mathcal{S}(C_\xi)\}$ and $\mathcal{S}_\xi(D_\eta) = \{\phi : D_\eta \to \mathbb{C} \,|\, q^{-2\langle \xi,\lambda\rangle}\phi(\lambda) \in \mathcal{S}(D_\eta)\}$. Obviously, these spaces are canonically isomorphic to $\mathcal{S}(C_\xi)$ and $\mathcal{S}(D_\eta)$. The modified Fourier transform $f(x) \mapsto \hat{f}(\lambda) := \int_{C_\xi} q^{2\langle \lambda, x\rangle} f(x)\,dx$ defines an isomorphism $\mathcal{S}_\eta(C_\xi) \to \mathcal{S}_\xi(D_\eta)$. The inverse transform $\hat{f}(\lambda) \mapsto f(x)$ is given by the formula $f(x) = \int_{D_\eta} q^{-2\langle \lambda, x\rangle}\hat{f}(\lambda)\,d\lambda$. This fixes uniquely a normalization of the Lebesgue measure $d\lambda$ on D_η, which will be used from now on.

Consider two integral transformations

$$K_{\mathrm{Im}} : \mathcal{S}_\eta(C_\xi) \to \mathcal{S}_\xi(D_\eta)\,, \quad f(x) \mapsto \int_{C_\xi} F(\lambda, -x) f(x)\,dx\,,$$

and

$$K_{\mathrm{Re}} : \mathcal{S}_\xi(D_\eta) \to \mathcal{S}_\eta(C_\xi)\,, \quad f(\lambda) \mapsto \int_{D_\eta} F(\lambda, x) Q(\lambda) f(\lambda)\,d\lambda\,,$$

where Q is given by (5.14).

Theorem 5.5 (cf. [EV2, Theorem 2.4])**.** *Assume that $\xi \in V_\mathbb{R}$ is big and $\eta \in V_\mathbb{R}$ is regular in a sense that $\Delta'(\lambda)\Delta'(-\lambda)$ is non-vanishing on D_η. Then the integral transforms are well defined, continuous in the Schwartz topology, and are inverse to each other,*

$$K_{\mathrm{Im}}\, K_{\mathrm{Re}} = Id\,, \quad K_{\mathrm{Re}}\, K_{\mathrm{Im}} = Id\,.$$

5.3. **Cherednik–Macdonald–Mehta integral over real cycle.** In case **b**, we can use Theorem 5.5 to derive a 'real' counterpart of Theorem 5.1, similarly to [EV2]. Namely, formula (5.15) says that for a fixed generic μ one has

$$K_{\mathrm{Im}}\left(F(\mu, x) q^{-(|x|^2+|\mu|^2)/2}\right) = (-1)^M C^{-1/2} q^{|\lambda|^2/2} F(\lambda, \mu)\,.$$

Applying K_{Re} to both sides, we obtain

$$F(\mu, x) q^{-(|x|^2+|\mu|^2)/2} = (-1)^M C^{-1/2} \int_{D_\eta} F(\lambda, x) F(\lambda, \mu) Q(\lambda) q^{|\lambda|^2/2}\,d\lambda\,.$$

Expressing everything back in terms of ψ, we obtain

$$\int_{D_\eta} \frac{\psi(\lambda, -x)\psi(\lambda, -\mu)}{\Delta(\lambda)\Delta(-\lambda)} q^{|\lambda|^2/2}\,d\lambda = C^{1/2}\psi(\mu, -x) q^{-(|x|^2+|\mu|^2)/2}\,.$$

In the derivation of this formula we assumed that $x \in C_\xi$ and μ is generic. However, since both sides are obviously analytic in μ and x, the formula remains valid for all $\mu, x \in V_\mathbb{C}$. After rearranging and using that $\psi(\lambda, x) = \psi(x, \lambda)$, we get the following result.

Theorem 5.6 (cf. [EV2, Theorem 2.6])**.** *Let $D_\eta = i\eta + V_\mathbb{R}$ with η regular in the sense of Theorem 5.5. Then for any $\mu, \nu \in V_\mathbb{C}$ we have*

$$\int_{D_\eta} \frac{\psi(\mu, \lambda)\psi(\nu, -\lambda)}{\Delta(\lambda)\Delta(-\lambda)}\, q^{|\lambda|^2/2}\,d\lambda = C^{1/2} q^{-(|\nu|^2+|\mu|^2)/2}\psi(\mu, \nu)\,,$$

where C is the constant (2.22).

5.4. **Symmetric version.** Similarly to Section 4.2, we can use the generalized Weyl formula to derive the analogues of Theorems 5.1, 5.4, 5.6 for Macdonald polynomials p_λ in case **b**. This gives a simple proof of the identities proved by Cherednik in [C3] using the double affine Hecke algebras.

Let p_λ and ∇ denote the Macdonald polynomials and weight function, respectively, in case **b** with $t = q^{m+1}$. For $\lambda, \mu \in P_+$ let us put $\widetilde{\lambda} = \lambda + \widetilde{\rho}$, $\widetilde{\mu} = \mu + \widetilde{\rho}$ in the notations of Sections 3.2 and 4.2. Also, put

$$\widetilde{\Delta}(x) := C\delta(x)/\Delta(x) = C\Delta(-x)\delta_0(x)\,, \tag{5.16}$$

where C is the constant (2.22).

Theorem 5.7 (cf. [C3, Theorems 1.1 and 1.2])**.** *We have the following identities:*

$$\int_{iV_\mathbb{R}} p_\lambda(x)p_\mu(x)q^{-|x|^2/2}\nabla(x)\,dx = (-1)^M C^{-1/2}\,|W|q^{(|\widetilde{\lambda}|^2+|\widetilde{\mu}|^2)/2}\widetilde{\Delta}(\widetilde{\mu})p_\lambda(\widetilde{\mu})\,,$$

$$\int_{iV_\mathbb{R}/\kappa Q^\vee} p_\lambda(x)p_\mu(x)\theta(x)\nabla(x)\,dx = (-1)^M C^{-1/2}\,|W|q^{(|\widetilde{\lambda}|^2+|\widetilde{\mu}|^2)/2}\widetilde{\Delta}(\widetilde{\mu})p_\lambda(\widetilde{\mu})\,,$$

$$\int_{V_\mathbb{R}} p_\lambda(x)p_\mu(-x)q^{|x|^2/2}\nabla(x)\,dx = C^{1/2}|W|q^{-(|\widetilde{\lambda}|^2+|\widetilde{\mu}|^2)/2}\widetilde{\Delta}(\widetilde{\mu})p_\lambda(\widetilde{\mu})\,.$$

Here C is the constant (2.22) *and $\theta(x)$ is the theta-function* (5.3).

Proof. The first two formulas are obviously equivalent. We will only derive the first identity, since the third one is entirely similar.

