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Square-Free Perfect Graphs

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Abstract

We prove that square-free perfect graphs are bipartite graphs or line graphs of bipartite graphs or have a 2-join or a star cutset.

1 Introduction

In this paper, all graphs are simple. A *hole* is a chordless cycle with at least four nodes. A *k-hole* is a hole with k nodes and a 4-hole is also called a *square*. Our main result is a decomposition theorem for graphs that contain no odd hole and no 4-hole. (We say that a graph G contains H if H is an induced subgraph of G). Partial results in this direction were obtained by Xue [31], [32], Fouquet [18] and recently by Linhares-Sales and Maffray [21].

A graph G is *perfect* if, in every induced subgraph, the size of a largest clique equals the chromatic number. A graph G is *minimally imperfect* if G is not perfect, but every proper induced subgraph of G is perfect. The Strong Perfect Graph Conjecture (SPGC), due to Berge [1], states that if G is a minimally imperfect graph, then either G or the complement of G is an odd hole. This conjecture was proved recently by Chudnovsky, Robertson, Seymour and Thomas [2]. In the 70's and 80's, the SPGC was proved for H-free graphs for quite a number of small graphs H. In fact, the class of square-free graphs was the last class of H-free graphs with $|V(H)| = 4$ for which the SPGC was not known: Seinsche [26] proved it in 1974 for P_4-free graphs, Meyniel [22], [23] in 1976 for paw-free graphs (see also Olariu [24]), Parthasarathy and Ravindra [25] in 1976 for claw-free graphs, Tucker [28] in 1977 for K_4-free graphs, Conforti and Rao [13], [14], Fonlupt and Zemirline [17] and Tucker [29] in 1987 for diamond-free graphs (see also [4]).

Given a graph G and a subset S of its nodes, $G \setminus S$ denotes the subgraph of G obtained by removing the nodes in S and the edges with at least one node in S. A node set S is a *cutset* of G if the graph $G \setminus S$ has more connected components than G.

A node set S is a *star* if it consists of a node x and neighbors of x. Chvátal [3] showed that a minimally imperfect graph cannot contain a star cutset.

A graph G has a *2-join*, denoted by $H_1|H_2$, with special sets A_1, B_1, A_2, B_2 that are nonempty and disjoint, if the nodes of G can be partitioned into sets H_1 and H_2 so that $A_1, B_1 \subseteq H_1$, $A_2, B_2 \subseteq H_2$, all nodes of A_1 are adjacent to all nodes of A_2, all nodes of B_1 are adjacent to all nodes of B_2 and these are the only adjacencies between H_1 and H_2. Also, for $i = 1, 2$, H_i has at least one path from A_i to B_i and if A_i and B_i are both of cardinality 1, then the graph induced by H_i is not a chordless path. Cornuéjols and Cunningham [16] showed that a minimally imperfect graph cannot contain a 2-join.

The *line graph* of a graph G is the graph H whose nodes are the edges of G and two nodes r, s of H are adjacent in H if and only if the edges r, s of G are incident to a common node of G. It is well known and easy to prove that that bipartite graphs and line graphs of bipartite graphs are perfect.

The main result of this paper is the following.

Theorem 1.1 *A square-free graph that contains no odd hole is bipartite or the line graph of a bipartite graph or has a star cutset or a 2-join.*

Theorem 1.1 can be used to prove the validity of the SPGC for square-free graphs.

Theorem 1.2 *Let G be a minimally imperfect square-free graph. Then G is an odd hole.*

Proof: Let G be a minimally imperfect graph. Chvátal [3] showed that G cannot contain a star cutset and Cornuéjols and Cunningham [16] showed that G cannot contain a 2-join (see also Cornuéjols [15]). Now if G is a square-free graph, by Theorem 1.1, G must be an odd hole. □

In fact, we prove the following result which is more general than Theorem 1.1. We *sign* a graph by assigning 0,1 weights to its edges. A graph G is *even-signable* if there exists a signing such that, in every triangle the sum of the weights of its edges is odd and in every hole the sum of the weights is even. It is obvious that G contains no odd holes if and only if G is even-signable and the above property is satisfied by assigning to all the edges of G the weight 1.

Theorem 1.3 *A square-free even-signable graph is triangle-free or the line graph of a triangle-free graph or has a star cutset or a 2-join.*

In the first draft of this paper (February 2001) we made the following observations: "The approach followed in this paper to decompose square-free perfect graphs might be used as a template for the decomposition of general perfect graphs in the same way as the work of Conforti and Rao [12] on linear balanced matrices was a template for the decomposition of general balanced matrices [10] and [8], [9]. For general perfect graphs, it is natural to consider as basic not only the bipartite graphs and the line graphs of bipartite graphs, but also their complements. In terms of decompositions, star cutsets need to be replaced by more general cutsets such as T-cutsets and U-cutsets (Hoàng [20]) or skew partitions (Chvátal [3])." Interestingly, Chudnovsky, Robertson, Seymour and Thomas [2] followed such an approach to prove the SPGC.

1.1 Even-signable graphs

A *wheel*, denoted by (H, x), is a graph induced by a hole H and a node $x \notin V(H)$ having at least three neighbors in H. Node x is the *center* of the wheel. If x has k neighbors in H, the wheel is called a k-wheel. A wheel (H, x) is *odd* if it contains an odd number of triangles and is *even* if it contains an even number of triangles: That is, (H, x) is odd (even) if the set of edges of H having both endnodes adjacent to x has odd (even) cardinality.

A $3PC(x_1x_2x_3, y)$ is a graph induced by three chordless paths $P^1 = x_1, \dots, y$, $P^2 = x_2, \dots, y$ and $P^3 = x_3, \dots, y$, having no common nodes other than y and such that the only adjacencies between nodes of $P^i \setminus y$ and $P^j \setminus y$, for $i, j \in \{1, 2, 3\}$ distinct, are the edges of the triangle induced by $\{x_1, x_2, x_3\}$. Also, at most one of the paths P^1, P^2, P^3 has length 1. We say that a graph G contains a $3PC(\Delta, .)$ if it contains a $3PC(x_1x_2x_3, y)$ for some $x_1, x_2, x_3, y \in V(G)$.

It is immediate to check that if G contains an odd wheel or a $3PC(\Delta, .)$, G is not even-signable and hence G contains an odd hole. Our proofs use the following characterization of even-signable graphs, which can be derived from a theorem of Truemper [27]. (See also [11]).

Theorem 1.4 [6] *A graph is even-signable if and only if it does not contain an odd wheel nor a $3PC(\Delta, .)$.*

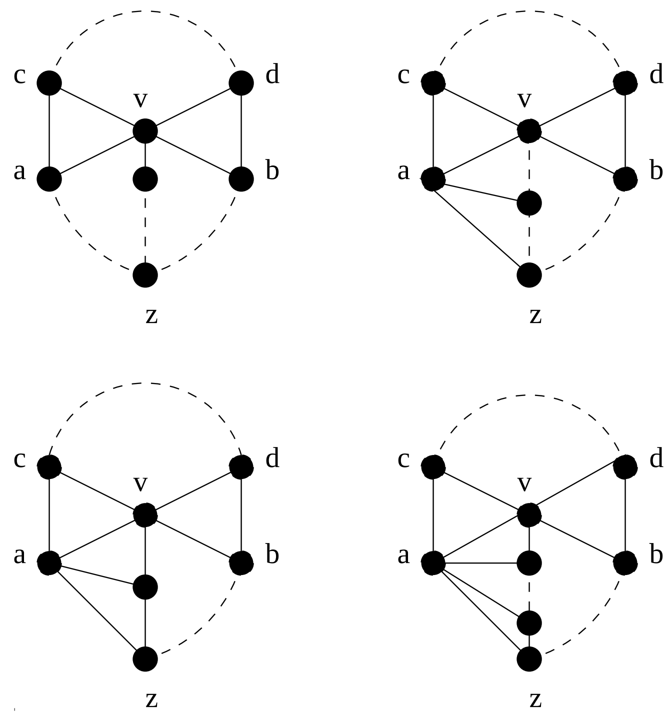

Figure 1: L-parachutes

2 WP-Free Graphs

In this section, we introduce a result proven in [5].

A *line wheel* is a 4-wheel (H, v) that contains exactly two triangles and these two triangles have only the center v in common. Note that if G is a line wheel, then G is the line graph of a triangle-free graph. A *twin wheel* is a 3-wheel containing exactly two triangles. A *universal wheel* is a wheel (H, v) where the center v is adjacent to all the nodes of H. A *triangle-free wheel* is a wheel containing no triangle. A *proper wheel* is a wheel that is not any of the above four types.

Definition 2.1 *An* L-parachute $LP(ca, db, v, z)$ *is a graph induced by a line wheel* (H, v) *where* $H = a, \ldots, z, \ldots, b, d, \ldots, c, a$*, where* a*,* b*,* c*,* d *are the neighbors of* v *in* H*, together with a chordless path* $P = v, \ldots, z$ *of length greater than one. No node of* $H \setminus \{z, a, b\}$ *and at most one node among* a*,* b*, may be adjacent to an interior node of* P*.*

Definition 2.2 *A* T-parachute $TP(t, v, a, b, z)$ *is a graph induced by a twin wheel* (H, v) *where* $H = a, t, b, \ldots, z, \ldots, a$*, where* t*,* a*,* b *are the neighbors of* v *in* H*, together with a chordless path* $P = v, \ldots, z$ *of length greater than one. No node of* $H \setminus \{z, a, b\}$ *and at most one node among* a*,* b *may be adjacent to an interior node of* P*.*

A *parachute* is either an L-parachute or a T-parachute. A graph G is *WP-free* if it contains neither a proper wheel nor a parachute. The results in [5] imply the following:

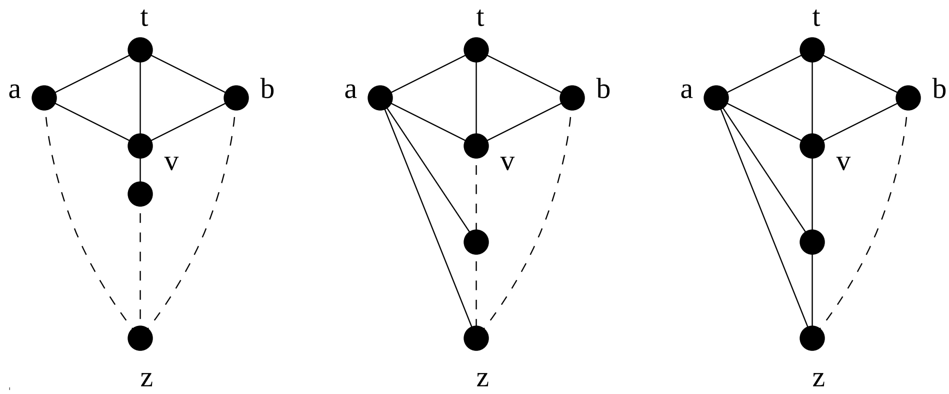

Figure 2: T-parachutes

Theorem 2.3 *Let G be a square-free even-signable graph. If G is WP-free, then G is a triangle-free graph, or the line graph of a triangle-free graph, or G has a star cutset or a 2-join.*

3 Outline of the proof

Two graphs play an important role in our proof of Theorem 1.3:

A $3PC(a_1a_2a_3, b_1b_2b_3)$ is the graph containing node disjoint triangles a_1, a_2, a_3 and b_1, b_2, b_3, plus three chordless paths, $P^1 = a_1, \ldots, b_1$, $P^2 = a_2, \ldots, b_2$ and $P^3 = a_3, \ldots, b_3$, having no common nodes and such that the only adjacencies between the nodes of distinct paths are the edges of the two triangles. A $3PC(a_1a_2a_3, b_1b_2b_3)$ is also referred to as a $3PC(\Delta, \Delta)$.

A *beetle* is a 4-wheel (H, v) containing exactly two triangles and these triangles have two nodes in common, the center v and a node of H.

Now as a consequence of Theorem 2.3, in the proof of Theorem 1.3, we may assume that G is a square-free graph that contains a proper wheel or a parachute.

-We first prove the theorem when G contains a proper wheel that is not a beetle (Theorem 3.6), and show that if G contains a beetle, then G contains a $3PC(\Delta, \Delta)$ plus additional nodes adjacent to it (Section 4, Theorem 3.4).

-We then prove the theorem when G contains an L-parachute (Section 5, Theorem 4.3).

In all the results that we obtain from Section 6 on, make use of the fact that the theorem holds whenever G contains an L-parachute, or a proper wheel that is not a beetle.

- In Section 7 we show that if G contains a beetle or a T-parachute, then G contains a $3PC(\Delta, \Delta)$ plus some additional nodes adjacent to it.

- From Section 8 on, we show that the theorem holds when G contains a $3PC(\Delta, \Delta)$.

4 Proper Wheels

Given a subgraph H of a graph G, a node $v \notin V(H)$ is *strongly adjacent* to H if $|N(v) \cap V(H)| \geq 2$, where $N(v)$ denotes the set of neighbors of x in G.

Following the terminology of West [30], a *path* P is a sequence of distinct nodes $x_1, \ldots, x_n$, $n \geq 1$, such that $x_i x_{i+1}$ is an edge, for all $1 \leq i < n$. For $i \leq l$, the path $x_i, x_{i+1}, \ldots, x_l$ is called the *$x_i x_l$-subpath* of P and is denoted by $P_{x_i x_l}$. The *length* of a path or a cycle is its number of edges.

Let A, B, C be three disjoint node sets such that no node of A is adjacent to a node of B. A path $P = x_1, \ldots, x_n$ *connects A and B* if either $n = 1$ and x_1 has neighbors in A and B, or $n > 1$ and one of the two endnodes of P is adjacent to at least one node in A and the other is adjacent to at least one node in B. The path P is a *direct connection between A and B* if, in the subgraph induced by the node set $V(P) \cup A \cup B$, no path connecting A and B is shorter than P. A direct connection P between A and B *avoids* C if $V(P) \cap C = \emptyset$. The direct connection P is said to be *from A to B* if x_1 is adjacent to some node in A and x_n to some node in B.

Lemma 4.1 *Let (H, x) be a proper even wheel and let y be a node that is not adjacent to x but has at least 2 neighbors in $N(x) \cap H$. Then y has exactly 2 neighbors in H and they are adjacent.*

Proof: Since G is square-free and y is not adjacent to x, all the neighbors of y in $N(x)$ are adjacent. So y has exactly two neighbors in $N(x) \cap H$, say y_1, y_2 and y_1, y_2 are adjacent. It remains to show that y has no other neighbor in H.

Claim 1: *Let Q be a subpath of H whose endnodes are in $N(x)$ at least one of which is distinct from y_1, y_2 and no interior node of Q is in $N(x)$. Then $Q \cup \{y\}$ contains an even number of triangles.*

Proof of Claim 1: Assume H contains a subpath Q with the above properties such that $Q \cup \{y\}$ contains an odd number of triangles. Then this number must be 1 and y has exactly two neighbors in Q, else $Q \cup \{x, y\}$ induces an odd wheel with center y. Let x_i, x_j be the endnodes of Q and y_i, y_j be the adjacent neighbors of y in Q, closest to x_i, x_j respectively. Since G is square-free and the pair y_i, y_j is distinct from y_1, y_2, we may assume w.l.o.g. that x_i and y_i are distinct.

Let x'_i, x'_j be the neighbors of x_i, x_j in $H \setminus Q$. Since (H, x) is a proper even wheel, x has a neighbor in $H \setminus \{x'_i, x_i, x_j, x'_j\}$. So if y also has a neighbor in $H \setminus (Q \cup \{x'_i, x'_j\})$, we have a $3PC(y y_i y_j, x)$. Therefore $N(y) \cap H \subseteq Q \cup \{x'_i, x'_j\}$. Since (H, y) has a positive even number of triangles, y_i, y_j are the only two neighbors of y in Q and $x_i \neq y_i$, it follows that $x_j = y_j$. So both x and y are adjacent to x'_j. Let H' be the hole $x_i, x, x'_j, y, y_i, Q_{y_i x_i}$. Then (H', x_j) is an odd wheel. This concludes Claim 1.

Since y is adjacent to y_1, y_2, it follows by Claim 1 that if y has at least three neighbors in H, then (H, y) contains an odd number of triangles and is an odd wheel. □

Let (H, x) be a wheel that contains at least one triangle. A *sector* of (H, x) is a maximal subpath Q of H such that $Q \cup \{x\}$ is a triangle-free graph. Two sectors Q_i and Q_j are *adjacent* if $Q_i \cup Q_j \cup \{x\}$ contains at least one triangle. Note that a sector may have length zero.

A *bicoloring* of a proper even wheel (H,x) is a partition of the nodes in H into nonempty sets R and B (the red and blue nodes) so that nodes in the same sector have the same color while nodes in adjacent sectors have distinct colors.

Lemma 4.2 *Let (H,x) be a proper even wheel that is bicolored and let y be a node that is not adjacent to x but has at least two neighbors in H painted with distinct colors. Then y has exactly 2 neighbors in H and these nodes are endnodes of adjacent sectors of (H,x).*

Proof: Let y_1, y_2 be neighbors of y in H having distinct colors, such that one of the y_1y_2-subpaths of H, say Q, contains no other neighbor of y. Since y_1, y_2 have distinct colors, then $Q \cup \{x\}$ contains an odd number of triangles. This number must be 1 and x has exactly two neighbors in Q, else $Q \cup \{x,y\}$ induces an odd wheel with center x. Let x_1, x_2 be the neighbors of x in Q, closest to y_1, y_2 respectively.

If x and y have two common neighbors in H, we are done by Lemma 4.1. So we assume w.l.o.g. that $y_2 \neq x_2$. Let y_1', y_2' be the neighbors of y_1, y_2 in $H \setminus Q$.

Claim 1: *Node x has a neighbor in $H \setminus (Q \cup \{y_1', y_2'\})$.*

Proof of Claim 1: Assume not. Since (H,x) is a proper even wheel, then $y_1 = x_1$, node x is adjacent to both y_1', y_2' and $y_1'y_2'$ is not an edge. Node y has at least three neighbors in H, else we have an odd wheel with center x, but y is not adjacent to y_1' or y_2' else we are done by Lemma 4.1. Let y_3 be the neighbor of y in $H \setminus Q$ closest to y_1 and let y_2'' be the neighbor of y_2' in H, distinct form y_2.

If $y_3 \neq y_2''$, there is an odd wheel (H', y_1) where $H' = y, y_2, y_2', x, H_{y_1'y_3}, y$. So $y_3 = y_2''$ and we have a $3PC(y_1'y_1x, y_2'')$. This completes the proof of Claim 1.

If y has a neighbor in $H \setminus (Q \cup \{y_1', y_2'\})$, by Claim 1, we have a $3PC(x_1x_2x, y)$. So all the neighbors of y in H are included in $\{y_1', y_1, y_2, y_2'\}$. Now $y_1 = x_1$ and both x and y are adjacent to y_1', else we have an odd wheel with center x. But now we are done by Lemma 4.1. □

Definition 4.3 *A* diamond *is a graph with four nodes and five edges.* Connected diamonds *$D(a_1a_2a_3a_4, b_1b_2b_3b_4)$ consist of two node disjoint sets $\{a_1, \dots, a_4\}$ and $\{b_1, \dots, b_4\}$ each of which induces a diamond such that a_1a_4 and b_1b_4 are not edges, together with four paths $P^1, \dots, P^4$ such that for $i = 1, \dots, 4$, P^i is an a_ib_i-path. Paths $P^1, \dots, P^4$ are node disjoint and the only adjacencies between them are the edges of the two diamonds.*

Lemma 4.4 *Let R, B be a bicoloring of a proper even wheel (H,x) and let $P = y_1, \dots, y_n$, $n > 1$, be a chordless path with the following properties:*

1) y_1 is adjacent to a node in B and to no node in R, y_n is adjacent to a node in R and no node in B, and y_1, y_n have nonadjacent neighbors in H.

2)No node of P is adjacent to x and no interior node of P is adjacent to a node of H.

Then (H,x) is a beetle and $(H,x) \cup P$ induces connected diamonds.

Proof: Let t_1, t_2 be blue and red neighbors of y_1, y_n in H such that one of the t_1t_2-subpaths, say Q_{12} of H is shortest. Then $Q_{12} \cup \{x\}$ contains an odd number of triangles and therefore

x has exactly two neighbors in Q_{12}, else there is an odd wheel. Let x_1, x_2 be these two neighbors, closest to t_1, t_2 in Q_{12} respectively. Since (H,x) is a proper wheel and y_1, y_n have nonadjacent neighbors in H, then y_1 or y_n has at least one other neighbor in H, else there is an odd wheel with center x.

Assume w.l.o.g. that y_1 has at least one other neighbor and let t_3 be such node, closest to t_2 in $H \setminus Q_{12}$. Let t_4 be the neighbor of y_n, closest to t_3 in the t_3t_2-subpath of $H \setminus t_1$. (Possibly $t_4 = t_2$). Let Q_{34} be the t_3t_4-subpath of $H \setminus t_1$. By the above argument, x has exactly two neighbors in Q_{34}. Let x_3, x_4 be these two neighbors, closest to t_3, t_4 in Q_{34} respectively. We use the notation $H = t_1, Q_{12}, t_2, Q_{24}, t_4, Q_{34}, t_3, Q_{13}, t_1$. If t_2 and t_4 coincide, Q_{24} has length 0.

Now either t_1 and t_3 are adjacent or $t_2 = t_4 = x_4$, else we have a $3PC(x_1x_2x, y_1)$. Assume the second case does not hold: So t_1 and t_3 are adjacent. Since t_1 t_3 are both blue nodes, x cannot be adjacent to both. So, since (H,x) is not a line wheel, x has at least one neighbor x^* in $H \setminus \{x_1, x_2, x_3, x_4\}$ and by construction x^* is an interior node of Q_{24}. This implies that t_2 and t_4 are distinct and nonadjacent and in $H \cup P$ there is a $3PC(y_1t_1t_3, y_n)$.