Consider the integral

$$\int_{C_\xi} \frac{\Phi_-(\widetilde{\lambda},x)\Phi_-(\widetilde{\mu},x)}{\Delta(x)\Delta(-x)}q^{-|x|^2/2}\,dx\,,$$

where $\Phi_-(\widetilde{\lambda},x)$, $\Phi_-(\widetilde{\mu},x)$ are as in (3.24). Expanding Φ_- in terms of ψ and applying formula (5.1), we conclude that the integral equals

$$(-1)^M C^{-1/2}q^{(|\widetilde{\lambda}|^2+|\widetilde{\mu}|^2)/2}\sum_{w,w'\in W}(-1)^{ww'}\psi(w\widetilde{\lambda},w'\widetilde{\mu})\,.$$

Using Lemma 3.4(i), we get that

$$\sum_{w,w'\in W}(-1)^{ww'}\psi(w\widetilde{\lambda},w'\widetilde{\mu}) = |W|\sum_{w\in W}(-1)^w\psi(w\widetilde{\lambda},\widetilde{\mu}) = |W|\Phi_-(\widetilde{\lambda},\widetilde{\mu})\,.$$

Therefore,

$$\int_{C_\xi} \frac{\Phi_-(\widetilde{\lambda},x)\Phi_-(\widetilde{\mu},x)}{\Delta(x)\Delta(-x)}q^{-|x|^2/2}\,dx = (-1)^M C^{-1/2}|W|q^{(|\widetilde{\lambda}|^2+|\widetilde{\mu}|^2)/2}\Phi_-(\widetilde{\lambda},\widetilde{\mu})\,.$$

After substituting expression (3.24) for Φ_- and rearranging, we get

$$\int_{C_\xi} p_\lambda(x)p_\mu(x)\frac{\delta(x)\delta(x)}{\Delta(x)\Delta(-x)}q^{-|x|^2/2}\,dx$$
$$= (-1)^M C^{-1/2}|W|q^{(|\widetilde{\lambda}|^2+|\widetilde{\mu}|^2)/2}\frac{\delta(\widetilde{\mu})}{\Delta(\widetilde{\mu})}p_\lambda(\widetilde{\mu})\,.$$

It follows from (3.21), (4.11) that

$$\frac{\delta(x)\delta(x)}{\Delta(x)\Delta(-x)} = C^{-1}\nabla(x)\,.$$

As a result, we obtain that

$$\int_{C_\xi} p_\lambda(x)p_\mu(x)q^{-|x|^2/2}\nabla(x)\,dx = (-1)^M C^{-1/2}|W|q^{(|\widetilde{\lambda}|^2+|\widetilde{\mu}|^2)/2}\widetilde{\Delta}(\widetilde{\mu})p_\lambda(\widetilde{\mu})\,.$$

Since the integrand in the left-hand side is non-singular, we can shift the contour back to $iV_{\mathbb{R}}$, and this leads to the required result. □

5.5. **q-Macdonald–Mehta integral.** Putting $\lambda=\mu=0$ in Theorem 5.7 gives us different variants of the q-analogue of the Macdonald–Mehta integral [M3], due to Cherednik [C3]. For instance, we have

$$\int_{iV_{\mathbb{R}}} q^{-|x|^2/2}\nabla(x;q,q^{m+1})\,dx = (-1)^M C^{1/2}|W|q^{|\widetilde{\rho}|^2}\widetilde{\Delta}(\widetilde{\rho})\,. \tag{5.17}$$

If we denote $k:=m+1$ and $\rho_k := \frac{1}{2}\sum_{\alpha\in R_+}k_\alpha\alpha$, then (5.17) can be written as

$$\int_{iV_{\mathbb{R}}} q^{-|x|^2/2}\nabla(x;q,q^k)\,dx = |W|\prod_{\alpha\in R_+}\frac{(q^{\langle\alpha,\rho_k\rangle};q_\alpha)_\infty}{(q_\alpha^{k_\alpha}q^{\langle\alpha,\rho_k\rangle};q_\alpha)_\infty}\,. \tag{5.18}$$

This makes it equivalent to the q-Macdonald–Mehta integral from [C3]. Each quantity $q^{\langle\beta,\rho_k\rangle}$ with $\beta\in R_+$ can be expressed as a polynomial in $t_\alpha = q_\alpha^{k_\alpha}$, after which the right-hand side of (5.18) allows analytic continuation to all complex values of t_α. According to [C3], (5.18) remains true for any $k_\alpha>0$. This, however, does not allow $t=q^{-m}$ with $m_\alpha\in\mathbb{Z}_+$, so it is not clear from the results of [C3] how to extend the formula (5.18) to such values.

On the other hand, Theorem 5.1 allows us to evaluate directly an integral of Macdonald–Mehta type for $t=q^{-m}$. Namely, let us put $\lambda=\mu=\rho$ in (5.1). By Proposition 3.11, ψ in that case becomes a nonzero constant $\Delta(-\rho)$. We therefore obtain

$$\int_{C_\xi} (\Delta(x)\Delta(-x))^{-1}\,q^{-|x|^2/2}\,dx = (-1)^M C^{-1/2}q^{|\rho|^2}\Delta^{-1}(-\rho)\,. \tag{5.19}$$

Here $-\rho=\rho_{-m}$ in the above notation. This identity can be written as

$$\int_{\xi+iV_{\mathbb{R}}} q^{-|x|^2/2}\nabla(x;q,q^{-m})\,dx = \prod_{\alpha\in R_+}\frac{(q^{\langle\alpha,\rho_{-m}\rangle};q_\alpha)_\infty}{(q_\alpha^{-m_\alpha}q^{\langle\alpha,\rho_{-m}\rangle};q_\alpha)_\infty}\,, \tag{5.20}$$

where the expression in the right-hand side is to be taken formally:

$$\prod_{\alpha\in R_+}\frac{(q^{\langle\alpha,\rho_{-m}\rangle};q_\alpha)_\infty}{(q_\alpha^{-m_\alpha}q^{\langle\alpha,\rho_{-m}\rangle};q_\alpha)_\infty}=\prod_{\alpha\in R_+}\prod_{j=1}^{m_\alpha}\left(1-q_\alpha^{-j}q^{\langle\alpha,\rho_{-m}\rangle}\right)^{-1}.$$

One can check that this expression coincides with the right-hand side of (5.18) evaluated at $k_\alpha=-m_\alpha\in\mathbb{Z}_-$, cf. Remark 3.14. (This is not entirely trivial, cf. [M1] where expressions similar to (5.18) are evaluated at $k_\alpha=0$.) Thus, (5.20) can be viewed as an analytic continuation of (5.18), which justifies C_ξ being a correct contour in the case $t=q^{-m}$.

Remark 5.8. An alternative approach would be to keep the same contour, but add corrections by taking into account the residues of the integrand between $iV_\mathbb{R}$ and C_ξ. This looks more complicated but has an advantage of handling the case of $t=q^{-m}$ with non-integer m. The results of [KS] seem to indicate such a possibility (at least, in rank one), see also [C4]. On the other hand, we note that in Theorems 4.6 and 5.6 the integration is performed over a real cycle which does not depend of m. Therefore, we expect these statements to remain valid (by analytic continuation in m) for non-integer m, with a suitably defined $\psi(\lambda,x)$. The same remark applies to the summation formula (6.1) below.

Remark 5.9. BA functions can be also defined and constructed in the rational and trigonometric settings, see [CFV2, Ch1]. They can be viewed as suitable limits of $\psi(\lambda,x)$ when $q\to1$, so some of the above results survive in such a limit. For example, the orthogonality relations can be stated and proved in a similar fashion. Also, the Cherdnik–Macdonald–Mehta integral survives in the rational (but not trigonometric) limit. Note that in the rational case $\psi(\lambda,x)$ exists also in non-crystallographic cases (for instance, for the dihedral groups). However, our proof of (5.1) does not work in the rational case, so by allowing $q\to1$ we can only obtain the result for the Weyl groups. It would be therefore interesting to find a direct proof of Cherednik–Macdonald–Mehta integral for BA functions in the rational setting, cf. [E] where the Macdonald–Mehta–Opdam integral is computed for all Coxeter groups in a uniform fashion.

Appendix
by Oleg Chalykh

6. Summation formulas

In [C3] Cherednik gives a version of Theorem 5.7 with integration replaced by summation. Here we prove a similar result for BA functions, which leads to new identities of Cherednik type. This also gives an elementary proof of Cherednik's results [C3, Theorem 1.3].

We will consider case **b**, so (R,m) is a reduced irreducible root system with W-invariant multiplicities $m_\alpha\in\mathbb{Z}_+$, and $(R',m')=(R,m)$. Generalizations to cases **a** and **c** are considered in Section 8.5. Throughout this section $|q|<1$.

For any $f(x)$ and $\xi\in V_\mathbb{C}$, define $\langle f\rangle_\xi$ as

$$\langle f\rangle_\xi=\sum_{\gamma\in P} f(\xi+\gamma)\,,$$

assuming convergence. For instance,

$$\langle q^{|x|^2/2}\rangle_\xi=q^{|\xi|^2/2}\theta(\xi)\,,$$

where $\theta(x)$ is the theta function (5.3).

Theorem 6.1 (cf. [C3, Theorem 1.3])**.** *For any $\lambda,\mu\in V_\mathbb{R}$ and $\xi\in V_\mathbb{C}$ we have*

$$\left\langle \frac{\psi(\lambda,x)\psi(\mu,-x)}{\Delta(x)\Delta(-x)}q^{|x|^2/2}\right\rangle_\xi=C^{1/2}q^{-\frac{|\lambda|^2+|\mu|^2}{2}}\psi(\lambda,\mu)\langle q^{\frac{|x+\lambda-\mu|^2}{2}}\rangle_\xi\,. \qquad (6.1)$$

where C is the constant (2.22)*. In particular, for $\lambda-\mu\in P$ we get*

$$\left\langle \frac{\psi(\lambda,x)\psi(\mu,-x)}{\Delta(x)\Delta(-x)}q^{|x|^2/2}\right\rangle_\xi=C^{1/2}q^{-\frac{|\lambda|^2+|\mu|^2}{2}}\psi(\lambda,\mu)q^{|\xi|^2/2}\theta(\xi)\,. \qquad (6.2)$$

We assume that ξ is generic so that the left-hand side of (6.1)*,* (6.2) *is well-defined.*

Proof. Denote

$$F(\lambda,\mu;x)=\frac{\psi(\lambda,x)\psi(\mu,-x)}{w(x)}q^{|x|^2/2}\,,\qquad w(x):=\Delta(x)\Delta(-x)\,. \qquad (6.3)$$

Using (3.3), (3.15) and duality, one easily checks that for every $v\in P^\vee=P(R^\vee)$ we have

$$F(\lambda,\mu;x+\kappa v)=e^{2\pi i\langle x+\lambda-\mu,v\rangle}e^{\pi i\kappa|v|^2}F(\lambda,\mu;x)\,, \qquad (6.4)$$

$$F(\lambda+\kappa v,\mu;x)=F(\lambda,\mu+\kappa v;x)=e^{2\pi i\langle x+\rho,v\rangle}F(\lambda,\mu;x)\,. \qquad (6.5)$$

Below we mostly write $F(x)$ for $F(\lambda,\mu;x)$.

The sum $\langle F(x)\rangle_\xi = \sum_{x\in\xi+P} F(x)$ is well-defined if ξ belongs to the following set:

$$V_{\mathbb{C}}^{\mathrm{reg}} = \{\xi \in V_{\mathbb{C}} \,|\, w(\xi+\gamma) \neq 0 \quad \forall\, \gamma \in P\}\,.$$

The complement $V_{\mathbb{C}} \setminus V_{\mathbb{C}}^{\mathrm{reg}}$ is a union of hyperplanes, each given locally by $q_\alpha^s q^{\langle\alpha,x\rangle} = 1$ for some $\alpha \in R$ and $s \in \mathbb{Z}$. This set of hyperplanes is locally finite and P-invariant, thus for every $\xi \in V_{\mathbb{C}}^{\mathrm{reg}}$ there exist a constant $\epsilon = \epsilon(\xi) > 0$ such that $|w(x)| > \epsilon$ for all $x \in \xi + P$. For such ξ the sum $\sum_{x\in\xi+P} F(x)$ is absolutely convergent, due to the exponentially-quadratic factor $q^{|x|^2/2}$ and the fact that $1/w(x)$ remains bounded. Therefore, $f(\xi) := \langle F(x)\rangle_\xi$ is holomorphic on $V_{\mathbb{C}}^{\mathrm{reg}}$. We claim that $f(\xi)$ extends to an entire function on $V_{\mathbb{C}}$.

To see that, let us look at the behaviour of $f(\xi)$ near the hypersurface

$$\pi_{\alpha,s} := \{\xi \in V_{\mathbb{C}} \,|\, q_\alpha^s q^{\langle\alpha,\xi\rangle} = 1\}\,.$$

We have

$$f(\xi)(1 - q_\alpha^s q^{\langle\alpha,\xi\rangle}) = \sum_{x\in\xi+P} \psi(\lambda,x)\psi(-\mu,x)q^{|x|^2/2}\frac{1 - q_\alpha^s q^{\langle\alpha,\xi\rangle}}{w(x)}\,. \tag{6.6}$$

Choose $\xi_0 \in \pi_{\alpha,s}$ away from the hyperplanes $\pi_{\beta,r}$ with $\beta \neq \alpha$. Then there exist a constant C such that for all ξ near ξ_0

$$\left|\frac{1 - q_\alpha^s q^{\langle\alpha,\xi\rangle}}{w(x)}\right| < C \quad \text{for all } x \in \xi + P\,.$$

As a result, the sum (6.6) converges absolutely and uniformly for all ξ near ξ_0. This implies that $f(\xi)$ has at most first order pole along $\pi_{\alpha,s}$, and its residue is the (absolutely convergent) sum of the residues of the terms $F(\xi+\gamma)$. In every subsum $\sum_{r\in\mathbb{Z}} F(\xi+\gamma_0+r\alpha)$ there are exactly $2m_\alpha$ terms with a pole along $\pi_{\alpha,s}$, and their residues sum to zero due to Lemma 4.3 and (5.2). As a result, $f(\xi)$ has a removable pole along $\pi_{\alpha,s}$, as needed.

Having established analyticity of $f(\xi) = \langle F(x)\rangle_\xi$, we now look at its translation properties. It is clearly periodic with respect to P. It follows from (6.4) that for $v \in Q^\vee$

$$f(\xi+\kappa v) = f(\xi)\, e^{2\pi i\langle\xi+\lambda-\mu,v\rangle} e^{\pi i\kappa|v|^2}\,.$$

Now a simple check shows that the function $\langle q^{\frac{|x+\lambda-\mu|^2}{2}}\rangle_\xi$ has the same translation properties in ξ-variable. A standard simple fact from the theory of theta-functions tells us that these two functions must differ by some factor independent of ξ. We record this in the following form:

$$\langle F(x)\rangle_\xi = \varphi(\lambda,\mu)q^{-|\lambda-\mu|^2/2}\langle q^{\frac{|x+\lambda-\mu|^2}{2}}\rangle_\xi\,, \tag{6.7}$$

for some entire function $\varphi(\lambda,\mu)$. It remains to relate φ to $\psi(\lambda,\mu)$.