So the second case must hold, i.e. $t_2 = t_4 = x_4$. Furthermore since x_1, x_2 are the unique neighbors of x in Q_{12}, we must have $x_2 = t_2 = t_4 = x_4$.

Since (H,x) is not a twin wheel, x has at least one other neighbor, say x^*, and by construction x^* is an interior node of Q_{13}. Now x^* must be adjacent to both t_1, t_3, else there is a $3PC(x_1x_2x, y_1)$ or a $3PC(x_3x_4x, y_1)$. Finally $t_1 \neq x_1$ and $t_3 \neq x_3$, else (H,x) is either an odd wheel or an universal wheel. So (H,x) is a beetle. Finally x^* is adjacent to y_1, else we have a $3PC(x_1x_2x, t_1)$. Therefore $(H,x) \cup P$ induces connected diamonds. □

Theorem 4.5 *Let G be a square-free even-signable graph that contains a proper even wheel (H,x). Furthermore if (H,x) is a beetle, assume that no connected diamonds contain (H,x). Let R, B be a bicoloring of the nodes in H and assume w.l.o.g. that $B \setminus N(x)$ is nonempty. Then $(N(x) \cup x) \setminus R$ is a star cutset of G, separating R from $B \setminus N(x)$.*

Proof: Assume not and let $P = y_1, \ldots, y_n$ be a direct connection from R to $B \setminus N(x)$ in $G \setminus (N(x) \cup x \cup H)$. Let y_k be the node of lowest index with a neighbor in B that is not adjacent to all the neighbors of y_1 in R. (Possibly $k = n$). By Lemma 4.2, $k > 1$. So y_k has no neighbor in R. If no node of P with index smaller than k is adjacent to a node in B, the theorem holds as a consequence of Lemma 4.4. Let $r \in R$ be adjacent to y_1 and let y_j, $j < k$, be the node of lowest index adjacent to a node $b \in B$. Since $j < k$, b and r are adjacent. If $j = 1$, by Lemma 4.2, b and r are the unique neighbors of y_1 in H and if $j > 1$, since G is square-free and b is adjacent to all the neighbors of y_1 in H, r is the unique neighbor of y_1 in H. Let z be the neighbor of r in $H \setminus \{b\}$.

Since (H,x) is a proper even wheel and $B \setminus N(x)$ is nonempty, x has a neighbor $b_1 \in B$ that is adjacent to neither b nor z. Let b_2 be a neighbor of y_k in $H \setminus \{z, r, b\}$, so that the b_1b_2-path T in $H \setminus \{z, r, b\}$ is shortest and let $Q = y_k, b_2, T, b_1, x$. Then $H' = x, r, y_1, P_{y_1y_k}, y_k, Q, x$ is a hole.

Claim 1: *If $z \in B$, then z has no neighbor in $H' \setminus \{x, r\}$.*

Proof of Claim 1: Assume not. Then both (H', b) and (H', z) are wheels. If (H', b) is a proper wheel, the claim follows by Lemma 4.2 applied to z and a bicoloring of (H', b). Furthermore if (H', z) is a proper wheel, then b contradicts Lemma 4.2. So (H', b) and (H', z) are either line wheels or twin wheels. There are three possibilities depending on whether both are line wheels, both are twin wheels or one is a line wheel and the other is a twin wheel. Since b and z are not adjacent to b_1, it can be verified that in all three cases there is an odd wheel or a square and this proves Claim 1.

Let b' be the first neighbor of y_k, encountered when traversing $H \setminus \{r\}$ from z to b and let S be the rb'-subpath of $H \setminus \{b\}$. By Claim 1, z has no neighbor in $P_{y_1 y_k}$, so $r, y_1, P_{y_1 y_k}, y_k, b', S, r$ is a hole. Since $b' \in B$ and $r \in R$, $S \cup \{x\}$ contains an odd number of triangles. Therefore x is adjacent to r, z and to no other node of S, else there in an odd wheel with center x. In particular, by Claim 1, $b' \neq z$, so $b' \notin N(x)$ and $y_k = y_n$.

Let x' be the first neighbor of x, encountered when traversing $H \setminus \{r\}$ from b' to b and let T be the $b'x'$-subpath of $H \setminus \{b\}$. If b' is the unique neighbor of y_k in T, there is a $3PC(xrz, b')$. If y_k has exactly two neighbors in T, say b', b'', and b', b'' are adjacent, there is a $3PC(y_k b' b'', x)$. Finally if y_k has two nonadjacent neighbors in T, there is a $3PC(xrz, y_k)$. □

The following result is an immediate consequence of the above theorem.

Theorem 4.6 *Let G be a square-free even-signable graph. If G contains a proper wheel that is not a beetle, then G has a star cutset.*

5 L-Parachutes

In this section, we assume that G is a square-free even-signable graph that contains an L-parachute $\Pi = LP(ca, db, v, z)$.

We use the following notation. The two triangles are acv and bdv. The *top path* is P_{cd} (the cd-path of $H \setminus b$) and we indicate with c', d' the neighbors of c and d in P_{cd}. The *bottom path* is P_{ab} (the ab-path of $H \setminus d$). The *middle path* is P_{mz}, where z in an interior node in P_{ab} and m is a neighbor of v in P.

Lemma 5.1 *Let y be a node not adjacent to v but adjacent to at least two nodes in $\{a, b, c, d\}$. Then $N(y) \cap \Pi$ is either $\{a, c\}$ or $\{b, d\}$.*

Proof: Since G is square-free, $N(y) \cap \{a, b, c, d\}$ is either $\{a, c\}$ or $\{b, d\}$ and we assume w.l.o.g. that $N(y) \cap \{a, b, c, d\} = \{a, c\}$.

Claim 1: $P_{cd} \cup \{y\}$ *contains an even number of triangles.*

Proof of Claim 1: Assume not. If y contains a neighbor in $P_{cd} \setminus \{c, c'\}$, since y is adjacent to c, $P_{cd} \cup \{v, y\}$ induces an odd wheel with center y. So c, c' are the only neighbors of y in P_{cd} and $P_{cd} \cup \{v, a, y\}$ induces an odd wheel with center c. This proves Claim 1.

The argument used in Claim 1 also shows that $P_{ab} \cup y$ contains an even number of triangles. So a, c are the only neighbors of y in $H = P_{cd} \cup P_{ab}$, else (H, y) is an odd wheel. Suppose y has a neighbor in P_{mz}. Note that y is not adjacent to m since otherwise v, c, y, m induces a square. Let R be a shortest path from y to b in $y \cup P_{mz} \cup P_{bz} \setminus m$. Then $(R \cup P_{cd} \cup y, v)$ is an odd wheel. □

Lemma 5.2 *Let y be a node not adjacent to v but adjacent to a node in P_{cd} and to a node in $P_{ab} \cup P_{mz}$. Then $N(y) \cap \Pi$ is either $\{a, c\}$ or $\{b, d\}$.*

Proof: By Lemma 5.1, it suffices to assume that y has at most one neighbor in $\{a, b, c, d\}$ and derive a contradiction.

Claim 1: *Node y has exactly two neighbors in P_{cd} and they are adjacent.*

Proof of Claim 1: Assume not, then y has either 1 neighbor or 2 nonadjacent neighbors in P_{cd}.

If y_1 is the unique neighbor of y in P_{cd}, assume w.l.o.g. that $y_1 \neq c$. When y has a neighbor in $P_{ab} \cup P_{mz} \setminus \{b, m\}$ there is a $3PC(acv, y_1)$. Since G is square-free, y cannot be adjacent to both b and m. If y is adjacent to b then it is not adjacent to d and there is a $3PC(acv, b)$. So m is the unique neighbor of y in $P_{ab} \cup P_{mz}$. If $y_1 = d$ then v, m, d, y induces a square. So $y_1 \neq d$. W.l.o.g. a does not have a neighbor in $P_{mz} \setminus z$. But then there is a $3PC(acv, m)$.

Assume y has nonadjacent neighbors in P_{cd}. Since y has at most one neighbor in $\{a, b, c, d\}$, there is a $3PC(acv, y)$ or a $3PC(bdv, y)$ whenever y has a neighbor in $P_{ab} \cup P_{mz} \setminus \{m\}$. So m is the unique neighbor of y in $P_{ab} \cup P_{mz}$ and assume w.l.o.g. that a does not have a neighbor in $P_{mz} \setminus z$. Then there is a $3PC(acv, m)$ and this completes Claim 1.

So $P_{cd} \cup \{y\}$ contains a unique triangle, say yy_1y_2. If y has a neighbor in $P_{ab} \cup P_{mz} \setminus \{a, b\}$, there is a $3PC(yy_1y_2, v)$. So y is adjacent to a or b and no other node of P_{ab}. Since y has at most one neighbor in $\{a, b, c, d\}$, $P_{cd} \cup P_{ab} \cup y$ induces an odd wheel with center y. □

Theorem 5.3 *Let G be a square-free even-signable graph. If G contains an L-parachute, then G has a star cutset.*

Proof: We show that, if G contains an L-parachute $\Pi = LP(ca, db, v, z)$ then $S = v \cup N(v) \setminus \{a, b, m\}$ is a star cutset of G, separating $P_{cd} \setminus \{c, d\}$ from $P_{ab} \cup P_{zm}$.

Assume not and let $P = y_1, \ldots, y_n$ be a direct connection from $P_{cd} \setminus \{c, d\}$ to $P_{ab} \cup P_{mz}$ in $G \setminus S$. We assume w.l.o.g. that Π and P are chosen so that the cardinality of the node set of $\Pi \cup P$ is minimized. By Lemma 5.2 $n > 1$. It also follows from our assumption that at most one of c, d has a neighbor in $P \setminus \{y_n\}$ and if c is adjacent to a node in $P \setminus \{y_n\}$, then c' and possibly c are the only neighbors of y_1 in Π. From now on, we assume that d has no neighbor in $P \setminus \{y_n\}$.

Claim 1: *No node of $P \setminus \{y_n\}$ is adjacent to c.*

Proof of Claim 1: Assume not. Then c' and possibly c are the only neighbors of y_1 in Π.

Case 1: Node y_1 is the only neighbor of c in $P \setminus \{y_n\}$.

If y_n has a neighbor in $P_{ab} \cup P_{zm} \setminus \{a, b\}$, by Lemma 5.2, y_n is not adjacent to c or d and there is a $3PC(y_1cc', v)$. If y_n is adjacent to a, possibly c and to no other node of Π there is an odd wheel with center c. If y_n is adjacent to b, d and to no other node of Π there is a $3PC(y_1cc', d)$. Finally, if y_n is adjacent to b and to no other node of Π there is a $3PC(y_1cc', b)$. By Lemma 5.2, these are all the possibilities.

Case 2: Node c has a neighbor in $P \setminus \{y_1, y_n\}$.

If y_n has a neighbor in $P_{ab} \cup P_{zm} \setminus \{m, b\}$, by Lemma 5.2, y_n is not adjacent to d. Then $P_{ab} \cup P_{zm} \cup \{y_n\} \setminus \{m, b\}$ contains a $y_n a$-path Q_1 and $P_{ab} \cup P_{zm} \cup \{y_n\}$ contains a $y_n b$-path Q_2 such that $H_1 = d, P_{dc'}, c', y_1, P, y_n, Q_1, a, v, d$ and $H_2 = d, P_{dc'}, c', y_1, P, y_n, Q_2, b, d$ are both holes. Now one of (H_1, c), (H_2, c) is an odd wheel. If y_n is adjacent to b and has no neighbor in $P_{ab} \cup P_{zm} \setminus \{m, b\}$ there is a $3PC(acv, b)$. Finally if m is the only neighbor of y_n in $P_{ab} \cup P_{zm}$ there is a $3PC(acv, m)$ or a $3PC(bdv, m)$. This completes Claim 1.

By the minimality of $\Pi \cup P$, y_1 has either a unique neighbor or two adjacent neighbors in P_{cd}.

Assume that y_1 has a unique neighbor, say y^*, in P_{cd}. If y_n is adjacent to c or d, say d, by Lemma 5.2, y_n is adjacent to b, d and to no other node of Π and there is a $3PC(y_n bd, y^*)$. If y_n has a neighbor in $P_{ab} \cup P_{zm} \setminus \{m\}$ there is a $3PC(acv, y^*)$ or a $3PC(bdv, y^*)$. So m is the unique neighbor of y_n in Π and there is a $3PC(acv, m)$ or a $3PC(bdv, m)$.

So y_1 has two neighbors, say y', y'' in P_{cd} and y', y'' are adjacent. By Claim 1, y_1 is not adjacent to c or d. If y_n has a neighbor in $P_{ab} \cup P_{zm} \setminus \{a, b\}$, by Lemma 5.2, y_n is not adjacent to c or d and hence there is a $3PC(y'y''y_1, v)$. By Lemma 5.2 y_n is not adjacent to both a and b. So w.l.o.g. a is the unique neighbor of y_n in $P_{ab} \cup P_{zm}$. By Lemma 5.2, y_n is not adjacent to d. If y_n is not adjacent to c there is a $3PC(y'y''y_1, a)$ and otherwise there is a $3PC(y'y''y_1, c)$. □

6 Nodes Adjacent to a $3PC(\Delta, \Delta)$

We denote by Σ a $3PC(a_1a_2a_3, b_1b_2b_3)$ with the three paths $P^1 = P_{a_1b_1}$, $P^2 = P_{a_2b_2}$ and $P^3 = P_{a_3b_3}$. For $i = 1, 2, 3$, we denote by a'_i the neighbor of a_i in P^i and by b'_i the neighbor of b_i in P^i. For distinct $i, j \in \{1, 2, 3\}$, we denote by H_{ij} the hole induced by $P^i \cup P^j$.

Lemma 6.1 *Let G be an even-signable graph and let Σ be a $3PC(\Delta, \Delta)$. If node u is adjacent to Σ, then it is one of the following types.*

Type tj for $j = 1, 2, 3$: *Node u has exactly j neighbors in Σ and they are either all contained in $\{a_1, a_2, a_3\}$ or all in $\{b_1, b_2, b_3\}$.*

Type p1: *Node u has exactly one neighbor in Σ and u is not of Type t1.*

Type p2: *Node u has exactly two neighbors in Σ, which are furthermore adjacent and contained in P^i for some $i \in \{1, 2, 3\}$.*

Type p3: *Node u has at least two nonadjacent neighbors in Σ, and all the neighbors of u in Σ are contained in P^i, for some $i \in \{1, 2, 3\}$.*

Type p4: *Node u has exactly four neighbors in Σ, u_1, u_2, u_3 and u_4, where u_1u_2 is an edge that belongs to some P^i, $i \in \{1, 2, 3\}$, and u_3u_4 is an edge that belongs to some P^j, $j \in \{1, 2, 3\} \setminus \{i\}$. Furthermore, u is not adjacent to both a_i and a_j, and it is not adjacent to both b_i and b_j.*

Type t2p: *For distinct indices $i, j, k \in \{1,2,3\}$ and for $z \in \{a, b\}$, u is adjacent to z_i and z_j, it has at least one neighbor in $P^k \setminus \{z_k\}$, and is not adjacent to any node in $(P^i \cup P^j \cup \{z_k\}) \setminus \{z_i, z_j\}$.*

Type t3p: *Node u has at least four neighbors in Σ. For some $z \in \{a, b\}$, u is adjacent to z_1, z_2 and z_3, and all the other neighbors of u in Σ belong to P^i for some $i \in \{1,2,3\}$.*

Type tj for $j = 4, 5, 6$: *Node u is adjacent to j nodes in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ and possibly other nodes of Σ. Furthermore, if u is of Type t4, then u has two neighbors in $\{a_1, a_2, a_3\}$ and two in $\{b_1, b_2, b_3\}$.*

Proof: Assume that u is not of Type p2 or p3. Then, w.l.o.g. u has neighbors in both P^1 and P^2.

Case 1: u does not have a neighbor in P^3.

First assume that u has a unique neighbor in P^1 or P^2, say P^1. Let u_1 be the neighbor of u in P^1, and w.l.o.g. assume that $u_1 \neq a_1$. Let u_2 be the neighbor of u in P^2 that is closest to a_2. If $u_2 \neq b_2$, then the node set $P^1 \cup P^2_{a_2u_2} \cup P^3 \cup \{u\}$ induces a $3PC(a_1a_2a_3, u_1)$. If $u_2 = b_2$, then either u is of Type t2 or the node set $P^1_{a_1u_1} \cup P^2 \cup P^3 \cup \{u\}$ induces a $3PC(a_1a_2a_3, u_2)$.

Now assume that u has at least two neighbors in both P^1 and P^2. Let u_1 (resp. v_1) be the neighbor of u in P^1 that is closest to a_1 (resp. b_1). Let u_2 (resp. v_2) be the neighbor of u in P^2 that is closest to a_2 (resp. b_2). First suppose that both u_1v_1 and u_2v_2 are edges. If u is adjacent to both a_1 and a_2, then $P^2 \cup P^3 \cup \{u, a_1\}$ induces an odd wheel with center a_2. So u is not adjacent to both a_1 and a_2, and similarly u is not adjacent to both b_1 and b_2. Hence u is of Type p4. Now assume w.l.o.g. that u_1v_1 is not an edge. If u is not adjacent to all four of the nodes a_1, a_2, b_1 and b_2, then either $P^1_{a_1u_1} \cup P^1_{v_1b_1} \cup P^2_{a_2u_2} \cup P^3 \cup \{u\}$ or $P^1_{a_1u_1} \cup P^1_{v_1b_1} \cup P^2_{v_2b_2} \cup P^3 \cup \{u\}$ induces a $3PC(\Delta, u)$. So u is adjacent to a_1, a_2, b_1 and b_2, and hence it is of Type t4.

Case 2: u has a neighbor in P^3.

For $i \in \{1,2,3\}$, let u_i (resp. v_i) be the neighbor of u in P^i that is closest to a_i (resp. b_i). If u is adjacent to at most one node in $\{a_1, a_2, a_3\}$ and at most one node in $\{b_1, b_2, b_3\}$, then the node set $P^1_{v_1b_1} \cup P^2_{v_2b_2} \cup P^3_{v_3b_3} \cup \{u\}$ induces a $3PC(b_1b_2b_3, u)$. So assume w.l.o.g. that u is adjacent to b_1 and b_2. If u does not have a neighbor in $(P^1 \cup P^2) \setminus \{b_1, b_2\}$, then u is of Type t2p, t3 or t3p. So assume w.l.o.g. that $u_1 \neq b_1$. Suppose u is not of Type t4, t5 ot t6. Then u is adjacent to at most one node of $\{a_1, a_2, a_3\}$. If $u_2 = b_2$ and $u_3 = b_3$, then u is of Type t3p. Otherwise, $P^1_{a_1u_1} \cup P^2_{a_2u_2} \cup P^3_{a_3u_3} \cup \{u\}$ induces a $3PC(a_1a_2a_3, u)$. □

Nodes adjacent to Σ are further classified as follows.

Type t6a: A node u that is of Type t6 w.r.t. Σ, such that none of the paths of Σ is an edge and u has no neighbors in the interior of any of the paths of Σ.

Type t6b: A node u that is of Type t6 w.r.t. Σ but is not of Type t6a w.r.t. Σ.

Type t4d: For distinct $i, j \in \{1,2,3\}$, $N(u) \cap \{a_1, a_2, a_3, b_1, b_2, b_3\} = \{a_1, a_2, a_3, b_1, b_2, b_3\} \setminus \{a_i, b_j\}$.