Using (6.5) and (6.7), it is easy to see that

$$\varphi(\lambda+\kappa v,\mu) = \varphi(\lambda,\mu+\kappa v) = e^{2\pi i\langle\rho,v\rangle}\varphi(\lambda,\mu) \quad \forall\, v \in P^\vee\,.$$

As a result, φ can be presented as a convergent (Fourier) series of the following form:

$$\varphi(\lambda,\mu)=q^{\langle\lambda+\mu,\rho\rangle}\sum_{\nu,\nu'\in Q}a_{\nu\nu'}q^{\langle\lambda,\nu\rangle}q^{\langle\mu,\nu'\rangle}\,. \tag{6.8}$$

We want to show that this series is finite. For that we will look at the asymptotics of φ as $\lambda,\mu\to\infty$. To get the asymptotics for $\varphi(\lambda,\mu)$, we check the behaviour of the left-hand side in (6.7).

Switching x,λ in (3.3) and (3.15), we present ψ as a finite sum of the form

$$\psi(\lambda,x)=q^{\langle\lambda,x+\rho\rangle}\sum_{\nu\in Q_-}\Gamma_\nu(x)q^{\langle\nu,\lambda\rangle}\,, \tag{6.9}$$

with $\Gamma_0=\psi_\rho=\Delta(x)$ and $\Gamma_\nu=\psi_{\nu+\rho}(\lambda)$. Since the support of $\psi(\lambda,x)$ in the x-variable is $\lambda+\mathcal{N}$, we have that $\operatorname{supp}\Gamma_\nu\subseteq\mathcal{N}$ for all ν.

Let $D_\eta=i\eta+V_{\mathbb{R}}$ for some generic $\eta\in V_{\mathbb{R}}$. Then the same arguments as in [EV2, Lemma 8.1] prove the following result.

Lemma 6.2. For all ν, Γ_ν/Γ_0 is bounded from above when restricted to D_η. □

This lemma and (6.9) have the following consequence.

Corollary 6.3. *Let $c(\lambda)=\max_{\alpha\in R_+}\langle\alpha,\lambda\rangle$. We have uniformly for all $x\in D_\eta$:*

$$\psi(\lambda,x)=q^{\langle\lambda,x+\rho\rangle}\Delta(x)(1+O(q^{-c(\lambda)}))\quad \textit{as } c(\lambda)\to-\infty\,.$$

□

Using this result we obtain a uniform asymptotics for the function (6.3) on D_η:

$$F(x)=q^{\langle\lambda-\mu,x\rangle}q^{\langle\lambda+\mu,\rho\rangle}q^{|x|^2/2}(1+O(q^{-c}))$$

as $c:=\max\{c(\lambda),c(\mu)\}$ tends to $-\infty$.

It follows that for $\xi\in D_\eta$

$$\langle F(x)\rangle_\xi=q^{\langle\lambda+\mu,\rho\rangle}\langle q^{\langle\lambda-\mu,x\rangle}q^{|x|^2/2}\rangle_\xi(1+O(q^{-c}))\,.$$

Substituting this in (6.7) and assuming $\lambda,\mu\in P$, we conclude that

$$q^{-\langle\lambda+\mu,\rho\rangle}\varphi(\lambda,\mu)=1+O(q^{-c})\,.$$

Since λ,μ tend to infinity independently, this implies that

$$\varphi(\lambda,\mu)=q^{\langle\lambda+\mu,\rho\rangle}\sum_{\nu,\nu'\in Q_-}a_{\nu\nu'}q^{\langle\lambda,\nu\rangle}q^{\langle\mu,\nu'\rangle})\,,\qquad a_{00}=1\,.$$

Taking into account asymptotics in various Weyl chambers, we obtain that

$$\varphi(\lambda,\mu)=\sum_{\nu,\nu'\in P\cap\mathcal{N}}\varphi_{\nu\nu'}q^{\langle\lambda,\nu\rangle}q^{\langle\mu,\nu'\rangle}\,,$$

with $\varphi_{\rho\rho}=1$. Therefore, the function $\widetilde{\psi}(\lambda,\mu):=q^{\langle\lambda,\mu\rangle}\varphi(\lambda,\mu)$ will have the form as in (3.10).

Note that by (6.7) we have

$$\langle F(x)\rangle_\xi = \widetilde{\psi}(\lambda,\mu)q^{-(|\lambda|^2+|\mu|^2)/2}\langle q^{\frac{|x+\lambda-\mu|^2}{2}}\rangle_\xi\,.$$

The left hand-side obviously inherits from ψ the properties (3.4) in λ, μ. Also, the expression $\langle q^{\frac{|x+\lambda-\mu|^2}{2}}\rangle_\xi$ in the right-hand side is P-periodic in λ, μ, so it satisfies (3.4) trivially. As a result, $\widetilde{\psi}(\lambda,\mu)$ must have properties (3.4) as well. Note that, by construction, we have $\psi(\lambda,\mu)=\psi(\mu,\lambda)$.

We see that $\widetilde{\psi}$ has the same properties as the normalized BA function ψ, therefore they differ by a constant factor. The normalized ψ has $\psi_{\rho\rho}=C^{-1/2}$, while $\widetilde{\psi}_{\rho\rho}=1$. Thus, $\widetilde{\psi}=C^{1/2}\psi$. This finishes the proof of the theorem.

□

Remark 6.4. The above theorem and its proof refer to the lattices P, Q, $P^\vee$ and $Q^\vee$. Analyzing the proof, we see that the only requirement for these is that $R\subset Q\subseteq P$, $Q^\vee$ is dual to P and $P^\vee$ is dual to Q. Therefore, the result works if we replace P by $\mathcal{L}=Q(R)$, in which case $Q^\vee$ in the proof would be replaced by $P^\vee=P(R^\vee)$.

We can use the generalized Weyl formula (3.24) to obtain a symmetric version of the above theorem, thus recovering Cherednik's result [C3, Theorem 1.3]. We will use the notation of Theorem 5.7.

Theorem 6.5 (cf. [C3, Theorem 1.3])**.** *Let* $\nabla(x)=\nabla(x;q,q^{m+1})$ *and* $p_\lambda(x)=p_\lambda(x;q,q^{m+1})$ *in case* **b**. *Then for any* $\lambda,\mu\in P_+$ *and any* $\xi\in V_{\mathbb{C}}$ *we have*

$$\sum_{x\in\xi+P} p_\lambda(x)p_\mu(-x)q^{|x|^2/2}\nabla(x)$$
$$=(-1)^{\widetilde{M}}C^{1/2}|W|q^{-\frac{|\widetilde{\lambda}|^2+|\widetilde{\mu}|^2}{2}}\widetilde{\Delta}(\widetilde{\mu})p_\lambda(\widetilde{\mu})q^{|\xi|^2/2}\theta(\xi)\,,$$

where $\theta(x)$ *is the theta function* (5.3), $\widetilde{\lambda}=\lambda+\widetilde{\rho}$, $\widetilde{\mu}=\mu+\widetilde{\rho}$ *and* $\widetilde{M}=\sum_{\alpha\in R_+}(m_\alpha+1)$.

This is checked in the same way as Theorem 5.7. □

7. Deformed root system $A_n(m)$

In [CFV1, CFV2] certain deformations of root systems were found, which admit BA function (in the rational setting of [CFV2]). Some of these BA functions have also trigonometric and the q-versions, see [Ch2, F]. These will be treated systematically in a separate publication [Ch4]. Here, as an illustration, let us consider one example, namely, the system $R=A_n(m)$.

This system depends on an integer parameter m and it is the union of two subsets R_0 and R_1 in $V_{\mathbb{R}} = \mathbb{R}^{n+1}$:

$$R_0 = \{\pm(e_i - e_j)\}_{1\le i<j\le n}\,, \quad R_1 = \{\pm(e_i - \sqrt{m}e_{n+1})\}_{i=1,\dots,n}\,. \tag{7.1}$$

In the case $m = 1$ this is the root system of type A_n in $\mathbb{R}^{n+1}$. We will call these vectors roots also in the case $m \neq 1$ when R is no longer a root system in the usual sense. A positive half of the system (7.1) can be chosen as

$$R_+ = \{e_i - e_j\}_{1\le i<j\le n} \cup \{e_i - \sqrt{m}e_{n+1}\}_{i=1,\dots,n}\,.$$

We define q_α in the same way as in case **b** previously, thus

$$q_\alpha = \begin{cases} q & \text{for } \alpha \in R_0\,, \\ q^{\frac{m+1}{2}} & \text{for } \alpha \in R_1\,. \end{cases}$$

The multiplicity function $m\,:\ R \to \mathbb{Z}_+$ is defined as follows:

$$m_\alpha = \begin{cases} m & \text{for } \alpha \in R_0\,, \\ 1 & \text{for } \alpha \in R_1\,. \end{cases} \tag{7.2}$$

Let ρ be the vector (2.10). Explicitly, we have

$$\rho = \frac{m}{2}(n-1, n-3, \dots, -n+1, 0) + \frac{1}{2}(1, \dots, 1, -\sqrt{m}n)\,. \tag{7.3}$$

As a substitute for the weight lattice P, we can use any lattice that contains R and ρ (see Remark 3.2); we take

$$P = \frac{1}{2}\mathbb{Z}e_1 \oplus \dots \oplus \frac{1}{2}\mathbb{Z}e_n \oplus \frac{1}{2}\sqrt{m}\mathbb{Z}e_{n+1}\,. \tag{7.4}$$

Below we will also need the lattice $Q^\vee$ dual to P, i.e.

$$Q^\vee = \{v \in V_{\mathbb{R}} \mid \langle \pi, v\rangle \in \mathbb{Z}\ \forall \pi \in P\}\,.$$

Explicitly, we have

$$Q^\vee = 2\mathbb{Z}e_1 \oplus \dots \oplus 2\mathbb{Z}e_n \oplus \frac{2}{\sqrt{m}}\mathbb{Z}e_{n+1}\,.$$

(Warning: $Q^\vee$ in this case is *not* the lattice generated by the coroots $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$.)

Note that this case should be viewed as a type **b** case, so when referring to the formulas or results from previous sections, one should assume that $(R', m') = (R, m)$.

A Baker–Akhiezer function is now a function of the form

$$\psi(\lambda, x) = q^{\langle\lambda,x\rangle} \sum_{\nu\in\mathcal{N}\cap P} \psi_\nu(\lambda) q^{\langle\nu,x\rangle}\,,$$

where $\mathcal{N}$ is the polytope (3.1). By definition, ψ must satisfy conditions (3.4), which defines it uniquely up to a factor depending on λ, cf. [Ch2, Section 7]. The normalized BA function is obtained by fixing one of the coefficients. Namely, ψ_ν with $\nu = \rho$ is set to be $\Delta(\lambda)$ as defined in (2.17). The following result was proved in [Ch2].

Theorem 7.1 ([Ch2, Theorems 7.3, 7.4])**.** *For the deformed root system* (7.1)*, the normalized Baker–Akhiezer function* $\psi(\lambda, x)$ *exists and it is unique. The function* $\psi(\lambda, x)$ *is symmetric in* λ, x*, namely,* $\psi(\lambda, x) = \psi(x, \lambda)$*.*

The function ψ serves as an eigenfunction for a commutative family of difference operators that are a deformation of the Ruijsenaars operators (2.33), see [Ch2, FS].

With this at hand, we can now generalize the results of Sections 4 and 5.

Theorem 7.2. *With the above notation, the statements of Theorems 4.1, 4.6, 5.1, 5.4, 5.5, 5.6 and 6.1 remain true for the deformed system* $R = A_n(m)$*.*

The proofs are identical to those in Sections 4 and 5. Reference to the Weyl chambers should be replaced by referring to the connected components of the complement to the hyperplanes $\langle \alpha, x\rangle = 0$, $\alpha \in R$. For the proof of Theorem 6.1, it is important to note that $\langle \alpha^\vee, \beta\rangle \in \mathbb{Q}$ for all $\alpha, \beta \in R$, as a simple check shows. This implies that the poles of $F(\xi + \gamma)$, $\gamma \in P$, will form a locally finite set of hyperplanes. The rest of the proofs remain the same. The only other difference appears in the proof of Lemma 4.3, since we cannot rely upon symmetry arguments. Crucially, we still have the following result.

Lemma 7.3. With the above notation, the statement of Lemma 4.3 remains true for the deformed system $A_n(m)$.

Proof. For $\alpha \in R_0$ this follows from the same arguments using the symmetry of the denominator $\Delta(x)\Delta(-x)$ under the reflection s_α. For $\alpha \in R_1$ the statement can be checked by a direct computation and is left to the reader. □

We finish this section by obtaining a version of the singular q-Macdonald–Mehta integral and sum for the system $A_n(m)$. To do this, we need an analogue of Proposition 3.11. First, let us describe the vertices of the polytope (3.1), following [Ch2, Section 7.2]. We have $(n+1)!$ ways to choose a positive half of R. Namely, for any permutation $\tau \in S_{n+1}$ take a vector

$$v^\tau := \left(\tau(1), \dots, \tau(n), \frac{1}{\sqrt{m}}\tau(n+1)\right).$$

Now denote by R_+^τ the following subset of R:

$$R_+^\tau = \{\alpha \in R \mid \langle \alpha, v^\tau\rangle < 0\},$$

and introduce ρ^τ as

$$\rho^\tau = \frac{1}{2}\sum_{\alpha \in R_+^\tau} m_\alpha \alpha.$$

The vectors $\{\rho^\tau\}$, $\tau \in S_{n+1}$, are the vertices of the polytope (3.1). For instance, taking $\tau = \mathrm{id}$ gives ρ and taking $\tau = (n+1, n, \dots, 1)$ gives $-\rho$.

The following non-obvious property of ρ^τ can be checked by a direct computation.

Lemma 7.4. The vectors ρ^τ, $\tau \in S_{n+1}$, have equal length.

The lemma implies that any two adjacent vertices ρ, ρ' of the polytope $\mathcal{N}$ are related by $\rho' = s_\alpha \rho$ for a suitable α. Now all the arguments used to prove Proposition 3.11 apply verbatim, leading to the following result.

Proposition 7.5. *For $\lambda = \rho^\tau$, $\tau \in S_{n+1}$, the normalized BA function $\psi(\lambda, x)$ does not depend on x and is equal to $\Delta(-\rho) \neq 0$.*

Finally, putting $\lambda = \mu = \rho$ in (5.1) leads, similarly to (5.20), to an explicit evaluation of a singular q-Macdonald–Mehta integral for the deformed root system $A_n(m)$.

Proposition 7.6. *For any big $\xi \in V_{\mathbb{R}}$, formula* (5.19) *remains true for the deformed root system $A_n(m)$* (7.1)–(7.2)*.*

We can also substitute $\lambda = \mu = \rho$ into (6.1) to compute the sum

$$\sum_{x \in \xi + P} \frac{1}{\Delta(x)\Delta(-x)}\,.$$

In fact, by Remark 6.4, we may replace the lattice P by

$$\mathcal{L} = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n \oplus \sqrt{m}\mathbb{Z}e_{n+1} \subset \mathbb{R}^{n+1}\,. \tag{7.5}$$

Then we obtain the following summation identity of Cherednik's type in the deformed case.