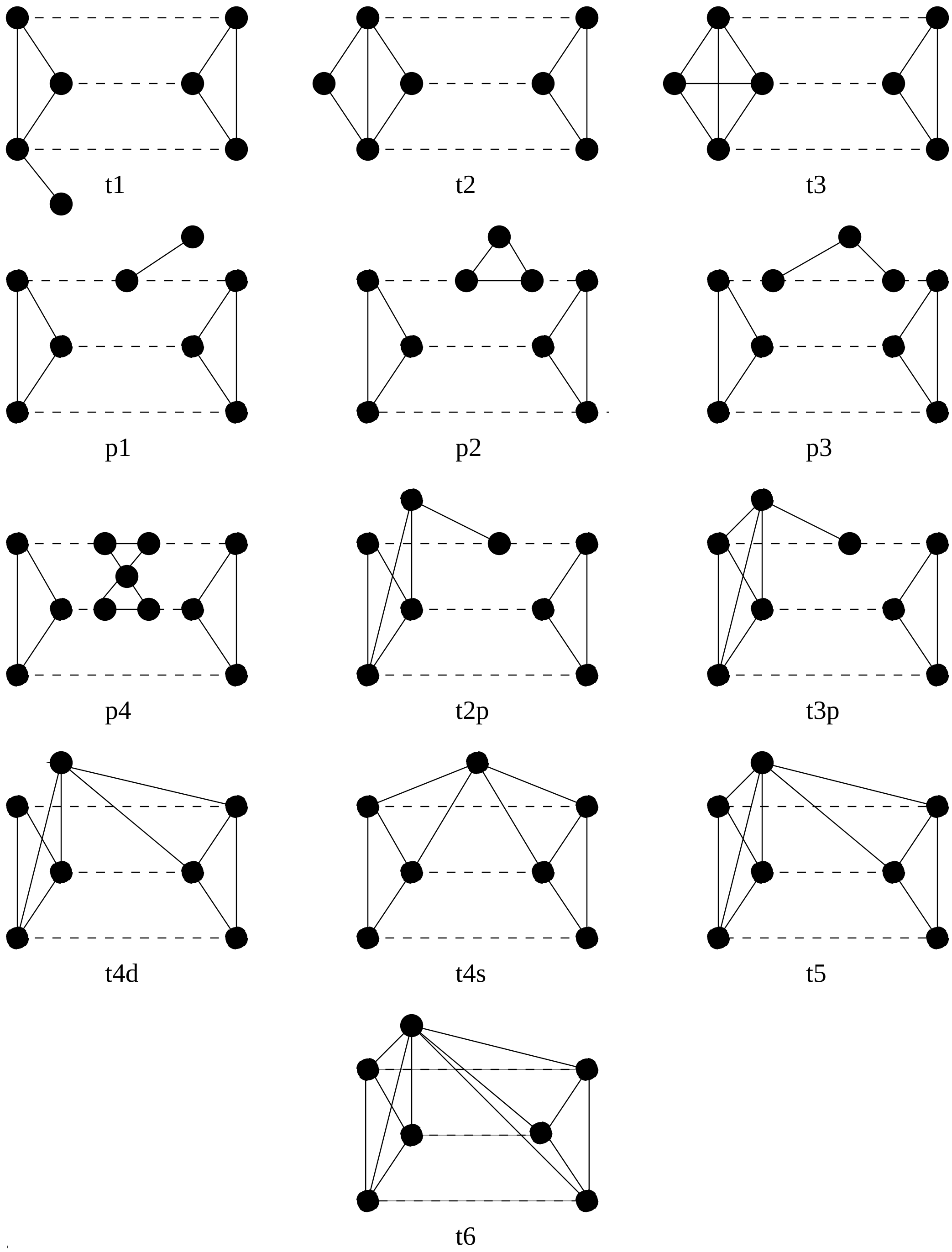

Figure 3: Strongly Adjacent Nodes to a $3PC(\Delta, \Delta)$

Type t4s: For some $i \in \{1,2,3\}$, $N(u) \cap \{a_1,a_2,a_3,b_1,b_2,b_3\} = \{a_1,a_2,a_3,b_1,b_2,b_3\} \setminus \{a_i,b_i\}$.

Lemma 6.2 *Let G be a square-free even-signable graph with no star cutset. If $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ then the following holds.*

(i) If u is of Type t6b w.r.t. Σ, then u is adjacent to all nodes of Σ.

(ii) If u is of Type t5 w.r.t. Σ, say not adjacent to a_3, then none of the paths of Σ is an edge and $N(u) \cap \Sigma = \{a_1,a_2,b_1,b_2,b_3,b'_3\}$.

(iii) If u is of Type t4d w.r.t. Σ, say not adjacent to a_3 and b_2, and a_1b_1 is not an edge, then $N(u) \cap \Sigma = \{a_1,a_2,a'_2,b_1,b_3,b'_3\}$, $\{a_1,a_2,b_1,b_3,b'_1\}$ or $\{a_1,a_2,a'_1,b_1,b_3\}$.

(iv) If u is of Type t4s w.r.t. Σ, say not adjacent to a_3 and b_3, then a_1b_1 and a_2b_2 are not edges.

Proof: Let u be of Type t6 w.r.t. Σ. For $i,j \in \{1,2,3\}$, (H_{ij},u) must be a line wheel or a universal wheel. If (H_{12},u) is a line wheel, then so is (H_{13},u), and hence none of the paths of Σ is an edge and u has no neighbors in the interior of any of the paths of Σ, i.e. u is of Type t6a w.r.t. Σ. If (H_{12},u) is a universal wheel, then so is (H_{13},u), and hence u is adjacent to all nodes of Σ.

Let u be of Type t5 w.r.t. Σ, say not adjacent to a_3. Suppose that (H_{12},u) is a universal wheel. Since G is square-free, a_1b_1 and a_2b_2 cannot both be edges. W.l.o.g. a_1b_1 is not an edge. But then (H_{13},u) is a proper wheel that is not a beetle, contradicting Theorem 4.6. So (H_{12},u) cannot be a universal wheel, and hence it must be a line wheel. But then (H_{13},u) must be a beetle, and so (ii) holds.

Let u be of Type t4d w.r.t. Σ, say not adjacent to a_3 and b_2. Suppose a_1b_1 is not an edge. (H_{12},u) must be a beetle or a line wheel. Suppose (H_{12},u) is a line wheel. Then $N(u) \cap (P^1 \cup P^2) = \{a_1,a_2,b_1,b'_1\}$. If u has a neighbor in $P^3 \setminus b_3$, then (H_{13},u) is a proper wheel that is not a beetle, contradicting Theorem 4.6. So now assume that (H_{12},u) is a beetle. Suppose that u is adjacent to a'_1. Then $N(u) \cap (P^1 \cup P^2) = \{a_1,a_2,a'_1,b_1\}$. If u has a neighbor on $P^3 \setminus b_3$, then (H_{13},u) is a proper wheel that is not a beetle, contradicting Theorem 4.6. Finally suppose that u is adjacent to a'_2. Then $N(u) \cap (P^1 \cup P^2) = \{a_1,a_2,a'_2,b_1\}$. But then (H_{23},u) must be a line wheel and hence $N(u) \cap P^3 = \{b_3,b'_3\}$.

Let u be of Type t4s w.r.t. Σ, say not adjacent to a_3 and b_3. Suppose a_1b_1 is an edge. Since G is square-free, a_2b_2 is not an edge. Node u must have a neighbor in P^3, since otherwise $P^3 \cup \{a_1,a_2,b_1,u\}$ induces an odd wheel with center a_1. So (H_{13},u) must be a line wheel. But then (H_{23},u) is a proper wheel that is not a beetle, contradicting Theorem 4.6. So a_1b_1 is not an edge, and similarly neither is a_2b_2. $\Box$

If node u is of Type p3, t2p or t3p w.r.t. Σ, then a subset of the node set $\Sigma \cup \{u\}$ induces a $\Sigma' = 3PC(\Delta,\Delta)$ that contains u. We say that Σ' is obtained by *substituting u into* Σ. If u is of Type t2p or t3p w.r.t. Σ, and for some $z \in \{a,b\}$ and $i \in \{1,2,3\}$, Σ' does not contain z_i, then we say that u is a *sibling* of z_i.

7 Beetles and T-Parachutes

To prove the main result of this section, we need the following lemma, whose proof appears in [5].

Lemma 7.1 *Let G be a T-parachute $TP(t,v,a,b,z)$ that is not an L-parachute and such that no proper subgraph of G is a parachute or a proper wheel. Then G is one of the following graphs, see Figure 2.*

Type a: *No interior node of P is adjacent to a or b.*

Type b: *An interior node of P is adjacent to a or b, say a, (C_b, a) is a triangle-free wheel and a is adjacent to z.*

Type c: *An interior node of P is adjacent to a or b, say a, (C_b, a) is a twin wheel and a is adjacent to z.*

Theorem 7.2 *Let G be a square-free even-signable graph. Assume that G has no $3PC(\Delta,\Delta)$ with a Type t2, t2p or t4 node. Then G is a triangle-free graph, or the line graph of a triangle-free graph, or G has a star cutset or a 2-join.*

Proof: By Theorem 2.3, the result holds for WP-free graphs. By Theorem 4.6, the result holds when G contains a proper wheel that is not a beetle. By Theorem 5.3, the result holds when G contains an L-parachute. So we may assume that G contains a beetle or a T-parachute.

Let Σ be a beetle or a T-parachute $TP(t,v,a,b,z)$. If Σ is a T-parachute, by Lemma 7.2, we assume that Σ is of Type a, b or c of Figure 2. If Σ is a beetle (H,v), denote the neighbors of v on the hole by a,t,b,z where atv and tbv are the triangles. If Σ is a T-parachute, denote by (H,v) its twin wheel with center v. We assume w.l.o.g. that if Σ is a T-parachute of Type c, then G contains no beetle and no T-parachute of Type a or b. We denote by P the path of Σ from v to z that uses no edge of H and by H_{za} and H_{zb} the subpaths of H from z to a and from z to b that do not contain node t. W.l.o.g. b has no neighbor in the interior of P. Let C be the hole of Σ containing b,v,z. Consider the star $S=(v\cup N(v))\setminus\{t,m\}$ where m is the neighbor of v in Σ distinct from a,b,t. Assume that S is not a cutset separating t from $B=V(\Sigma)\setminus\{a,b,v,t\}$ and let $Q=x_1,\dots,x_n$ be a direct connection from t to B in $G\setminus S$. No node of Q is adjacent to both a and b since G is square-free.

Case 1: $n=1$, or $n>1$ and no node of $Q_{x_1x_{n-1}}$ is adjacent to a or b.

Node x_n has at least one neighbor in C if Σ is a T-parachute of Type b or c and, by symmetry, we assume w.l.o.g. that this is also the case when Σ is a beetle or a T-parachute of Type a.

If x_n has exactly one neighbor p in C, then there is a $3PC(bvt,p)$ since p is distinct from b and v.

Assume x_n has exactly two adjacent neighbors in C. Assume first that one of these neighbors is b and the other is b' adjacent to b in H_{zb}. If $n=1$, there is an odd wheel with center b. So $n>1$. If x_n has no neighbor in $H_{za}\setminus z$, there is a $3PC(x_nbb',t)$. Let z' be the neighbor of z in H_{za}. If x_n has a neighbor in $H_{za} \setminus \{z, z'\}$, then there is a $3PC(x_nbb', v)$. Therefore x_n is adjacent to z'. If $b' \neq z$, (H, x_n) is an odd wheel. So $b' = z$ and (H, x_n) is a twin wheel. Now $Q \cup \Sigma \setminus b$ induces a $3PC(\Delta, \Delta)$ and b is a Type t4s node with respect to it, so the result holds.

Now assume that both neighbors of x_n in C are distinct from b. Then there is a $3PC(\Delta, \Delta)$ with a node of Type t2 (when Σ is a beetle or of Type a) or of Type t2p (when Σ is of Type b) or of Type t4d (when Σ is of Type c). Therefore the result holds.

Assume x_n has two nonadjacent neighbors in C. Then there is a $3PC(bvt, x_n)$ if $n > 1$ or if x_n is not adjacent to b. So assume $n = 1$ and x_1 is adjacent to b. If x_1 has a neighbor in $H_{za} \setminus z$, there is an odd wheel with center t. So all the neighbors of x_1 in Σ are in $C \cup t$. If Σ is a beetle, there is a $3PC(avt, z)$. So Σ is a T-parachute. Assume first that Σ is of Type a. If x_1 has no neighbor in P, there is a $3PC(avt, z)$. If x_1 has a unique neighbor p in P, there is a $3PC(avt, p)$. Therefore x_1 has several neighbors in P. Node z is adjacent to b since, otherwise, there is an odd wheel with center t. Let p be the neighbor of x_1 in P that is closest to z. Then $H_{za} \cup P_{zp} \cup \{t, x_1\}$ induces a hole H_1 and, if $p \neq z$, (H_1, b) is an odd wheel. So $p = z$. Let z' be the neighbor of z in P. If x_1 has a neighbor in $P \setminus \{z, z'\}$, there is a $3PC(bzx_1, v)$. So the neighbors of x_1 in P are exactly z and z' and therefore (C, x_1) is a twin wheel. Then $(\Sigma \setminus b) \cup x_1$ induces a $3PC(\Delta, \Delta)$ and b is a Type t4d strongly adjacent node with respect to it, so the result holds. Now assume that Σ is a T-parachute of Type b or c. Node x_1 is not adjacent to m since, otherwise, m, v, t, x_1 would be a square. Since x_1 has a neighbor in C distinct from b and its neighbor in C, there is an odd wheel with center t.

Case 2: $n > 1$ and some node of $Q_{x_1x_{n-1}}$ is adjacent to a or b.

Let x_j be such a node with lowest index.

Assume first that $j \geq 2$. Denote by d the node among a, b that is adjacent to x_j. Let R be a shortest path from x_n to v in $\Sigma \setminus \{a, b, t\}$ and let R' be a shortest path in $\Sigma \setminus \{t, v, d\}$ from x_n to the node d' among a, b that is distinct from d. Let H_1 be the hole induced by $R \cup Q \cup t$ and H_2 the hole induced by $R' \cup Q \cup t$. We will show that the wheel (H_1, d) is a proper wheel that is not a beetle. (H_1, d) is not a universal wheel since d is not adjacent to x_1 nor a triangle-free wheel since d is adjacent to v and t, nor a twin wheel since x_j is adjacent to d but not to t or v. Suppose (H_1, d) is a line wheel. Then (H_2, d) is a proper wheel. Suppose (H_2, d) is a beetle. Then $j = n-1$ and z is adjacent to both x_n and d, and x_n has a neighbor in $P_{vz} \setminus z$. Note that if $d = b$, then az cannot be an edge (since bz is an edge and H cannot be a square) and hence Σ cannot be a T-parachute of Type a or b. If x_n has a neighbor in $P \setminus \{z, z'\}$ (where z' is the neighbor of z in P), then there is a $3PC(tvd', x_n)$. So z and z' are the only neighbors of x_n in P, and hence there is a $3PC(\triangle, \triangle)$ with a Type t4d node and the result holds. Finally, suppose (H_1, d) is a beetle. Then $d = a$, Σ is of Type c and x_n is adjacent to m. If x_n has no other neighbor in Σ, there is a $3PC(tvd', m)$. If x_n has a neighbor distinct from m and z in C, there is a $3PC(tvd', x_n)$. If x_n has exactly m and z as neighbors, there is a $3PC(\Delta, \Delta)$ with node d being of Type t4d relative to it. So the result holds. Therefore (H_1, d) is a proper wheel that is not a beetle. So the result holds by Theorem 4.6.

Assume now that $j = 1$. Denote by d the node among a, b that is adjacent to x_1.

Case 2.1: No node of $Q_{x_2x_n}$ is adjacent to a or b.

If every chordless path from x_n to t in $\Sigma \setminus \{d, v\}$ contains m, then let R_1 be such a path and let H_3 be the hole $R_1 \cup Q$. Then (H_3, v) is an odd wheel unless $d = b$ and Σ is a T-parachute of Type c. But then $Q \cup \Sigma \setminus v$ induces a $3PC(\Delta, \Delta)$ and v is a strongly adjacent node of Type t4d with respect to it. So the result holds.

Now assume some chordless path R_2 from x_n to t in $\Sigma \setminus \{d, v\}$ does not contain m. If R_2 does not contain neighbors of d, then let H_4 be the hole $(R_2 \setminus t) \cup Q \cup \{v, d\}$. Then (H_4, t) is an odd wheel. So R_2 contains a neighbor of d. Let H_5 be the hole $R_2 \cup Q$. Then (H_5, d) is an odd wheel.

Case 2.2: Some node of $Q_{x_2 x_n}$ is adjacent to a or b.

Let x_k be such a node with lowest index. If x_k is not adjacent to d, then $Q_{x_1 x_k} \cup \{a, v, b\}$ induces a hole H_6 and (H_6, t) is an odd wheel. So x_k is adjacent to d. Let H_7 be a hole in $Q \cup \Sigma \setminus \{a, b\}$ that contains v. Since d is adjacent to v, t, x_1 and x_k, the wheel (H_7, d) is a proper wheel or a universal wheel.

Case 2.2.1: (H_7, d) is a proper wheel.

If (H_7, d) is not a beetle, the result holds by Theorem 4.6. So assume (H_7, d) is a beetle. If $d = a$ then Σ cannot be a T-parachute of Type c. The node in $\{a, b\} \setminus \{d\}$ has no neighbor in Q, since otherwise a subpath of Q together with a, t and b induces an odd wheel with center d. Let R be a chordless path from x_n to t in $\Sigma \setminus \{d, v\}$. Some interior node of R must be adjacent to d since, otherwise, there is an odd wheel with center d. x_n must have a neighbor in Σ distinct from the neighbor d' of d in H_{zd} since, if d' were the only neighbor of x_n in Σ, the assumption that (H_7, d) is a beetle would be contradicted. This implies that $d' = z$ and that x_n has all its neighbors in P. So Σ is not a beetle. Let H_8 be the hole $R \cup Q$. (H_8, d) is a wheel. Since d is adjacent to t and x_1 but not its neighbors on H_8, it is not a beetle. So if (H_8, d) is a proper wheel, the result holds by Theorem 4.6. If (H_8, d) is not a proper wheel, it must be a line wheel. This implies that $k = n$ and that x_n is adjacent to z. If $d = b$ then, since bz is an edge, az cannot be an edge (else H is a square) and so Σ cannot be a T-parachute of Type c. Σ cannot be a T-parachute of Type b since, otherwise, $d = a$ and there is a $3PC(azx_n, v)$. So Σ is a T-parachute of Type a. Since (H_7, d) is a beetle, x_n has a neighbor in $P \setminus z$. If x_n has a neighbor in $P \setminus z$ distinct from the neighbor z' of z, there is a $3PC(dzx_n, v)$. So x_n has exactly three neighbors in Σ, namely d, z and z'. In this case, $Q \cup \Sigma \setminus d$ induces a $3PC(\Delta, \Delta)$ and the node d is a strongly adjacent node of Type t4d with respect to it. So the result holds.

Case 2.2.2: (H_7, d) is a universal wheel.

Then Σ is a T-parachute of Type c and $d = a$. Let H_9 be the hole in $Q \cup \Sigma \setminus \{a, v\}$ that contains b. Then (H_9, a) is a proper wheel that is not a beetle unless $n = 2$ and x_n has a neighbor in $C \setminus \{z, m, v, b\}$. Since $n = 2$, x_2 is not adjacent to m (otherwise there is a 5-wheel with center a) but x_2 is adjacent to z since (H_7, a) is a universal wheel. Let z' be the neighbor of z on C distinct from m and p the neighbor of x_2 closest to b in $H \setminus \{a, z\}$. Suppose $p \neq z'$ and let H_{10} be the hole induced by $H_{bp} \cup \{v, m, z, x_2\}$. Then (H_{10}, a) is an odd wheel. So $p = z'$. But now $\Sigma \setminus a \cup \{x_1, x_2\}$ induces a $3PC(\Delta, \Delta)$ and node a is a Type t4s node. So the result holds. $\square$

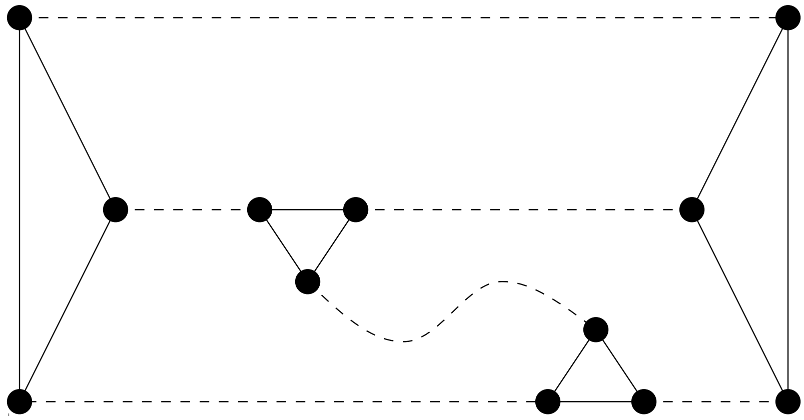

Figure 4: Crosspath

8 Different Paths

In this section, we assume that G is an even-signable graph.

Definition 8.1 *A* crosspath *w.r.t.* $\Sigma = 3PC(\Delta, \Delta)$ *is a chordless path* $P = x_1, \ldots, x_n$ *in* $G \setminus \Sigma$ *that satisfies one of the following:*

- $n = 1$ *and* x_1 *is of Type p4 w.r.t.* Σ*, or*
- $n > 1$*,* x_1 *and* x_n *are of Type p2 w.r.t.* Σ*, with neighbors in different paths of* Σ*, and no interior node of* P *has a neighbor in* Σ*. See Figure 4*

If x_1 *or* x_n *has neighbors in a path* P^i *of* Σ*, we say that* P *is a* P^i-crosspath *w.r.t.* Σ.

Definition 8.2 *A* weak connection *w.r.t.* $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ *is a chordless path* $P = x_1, \ldots, x_n$ *in* $G \setminus \Sigma$ *such that no interior node of* P *has a neighbor in* Σ*,* $N(x_1) \cap \Sigma = p$ *and* $N(x_n) \cap \Sigma = q$*,* $p \neq q$*, and either both* p, q *are contained in* $\{a_1, a_2, a_3\}$ *or both are contained in* $\{b_1, b_2, b_3\}$*.*

Theorem 8.3 *Let* G *be an even-signable graph, let* $\Sigma = 3PC(\Delta, \Delta)$ *and let* $P = x_1, \ldots, x_n$*,* $n > 1$*, be a chordless path in* $G \setminus \Sigma$*. If* $\emptyset \neq N(x_1) \cap \Sigma \subseteq P^i$*,* $\emptyset \neq N(x_n) \cap \Sigma \subseteq P^j$*,* $i \neq j$*, and no interior node of* P *has a neighbor in* Σ*, then* P *is a crosspath or a weak connection w.r.t.* Σ*.*

If, furthermore, G *is square-free and has no star cutset, then* P *is always a crosspath.*

Proof: Assume w.l.o.g. that $i = 1$ and $j = 2$. Since $N(x_1) \cap \Sigma \subseteq P^1$ and $N(x_n) \cap \Sigma \subseteq P^2$, x_1 and x_n are of Type t1, p1, p2 or p3 w.r.t. Σ. Suppose that P is not a crosspath. Then w.l.o.g. x_1 is of Type t1, p1 or p3 w.r.t. Σ. If x_1 is of Type p3 w.r.t. Σ, then we consider the $3PC(\Delta, \Delta)$ obtained by substituting x_1 into Σ. It follows from Lemma 6.1 that x_n is of Type t1, p1, p2 or p3 w.r.t. this new $3PC(\Delta, \Delta)$. So we may assume that x_1 is of Type t1 or p1 w.r.t. Σ and similarly that x_n is of Type t1, p1 or p2 w.r.t. Σ. If x_n is of Type p2 w.r.t. Σ, then the node set $P^1 \cup P^2 \cup P$ induces a $3PC(\Delta, .)$. Hence x_n is also of Type t1 or p1 w.r.t. Σ. Let u_1 (resp. u_2) be the unique neighbor of x_1 (resp. x_n) in Σ. W.l.o.g. $u_1 \neq a_1$. If $u_1 = b_1$ and $u_2 = b_2$, then P is a weak connection, and otherwise the node set $P^1 \cup P^2_{a_2u_2} \cup P^3 \cup P$ induces a $3PC(a_1a_2a_3, u_1)$.