Proposition 7.7. *For the deformed root system $A_n(m)$* (7.1)–(7.2)*, we have*

$$\sum_{x \in \xi + \mathcal{L}} \frac{1}{\Delta(x)\Delta(-x)} = C^{1/2} \frac{q^{-|\rho|^2}}{\Delta(-\rho)} q^{|\xi|^2/2} \theta(\xi)\,,$$

where $\theta(x)$ is the theta function (5.3)*, associated to the lattice* (7.5)*. Explicitly,*

$$\theta(x) = \theta_0(x_1; q) \dots \theta_0(x_n; q) \theta_0\left(\frac{x_{n+1}}{\sqrt{m}}; q^m\right)\,, \quad \theta_0(z; q) := \sum_{n \in \mathbb{Z}} q^{nz} q^{n^2/2}\,.$$

Remark 7.8. In a recent paper [FS], an interesting method to produce deformed systems of Macdonald–Ruijsenaars type is proposed, which is based on studying special representations of double affine Hecke algebras. It produces commuting difference operators, but does not lead to a simple construction for BA functions. Also, the method of [FS] does not cover all known examples of integrable systems of Macdonald–Ruijsenaars type.

8. Gaussian integrals, twisted BA functions and twisted Macdonald–Ruijsenaars model

Let us consider what happens if we replace the Gaussian $q^{-|x|^2/2}$ in Theorem 5.1 by $q^{-a|x|^2/2}$ with $a > 0$. For Proposition 5.2 and the cancelation of residues to work, we need the function $g(x) = q^{-a|x|^2/2}$ to take equal values along the shifted hyperplanes:

$$g(x - \frac{1}{2}j\alpha) = g(x + \frac{1}{2}j\alpha) \quad \text{for } q^{\langle \alpha, x\rangle} = 1 \quad \text{and } j \in \mathbb{Z}\,. \tag{8.1}$$

We have

$$g(x - \frac{1}{2}j\alpha)/g(x + \frac{1}{2}j\alpha) = q^{aj\langle \alpha, x\rangle}\,.$$

Therefore, (8.1) will hold as soon as $a \in \mathbb{N}$.

So, let us take $a = \ell \in \mathbb{N}$ and consider the integral

$$\int_{C_\xi} \frac{\psi(\lambda, x)\psi(\mu, x)}{\Delta(x)\Delta(-x)} q^{-\ell|x|^2/2}\, dx\,. \tag{8.2}$$

It turns out that this integral is still 'computable', but for $\ell > 1$ the result will be expressed in terms of a new function ψ_ℓ whose properties are similar to those of ψ. This 'twisted' BA function ψ_ℓ will be a common eigenfunction for a certain quantum integrable model given by commuting W-invariant difference operators that generalize the Macdonald operators D^π. To the best of our knowledge, this model is new; in the case $R = A_n$ it generalizes the trigonometric Ruijsenaars model [R1].

8.1. **Twisted BA functions.** We keep the notation of Section 3. In this section we consider case **b** only, so all the notation of Section 3 applies with $(R', m') = (R, m)$ and $q_\alpha = q^{|\alpha|^2/2}$.

For a reduced irreducible root system R, a W-invariant set of labels $m_\alpha \in \mathbb{Z}_+$, and an integer $\ell \in \mathbb{N}$, a twisted BA function ψ_ℓ (of type **b**) has the following form:

$$\psi_\ell(\lambda, x) = q^{\langle \lambda, x\rangle/\ell} \sum_{\nu \in \mathcal{N} \cap \ell^{-1}P} \psi_\nu(\lambda) q^{\langle \nu, x\rangle}\,, \tag{8.3}$$

where $\mathcal{N}$ is the polytope (3.1).

The function ψ_ℓ must also satisfy further conditions, similar to (3.4). Namely, we require that for each $\alpha \in R$, $j = 1, \ldots, m_\alpha$ and any ϵ with $\epsilon^\ell = 1$ we have

$$\psi_\ell\left(\lambda,\, x - \frac{1}{2}j\alpha\right) = \epsilon^j \psi_\ell\left(\lambda,\, x + \frac{1}{2}j\alpha\right) \quad \text{for} \quad q^{\langle \alpha, x\rangle/\ell} = \epsilon\,. \tag{8.4}$$

Definition. A function $\psi_\ell(\lambda, x)$ with the properties (8.3)–(8.4) is called a twisted Baker–Akhiezer function associated to the data $\{R, m, \ell\}$.

For $\ell = 1$ this is the definition of Section 3. Our goal is to prove the following two results.

Theorem 8.1. *(1) A twisted Baker–Akhiezer function $\psi_\ell(\lambda, x)$ exists and is unique up to multiplication by a factor depending on λ.*

(2) Let us normalize ψ_ℓ by requiring (3.8) *(recall that $\Delta' = \Delta$ in case* **b***). Then we have*

$$\psi_\ell(\lambda, x) = \psi_\ell(x, \lambda)\,.$$

(3) As a function of x, ψ_ℓ is a common eigenfunction of certain pairwise commuting W-invariant difference operators D_ℓ^π, $\pi \in P_+$, namely,

$$D_\ell^\pi \psi_\ell = \mathfrak{m}_\pi(\lambda)\psi_\ell\,, \qquad \mathfrak{m}_\pi(\lambda) = \sum_{\tau\in W\pi} q^{\langle\tau,\lambda\rangle}\,.$$

The operators D_ℓ^π have the same leading terms as $(D^\pi)^\ell$, i.e. they are lower-term perturbations of the Macdonald operators raised to the ℓth power.

Theorem 8.2. *For any $\ell \in \mathbb{N}$, any $\lambda, \mu \in V_\mathbb{C}$ and big $\xi \in V_\mathbb{R}$ we have*

$$\int_{C_\xi} \frac{\psi(\lambda, x)\psi(\mu, x)}{\Delta(x)\Delta(-x)} q^{-\ell|x|^2/2}\,dx = (-1)^M C^{-1/2} q^{\frac{|\lambda|^2+|\mu|^2}{2\ell}} \psi_\ell(\lambda, \mu)\,, \qquad (8.5)$$

where ψ_ℓ is the normalized twisted BA function and C, M are the same as in Theorem 5.1.

Theorem 8.1 is analogous to Theorem 3.1, but we cannot use the same method to prove it. The reason is that the arguments of [Ch2] exploit in an essential way Macdonald operators and their properties. In the twisted case there exist certain analogues of these operators (these are D_ℓ^π appearing in Theorem 8.1), but we cannot write them down explicitly. In fact, the existence of D_ℓ^π will be established only once we know the existence of ψ_ℓ. So we change our tack: we will instead *define* ψ_ℓ by the formula (8.5) and from that we will derive the required properties (8.3)–(8.4).

8.2. **Proof of Theorem 8.1.** Let $I_\ell(\xi)$ denote the integral (8.2). Since the function $g(x) = q^{-\ell|x|^2/2}$ satisfies (8.1), the residues of the integrand cancel as in Lemma 4.3, therefore, $I_\ell(\xi)$ will not depend on ξ provided that it is big.

Now we compute the integral using series expansion as in Section 5.1. Assuming ξ belongs to the negative Weyl chamber, we get similarly to (5.8) that

$$I_\ell(\xi) = \sum_{\gamma\in Q_-} f_\gamma \int_{C_\xi} q^{\langle\lambda+\mu+\gamma,x\rangle} q^{-\ell|x|^2/2}\,dx =$$

$$\sum_{\gamma\in Q_-} f_\gamma q^{\frac{1}{2\ell}|\lambda+\mu+\gamma|^2} = q^{\frac{1}{2\ell}|\lambda+\mu|^2} \sum_{\gamma\in Q_-} f_\gamma q^{\frac{1}{\ell}\langle\gamma,\lambda+\mu\rangle} q^{\frac{1}{2\ell}|\gamma|^2}\,, \quad (8.6)$$

where the coefficients $f_\gamma(\lambda, \mu)$ are exactly the same as in (5.8).