Now assume that G is square-free and has no star cutset. Suppose $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ has a weak connection $P = x_1, \ldots, x_n$ from a_1 to a_2. Let $S = (N(a_1) \cup a_1) \setminus \{x_1, a'_1\}$. Since S is not a star cutset, there exists a direct connection $Q = y_1, \ldots, y_m$ from P to $\Sigma \setminus S$ in $G \setminus S$. The nodes of $P \cup P^1 \cup P^3 \cup a_2$ induce a Mickey Mouse as defined in [6]. By the Mickey Mouse theorem [6], some node of Q is adjacent to both a_2 and a_3. Choose P, Q such that Q is as short as possible subject to the condition that P is a weak connection with one endnode adjacent to a_1 and the other adjacent to either a_2 or a_3. Let y_k be the node of lowest index adjacent to both a_2 and a_3.

Claim 1: *If $k \geq 2$, no node $y_1, \ldots, y_{k-1}$ is adjacent to a_2 or a_3.*

Proof of Claim 1: Suppose not and let y_j be the node of lowest index adjacent to a_2 or a_3 (note that $j = 1$ is possible). Node y_1 has a unique neighbor on P and this neighbor is x_n since, otherwise, our choice of P, Q would be violated. If y_j is adjacent to a_3, there is a $3PC(a_1a_2a_3, x_n)$. So y_j is adjacent to a_2. Let y_i, $j < i \leq k$, be the node of lowest index adjacent to a_3. Then $W = (Q_{y_1y_i}a_3a_1P, a_2)$ is a wheel that is not triangle-free, universal, a twin wheel or a beetle. Suppose it is a line wheel with triangles $a_2x_ny_1$ and $a_1a_2a_3$. Then $i < k$ and therefore there is an L-parachute with middle path $a_2, y_k, y_{k-1}, \ldots, y_i$. But then G has a star cutset by Theorem 5.3, a contradiction. So W is a proper wheel that is not a beetle. But then G has a star cutset by Theorem 4.6, a contradiction. This completes the proof of Claim 1.

Let $S' = (N(a_2) \cup a_2) \setminus \{x_n, a'_2\}$. Since G has no star cutset, there is a direct connection $Q' = z_1, \ldots, z_p$ from P to $\Sigma \setminus S'$ in $G \setminus S'$. By the Mickey Mouse theorem applied to the Mickey Mouse induced by the nodes $P \cup P^2 \cup P^3 \cup a_1$, there is node of Q' adjacent to both a_1 and a_3. Let z_l be the node of lowest index in Q' adjacent to both a_1 and a_3. Let z_t be the node of lowest index in Q' that is adjacent to a_1 or a_3.

Case 1: $t < l$

Suppose first that z_t is adjacent to a_3. Then z_1 is adjacent to a_1, P in a triangle since, otherwise, there is a $3PC(a_1a_2a_3, .)$. Let u, v be the neighbors of z_1 on P. If $u, v \neq a_1$, then there is a $3PC(z_1uv, a_1)$. So u or v coincides with a_1. But then there is an L-parachute induced by a subpath of $P \cup Q'_{z_1z_l} \cup \{a_1, a_2, a_3\}$. So, by Theorem 5.3, there is a star cutset, a contradiction.

So z_t is adjacent to a_1. Let z_r be the node of Q' with lowest index adjacent to a_3. Clearly $t < r \leq l$. Let H be the hole passing through a_3 in $Q' \cup P \cup a_3$. Then (H, a_1) is a wheel that is not triangle-free, universal or a twin wheel. If $r < l$, then the wheel (H, a_1) is not a beetle. Suppose it is a line wheel with triangles $a_1x_1z_t$ and $a_1a_2a_3$. Then there is an L-parachute with middle path being a subpath of $a_1, z_l, z_{l-1}, \ldots, z_t$. If (H, a_1) is a line wheel with triangles $a_1z_tz_{t+1}$ and $a_1a_2a_3$, then there is an L-parachute with middle path being a subpath of $a_1, z_l, \ldots, z_r$. But then G has a star cutset by Theorem 5.3, a contradiction. So

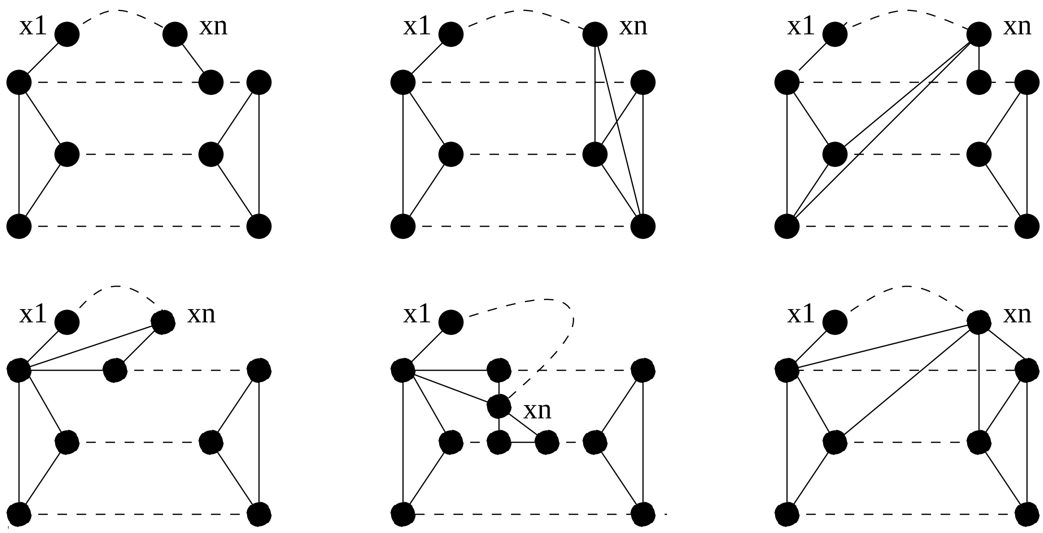

Figure 5: Paths from a Type t1 node

W is a proper wheel that is not a beetle. But then G has a star cutset by Theorem 4.6, a contradiction. Therefore $r = l$. Now (H, a_1) is not an L-wheel. Since G does not have a star cutset, (H, a_1) cannot be a proper wheel by Theorem 4.6, so it must be a beetle. Note that Q is a path from the top of the beetle to the bottom that does not induce connected diamonds with the nodes of the beetle (since y_k is adjacent to a_2), contradicting Lemma 4.4.

Case 2: $t = l$

Let R be a shortest path from y_k to z_l in $P \cup Q \cup Q'$. If R contains neither x_1 nor x_n, then $R \cup \{a_1, a_2, a_3\}$ induces an odd wheel with center a_3. So R contains x_1 or x_n and Q, Q' have no adjacent nodes. W.l.o.g. assume that y_1 is adjacent to x_1 or x_n. Then there is a $3PC(a_1a_3z_l, x_1)$ or a $3PC(a_2a_3y_k, x_n)$. □

Lemma 8.4 *Let $P = x_1, \ldots, x_n$ be a chordless path in $G \setminus \Sigma$ such that x_1 is of Type t1 w.r.t. Σ, say adjacent to a_1, x_n has a neighbor in $\Sigma \setminus \{a_1, a_2, a_3\}$, and no interior node of P has a neighbor in Σ. Then one of the following holds.*

(i) x_n is of Type t1, p1 or p3 w.r.t. Σ, with a neighbor in P^1.

(ii) x_n is of Type t2, t2p or t3p w.r.t. Σ and $N(x_n) \cap (P^2 \cup P^3) = \{b_2, b_3\}$.

(iii) x_n is of Type t2p or t3p w.r.t. Σ with at least two neighbors in $\{a_1, a_2, a_3\}$.

(iv) x_n is of Type p2 or p4 w.r.t. Σ and it is adjacent to a_1.

(v) x_n is of Type t4, t5 or t6 w.r.t. Σ.

Proof: Suppose x_n is of Type t1 or p1 with a unique neighbor u in Σ. If u is not in P^1, then $P^2 \cup P^3 \cup P \cup x$ induces a $3PC(a_1a_2a_3, u)$. Similarly, if x_n is of Type p3, then it must satisfy

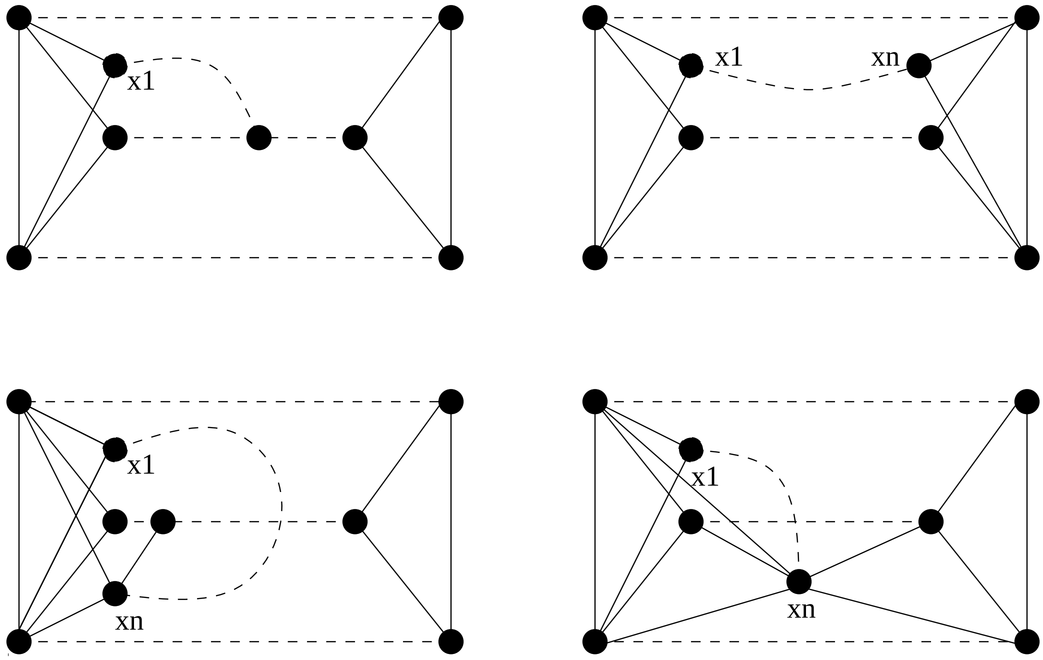

Figure 6: Paths from a Type t2 node

(i), else there is a $3PC(a_1a_2a_3, x_n)$. Suppose x_n is of Type p2, with neighbors u and v in Σ, and w.l.o.g. assume that u and v are not in P^3. If x_n does not satisfy (iv), then $P^1 \cup P^2 \cup P$ induces a $3PC(uvx_n, a_1)$. If x_n is of Type t2 and it does not satisfy (ii), then w.l.o.g. we may assume that it is adjacent to b_1 and b_3, and hence $P \cup P^2 \cup P^3$ induces a $3PC(a_1a_2a_3, b_3)$. Suppose x_n is of Type t2p or t3p and it does not satisfy (ii) or (iii). W.l.o.g. x_n is adjacent to b_1, b_3 and it has a neighbor in $P^2 \setminus b_2$. Then $(P \cup P^1 \cup P^2) \setminus b_2$ contains a $3PC(a_1a_2a_3, x_n)$. If x_n is of Type t3, then it is adjacent to b_1, b_2, b_3 and $P \cup P^1 \cup P^2$ induces a $3PC(b_1b_2x_n, a_1)$. Finally assume that x_n is of Type p4. If the neighbors of x_n in Σ are contained in $P^2 \cup P^3$, then $((P^2 \cup P^3) \setminus \{b_2, b_3\}) \cup P \cup a_1$ contains a $3PC(a_1a_2a_3, x_n)$. Else, we may assume w.l.o.g. that the neighbors of x_n in Σ are contained in $P^1 \cup P^2$. If x_n does not satisfy (v), then $(\Sigma \setminus \{b_2, a_1'\}) \cup P$ contains a $3PC(a_1a_2a_3, x_n)$. □

Lemma 8.5 *Let $P = x_1, \ldots, x_n$ be a chordless path in $G \setminus \Sigma$ such that x_1 is of Type t2 w.r.t. Σ, say adjacent to a_1 and a_3, x_n has a neighbor in $\Sigma \setminus \{a_1, a_2, a_3\}$, and no interior node of P has a neighbor in Σ. Then one of the following holds.*

(i) x_n is of Type t1, p1 or p3 w.r.t. Σ, with a neighbor in P^2.

(ii) x_n is of Type t2, t2p or t3p w.r.t. Σ and $N(x_n) \cap (P^1 \cup P^3) = \{b_1, b_3\}$.

(iii) x_n is of Type t2p or t3p w.r.t. Σ with at least two neighbors in $\{a_1, a_2, a_3\}$.

(iv) x_n is of Type t4, t5 or t6 w.r.t. Σ

Proof: Suppose x_n is of Type t1 or p1 with u being its unique neighbor in Σ and (i) does not hold. Then w.l.o.g. u is in P^1 and hence $P^1 \cup P^3 \cup P$ induces a $3PC(x_1a_1a_3, u)$. Suppose x_n is of Type p3 and it does not satisfy (i). W.l.o.g. the neighbors of x_n in Σ are in P^1. If $n = 2$ and x_n is adjacent to a_1, then $P^1 \cup P^2 \cup P \cup a_3$ contains an odd wheel with center a_1, and otherwise $P^1 \cup P^3 \cup P$ contains a $3PC(x_1a_1a_3, x_n)$. Suppose x_n is of Type p2, with neighbors u and v in Σ, and w.l.o.g. assume that u and v are not in P^3. If x_n is not adjacent to a_1, then $P^1 \cup P^2 \cup P$ induces a $3PC(x_nuv, a_1)$. So x_n is adjacent to a_1, and hence $P^1 \cup P^2 \cup P \cup a_3$ induces an odd wheel with center a_1 when $n > 2$ and $P^1 \cup P^3 \cup P$ induces an odd wheel with center a_1 when $n = 2$. If x_n is of Type t2, adjacent to b_2 and say b_1, then $P^1 \cup P^2 \cup P$ induces a $3PC(x_nb_1b_2, a_1)$. So, if x_n is of Type t2, then it satisfies (ii). Similarly, if x_n is of Type t3, then there is a $3PC(x_nb_1b_2, a_1)$. If x_n is of Type t2p or t3p, and it does not satisfy (ii) or (iii), then w.l.o.g. we may assume that x_n is adjacent to b_1, b_2 and a node of $P^3 \setminus b_3$. But then $P^1 \cup P^2 \cup P$ induces a $3PC(x_nb_1b_2, a_1)$. Finally assume that x_n is of Type p4. If the neighbors of x_n in Σ are contained in $P^1 \cup P^3$, then w.l.o.g. x_n is not adjacent to a_3 and hence $(\Sigma \setminus \{a_1, a_3'\}) \cup P$ contains a $3PC(b_1b_2b_3, x_n)$. Otherwise, w.l.o.g. we may assume that the neighbors of x_n in Σ are contained in $P^1 \cup P^2$, and hence $(\Sigma \setminus \{a_1, a_2\}) \cup P$ contains a $3PC(b_1b_2b_3, x_n)$ □

Lemma 8.6 *Let $P = x_1, \ldots, x_n$ be a chordless path in $G \setminus \Sigma$ such that x_1 is of Type t3 w.r.t. Σ, say adjacent to a_1, a_2 and a_3, x_n has a neighbor in $\Sigma \setminus \{a_1, a_2, a_3\}$, and no interior node of P has a neighbor in Σ. Then one of the following holds.*

(i) x_n is of Type p2 or t3 w.r.t. Σ.

(ii) $n = 2$, x_n is of Type t2p or t3p w.r.t. Σ with at least two neighbors in $\{b_1, b_2, b_3\}$ and x_n is adjacent to a_1, a_2 or a_3.

(iii) x_n is of Type t2p or t3p w.r.t. Σ with at least two neighbors in $\{a_1, a_2, a_3\}$.

(iv) $n = 2$, x_n is of Type p3 w.r.t. Σ adjacent to a_1, a_2 or a_3.

(v) x_n is of Type t4, t5 or t6 w.r.t Σ.

Proof: If x_n is of Type t1 ot p1 with its neighbor u in say P_1, then there is a $3PC(a_1a_2x_1, u)$. If x_n is of Type p3, with neighbors in say P^1, and (iv) does not hold, then $P^2 \cup P^3 \cup P$ contains a $3PC(x_1a_1a_2, x_n)$. By Lemma 8.5, x_n cannot be of Type t2 w.r.t. Σ. Suppose x_n is of Type t2p or t3p, but (ii) and (iii) do not hold. Then w.l.o.g. x_n is a sibling of b_1, and hence $(P^1 \cup P^2 \cup P) \setminus b_1$ contains a $3PC(x_1a_1a_2, x_n)$. Finally assume that x_n is of Type p4, with neighbors w.l.o.g. in $P^2 \cup P^3$. Then $(\Sigma \cup P) \setminus \{a_1, a_3\}$ contains a $3PC(b_1b_2b_3, x_n)$. □

9 Type t4, t5 and t6b Nodes

Theorem 9.1 *Let G be a square-free even-signable graph. If G contains a $\Sigma = 3PC(\Delta, \Delta)$ with a Type t4, t5 or t6b node then G has a star cutset.*

Proof: Assume G has no star cutset. Then by Theorem 4.6, G contains no proper wheel that is not a beetle.

Let $\mathcal{C}$ be the set of all ordered pairs Σ, u such that $\Sigma = 3PC(\Delta, \Delta)$ and u is of Type t4, t5 or t6b w.r.t. Σ.

For $\Sigma, u \in \mathcal{C}$, we assume w.l.o.g. that if u is of Type t5 w.r.t. Σ then u is not adjacent to a_3, if u is of Type t4d w.r.t. Σ then u is not adjacent to a_3 and b_2, and if u is of Type t4s w.r.t. Σ then u is not adjacent to a_3 and b_3.

For $\Sigma, u \in \mathcal{C}$ define the corresponding sets S as follows. If u is of Type t5 or t6 w.r.t. Σ, then let $S = (N(u) \cup u) \setminus (\Sigma \setminus \{a_1, a_2, b_2, b_3\})$. If u is of Type t4d w.r.t. Σ, then let $S = (N(u) \cup u) \setminus (\Sigma \setminus \{a_1, a_2, b_1, b_3\})$. If u is of Type t4s w.r.t. Σ, then let $S = (N(u) \cup u) \setminus (\Sigma \setminus \{a_1, a_2, b_1, b_2\})$. Since S is not a star cutset, there exists a direct connection $P = x_1, \dots, x_n$ in $G \setminus S$ from $P^1 \cup P^2$ to P^3. Let Σ, u be chosen from $\mathcal{C}$ so that the cardinality of $N(u) \cap \Sigma$ is minimized and, subject to this, the size of the corresponding P is minimized.

Claim 1: *No node of P is of Type t4, t5 or t6 w.r.t. Σ.*

Proof of Claim 1: It is enough to show that if v and w are both of Type t4, t5 or t6 w.r.t. Σ, then vw is an edge. Suppose not. Since v and w both have at least two neighbors in each of the sets $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, for some $i, j \in \{1, 2, 3\}$, a_i and b_j are common neighbors of v and w. Since $\{a_i, b_j, u, v\}$ cannot induce a square, $a_i b_j$ is an edge. W.l.o.g. $i = j = 2$, $N(v) \cap \{a_1, a_2, a_3\} = \{a_1, a_2\}$ and $N(w) \cap \{a_1, a_2, a_3\} = \{a_2, a_3\}$. But then $\{a_1, a_2, a_3, b_2, v, w\}$ induces an odd wheel with center a_2. This completes the proof of Claim 1.

Claim 2: *No node of P is of Type p3 w.r.t. Σ.*

Proof of Claim 2: Suppose x_i is of Type p3 w.r.t. Σ. Let Σ' be obtained from Σ by substituting x_i into Σ. Since $x_i \in G \setminus S$, node x_i is not adjacent to u. Therefore $|N(u) \cap \Sigma'| \leq |N(u) \cap \Sigma|$. Therefore Σ', u and P', where $P' = P_{x_1 x_{i-1}}$ or $P' = P_{x_{i+1} x_n}$, contradict our choice of Σ, u and P. This completes the proof of Claim 2.

Claim 3: *No node of P is of Type t2p or t3p w.r.t. Σ.*

Proof of Claim 3: Suppose that x_i is of Type t2p or t3p w.r.t. Σ and let Σ' be obtained from Σ by substituting x_i for its sibling. Note that u cannot be of Type t6 w.r.t. Σ, since otherwise u is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. In particular, u is not adjacent to a_3.

Suppose x_i is a sibling of a_1. Since u is adjacent to a_2, b_1 and at least one of b_2, b_3, and it is not adjacent to x_i and a_3, node u must be of Type t2p or t3p w.r.t Σ', adjacent to b_3. In particular, $N(u) \cap P^3 = \{b_3\}$ and b_1 is the unique neighbor of u in the $x_i b_1$-path of Σ'. Node x_i is not adjacent to b_1, else $\{a_2, b_1, x_i, u\}$ induces a square. Hence (H_{13}, u) must be a line wheel with u adjacent to a_1'. But then $(P^1 \setminus \{a_1, b_1\}) \cup P^3 \cup \{a_2, u, x_i\}$ contains a $3PC(x_i a_2 a_3, u)$.

Suppose x_i is a sibling of a_2. Since u is adjacent to a_1, b_1 and at least one of b_2, b_3, and it is not adjacent to a_3 and x_i, it must be of Type t3p w.r.t. Σ', and hence of Type t5 w.r.t. Σ. Since u is of Type t3p w.r.t. Σ', $N(u) \cap P^3 = \{b_3\}$. But this contradicts Lemma 6.2 applied to Σ and u.

Suppose x_i is a sibling of a_3. Then $i > 1$ and u is of the same type w.r.t. Σ' as it is w.r.t. Σ. But then Σ', u and $P_{x_1 x_{i-1}}$ contradict our choice of Σ, u and P.

Suppose x_i is a sibling of b_1. Then u is of Type t2p or t4d w.r.t. Σ', adjacent to b_3. If u is of Type t4d w.r.t. Σ', then Σ', u contradict our choice of Σ, u. So u is of Type t2p w.r.t. Σ'. In particular, $N(u) \cap P^2 = \{a_2\}$. So u must be of Type t4d w.r.t. Σ. Node x_i cannot be adjacent to a_1, else $\{a_1, b_3, x_i, u\}$ induces a square. So $a_1 b_1$ is not an edge. By Lemma 6.2, $N(u) \cap P^3 = \{b_3\}$ and u has a neighbor in $P^1 \setminus \{a_1, b_1\}$. So there is a subpath P' of $P^1 \setminus \{a_1, b_1\}$ such that one endnode of P' is adjacent to u, the other to x_i and no proper subpath of P' has this property. But then $P^2 \cup P' \cup \{b_3, x_i, u\}$ induces a $3PC(b_2 b_3 x_i, u)$.

Suppose x_i is a sibling of b_2. Then u must be of Type t4d w.r.t. Σ'. Node x_i is not adjacent to a_2, else $\{a_2, b_3, x_i, u\}$ induces a square. So $i = 1$. Since u is adjacent to b_3, node b_3 is in S and hence $n > 1$. But then Σ', u and $P_{x_2 x_n}$ contradict our choice of Σ, u and P.

Finally suppose that x_i is a sibling of b_3. Then u is of Type t4s w.r.t. Σ'. If u is of Type t5 w.r.t. Σ, then Σ', u contradict our choice of Σ, u. So u is of Type t4s w.r.t. Σ. Hence $i = n$ and $n > 1$. But then Σ', u and $P_{x_1 x_{i-1}}$ contradict our choice of Σ, u and P. This completes the proof of Claim 3.

Claim 4: *If x_i is of Type p4 w.r.t. Σ, then $i = 1$ and the neighbors of x_i in Σ are contained in $P^1 \cup P^2$.*

Proof of Claim 4: Suppose x_i is of Type p4 w.r.t. Σ. If the neighbors of x_i in Σ are contained in $P^1 \cup P^2$ then $i = 1$.

Suppose that the neighbors of x_i in Σ are contained in $P^k \cup P^3$ for $k = 1$ or 2. For $j = k, 3$, let u_j (resp. v_j) be the neighbor of x_i is P^j that is closest to a_j (resp. b_j). If u is of Type t6b w.r.t. Σ, then by Lemma 6.2, u is adjacent to all nodes of Σ and hence $\{u, x_i, v_k, u_3\}$ induces a square. Therefore, u is not adjacent to a_3.

First suppose that x_i is adjacent to a_3. Then x_i is not adjacent to a_k and so $(\Sigma \cup x_i) \setminus P^3_{v_3 b_3}$ induces a $\Sigma' = 3PC(a_1 a_2 a_3, u_k v_k x_i)$. Since u is adjacent to a_1, a_2, b_1 and it is not adjacent to a_3, x_i, it must be of Type t4s w.r.t. Σ'. Hence u is adjacent to u_k and v_k. Node u cannot have neighbors in P^3, since otherwise Σ', u would contradict the choice of Σ, u. So u is of Type t4s w.r.t. Σ. Since u is adjacent to $u_k \neq a_k$, (H_{12}, u) must be a universal wheel. Then by Lemma 6.2, $a_1 b_1$ and $a_2 b_2$ are not edges and so both (H_{13}, u) and (H_{23}, u) must be twin wheels. Hence both P^1 and P^2 are of length 2. But then $\{a_k, a_3, u_k, x_i\}$ induces a square. Therefore x_i is not adjacent to a_3.

Let $\Sigma' = 3PC(a_1 a_2 a_3, x_i v_3 u_3)$ induced by $(\Sigma \cup x_i) \setminus P^k_{v_k b_k}$. Since u is adjacent to a_1, a_2 and at least one of b_2, b_3 (i.e. it has a neighbor in the $a_3 v_{3-k}$-path of Σ'), and it is not adjacent to a_3 and x_i, it must be of Type t4d w.r.t. Σ'. If $k = 1$, Σ', u contradicts our choice of Σ, u since b_1 is adjacent to u and belongs to $\Sigma \setminus \Sigma'$. So $k = 2$. If $a_1 b_1$ is an edge, then $\{u, x_i, u_3, b_1\}$ induces a square. But then by Lemma 6.2, $v_3 = b_3$ and $N(u) \cap \Sigma = \{a_1, a_2, b_1, b_3, b'_3\}$. Hence $P^2 \cup \{u, x_i, b_1, b'_3\}$ induces a $3PC(u_2 v_2 x_i, u)$. This completes the proof of Claim 4.

Claim 5: *$n > 1$, x_1 is of Type t1, p1, p2, t2, t3 or p4 w.r.t. Σ, and x_n is of Type t1, p1,* *p2, t2 or t3 w.r.t.* Σ.

Proof of Claim 5: Follows from Claims 1, 2, 3 and 4.

Claim 6: *If* a_1 *(resp.* a_2*) has a neighbor in the interior of* P*, then* b_2 *and* b_3 *(resp.* b_1 *and* b_3*) do not.*

Proof of Claim 6: Suppose that x_i and x_j are nodes of the interior of P such that x_i is adjacent to a_1, x_j is adjacent to b_2 or b_3, and no proper subpath of $P_{x_ix_j}$ has this property.

First suppose that x_j is adjacent to both b_2 and b_3. Then, by the definition of S, b_1 has no neighbor in the interior of P and u is of Type t5 or t6 w.r.t. Σ. By Claims 1 and 3, x_j is of Type t2 w.r.t. Σ. By Lemma 8.5 applied to a subpath of $P_{x_ix_j}$, a_2 has no neighbor in $P_{x_ix_j}$. Then $P^2 \cup P^3 \cup P_{x_ix_j} \cup a_1$ induces a $\Sigma' = 3PC(a_1a_2a_3, x_jb_2b_3)$. If u is of Type t6 (resp. t5) w.r.t. Σ then it is of Type t5 (resp. t4d) w.r.t. Σ', contradicting our choice of Σ, u.

Next suppose that x_j is adjacent to b_2 and not to b_3. If b_1 does not have a neighbor in $P_{x_ix_j}$, then $P^1 \cup P^3 \cup P_{x_ix_j} \cup b_2$ induces a $3PC(b_1b_2b_3, a_1)$. If a_2 does not have a neighbor in $P_{x_ix_j}$, then $P^2 \cup P^3 \cup P_{x_ix_j} \cup a_1$ induces a $3PC(a_1a_2a_3, b_2)$. So both a_2 and b_1 have a neighbor in $P_{x_ix_j}$. Let x_k and x_l be nodes of $P_{x_ix_j}$ such that x_k is adjacent to a_2, x_l to b_1 and no proper subpath of $P_{x_kx_l}$ has this property. If $j \neq k, l$ then $P^2 \cup P^3 \cup P_{x_kx_l} \cup b_1$ induces a $3PC(b_1b_2b_3, a_2)$. If $i \neq k, l$ then $P^1 \cup P^3 \cup P_{x_kx_l} \cup a_2$ induces a $3PC(a_1a_2a_3, b_1)$. By Claim 1, $i \neq j$. If $j = k$ and $i = l$, then by Claim 2, $\{a_1, a_2, b_1, b_2\}$ induces a square. So $j = l$ and $i = k$. Hence $P^1 \cup P^2 \cup P_{x_ix_j}$ induces a $3PC(a_1a_2x_i, b_1b_2x_j)$. Since $b_1, b_2 \in S$, u is of Type t4s w.r.t. Σ and hence w.r.t. Σ' as well. If $i < j$ let $P' = P_{x_1x_{i-1}}$ and otherwise let $P' = P_{x_1x_{j-1}}$. Then Σ', u and P' contradict our choice of Σ, u and P.

Finally suppose that x_j is adjacent to b_3 and not to b_2. So b_2 has no neighbor in $P_{x_ix_j}$. If a_2 has no neighbor in $P_{x_ix_j}$, then $P^2 \cup P^3 \cup P_{x_ix_j} \cup a_1$ induces a $3PC(a_1a_2a_3, b_3)$. So a_2 has a neighbor in $P_{x_ix_j}$. Let x_k be such a neighbor that is closest to x_j. By Claims 1 and 3, $i \neq j$. Suppose $i \neq k$. Then by Lemma 8.4 applied to a subpath of $P_{x_kx_j}$, b_1 has a unique neighbor x_j in $P_{x_kx_j}$. Since $b_1, b_3 \in S$, u is of Type t4d w.r.t. Σ. Let $\Sigma' = 3PC(a_1a_2a_3, b_1x_jb_3)$ induced by $P^1 \cup P^3 \cup P_{x_kx_j} \cup a_2$. Then u is of Type t4d w.r.t. Σ' as well. If $i > j$ let $P' = P_{x_{k+1}x_n}$ and if $i < j$ let $P' = P_{x_{j+1}x_n}$. Then Σ', u and P' contradict our choice of Σ, u and P. So $i = k$ and hence x_i is of Type t2 w.r.t. Σ. By Lemma 8.5 applied to a subpath of $P_{x_ix_j}$, b_1 does not have a neighbor in $P_{x_ix_j}$. Hence $P^1 \cup P^2 \cup P_{x_ix_j} \cup b_3$ induces a $\Sigma' = 3PC(a_1a_2x_i, b_1b_2b_3)$. Note that u is of Type t5 or t4d w.r.t. Σ'. If $i < j$ let $P' = P_{x_1x_{i-1}}$ and otherwise let $P' = P_{x_1x_{j-1}}$. Then Σ', u and P' contradict our choice of Σ, u and P.

Therefore, if a_1 has a neighbor in the interior of P, then b_2 and b_3 do not.

Now suppose that a_2 and at least one of b_1, b_3 has a neighbor in the interior of P. Let x_i and x_j be nodes of the interior of P such that x_i is adjacent to a_2, x_j is adjacent to b_1 or b_3, and no proper subpath of $P_{x_ix_j}$ has this property.

First suppose that x_j is adjacent to both b_1 and b_3. Then a_1 has no neighbor in the interior of P by the first part of Claim 6 and u is of Type t4d w.r.t. Σ. Let $\Sigma' = 3PC(a_1a_2a_3, b_1x_jb_3)$ induced by $P^1 \cup P^3 \cup P_{x_ix_j} \cup a_2$. Then u is of Type t4d w.r.t. Σ'. If $i < j$ let $P' = P_{x_{j+1}x_n}$ and otherwise let $P' = P_{x_{i+1}x_n}$. Then Σ', u and P' contradict our choice of Σ, u and P.

Next suppose that x_j is adjacent to b_1 and not to b_3. If a_1 does not have a neighbor in $P_{x_ix_j}$, then $P^1 \cup P^3 \cup P_{x_ix_j} \cup a_2$ induces a $3PC(a_1a_2a_3, b_1)$. So a_1 has a neighbor in $P_{x_ix_j}$, and hence b_2 does not by the first part of Claim 6. But then $P^2 \cup P^3 \cup P_{x_ix_j} \cup b_1$ induces a

$3PC(b_1b_2b_3, a_2)$.

Finally suppose that x_j is adjacent to b_3 and not to b_1. Then a_1 does not have a neighbor in the interior of P by the first part of Claim 6, and hence $P^1 \cup P^3 \cup P_{x_ix_j} \cup a_2$ induces a $3PC(a_1a_2a_3, b_3)$. This completes the proof of Claim 6.

Claim 7: *If a_1 has a neighbor in the interior of P, then b_1 does not.*

Proof of Claim 7: Suppose both a_1 and b_1 have a neighbor in the interior of P. Then, by Claim 6, a_2, b_2 and b_3 do not. Since $b_1 \in S$, u is of Type t4 w.r.t. Σ.

Suppose a_1b_1 is not an edge. Let x_i and x_j be nodes of the interior of P such that x_i is adjacent to a_1, x_j to b_1, and no proper subpath of $P_{x_ix_j}$ has this property. Let $\Sigma' = 3PC(a_1a_2a_3, b_1b_2b_3)$ induced by $P^2 \cup P^3 \cup P_{x_ix_j} \cup \{a_1, b_1\}$. If $i > j$ let $P' = P_{x_{i+1}x_n}$ and otherwise let $P' = P_{x_{j+1}x_n}$. Note that u is of the same type w.r.t. Σ' as it is w.r.t. Σ. But then Σ', u and P' contradict our choice of Σ, u and P.

Therefore, a_1b_1 is an edge. By Lemma 6.2, u is of Type t4d w.r.t. Σ and hence x_n has a neighbor in $P^3 \setminus b_3$. Let x_j (resp. x_i) be the node of the interior of P with highest index adjacent to b_1 (resp. a_1). Suppose x_n is of Type t1, p1 or p2 w.r.t. Σ. By Theorem 8.3 applied to $P_{x_ix_n}$ or $P_{x_jx_n}$, $i = j$ and x_n is of Type p2 w.r.t. Σ. Let u_3 (resp. v_3) be the neighbor of x_n in P^3 that is closest to a_3 (resp. b_3). Let $\Sigma' = 3PC(a_1b_1x_i, u_3v_3x_n)$ induced by $P^1 \cup P^3 \cup P_{x_ix_n}$. Since u is adjacent to a_1, b_1 and b_3, and to no node of $P_{x_ix_n}$, it must be of Type t4s w.r.t. Σ', adjacent to u_3 and v_3. But then Σ', u contradicts the choice of Σ, u since a_2 is a neighbor of u in Σ but not Σ'. Therefore, x_n is of Type t2 or t3 w.r.t. Σ, adjacent to a_3. If x_n is not adjacent to a_2, then $P^2 \cup P^3 \cup P_{x_jx_n} \cup b_1$ induces a $3PC(b_1b_2b_3, a_3)$. So x_n is adjacent to a_2.

Suppose that x_1 is of Type t1, p1 or p2 w.r.t. Σ. Since a_1b_1 is an edge and $a_1, b_1 \in S$, the neighbors of x_1 in Σ are contained in P^2. If x_1 is of Type t1 or p1, then $P^2 \cup P^3 \cup P$ induces a $3PC(a_2a_3x_n, \cdot)$. So x_1 is of Type p2. Let u_2 (resp. v_2) be the neighbor of x_1 in P^2 that is closest to a_2 (resp. b_2). Let x_k be the node of the interior of P with lowest index adjacent to a_1 or b_1. By Theorem 8.3 applied to $P_{x_1x_k}$, x_k is adjacent to both a_1 and b_1. But then $P^1 \cup P^2 \cup P_{x_1x_k}$ induces a $\Sigma' = 3PC(a_1b_1x_k, u_2v_2x_1)$. Since u is adjacent to a_1, b_1 and a_2, and no node of $P_{x_1x_k}$, it must be of Type t4s w.r.t. Σ'. But then Σ', u contradicts the choice of Σ, u since b_3 is a neighbor of u in Σ but not Σ'.

Suppose that x_1 is of Type p4 w.r.t. Σ. Define u_2 and v_2 as before. Then $u_2 \neq a_2$, and hence $P^2 \cup P^3 \cup P$ induces a $\Sigma' = 3PC(a_1a_3x_n, u_2v_2x_1)$. Since u is adjacent to a_2 and b_3, and to no node of $P \cup a_3$, it must be of Type p4 w.r.t. Σ'. In particular, u is adjacent to a_2'. But then (H_{12}, u) is a proper wheel that is not a beetle.

Therefore, x_1 is of Type t2 or t3 w.r.t. Σ. Since $b_1, b_3 \in S$, x_1 is adjacent to b_2. Node x_1 must be adjacent to b_3, else $P^2 \cup P^3 \cup P$ induces a $3PC(a_2a_3x_n, b_2)$. Then $P^2 \cup P^3 \cup P$ induces a $\Sigma' = 3PC(a_2a_3x_n, b_2b_3x_1)$. Since u is adjacent to a_2 and b_3 and to no node of $P \cup \{a_3, b_2\}$, it must be of Type p4 w.r.t. Σ'. In particular, u is adjacent to a_2'. But then (H_{12}, u) is a proper wheel that is not a beetle. This completes the proof of Claim 7.

Claim 8: *If a_2 has a neighbor in the interior of P, then b_2 does not.*

Proof of Claim 8: Suppose that both a_2 and b_2 have a neighbor in the interior of P. Then, by Claim 6, a_1, b_1 and b_3 do not. Since $b_2 \in S$, u cannot be of Type t4d w.r.t. Σ. By an analogous argument as in Claim 7, a_2b_2 is an edge. Hence, by Lemma 6.2, u is of Type t6 w.r.t. Σ adjacent to all nodes of Σ. Then $b_3 \in S$ and so b_3 cannot be the unique neighbor of x_n in P^3. Let x_i (resp. x_j) be the node of the interior of P with highest index adjacent to a_2 (resp. b_2).

Suppose that x_n is of Type t1, p1 or p2 w.r.t. Σ. By Theorem 8.3 applied to $P_{x_ix_n}$ or $P_{x_jx_n}$, $i = j$ and x_n is of Type p2 w.r.t. Σ. Hence $P^2 \cup P^3 \cup P_{x_ix_n}$ induces a $\Sigma' = 3PC(a_2b_2x_i, u_3v_3x_n)$, where u_3 and v_3 are the neighbors of x_n in P^3. Since u is adjacent to all nodes of Σ and no node of $P_{x_ix_n}$, it must be of Type t4s w.r.t. Σ', contradicting our choice of Σ, u.

Therefore, x_n is of Type t2 or t3 w.r.t. Σ, adjacent to a_3. Suppose x_n is adjacent to a_1. Then $P^1 \cup P^3 \cup P_{x_jx_n} \cup b_2$ induces a $\Sigma' = 3PC(a_1x_na_3, b_1b_2b_3)$. Since u is of Type t6 w.r.t. Σ, it is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. So x_n is not adjacent to a_1. But then $P^1 \cup P^3 \cup P_{x_jx_n} \cup b_2$ induces a $3PC(b_1b_2b_3, a_3)$. This completes the proof of Claim 8.

Claim 9: *If a_1 or a_2 has a neighbor in the interior of P, then $N(x_n) \cap \Sigma \subseteq \{a_1, a_2, a_3\}$.*

Proof of Claim 9: Let x_i be the node of the interior of P with highest index adjacent to a_1 or a_2, and suppose that x_n has a neighbor in $\Sigma \setminus \{a_1, a_2, a_3\}$. By Claims 6, 7 and 8, b_1, b_2 and b_3 do not have neighbors in the interior of P.

First suppose that x_i is adjacent to both a_1 and a_2. By Lemma 8.5 applied to $P_{x_ix_n}$, and since x_n has a neighbor in P^3 and is of Type t1, p1, p2, t2, or t3 by Claim 5, x_n must be of Type t1 or p1 w.r.t. Σ. Let $\Sigma' = 3PC(a_1a_2x_i, b_1b_2b_3)$ contained in $(\Sigma \setminus a_3) \cup P_{x_ix_n}$. If u is of Type t6 w.r.t. Σ, then it is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. So u is of the same type w.r.t. Σ' as it is w.r.t. Σ. But then Σ', u and $P_{x_1x_{i-1}}$ contradict our choice of Σ, u and P.

Next suppose that x_i is adjacent to a_2 and not to a_1. By Lemma 8.4 applied to $P_{x_ix_n}$, x_n is of Type t2 w.r.t. Σ, adjacent to b_1 and b_3. Let $\Sigma' = 3PC(a_1a_2a_3, b_1x_nb_3)$ contained in $(\Sigma \setminus b_2) \cup P_{x_ix_n}$. By definition of S, x_n must be of Type t4s w.r.t. Σ. But then x_n is a strongly adjacent node relative to Σ' that violates Lemma 6.1.

Finally suppose that x_n is adjacent to a_1 and not to a_2. By Lemma 8.4 applied to $P_{x_ix_n}$, x_n is of Type t2 w.r.t. Σ, adjacent to b_2 and b_3. Then u is of Type t4s w.r.t. Σ. Let $\Sigma' = 3PC(a_1a_2a_3, x_nb_2b_3)$ induced by $P^2 \cup P^3 \cup P_{x_ix_n} \cup a_1$. Then u violates Lemma 6.1 w.r.t. Σ'. This completes the proof of Claim 9.