Comparing such expansions for different chambers, we conclude (exactly in the same way as in Section 5.1) that I_ℓ is an elementary function of the form

$$I_\ell = q^{\frac{1}{2\ell}|\lambda+\mu|^2} \sum_{\nu\in\mathcal{N}\cap\ell^{-1}P} \psi_\nu(\lambda) q^{\langle\nu,\mu\rangle}\,.$$

Therefore, I_ℓ has the form $q^{\frac{1}{2\ell}(|\lambda|^2+|\mu|^2)}\psi_\ell(\lambda,\mu)$ where ψ_ℓ has the required form (8.3) (with x replaced by μ).

As a function of μ, I_ℓ has the properties (3.4). As a result, we obtain that for $j = 1,\dots,m_\alpha$ and for $q^{\langle\alpha,\mu\rangle} = 1$

$$q^{\frac{1}{2\ell}|\mu-\frac{1}{2}j\alpha|^2}\psi_\ell(\lambda,\mu-\frac{1}{2}j\alpha) = q^{\frac{1}{2\ell}|\mu+\frac{1}{2}j\alpha|^2}\psi_\ell(\lambda,\mu+\frac{1}{2}j\alpha)\,.$$

It is easy to see that these are equivalent to conditions (8.4) (again, with x replaced by μ).

This proves the existence of a function ψ_ℓ satisfying (8.3)–(8.4). Its uniqueness, up to a factor depending on λ, can be proved by exactly the same arguments as in [Ch2, Proposition 3.1].

Finally, I_ℓ is obviously symmetric in λ and μ, therefore, $\psi_\ell(\lambda,\mu)$ is also symmetric:

$$\psi_\ell(\lambda,\mu) = \psi_\ell(\mu,\lambda)\,.$$

It remains to show that the leading coefficient ψ_ρ equals $(-1)^M C^{-1/2}\,\Delta(\lambda)$. This follows from the formula (5.10), which still applies in this case. Thus, the Gaussian integral automatically gives us the normalized and symmetric ψ_ℓ. This finishes the proof of parts (1) and (2) of Theorem 8.1.

Part (3) follows from the uniqueness of ψ_ℓ by Krichever's argument [Kr1, Kr2], cf. [Ch2, Section 5.1]. Namely, recall the ring $\mathcal{Q}\subset\mathbb{R}[P]$ of all $f(x)\in\mathbb{R}[P]$ satisfying the conditions (3.7). Note that $\mathbb{R}[P]^W\subset\mathcal{Q}$. We have the following result.

Theorem 8.3 (cf. [Ch2, Theorem 5.1 and Proposition 5.3])**.** *(1) For each $f(x)\in\mathcal{Q}$ there exists a difference operator D_f in λ-variable on the lattice P such that $\psi_\ell(\lambda,x)$ is its eigenfunction: $D_f\psi_\ell = f(x)\psi_\ell$. All these operators pairwise commute.*

(2) For any dominant weight $\pi\in P_+$ and $f = \mathfrak{m}_\pi(x)$ the corresponding operator D_f is W-invariant and has the form

$$D_f = \sum_{\tau\in W\pi} a_\tau T^{\ell\tau} + l.o.t.\,,$$

where the leading coefficients a_τ are given by

$$a_\pi(\lambda) = \frac{\Delta(\lambda)}{\Delta(\lambda+\ell\pi)}\,,\quad a_{w\pi}(\lambda) = a_\pi(w^{-1}\lambda)\,.$$

This is proved in the same way as [Ch2, Theorem 5.1 and Proposition 5.3].

Part (3) of Theorem 8.1 now follows immediately from this result after switching between x and λ. This completes the proof of Theorem 8.1.

Remark 8.4. In the same way one constructs twisted BA functions and the corresponding twisted quantum integrable model for the deformed root system $A_n(m)$.

8.3. **Proof of Theorem 8.2.** This is now immediate from above: we have

$$I_\ell(\xi) = (-1)^M C^{-1/2} q^{\frac{1}{2\ell}(|\lambda|^2+|\mu|^2)} \psi_\ell(\lambda,\mu)\,,$$

where ψ_ℓ will satisfy all the properties of the normalized twisted BA function. □

8.4. **Summation formula.** We can also generalize the summation formula to the twisted case. Since the arguments are entirely analogous, we only formulate the result.

Theorem 8.5. *For any $\lambda, \mu \in V_{\mathbb{R}}$ and $\xi \in V_{\mathbb{C}}$ we have*

$$\sum_{x\in\xi+P} \frac{\psi(\lambda,x)\psi(\mu,-x)}{\Delta(x)\Delta(-x)} q^{\ell|x|^2/2} = C^{1/2} q^{-\frac{|\lambda|^2+|\mu|^2}{2\ell}} \psi_\ell(\lambda,\mu) \sum_{x\in\xi+P} q^{\frac{\ell}{2}|x+\frac{\lambda-\mu}{\ell}|^2}\,,$$

where C is the constant (2.22) *and ψ_ℓ is the twisted BA function associated to (R,m,ℓ).*

One can use Theorems 8.2, 8.5 and the generalized Weyl formula (3.24) to express, in terms of the twisted BA functions, the integrals and sums

$$\int_{iV_{\mathbb{R}}} p_\lambda(x)p_\mu(x)\nabla(x)q^{-\ell|x|^2/2}\,dx\,, \qquad \sum_{x\in\xi+P} p_\lambda(x)p_\mu(x)\nabla(x)q^{\ell|x|^2/2}$$

for $p_\lambda = p_\lambda(x;q,q^{m+1})$, $\nabla = \nabla(x;q,q^{m+1})$ in case **b**. In particular, this gives an expression for $\int_{iV_{\mathbb{R}}} \nabla(x)q^{-\ell|x|^2/2}\,dx$. In general, however, this does not seem to lead to a nice factorized form as in the case $\ell = 1$.

8.5. **Twisted BA functions in cases a and c.** Let us consider the Gaussian integrals for the remaining cases **a** and **c** of Macdonald's theory. Note that case **a** for $R = A, D, E$ is the same as case **b** if we choose the scalar product so that all roots have length $\sqrt{2}$. Thus, the only cases not covered by Theorems 5.1 and 8.2 are case **c** (when $R = C_n$) and case **a** for $R = B_n, C_n, F_4, F_4^\vee, G_2, G_2^\vee$. Also note that the cases $R = F_4$ and $R = F_4^\vee$ are equivalent because these roots systems are isomorphic, and the same is true for G_2, while the cases $R = B_n$ and $R = C_n$ can be obtained from case **c** by a suitable specialization of the parameters m_i.

Let $\psi = \psi_{R,m}$ be the corresponding normalized BA function. Consider the integral (8.2). As a starting point, we would like that integral to be independent of ξ (provided that it is big). To have the cancelation of residues as in Lemma 4.3, we need $g(x) = q^{-\ell|x|^2/2}$ to satisfy the properties

$$g(x - \frac{1}{2}j\alpha') = g(x + \frac{1}{2}j\alpha') \quad \text{for } q^{\langle\alpha,x\rangle} = 1\,,$$

where $\alpha \in R$ in case **a** or $\alpha \in R^2$ in case **c**. In addition to that, in case **c** we need that

$$g(x - se_i) = g(x + se_i) \quad \text{for } q^{x_i} = \pm 1\,, \tag{8.7}$$

where $s \in \frac{1}{2}\mathbb{Z}$.