Claim 10: *If b_1 or b_2 has a neighbor in the interior of P, then $N(x_n) \cap \Sigma \subseteq \{b_1, b_2, b_3\}$.*

Proof of Claim 10: Let x_i be the node of the interior of P with highest index adjacent to b_1 or b_2, and suppose that x_n has a neighbor in $\Sigma \setminus \{b_1, b_2, b_3\}$. By Claims 6, 7 and 8, a_1 and a_2 do not have neighbors in the interior of P.

First suppose that x_i is adjacent to both b_1 and b_2. Then u is of Type t4s w.r.t. Σ, and hence b_3 does not have a neighbor in the interior of P. By Lemma 8.5 applied to $P_{x_ix_n}$, x_n is of Type t1 or p1 w.r.t. Σ. Then $(\Sigma \setminus b_3) \cup P_{x_ix_n}$ contains a $\Sigma' = 3PC(a_1a_2a_3, b_1b_2x_i)$. Node u is of Type t4s w.r.t. Σ', and hence Σ', u and $P_{x_1x_{i-1}}$ contradict our choice of Σ, u and P.

Next suppose that x_i is adjacent to b_1 and not to b_2. If x_n is of Type t1, p1 or p2 w.r.t. Σ, then $P^1 \cup P^2 \cup (P^3 \setminus b_3) \cup P_{x_ix_n}$ contains a $3PC(a_1a_2a_3, b_1)$. So x_n is of Type t2 or t3 w.r.t. Σ, adjacent to a_3. Node x_n cannot be adjacent to both a_1 and a_2, else $P^1 \cup P^2 \cup P_{x_ix_n}$

induces a $3PC(a_1a_2x_n, b_1)$. Suppose x_n is adjacent to a_1. Then x_n is of Type t2 w.r.t. Σ, and so by Lemma 8.5, x_i is the unique neighbor of b_3 in $P_{x_ix_n}$. Hence $P^1 \cup P^3 \cup P_{x_ix_n}$ induces a $\Sigma' = 3PC(a_1x_na_3, b_1x_ib_3)$. Since $b_1, b_3 \in S$, u is of Type t4d w.r.t. Σ, and hence it violates Lemma 6.1 applied to Σ'. Hence x_n is of Type t2 w.r.t. Σ, adjacent to a_2 and a_3. If b_3 has a neighbor in $P_{x_ix_n}$, then a subpath of $P_{x_ix_n}$ contradicts Lemma 8.5 applied to Σ. So b_3 does not have a neighbor in $P_{x_ix_n}$ and hence $P^2 \cup P^3 \cup P_{x_ix_n} \cup b_1$ induces a $\Sigma' = 3PC(x_na_2a_3, b_1b_2b_3)$. If u is of Type t6 w.r.t. Σ then it is of Type t5 w.r.t. Σ', and hence Σ', u contradicts our choice of Σ, u. If u is of Type t4s or t5 w.r.t. Σ, then by Lemma 6.2, u and Σ' violate Lemma 6.1. So u is of Type t4d w.r.t. Σ, and hence of Type t2p w.r.t. Σ'. In particular, $N(u) \cap P^3 = \{b_3\}$. So b_2 has no neighbors in the interior of P. If b_3 has a neighbor in the interior of P, then $P^2 \cup P^3 \cup P_{x_2x_n}$ contains a $3PC(x_na_2a_3, b_3)$. So b_3 has no neighbors in the interior of P. Suppose a_1b_1 is not an edge. Then by Lemma 6.2 (iii), $P^1 \cup P_{x_ix_n}\{a_3, u\}$ induces an odd wheel with center u. So a_1b_1 is an edge, and hence x_1 has a neighbor in $P^2 \setminus a_2$. Let x_j be the node of the interior of P with lowest index adjacent to b_1. If x_1 is of Type t1, p1 or p2 w.r.t. Σ, then $P_{x_1x_j}$ contradicts Theorem 8.3. If x_1 is of Type p4 w.r.t. Σ, then $(P^2 \setminus b_2) \cup P^3 \cup \{x_1, u\}$ contains an odd wheel with center u. So x_1 is of Type t2 or t3 w.r.t. Σ adjacent to b_2. If x_1 is not adjacent to b_3 then $P \cup P^2 \cup P^3$ induces a $3PC(x_na_2a_3, b_2)$. So x_1 is adjacent to b_3, and hence $P \cup P^2 \cup P^3$ induces a $\Sigma" = 3PC(x_na_2a_3, x_1b_2b_3)$. But then u and $\Sigma"$ violate Lemma 6.1.

Finally suppose that x_i is adjacent to b_2 and not to b_1. If x_n is of Type t1, p1 or p2 w.r.t. Σ, then $(\Sigma \setminus b_3) \cup P_{x_ix_n}$ contains a $3PC(a_1a_2a_3, b_2)$. So x_n is of Type t2 or t3 w.r.t. Σ, adjacent to a_3. Node x_n cannot be adjacent to both a_1 and a_2, else $P^1 \cup P^2 \cup P_{x_ix_n}$ induces a $3PC(a_1a_2x_n, b_2)$. Suppose x_n is adjacent to a_2. Then x_n is of Type t2 w.r.t. Σ and so by Lemma 8.5, x_i is the unique neighbor of b_3 in $P_{x_ix_n}$. Let $\Sigma' = 3PC(a_2a_3x_n, b_2b_3x_1)$ induced by $P^2 \cup P^3 \cup P_{x_ix_n}$. Since $b_2, b_3 \in S$, u is of Type t5 or t6 w.r.t. Σ. If u is of Type t6 w.r.t. Σ, then u is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. If u is of Type t5 w.r.t. Σ, then u violates Lemma 6.1 applied to Σ'. Therefore x_n is adjacent to a_1 and not to a_2. By Lemma 8.5, b_3 cannot have a neighbor in $P_{x_ix_n}$. Then $P^1 \cup P^3 \cup P_{x_ix_n} \cup b_2$ induces a $\Sigma' = 3PC(a_1x_na_3, b_1b_2b_3)$. If u is of Type t6 w.r.t. Σ, then u is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. If u is of Type t5 w.r.t. Σ, then by Lemma 6.2, u is adjacent to b_3' and hence it violates Lemma 6.1 applied to Σ'. If u is of Type t4 w.r.t. Σ, then it violates Lemma 6.1 applied to Σ'. This completes the proof of Claim 10.

By Claim 5, we now consider the following cases.

Case 1: x_n is of Type t1, p1 or p2 w.r.t. Σ.
Case 1.1: x_1 is of Type t1, p1 or p2 w.r.t. Σ.

We first show that a_1 and a_2 do not have a neighbor in the interior of P. Assume not. Then by Claims 6, 7 and 8, b_1, b_2 and b_3 do not. By Claim 9, $N(x_n) \cap \Sigma = \{a_3\}$. W.l.o.g. we may assume that the neighbors of x_1 in Σ are contained in P^1. If a_2 does not have a neighbor in the interior of P, then $(\Sigma \setminus a_1) \cup P$ contains a $3PC(b_1b_2b_3, a_3)$. Let x_i be the node of P with lowest index adjacent to a_2. Then $(\Sigma \setminus a_1) \cup P_{x_1x_i}$ contains a $3PC(b_1b_2b_3, a_2)$.

Suppose that b_3 is the unique neighbor of x_n is Σ. Then u is of Type t4s w.r.t. Σ and so x_1 has a neighbor in $(P^1 \cup P^2) \setminus \{a_1, a_2, b_1, b_2\}$. We may assume w.l.o.g. that the neighbors of x_1 in Σ are contained in P^2. Then $(\Sigma \setminus b_2) \cup P$ contains either a $3PC(a_1a_2a_3, b_3)$ (if b_1 has no neighbors in the interior of P) or a $3PC(a_1a_2a_3, b_1)$ (otherwise). So b_3 is not the unique neighbor of x_n in Σ, and hence by Claim 10, b_1 and b_2 do not have neighbors in the interior of P.

Suppose b_3 has a neighbor in the interior of P, and let x_i be such a neighbor with lowest index. Then $P_{x_1x_i}$ contradicts Theorem 8.3. So b_3 does not have a neighbor in the interior of P. By Theorem 8.3 applied to P, x_1 and x_n must both be of Type p2 w.r.t. Σ.

Suppose that the neighbors of x_1 in Σ are contained in P^2. Let u_2 (resp. v_2) be the neighbor of x_1 in P^2 that is closest to a_2 (resp. b_2). Let u_3 (resp. v_3) be the neighbor of x_n in P^3 that is closest to a_3 (resp. b_3). Let Σ' be the $3PC(u_2v_2x_1, u_3v_3x_n)$ induced by $P^2 \cup P^3 \cup P$. Let $P'_{u_2u_3}$ be the u_2u_3-path of Σ' and similarly define $P'_{v_2v_3}$. Since u is adjacent to a_2, it has a neighbor in $P'_{u_2u_3} \setminus u_3$. Since u is adjacent to b_2 or b_3, it has a neighbor in $P'_{v_2v_3}$. Note that u cannot be of Type t2 w.r.t. Σ' since then u is of Type t4s w.r.t. Σ and a_2b_2 is not an edge by Lemma 6.2, a contradiction. If u is of Type t4s w.r.t. Σ', then our choice of Σ, u is contradicted. Therefore, u is of Type p4 w.r.t. Σ'. By Lemma 6.2, u is not of Type t6 w.r.t. Σ, and hence u is not adjacent to a_3. So the neighbors of u in $P'_{u_2u_3}$ are a_2 and a'_2. But then $P'_{u_2u_3} \cup P \cup \{u, a_1\}$ induces an odd wheel with center a_2.

Analogous argument holds when the neighbors of x_1 in Σ are contained in P^1.

Case 1.2: x_1 is of Type t2 or t3 w.r.t. Σ.

Then x_1 is adjacent to b_1 or b_2, and u is not of Type t4s w.r.t. Σ. So $b_3 \in S$.

Suppose that a_1 or a_2 has a neighbor in the interior of P. Then by Claims 6, 7 and 8, b_1, b_2 and b_3 do not. By Claim 9, $N(x_n) \cap \Sigma = \{a_3\}$. Let x_i (resp. x_j) be the node of P with lowest index adjacent to a_1 (resp. a_2). Suppose x_1 is adjacent to b_3, and say b_2. If a_2 has no neighbor in the interior of P then $P^2 \cup P^3 \cup P$ induces a $3PC(b_2b_3x_1, a_3)$, and otherwise $P^2 \cup P^3 \cup P_{x_1x_j}$ induces a $3PC(b_2b_3x_1, a_2)$. Hence x_1 is not adjacent to b_3, and so it is adjacent to b_1 and b_2. So x_1 is of Type t2 w.r.t. Σ and hence by Lemma 8.5, $i = j$. Let $\Sigma' = 3PC(a_1a_2x_i, b_1b_2x_1)$ induced by $P^1 \cup P^2 \cup P_{x_1x_i}$. If u is of Type t5 or t6 w.r.t. Σ, then u is of Type t4s w.r.t. Σ', contradicting our choice of Σ, u. If u is of Type t4d w.r.t. Σ, then u is a strongly adjacent node to Σ' violating Lemma 6.1. Therefore a_1 and a_2 do not have neighbors in the interior of P.

Since $b_3 \in S$, b_3 cannot be the unique neighbor of x_n in Σ, and so by Claim 10, b_1 and b_2 do not have a neighbor in the interior of P. Node x_1 must be adjacent to both b_1 and b_2, else $(\Sigma \setminus b_3) \cup P$ contains a $3PC(a_1a_2a_3, b_1)$ or a $3PC(a_1a_2a_3, b_2)$. Hence, $(\Sigma \setminus b_3) \cup P$ contains a $\Sigma' = 3PC(a_1a_2a_3, b_1b_2x_1)$. If u is of Type t5 or t6 w.r.t. Σ, then u is of Type t4s or t5 respectively w.r.t. Σ', contradicting our choice of Σ, u. If u is of Type t4d w.r.t. Σ, then u is a strongly adjacent node to Σ' that violates Lemma 6.1.

Case 1.3: x_1 is of Type p4 w.r.t. Σ.

Suppose that a_1 or a_2 has a neighbor in the interior of P. Then by Claims 6, 7 and 8, b_1, b_2 and b_3 do not. By Claim 9, $N(x_n) \cap \Sigma = \{a_3\}$. But then $(\Sigma \setminus \{a_1, a_2\}) \cup P$ contains a $3PC(b_1b_2b_3, x_1)$. Therefore a_1 and a_2 do not have neighbors in the interior of P.

Then $(\Sigma \cup P) \setminus \{b_1, b_2\}$ contains a $3PC(a_1a_2a_3, x_1)$.

Case 2: x_n is of Type t2 or t3 w.r.t. Σ, adjacent to a_3.

Then by Claim 10, b_1 and b_2 do not have neighbors in the interior of P. Suppose b_3 has a neighbor in the interior of P and let x_i be such a neighbor with highest index. By Claims 6 and 7, a_1 and a_2 do not have neighbors in the interior of P. Then $P_{x_i x_n}$ contradicts Lemma 8.4. Therefore, b_3 does not have a neighbor in the interior of P.

Case 2.1: x_1 is of Type t1, p1 or p2 w.r.t. Σ.

Suppose that the neighbors of x_1 in Σ are contained in P^1. If a_2 does not have a neighbor in $P_{x_2 x_n}$, then $(\Sigma \setminus a_1) \cup P$ contains a $3PC(b_1 b_2 b_3, a_3)$. Let x_i be the node of $P_{x_2 x_n}$ with lowest index adjacent to a_2. If $i \neq n$ then $(\Sigma \setminus a_1) \cup P_{x_1 x_i}$ contains a $3PC(b_1 b_2 b_3, a_2)$. So $i = n$ and hence $(\Sigma \setminus a_1) \cup P$ contains a $\Sigma' = 3PC(x_n a_2 a_3, b_1 b_2 b_3)$. If u is of Type t6 w.r.t. Σ, then it is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. If u is of Type t5 w.r.t. Σ, then by Lemma 6.2, $N(u) \cap \Sigma = \{a_1, a_2, b_1, b_2, b_3, b_3'\}$, and hence u is a strongly adjacent node to Σ' that violates Lemma 6.1. If u is of Type t4s w.r.t. Σ, then u violates Lemma 6.1 w.r.t. Σ'. So u is of Type t4d w.r.t. Σ, and hence it must be of Type t2p w.r.t. Σ'. In particular, u has no neighbor in $P^3 \setminus b_3$. Since x_1 has a neighbor in $P^1 \setminus S$, $a_1 b_1$ is not an edge. Hence, by Lemma 6.2, $P^3 \cup P \cup \{u, a_2\} \cup P^1 \setminus \{a_1\}$ contains a $3PC(x_n a_2 a_3, u)$.

Now suppose that the neighbors of x_1 in Σ are contained in P^2. If a_1 does not have a neighbor in $P_{x_2 x_n}$, then $(\Sigma \setminus a_2) \cup P$ contains a $3PC(b_1 b_2 b_3, a_3)$. Let x_i be the node of $P_{x_2 x_n}$ with lowest index adjacent to a_1. If $i \neq n$ then $(\Sigma \setminus a_2) \cup P_{x_1 x_i}$ contains a $3PC(b_1 b_2 b_3, a_1)$. So $i = n$ and hence $(\Sigma \setminus a_2) \cup P$ contains a $\Sigma' = 3PC(a_1 x_n a_3, b_1 b_2 b_3)$. If u is of Type t6 w.r.t. Σ, then it is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. If u is of Type t5 w.r.t. Σ then by Lemma 6.2, u is adjacent to b_3' and hence it violates Lemma 6.1 w.r.t. Σ'. If u is of Type t4 w.r.t. Σ, then it violates Lemma 6.1 w.r.t. Σ'.

Case 2.2: x_1 is of Type t2 or t3 w.r.t. Σ.

Then the neighbors of x_1 in Σ are contained in $\{b_1, b_2, b_3\}$, and u is not of Type t4s w.r.t. Σ. Let x_i be the node of $P_{x_2 x_n}$ with lowest index adjacent to a_1 or a_2.

Suppose x_1 is adjacent to b_1 and b_2. Node x_i must be adjacent to both a_1 and a_2, else $P^1 \cup P^2 \cup P_{x_1 x_i}$ induces a $3PC(b_1 b_2 x_1, \cdot)$. Then $P^1 \cup P^2 \cup P_{x_1 x_i}$ induces a $\Sigma' = 3PC(a_1 a_2 x_n, b_1 b_2 x_1)$. If u is of Type t5 or t6 w.r.t. Σ, then it is of Type t4s w.r.t. Σ', contradicting our choice of Σ, u. If u is of Type t4d w.r.t. Σ, then it violates Lemma 6.1 w.r.t. Σ'. Therefore, x_1 is not adjacent to both b_1 and b_2, and hence it is adjacent to b_3.

Suppose that x_1 is adjacent to b_1. Not both a_1 and a_2 can be adjacent to x_i, else $P^1 \cup P^2 \cup P_{x_1 x_i}$ induces a $3PC(a_1 a_2 x_i, b_1)$. Suppose a_1 is adjacent to x_i. By Lemma 8.5, $i = n$, and hence $P^1 \cup P^3 \cup P$ induces a $\Sigma' = 3PC(a_1 x_n a_3, b_1 x_1 b_3)$. If u is of Type t6 w.r.t. Σ, then it is of Type t4s w.r.t. Σ', contradicting our choice of Σ, u. If u is of Type t5 w.r.t. Σ, then it violates Lemma 6.1 w.r.t. Σ'. Since $N(x_1) \cap \Sigma = \{b_1, b_3\}$, u cannot be of Type t4d w.r.t. Σ. Therefore a_1 is not adjacent to x_i, and so a_2 is. By Lemma 8.5, $i \neq n$, and hence $P^1 \cup P^3 \cup P_{x_1 x_i}$ induces a $\Sigma' = 3PC(a_1 a_2 a_3, b_1 x_1 b_3)$. If u is of Type t6 (resp. t5) w.r.t. Σ, then it is of Type t5 (resp. t4d) w.r.t. Σ', contradicting our choice of Σ, u. Since $N(x_1) \cap \Sigma = \{b_1, b_3\}$, u cannot be of Type t4d w.r.t. Σ.

Therefore $N(x_1) \cap \Sigma = \{b_2, b_3\}$. Hence u must be of Type t4d w.r.t. Σ. If a_2 is not adjacent to a node of $P_{x_2 x_n}$, then $P^2 \cup P^3 \cup P$ induces a $3PC(x_1 b_2 b_3, a_3)$. Let x_j be the node of $P_{x_2 x_n}$ with lowest index adjacent to a_2. If $j \neq n$ then $P^2 \cup P^3 \cup P_{x_1 x_j}$ induces a $3PC(x_1 b_2 b_3, a_2)$. So $j = n$ and hence x_n is the unique neighbor of a_2 in P. Then $P^2 \cup P^3 \cup P$ induces a $\Sigma' = 3PC(x_n a_2 a_3, x_1 b_2 b_3)$. Since u is of Type t4d w.r.t. Σ, it must be of Type p4 w.r.t. Σ', adjacent to a_2' and b_3'. Note that $a_1 b_1$ cannot be an edge, else (H_{12}, u) induces an odd wheel. So, by Lemma 6.2, $N(u) \cap \Sigma = \{a_1, a_2, a_2', b_1, b_3, b_3'\}$. If $i \neq n$, then (H, u), where H is the hole induced by $P^3 \cup P_{x_1 x_i} \cup a_1$, is an odd wheel. So $i = n$. If a_1 is adjacent to x_n, then $P^1 \cup P^3 \cup P$ induces a $3PC(a_1 a_3 x_n, b_3)$. Hence a_1 has no neighbor in P. Let H be the hole induced by $P \cup \{a_1, a_3, b_1, b_2, u\}$. Then (H, a_2) is an odd wheel.

Case 2.3: x_1 is of Type p4 w.r.t. Σ.
Then $(\Sigma \cup P) \setminus \{a_1, a_2\}$ contains a $3PC(b_1 b_2 b_3, a_3)$.

Case 3: x_n is of Type t2 or t3 w.r.t. Σ, adjacent to b_3.
Then $b_3 \notin S$ and hence u is of Type t4s w.r.t. Σ. By Claim 9, a_1 and a_2 do not have neighbors in the interior of P.

Suppose that x_1 is of Type t1, p1 or p2 w.r.t. Σ. We may assume w.l.o.g. that the neighbors of x_1 in Σ are contained in P^1. Suppose b_2 has a neighbor in the interior of P, and let x_i be the node of P with lowest index adjacent to b_2. Then $(\Sigma \setminus b_1) \cup P_{x_1 x_i}$ contains a $3PC(a_1 a_2 a_3, b_2)$. Hence b_2 has no neighbors in the interior of P. If b_2 is not adjacent to x_n, then $(\Sigma \setminus b_1) \cup P$ contains a $3PC(a_1 a_2 a_3, b_3)$. Therefore b_2 is adjacent to x_n and hence $(\Sigma \setminus b_1) \cup P$ contains a $\Sigma' = 3PC(a_1 a_2 a_3, x_n b_2 b_3)$. Since u is of Type t4s w.r.t. Σ, it violates Lemma 6.1 w.r.t. Σ'.

Since u is of Type t4s w.r.t. Σ, $a_1, a_2, b_1, b_2 \in S$ and hence x_1 cannot be of Type t2 or t3 w.r.t. Σ. So x_1 is of Type p4 w.r.t. Σ. But then $(\Sigma \cup P) \setminus \{b_1, b_2\}$ contains a $3PC(a_1 a_2 a_3, x_1)$.
□

10 Attachments

In this section, we assume that G is a square-free even-signable graph. Furthermore, we assume that G has no star cutset. So by Theorem 9.1, there are no Type t4, t5 and t6b nodes.

Definition 10.1 *Let $\Sigma = 3PC(a_1 a_2 a_3, b_1 b_2 b_3)$ and let u be of Type t1 w.r.t. Σ, adjacent to say a_3. A chordless path $P = y_1, \ldots, y_m$ in $G \setminus (\Sigma \cup u)$ is an* attachment *of u to Σ if u is adjacent to y_1 and to no other node of P, no node of $P \setminus y_m$ has a neighbor in $\Sigma \setminus a_3'$ and one of the following holds.*

(i) y_m is of Type t1, p1 or p3 w.r.t. Σ, it is not adjacent to a_3 and it has a neighbor in $P^3 \setminus \{a_3, a_3'\}$.

(ii) y_m is of Type t2, t2p or t3p w.r.t. Σ, adjacent to b_1, b_2 and no node of $(P^1 \setminus b_1) \cup (P^2 \setminus b_2) \cup a_3$.

(iii) y_m is of Type p2 w.r.t. Σ, $N(a_3') \cap P = \{y_{m-1}, y_m\}$ and y_m is not adjacent to a_3.

Definition 10.2 *Let $\Sigma = 3PC(a_1 a_2 a_3, b_1 b_2 b_3)$ and let u be of Type t1 w.r.t. Σ, adjacent to say a_3. A chordless path $P = y_1, \ldots, y_m$ in $G \setminus (\Sigma \cup u)$ is a* bad connection *of u to Σ if u is adjacent to y_1 and to no other node of P, no node of $P \setminus y_m$ has a neighbor in $\Sigma \setminus a_3'$ and y_m is of Type t2 or t2p w.r.t. Σ, adjacent to a_1 and a_2.*

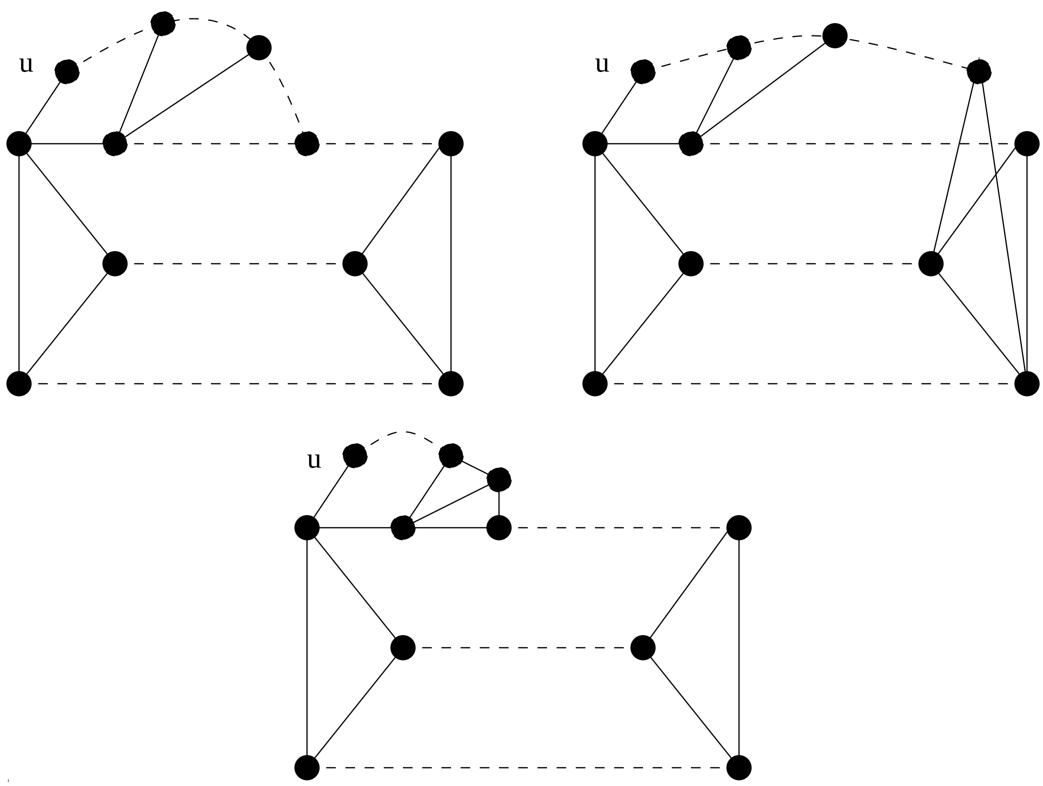

Figure 7: Attachments of a node of Type t1

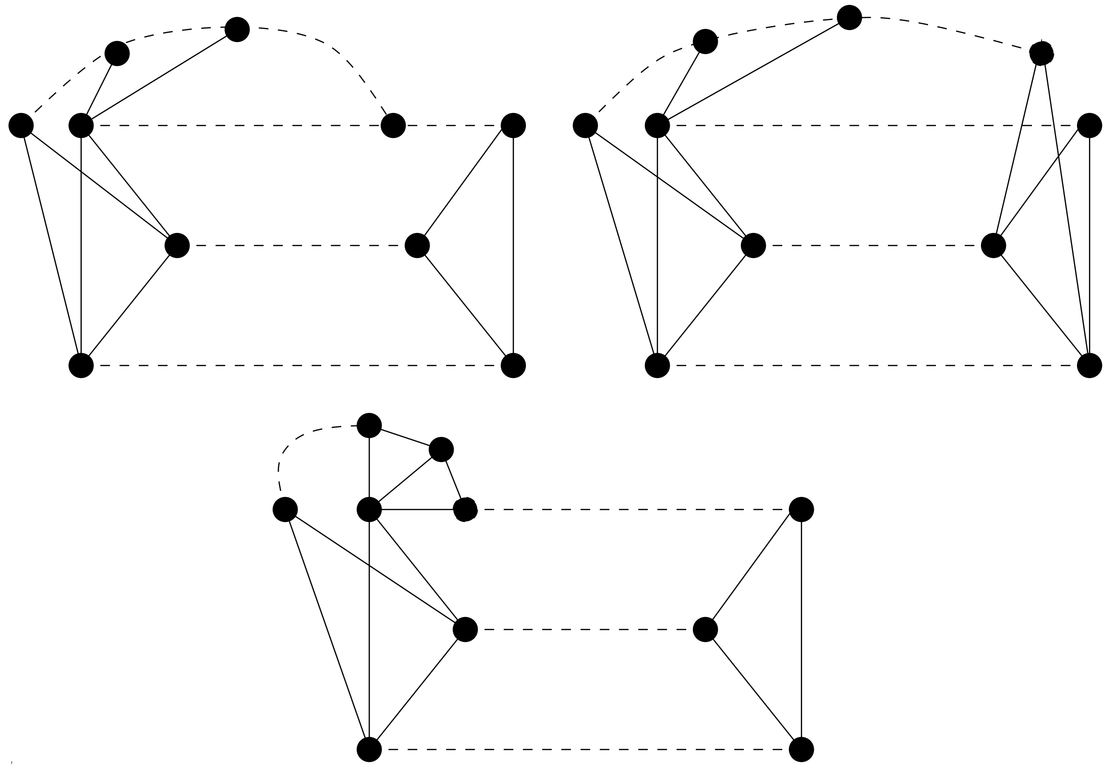

Figure 8: Attachments of a node of Type t2

Definition 10.3 *Let $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ and let u be of Type t2 w.r.t. Σ, adjacent to say a_2 and a_3. A chordless path $P = y_1, \ldots, y_m$ in $G \setminus (\Sigma \cup u)$ is an* attachment *of u to Σ if u is adjacent to y_1 and to no other node of P, no node of $P \setminus y_m$ has a neighbor in $\Sigma \setminus a_1$ and one of the following holds.*

(i) y_m is of Type t1, p1 or p3 w.r.t. Σ and it has a neighbor in $P^1 \setminus a_1$.

(ii) y_m is of Type t2, t2p or t3p w.r.t. Σ, adjacent to b_2, b_3 and no node of $(P^2 \cup P^3) \setminus \{b_2, b_3\}$.

(iii) y_m is of Type p2 w.r.t. Σ and $N(a_1) \cap P = \{y_{m-1}, y_m\}$.

Suppose u is of Type t1 or t2 w.r.t. Σ. If there exists an attachment of u to Σ, we say that u is *attached* to Σ. If P is an attachment of u to Σ, then a subset of $\Sigma \cup P \cup u$ induces a $\Sigma' = 3PC(\Delta, \Delta)$ that contains u and two of the paths of Σ. We say that Σ' is *obtained* from Σ by substituting u and P into Σ.

Lemma 10.4 *Every Type t1 node w.r.t. $\Sigma = 3PC(\Delta, \Delta)$ is either attached to Σ or it has a bad connection to Σ.*

Proof: Let $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ and assume that u is of Type t1 w.r.t. Σ, say adjacent to a_3. Let $S = (N(a_3) \cup a_3) \setminus u$ and let $x_1, \ldots, x_n$ be a direct connection from u to $\Sigma \setminus S$

in $G \setminus S$. Let $x_0 = u$ and $P = x_0, x_1, \ldots, x_n$. By definition of S, x_n cannot be of Type t6a w.r.t. Σ and the only nodes of Σ that can have a neighbor in $P \setminus \{x_0, x_n\}$ are a_1, a_2 and a_3'.

First suppose that a_1 or a_2 has a neighbor in $P \setminus x_n$. Let x_i be the node of $P \setminus x_n$ with lowest index adjacent to a_1 or a_2. Since G is square-free, x_i is not adjacent to a_3'. If x_i is adjacent to exactly one of a_1, a_2, then Theorem 8.3 is contradicted. So x_i is adjacent to both a_1 and a_2. Thus $P_{x_1 x_i}$ is a bad connection of u to Σ.

So now we may assume that a_1 and a_2 do not have neighbors in $P \setminus x_n$. If a_3' does not have a neighbor in $P \setminus x_n$, then the result follows from Lemma 8.4 applied to P. So we may assume that a_3' has a neighbor in $P \setminus x_n$. Let x_i be such a neighbor with highest index.

If x_n is of Type t1, p1, p2 or p3 with neighbors in Σ contained in $P^1 \cup P^2$, then $P_{x_i x_n}$ contradicts Theorem 8.3. If x_n is of Type t1, p1 or p3 with neighbors in Σ contained in P^3, then the result follows. Suppose x_n is of Type p2 and its neighbors in Σ are contained in P^3. Then x_n is adjacent to a_3', else $P^1 \cup P^3 \cup P_{x_i x_n}$ induces a $3PC(\Delta, a_3')$. But then $P^1 \cup P^3 \cup P$ must induce a beetle with center a_3', and hence the result follows.

Suppose x_n is of Type t2, t2p, t3 or t3p w.r.t. Σ and the result does not hold. Then we may assume w.l.o.g. that x_n is adjacent to b_1, b_3 and to no node of $(P^1 \cup P^3) \setminus \{b_1, b_3\}$. But then $P^1 \cup P^3 \cup P_{x_i x_n}$ induces a $3PC(b_1 b_3 x_n, a_3')$.

Suppose x_n is of Type p4 w.r.t. Σ. If the neighbors of x_n in Σ are contained in $P^1 \cup P^2$, then $(\Sigma \setminus \{a_1, a_2, a_3\}) \cup P_{x_i x_n}$ contains a $3PC(b_1 b_2 b_3, x_n)$. So we may assume w.l.o.g. that the neighbors of x_n is Σ are contained in $P^1 \cup P^3$. But then $(\Sigma \setminus \{a_1, a_3'\}) \cup P$ contains a $3PC(b_1 b_2 b_3, x_n)$. □

Lemma 10.5 *Every Type t2 node w.r.t.* $\Sigma = 3PC(\Delta, \Delta)$ *is attached to* Σ. *Furthermore, let* $\Sigma = 3PC(a_1 a_2 a_3, b_1 b_2 b_3)$ *and* u *be a Type t2 node adjacent to* a_2 *and* a_3. *Then every direct connection from* u *to* $\Sigma \setminus \{a_1, a_2, a_3\}$ *that contains no neighbor of* a_3 *is an attachment.*

Proof: Let $\Sigma = 3PC(a_1 a_2 a_3, b_1 b_2 b_3)$ and let u be a node of Type t2 w.r.t. Σ, say adjacent to a_2 and a_3. Let $S = (N(a_3) \cup a_3) \setminus \{u, a_3'\}$ and let $x_1, \ldots, x_n$ be a direct connection from u to $\Sigma \setminus S$ in $G \setminus S$. Let $x_0 = u$ and $P = x_0, x_1, \ldots, x_n$. By definition of S, x_n cannot be of Type t6a w.r.t. Σ and the only nodes of Σ that can have a neighbor in $P \setminus \{x_0, x_n\}$ are a_1 and a_2. Let x_j be the node of $P \setminus x_n$ with highest index adjacent to a_2. If a_1 has a neighbor in $P \setminus x_n$, then let x_i be such a neighbor with highest index.

Case 1: $j > 0$

We first show that a_2 is adjacent to x_1. Suppose not and let x_k be the node of $P \setminus u$ with lowest index adjacent to a_2. Let H be the hole in $(\Sigma \cup P) \setminus \{a_1, a_2\}$ that contains u. Since $j > 0$, $W = (H, a_2)$ is a wheel. Since a_2 is adjacent to a_3 and u but not to a_3' and x_1, W is either a proper wheel that is not a beetle, or a line wheel. In the former case, Theorem 4.6 is contradicted. So assume W is a line wheel. Then (W, a_2) belongs to an L-parachute with center path obtained by taking the shortest path from a_2 to W in $\Sigma \setminus \{a_1, a_3\}$. So Theorem 5.3 is contradicted. Therefore a_2 is adjacent to x_1.

Case 1.1: a_1 has a neighbor in $P \setminus x_n$.

Let x_k be such a neighbor with lowest index. If $k = 1$ then $\{a_1, a_3, x_0, x_1\}$ induces a square. So $k > 1$. $P_{x_0 x_k} \cup \{a_1, a_2, a_3\}$ must be a universal wheel with center a_2. In particular, a_2 is adjacent to x_2.

Let H be a hole that contains u in $P \cup \Sigma \cup \{a_1, a_2\}$. By construction, H contains a_3'. So (H, a_2) is a proper wheel that is not a beetle.

Case 1.2: a_1 does not have a neighbor in $P \setminus x_n$.

By Lemma 8.4 applied to $P_{x_j x_n}$, x_n is either of Type t1, p1 or p3 w.r.t. Σ with a neighbor in $P^2 \setminus a_2$, or of Type t2, t2p or t3p w.r.t. Σ adjacent to b_1, b_3 and no node of $(P^1 \cup P^3) \setminus \{b_1, b_3\}$, or of Type t2p w.r.t. Σ adjacent to a_1, a_2 and a node of $P^3 \setminus a_3$, or of Type p2 or p4 w.r.t. Σ adjacent to a_2. If x_n is of Type t1, p1, p2 or p3, then $(\Sigma \setminus a_2) \cup P$ contains a $3PC(b_1b_2b_3, a_3)$. If x_n is of Type t2, t2p or t3p adjacent to b_1 and b_3, then $P^1 \cup P^3 \cup P$ induces a $3PC(b_1x_nb_3, a_3)$. If x_n is of Type p4 with a neighbor in P^1, let u_1 be such a neighbor closest to a_1. Then $P^1_{a_1u_1} \cup P \cup \{a_2, a_3\}$ induces a proper wheel with center a_2 that is not a beetle. If x_n is of Type p4 with a neighbor in P^3, then $P^1 \cup P^2 \cup P \cup a_3$ induces a proper wheel with center a_2 that is not a beetle. So x_n is of Type t2p adjacent to a_1 and a_2. Note that $n > 1$ since otherwise a_1, a_3, u, x_1 induces a square. Let H be the hole induced by $P \cup \{a_1, a_3\}$. (H, a_2) must be a universal wheel. In particular, a_2 is adjacent to all nodes of P. Let v be the neighbor of x_n in P^3 that is closest to a_3. Let H' be the hole induced by $P^3_{a_3v} \cup P$. Then (H', a_2) is a proper wheel that is not a beetle.

Case 2: $j = 0$

Suppose a_1 does not have a neighbor in $P \setminus x_n$. By Lemma 8.5 applied to P, $P_{x_1x_n}$ is either an attachment of u to Σ, or x_n is of Type t2p w.r.t. Σ adjacent to a_1 and a_2. Suppose x_n is of Type t2p adjacent to a_1 and a_2. If $n = 1$ then $\{a_1, a_3, x_0, x_1\}$ induces a square. So $n > 1$. But then $P \cup \{a_1, a_2, a_3\}$ induces an odd wheel with center a_2.

So we may assume that a_1 has a neighbor in $P \setminus x_n$. By Lemma 8.4 applied to $P_{x_ix_n}$, x_n is either of Type t1, p1 or p3 w.r.t. Σ adjacent to a node of $P^1 \setminus a_1$, or of Type p2 or p4 w.r.t. Σ adjacent to a_1, or of Type t2, t2p or t3p w.r.t. Σ adjacent to b_2, b_3 and no node of $(P^2 \cup P^3) \setminus \{b_2, b_3\}$, or of Type t2p w.r.t. Σ adjacent to a_1 and a_2. If x_n is of Type t1, p1 or p3, or of Type t2, t2p or t3p adjacent to b_2 and b_3, then $P_{x_1x_n}$ is an attachment of u to Σ. If x_n is of Type p4, then w.l.o.g. x_n does not have a neighbor in P^2, and hence $(\Sigma \setminus \{a_1, a_3\}) \cup P$ contains a $3PC(b_1b_2b_3, x_n)$. If x_n is of Type p2, then $P^1 \cup P^2 \cup P$ must induce a beetle with center a_1, and hence $P_{x_1x_n}$ is an attachment of u to Σ. So we may assume that x_n is of Type t2p w.r.t. Σ, adjacent to a_1 and a_2. Let u_3 be the neighbor of x_n in P_3 that is closest to a_3. Then $P_{a_3u_3} \cup P$ induces a hole H and (H, a_2) is an odd wheel. □

Lemma 10.6 *Let $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ and let y be a Type t2 or t2p node w.r.t. Σ, adjacent to say b_2 and b_3. Then*

(i) there cannot exist a node x that is of Type t1 w.r.t. Σ adjacent to b_3 and y;

(ii) every node x of Type t2 or t2p w.r.t. Σ adjacent to b_1, b_2 is adjacent to y. Every sibling x of b_3 of Type t3p w.r.t. Σ is adjacent to y.

Proof: We first prove (i). By Lemma 10.5 let $P^y = y_1, \ldots, y_m$ be an attachment of y to Σ, and let Σ^y be obtained by substituting y and P^y into Σ. Assume there is a node x of Type t1 w.r.t. Σ, adjacent to b_3 and to y. By Lemma 6.1 applied to Σ^y, x is of Type t2 w.r.t. Σ^y. By Lemma 10.5, let $P^x = x_1, \ldots, x_n$ be an attachment of x to Σ^y.

First we show that no node of P^1 is adjacent to or coincident with a node of $P^x \setminus x_n$. Assume not and let x_i be the node of $P^x \setminus x_n$ with lowest index that is adjacent to a node of P^1. If b_2 has no neighbor in $P^x_{x_1x_i}$, then $x, P^x_{x_1x_i}$ contradicts Theorem 8.3 applied to Σ. So b_2 has a neighbor in $P^x_{x_1x_i}$, and let x_j be such a neighbor with lowest index. By Theorem 8.3 applied to $x, P^x_{x_1x_j}$ and Σ, $i = j$. By Lemma 6.1 applied to x_i and Σ, x_i is of Type t2 w.r.t. Σ, adjacent to b_1 and b_2. Let R be the shortest path from b_1 to y in $P^y \cup P^1 \cup \{y, a_3\}$. Then $R \cup P^x_{x_1x_i} \cup \{b_2, x\}$ induces an odd wheel with center b_2. Therefore, no node of P^1 is adjacent to or coincident with a node of $P^x \setminus x_n$.

If b_2 has a neighbor in $P^x \setminus x_n$, then $P^x_{x_1x_i}$ (where x_i is the neighbor of b_2 in $P^x \setminus x_n$ with lowest index) contradicts Theorem 8.3 applied to Σ. So b_2 has no neighbor in $P^x \setminus x_n$. In particular, x_n is not of Type p2 w.r.t. Σ^y and therefore the attachment P^x of x to Σ^y satisfies Definition 10.3(i) or (ii). Suppose that x_n is of type t1, p1 or p3 w.r.t. Σ^y. Then its neighbors in Σ^y are contained in P^2. By Lemma 8.4 applied to x, P^x and Σ, x_n is of Type t2 w.r.t. Σ, adjacent to a_1 and a_2 and y is a Type t2 node w.r.t. Σ with attachment P^y satisfying Definition 10.3(ii). But then $P^1 \cup P^x \cup \{x, y, b_2, b_3\}$ induces an odd wheel with center b_3. So x_n is of Type t2, t2p or t3p w.r.t. Σ^y. So x_n is adjacent to a_3, and if it is of Type t2p or t3p w.r.t. Σ^y then it has a neighbor in $P^2 \setminus a_2$. If x_n is adjacent to a_1, then by Lemma 6.1 x_n is of Type t2, t2p or t3p w.r.t. Σ, and hence x, P^x contradicts Lemma 8.4 applied to Σ. So x_n is not adjacent to a_1. By Lemma 6.1, x_n is of Type t1 w.r.t. Σ and y is of Type t2 w.r.t. Σ. But then $P^1 \cup P^x \cup \{x, y, b_2, b_3, a_3\}$ induces an odd wheel with center b_3.