This puts the following restrictions on ℓ in case **a**:

$$\ell \in \frac{1}{2}|\alpha|^2\mathbb{Z} \quad \text{for all } \alpha \in R\,. \tag{8.8}$$

If we assume that the *short* roots in R have length $\sqrt{2}$, then we have

$$\ell \in \begin{cases} \mathbb{Z} & R = A_n, D_n, E_{6-8}\,, \\ 2\mathbb{Z} & R = B_n, C_n, F_4\,, \\ 3\mathbb{Z} & R = G_2\,. \end{cases}$$

In general, let ν_R denote

$$\nu_R = \max_{\alpha \in R}\{|\alpha|^2/2\}\,,$$

then our conditions on ℓ can be written in all cases as

$$\ell \in \nu_R\mathbb{Z} \cap (\nu_{R^\vee})^{-1}\mathbb{Z}\,. \tag{8.9}$$

In case **c**, we obtain from (8.7) that $\ell \in 2\mathbb{Z}$. In fact, since we only need (8.7) to hold for certain half-integral s (see (3.5)–(3.6)), it is possible to choose $\ell \in \mathbb{Z}$ if either m_3 or m_4 is 1/2. We will ignore this option, and will always assume for simplicity that $\ell \in 2\mathbb{Z}$ in case **c**.

Our goal is to show that the integral (8.2) for such ℓ can be expressed in terms of a suitably defined BA function. What looks particularly peculiar in cases **a** , **c** is that the usual BA function $\psi(\lambda, x)$ is not self-dual, $\psi(\lambda, x) \neq \psi(x, \lambda)$, since one has also to switch from (R, m) to (R', m') under the duality. However, the twisted BA functions ψ_ℓ defined below are always self-dual, even for the case **c** with full five parameters $m_1, \ldots, m_5$.

So, let (R, m) be of type **a** or **c**, in the notation of Section 2.1. That is, in case **a** we consider a reduced root system R and W-invariant integers $m_\alpha \in \mathbb{Z}_+$, and put $(R', m') = (R^\vee, m)$. In case **c**, we take $R = R' = C_n$ with m, m' being (half-)integers m_i and m_i' (2.5).

Choose ℓ such that

$$\ell \in \nu_{R'}\mathbb{Z} \cap (\nu_{R'^\vee})^{-1}\mathbb{Z}\,. \tag{8.10}$$

(This is the choice, dual to (8.9). In case **c** this still means $\ell \in 2\mathbb{Z}$.)

A twisted BA function ψ_ℓ in cases **a** or **c** has the same form (8.3):

$$\psi_\ell(\lambda, x) = q^{\langle\lambda, x\rangle/\ell} \sum_{\nu \in \mathcal{N} \cap \ell^{-1}P} \psi_\nu(\lambda) q^{\langle\nu, x\rangle}\,.$$

It must also satisfy further conditions, similar to (8.4). Namely, for each $\alpha \in R$ (in case **a**) or $\alpha \in R^2$ (in case **c**), any $j = 1, \ldots, m_\alpha$ and any ϵ with $\epsilon^\ell = 1$ we have

$$\psi_\ell\left(\lambda,\, x - \frac{1}{2}j\alpha'\right) = \epsilon^j \psi_\ell\left(\lambda,\, x + \frac{1}{2}j\alpha'\right) \quad \text{for } \ q^{\langle\alpha, x\rangle/\ell} = \epsilon\,. \tag{8.11}$$

In case **c**, we require additionally for each $\alpha = e_i \in R^1$ the following: for any ϵ with $\epsilon^{2\ell} = 1$

(1) for all $0 < s \preccurlyeq (m_1, m_2)$

$$\psi(\lambda, x - se_i) = \epsilon^{2s}\psi(\lambda, x + se_i) \quad \text{for } q^{x_i/\ell} = \epsilon\,, \quad \text{provided } \epsilon^\ell = 1; \tag{8.12}$$

(2) for all $0 < s \preccurlyeq (m_3, m_4)$

$$\psi(\lambda, x - se_i) = \epsilon^{2s}\psi(\lambda, x + se_i) \quad \text{for } q^{x_i/\ell} = \epsilon\,, \quad \text{provided } \epsilon^\ell = -1\,. \tag{8.13}$$

Definition. Let ℓ be as in (8.10). A function $\psi_\ell(\lambda, x)$ of the form (8.3) satisfying conditions (8.11)–(8.13) is called a twisted Baker–Akhiezer function of type **a** or **c**, respectively, associated to the data $\{R, m, \ell\}$.

Now the same arguments as in case **b** prove the following results.

Theorem 8.6. *(1) A twisted Baker–Akhiezer function $\psi_\ell(\lambda, x)$ exists and is unique up to multiplication by a factor depending on λ.*

(2) Let us normalize ψ_ℓ by requiring (3.8)*. Then we have*

$$\psi_\ell(\lambda, x) = \psi_\ell(x, \lambda)\,.$$

(3) As a function of x, ψ_ℓ is a common eigenfunction of certain pairwise commuting W-invariant difference operators D_ℓ^π, $\pi \in P_+$, namely,

$$D_\ell^\pi \psi_\ell = \mathfrak{m}_\pi(\lambda)\psi_\ell\,, \qquad \mathfrak{m}_\pi(\lambda) = \sum_{\tau \in W\pi} q^{\langle \tau, \lambda \rangle}\,.$$

The operators D_ℓ^π have the same leading terms as $(D^\pi)^\ell$, i.e. they are lower-term perturbations of the Macdonald operators raised to the ℓth power.

Theorem 8.7. *Let $\psi(\lambda, x)$ be the normalized BA function associated to (R, m) in cases* **a** *or* **c***. Let ℓ be as in* (8.9)*. For any $\lambda, \mu \in V_{\mathbb{C}}$ and big $\xi \in V_{\mathbb{R}}$ we have*

$$\int_{C_\xi} \frac{\psi(\lambda, x)\psi(\mu, x)}{\Delta(x)\Delta(-x)} q^{-\ell |x|^2/2}\, dx = (-1)^M C^{-1/2} q^{\frac{|\lambda|^2 + |\mu|^2}{2\ell}} \psi'_\ell(\lambda, \mu)\,, \tag{8.14}$$

where C, M are the same as in Theorem 5.1 and $\psi'_\ell = \psi_{R', m', \ell}$ is the normalized twisted BA function associated to the dual data (R', m', ℓ).

We also have the related summation formulas, similar to Theorem 8.5 and proved in the same way.

Theorem 8.8. *Assume the notation of Theorem 8.7. For any $\lambda, \mu \in V_{\mathbb{R}}$ and $\xi \in V_{\mathbb{C}}$ we have*

$$\sum_{x \in \xi + P'} \frac{\psi(\lambda, x)\psi(\mu, -x)}{\Delta(x)\Delta(-x)} q^{\ell |x|^2/2} = C^{1/2} q^{-\frac{|\lambda|^2 + |\mu|^2}{2\ell}} \psi'_\ell(\lambda, \mu) \sum_{x \in \xi + P'} q^{\frac{\ell}{2}|x + \frac{\lambda - \mu}{\ell}|^2}\,,$$

where $P' = P(R')$ is the weight lattice of R' and ψ'_ℓ is the twisted BA function of type **a** *or* **c***, associated to (R', m', ℓ).*

Remark 8.9. In [S] some analogues of Cherednik–Macdonald–Mehta identities are obtained for Koornwinder polynomials, with suitably modified $\theta(x)$. This is different from the Gaussians used above.

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