Now we prove (ii). If x is of Type t2p or t3p w.r.t. Σ, let Σ' be obtained from Σ by substituting x for its sibling b_3. If x is of Type t2 w.r.t. Σ, then by Lemma 10.5, there is an attachment $Q = x_1, \ldots, x_n$ of x to Σ. In this case, let Σ' be obtained by substituting x and its attachment Q into Σ. Note that $P^1 \cup P^2 \subseteq \Sigma'$. Let P^3_x be the path of $\Sigma' \setminus (P^1 \cup P^2)$. Suppose that y is not adjacent to x. Then by Lemma 6.1 applied to Σ', y is of Type t1 w.r.t. Σ' and hence of Type t2 w.r.t. Σ. By Lemma 10.5, there is an attachment $P^y = y_1, \ldots, y_m$ of y to Σ. Let Σ^y be obtained by substituting y and P^y into Σ. If x is of Type t2p or t3p in Σ, then x violates Lemma 6.1 in Σ^y. So x is of Type t2 in Σ.

Let R be a shortest path from x to y in $P^y \cup \Sigma' \setminus (P^2 \cup \{b_1, b_3\})$. Then $R \cup b_2$ induces a hole H'. If b_1 has a neighbor in $R \setminus x$, then $W = (H', b_1)$ is a wheel that is not a twin wheel, a triangle-free wheel, a universal wheel or a beetle. Suppose W is a line wheel. Then there is an L-parachute with center path included in $P^1 \cup P^y \cup a_3$. But then, by Theorem 5.3, there is a star cutset. So W is a proper wheel that is not a beetle and by Theorem 4.6 there is a star cutset. So b_1 has no neighbor in $R \setminus x$. Similarly b_3 has no neighbor in $R \setminus x$. But then (Rb_3b_1, b_2) is an odd wheel. □

11 Type t6a Nodes

Theorem 11.1 *Let G be a square-free even-signable graph. Let $\Sigma = 3PC\ (a_1a_2a_3, b_1b_2b_3)$ and let u be a Type t6a node w.r.t. Σ. If for some $i \in \{1,2,3\}$, there is no P^i-crosspath w.r.t. Σ, then G has a star cutset.*

Proof: Assume there is no P^3-crosspath w.r.t. Σ. Suppose there is no star cutset. Then by Theorem 9.1, no node is of Type t4, t5 or t6b w.r.t. a $3PC(\Delta, \Delta)$. Let $S = (N(u) \cup u) \setminus (\Sigma \setminus$

$\{a_1, a_2, b_3\}$) and let $P = x_1, \ldots, x_n$ be a direct connection from $P^1 \cup P^2$ to P^3 in $G \setminus S$.

Claim 1: *No node of P is of Type t2, t2p, t3p or t6a w.r.t. Σ.*

Proof of Claim 1: If x_i is of Type t2p or t3p w.r.t. Σ, then let Σ' be obtained from Σ by substituting x_i for its sibling. If x_i is of Type t2 w.r.t. Σ, then by Lemma 10.5, x_i is attached to Σ. Let Σ' be obtained from Σ by substituting x_i and its attachment into Σ. Since u is not adjacent to x_i, it is of Type t5 or t4s w.r.t. Σ', a contradiction. Since G is square-free, every node of Type t6a w.r.t. Σ is adjacent to u. It follows from the definition of S that no node of P is of Type t6a w.r.t. Σ. This completes the proof of Claim 1.

Claim 2: *$n > 1$, x_1 is of Type t1, p1, p2, p3 or p4 w.r.t. Σ with neighbors contained in $P^1 \cup P^2$, or of Type t3 w.r.t. Σ adjacent to b_1, b_2 and b_3, and x_n is of Type t1, p1, p2 or p3 w.r.t. Σ with neighbors contained in P^3, or of Type t3 w.r.t. Σ adjacent to a_1, a_2 and a_3.*

Proof of Claim 2: Since there is no P^3-crosspath, if a node of P is of Type p4 w.r.t. Σ, then its neighbors are contained in $P^1 \cup P^2$. Now the result follows from Claim 1. This completes the proof of Claim 2.

Claim 3: *If a_1 or a_2 has a neighbor in the interior of P, then b_3 has no neighbor in the interior of P.*

Proof of Claim 3: Let x_i and x_j be nodes in the interior of P adjacent to a_1 or a_2, and to b_3 respectively so that the $P_{x_i x_j}$ subpath is shortest possible. W.l.o.g. assume that x_i is adjacent to a_1. By Claim 1, x_i is of Type t1 w.r.t. Σ, and hence $P_{x_i x_j} \cup P^2 \cup P^3$ induces a $3PC(a_1 a_2 a_3, b_3)$. This completes the proof of Claim 3.

Claim 4: *No interior node of P is adjacent to a_1, a_2 or b_3.*

Proof of Claim 4: Assume first that b_3 has a neighbor in the interior of P. Let x_i be a node with highest index in $P_{x_2 x_{n-1}}$ adjacent to b_3. By Claim 3, no interior node of P is adjacent to a_1 or a_2. So, by Lemma 8.4 applied to $P_{x_i x_n}$, x_n is of Type t1, p1 or p3 with neighbors in P^3 or of Type p2 adjacent to b_3. Similarly by Theorem 8.3 and Lemma 8.4, x_1 is of Type p4 with neighbors in $P^1 \cup P^2$, or of Type t3 adjacent to b_1, b_2, b_3. If x_1 is of Type p4, there is a $3PC(b_1 b_2 b_3, x_1)$ contained in $(P \cup \Sigma) \setminus \{a_1, a_2, a_3, x_n\}$. If x_1 is of Type t3 adjacent to b_1, b_2, b_3, then $(P \cup \Sigma) \setminus b_3$ contains a $3PC(a_1 a_2 a_3, b_1 b_2 x_1)$ and u is of Type t5 w.r.t. it, a contradiction.

Assume now that some interior node of P is adjacent to a_1 or a_2. Let x_i be the node with lowest index in $P_{x_2 x_{n-1}}$ adjacent to a_1 or a_2, say a_1. By Claim 1, x_i is of Type t1 w.r.t. Σ. By Lemma 8.4 applied to $P_{x_1 x_i}$, x_1 is of Type t1, p1 or p3 with neighbors in P^1 or of Type p2 or p4 adjacent to a_1. Suppose a_2 has a neighbor in $P_{x_{i+1} x_{n-1}}$ and let x_j be such a neighbor with lowest index. If x_1 is of Type t1, p1, p2 or p3 w.r.t. Σ, then $(\Sigma \cup P_{x_1 x_j}) \setminus a_1$ contains a $3PC(b_1 b_2 b_3, a_2)$. If x_1 is of Type p4 w.r.t. Σ, then $P^1 \cup P^3 \cup P_{x_1 x_j} \cup a_2$ induces a proper wheel with center a_1 that is not a beetle, contradicting Theorem 4.6. Therefore a_2 has no neighbor in $P_{x_2 x_{n-1}}$. Then, by Theorem 8.3, x_n is of Type t3 adjacent to a_1, a_2, a_3. If x_1 is of Type p4 w.r.t. Σ, then $(\Sigma \cup P) \setminus \{a_1, a_2\}$ contains a $3PC(b_1 b_2 b_3, x_1)$. So x_1 is not of Type p4 w.r.t. Σ and hence by a symmetric argument as in the previous paragraph, we obtain a contradiction. This completes the proof of Claim 4.

Case 1: x_n is of Type t1, p1, p2 or p3 w.r.t. Σ.

If x_1 is of Type t1, p1, p2 or p3 w.r.t. Σ, by Theorem 8.3, P is a P^3-crosspath, a contradiction. Suppose x_1 is of Type t3 w.r.t. Σ. Let $\Sigma' = 3PC(a_1a_2a_3, b_1b_2x_1)$ contained in $(\Sigma \setminus b_3) \cup P$. Then u is of Type t5 w.r.t. Σ', a contradiction. So x_1 is of Type p4 w.r.t. Σ. But then $(\Sigma \setminus \{b_1, b_2, b_3\}) \cup P$ contains a $3PC(a_1a_2a_3, x_1)$.

Case 2: x_n is of Type t3 w.r.t. Σ

Suppose x_1 is of Type t1, p1, p2 or p3 w.r.t. Σ, with neighbors w.l.o.g. in P^2. $(\Sigma \setminus a_2) \cup P$ contains a $\Sigma' = 3PC(a_1x_na_3, b_1b_2b_3)$. But then u is of Type t5 w.r.t. Σ', a contradiction.

Suppose x_1 is of Type t3 w.r.t. Σ. Then $P^1 \cup P^3 \cup P$ induces a $\Sigma' = 3PC(a_1x_na_3, b_1x_1b_3)$. Then u is of Type t4s w.r.t. Σ', a contradiction. So x_1 is of Type p4 w.r.t. Σ. But then $(\Sigma \setminus \{a_1, a_2\}) \cup P$ contains a $3PC(b_1b_2b_3, x_1)$. □

12 Type t2 and t2p Nodes

In this section, we assume that G is a square-free even-signable graph. Furthermore, we assume that G has no star cutset. So by Theorem 9.1, there are no Type t4, t5 and t6b nodes.

Definition 12.1 *A $3PC(a_1a_2a_3, b_1b_2b_3) = \Sigma$ in G is* decomposable *if there exists a node of Type t2 or t2p w.r.t. Σ, say adjacent to a_2 and a_3, but there is no P^1-crosspath w.r.t. Σ. Path P^1 of Σ is called the* middle path.

Denote by H the graph induced by a decomposable $3PC(a_1a_2a_3, b_1b_2b_3)$ together with a node a_4 of Type t2 or t2p adjacent to a_2, a_3. Let $H_1 = P^1 \cup a_4$ and $H_2 = P^2 \cup P^3$. Then $H_1|H_2$ is a 2-join of H with special sets $A_1 = \{a_1, a_4\}$, $B_1 = \{b_1\}$, $A_2 = \{a_2, a_3\}$ and $B_2 = \{b_2, b_3\}$. In this section, we show that the 2-join $H_1|H_2$ of H extends to a 2-join of G. First, we prove the following result.

Theorem 12.2 *If G contains a $3PC(\Delta, \Delta)$ with a Type t2 or t2p node, then G contains a decomposable $3PC(\Delta, \Delta)$.*

Proof: First suppose that G contains a connected diamonds $D(a_1a_2a_3a_4, b_1b_2b_3b_4)$. Let $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ (resp. $\Sigma' = 3PC(a_4a_2a_3, b_4b_2b_3)$) induced by paths P^1, P^2 and P^3 (resp. P^4, P^2 and P^3) of D. Suppose that $P = x_1, \ldots, x_n$ is a P^1-crosspath w.r.t. Σ. W.l.o.g. x_1 has a neighbor in P^1 and x_n has a neighbor in P^2. Let u_1 and v_1 be the neighbors of x_1 in P^1. If no node of P^4 has a neighbor in P, then $P^1 \cup (P^2 \setminus b_2) \cup b_3 \cup P^4 \cup P$ contains a $3PC(x_1u_1v_1, a_2)$. So a node of P^4 has a neighbor in P. Let x_i be such a neighbor with highest index. Let v be the neighbor of x_i in P^4 that is closest to a_4. By Lemma 6.1 and Theorem 8.3 applied to $P_{x_ix_n}$ and Σ', $P_{x_ix_n}$ is a P^4-crosspath w.r.t. Σ'. Hence $v \neq b_4$. If $i \neq 1$ then the path $P^4_{a_4v}, P_{x_ix_n}$ contradicts Lemma 8.5 applied to Σ. So $i = 1$ and hence the path $P^4_{a_4v}, x_1$ contradicts Lemma 8.5 applied to Σ. Therefore, there is no P^1-crosspath w.r.t. Σ, and hence Σ is a decomposable $3PC(\Delta, \Delta)$.

Now we may assume that G does not contain connected diamonds.

Let Σ be a $3PC(\Delta, \Delta)$ with a Type t2 or t2p node that has shortest middle path. Let $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ and w.l.o.g. let a_4 be a node of Type t2 or t2p adjacent to a_2 and a_3. Suppose Σ is not decomposable and let $P = x_1, \dots, x_n$ be a P^1-crosspath w.r.t. Σ. W.l.o.g. x_1 has a neighbor in P^1 and x_n in P^2. Let u_1 (resp. v_1) be the neighbor of x_1 in P^1 that is closest to a_1 (resp. b_1).

First suppose that $u_1 \neq a_1$. Let $\Sigma' = 3PC(a_1a_2a_3, u_1x_1v_1)$ contained in $(\Sigma \cup P) \setminus b_2$. Since a_4 cannot be of Type t4 or t5 w.r.t. Σ', it must be of Type t2 or t2p w.r.t. Σ'. Since Σ' has a shorter middle path than Σ, this contradicts our choice of Σ. Therefore, $u_1 = a_1$.

Suppose a_4 is of Type t2p w.r.t. Σ. We show that a_4 has no neighbor in P. Suppose it does. Let H be the hole contained in $(\Sigma \cup P) \setminus \{a_1, b_2\}$. Consider the wheel (H, a_4). Since a_4 is adjacent to a_2, a_3, (H, a_4) is not triangle-free. Since a_4 has a neighbor in P_1, it is not a twin wheel. Since a_4 is not adjacent to b_3, it is not a universal wheel. (H, a_4) is not a line wheel since, if a_4 were adjacent to a_1' and x_1, then a_1, a_2, a_4, a_1' would induce a square. (H, a_4) is not a beetle since if x_n were adjacent to a_2 and to a_4, there would be an odd wheel with center a_2 induced by $P^2 \cup P^3 \cup \{a_4, x_n\}$. Therefore (H, a_4) is a proper wheel that is not a beetle and by Theorem 4.6, G has a star cutset, a contradiction. Therefore, a_4 has no neighbor in P. Let H be the hole contained in $P \cup (P^1 \setminus a_1) \cup (P^2 \setminus b_2) \cup a_4$. Then (H, a_1) is an odd wheel.

Therefore, a_4 is of Type t2 w.r.t. Σ. By Lemma 10.5, let $Q = y_1, \dots, y_m$ be an attachment of a_4 to Σ. Let Σ' be obtained by substituting a_4 and Q into Σ. Suppose a_4 has a neighbor in P and let x_i be its neighbor in P with highest index. If $i = 1$ then a_4, x_1 contradicts Lemma 8.5 applied to Σ and otherwise $a_4, P_{x_ix_n}$ contradicts Lemma 8.5 applied to Σ. So a_4 does not have a neighbor in P. Next we show that no node of $Q \setminus y_m$ is adjacent to or coincident with a node of P. Suppose not and let y_i be the node of Q with lowest index adjacent to a node of P, and let x_j be the node of P with highest index adjacent to y_i. If $j = 1$, then $a_4, Q_{y_1y_i}, x_1$ must satisfy Lemma 10.5. Therefore y_i is adjacent to a_1. But then there is a $3PC(a_1y_ix_1, a_2)$ contained in $Q_{y_1y_i} \cup (P^2 \setminus b_2) \cup P \cup \{a_1, a_4\}$. If $j > 1$, then $a_4, Q_{y_1y_i}, P_{x_jx_n}$ violates Lemma 10.5 w.r.t. Σ. So no node of $Q \setminus y_m$ is adjacent to or coincident with a node of P.

Assume y_m is of Type p2 w.r.t. Σ. Node y_m cannot be adjacent to any node of $P \setminus x_1$ since, otherwise, there is a $3PC(a_1y_{m-1}y_m, a_2)$. Now y_m is adjacent to x_1 since, otherwise, there is an odd wheel with center a_1 contained in $Q \cup (P^2 \setminus b_2) \cup P \cup \{a_1, a_4\}$. But then there is a $3PC(a_2a_3a_4, x_1a_1'y_m)$ with a_1 a strongly adjacent node of Type t5, a contradiction.

Assume y_m is of Type t1, p1 or p3 w.r.t. Σ. We show that y_m does not have a neighbor in P. Suppose not and let x_i be the neighbor of y_m in P with highest index. Then $P_{x_ix_n}$ contradicts Theorem 8.3 applied to Σ' unless $i = 1$ and y_m is of Type p1 adjacent to a_1'. But then there is a $3PC(a_2a_3a_4, x_1a_1'y_m)$ and a_1 is a strongly adjacent node of Type t4s relative to it, a contradiction. Therefore y_m does not have a neighbor in P. Let v be the neighbor of y_m in $P^1 \setminus a_1$ that is closest to a_1'. Let H be the hole contained in $P \cup P^1_{a_1'v} \cup (P^2 \setminus b_2) \cup Q \cup a_4$. Then (H, a_1) is a proper wheel that is not a beetle unless y_m is a strongly adjacent node of Type p3 adjacent to a_1, a_1' and at least one other node of P^1. Since $P \cup a_1'$ is a crosspath w.r.t. Σ', the only other node of P^1 adjacent to y_m is the neighbor a_1'' of a_1'. But then a_1' is the center of an odd wheel with hole contained in $P \cup \{a_1, y_m\} \cup P_{a_1''b_1} \cup P^2$.

Therefore, y_m is of Type t2, t2p or t3p w.r.t. Σ. We show that y_m does not have a neighbor in P. Assume not and let x_i be the neighbor of y_m in P with largest index. If

$i = n$ and x_n is adjacent to b_2, then $P^2 \cup P^3 \cup \{x_n, y_m\}$ induces an odd wheel with center b_2. Otherwise, $P^2 \cup P_{x_i x_n} \cup Q \cup a_4$ induces a $3PC(\Delta, y_m)$. So y_m does not have a neighbor in P. If y_m is of Type t2 and a_1 has no neighbor in Q, $\Sigma \cup Q$ induces connected diamonds and the result holds. If y_m is of Type t2 and a_1 has at least one neighbor in Q, there is a $3PC(b_2 b_3 y_m, a_1)$ contained in $(P \setminus a_2) \cup P^3 \cup P \cup Q \cup a_1$. So y_m is of Type t2p or t3p w.r.t. Σ and hence it has a neighbor in $P^1 \setminus b_1$. If y_m is adjacent to a_1, then $(P^2 \setminus a_2) \cup P^3 \cup P \cup \{a_1, y_m\}$ induces a $3PC(b_2 b_3 y_m, a_1)$. So y_m is not adjacent to a_1. Let v be the neighbor of y_m in P^1 that is closest to a_1. Then $P^1_{a_1 v} \cup (P^2 \setminus b_2) \cup P \cup Q \cup a_4$ contains a proper wheel with center a_1 that is not a beetle. □

12.1 2-Joins and Blocking Sequences

In this section, we consider an induced subgraph H of G that contains a 2-join $H_1|H_2$. We say that a 2-join $H_1|H_2$ *extends* to G if there exists a 2-join of G, $H_1'|H_2'$ with $H_1 \subseteq H_1'$ and $H_2 \subseteq H_2'$. We characterize the situation in which the 2-join of H does not extend to a 2-join of G.

Definition 12.3 *A* blocking sequence *for a 2-join $H_1|H_2$ of a subgraph H of G is a sequence of distinct nodes $x_1, \ldots, x_n$ in $G \setminus H$ with the following properties:*

1. *i) $H_1|H_2 \cup x_1$ is not a 2-join of $H \cup x_1$,*

   ii) $H_1 \cup x_n|H_2$ is not a 2-join of $H \cup x_n$, and

   iii) if $n > 1$ then, for $i = 1, \ldots, n-1$, $H_1 \cup x_i|H_2 \cup x_{i+1}$ is not a 2-join of $H \cup \{x_i, x_{i+1}\}$.

2. *$x_1, \ldots, x_n$ is minimal with respect to Property 1, in the sense that no sequence $x_{j_1}, \ldots, x_{j_k}$ with $\{x_{j_1}, \ldots, x_{j_k}\} \subset \{x_1, \ldots, x_n\}$, satisfies Property 1.*

Blocking sequences with respect to a 1-join were introduced and studied by Geelen in [19]. Blocking sequences with respect to a 2-join were introduced in [7], where the following results are obtained.

Let H be an induced subgraph of G with 2-join $H_1|H_2$ and special sets A_1, B_1, A_2, B_2.

In the following remarks and lemmas, we let $S = x_1, \ldots, x_n$ be a blocking sequence for the 2-join $H_1|H_2$ of a subgraph H of G.

Remark 12.4 *$H_1|H_2 \cup u$ is a 2-join in $H \cup u$ if and only if $N(u) \cap H_1 = \emptyset, A_1$ or B_1. Similarly $H_1 \cup u|H_2$ is a 2-join in $H \cup u$ if and only if $N(u) \cap H_2 = \emptyset, A_2$ or B_2.*

Lemma 12.5 *If $n > 1$ then, for every node x_j, $j \in \{1, \ldots, n-1\}$, $N(x_j) \cap H_2 = \emptyset, A_2$ or B_2, and for every node x_j, $j \in \{2, \ldots, n\}$, $N(x_j) \cap H_1 = \emptyset, A_1$ or B_1.*

Lemma 12.6 *If $n > 1$ and $x_i x_{i+1}$ is not an edge, where $i \in \{1, \ldots, n-1\}$, then either $N(x_i) \cap H_2 = A_2$ and $N(x_{i+1}) \cap H_1 = A_1$, or $N(x_i) \cap H_2 = B_2$ and $N(x_{i+1}) \cap H_1 = B_1$.*

Theorem 12.7 *Let H be an induced subgraph of graph G that contains a 2-join $H_1|H_2$. The 2-join $H_1|H_2$ of H extends to a 2-join of G if and only if there exists no blocking sequence for $H_1|H_2$ in G.*

Lemma 12.8 *For $1 < i < n$, $H_1 \cup \{x_1, \ldots, x_{i-1}\} | H_2 \cup \{x_{i+1}, \ldots, x_n\}$ is a 2-join in $H \cup (S \setminus \{x_i\})$.*

Lemma 12.9 *If $x_i x_k$, $n \geq k > i+1 \geq 2$, is an edge then either $N(x_i) \cap H_2 = A_2$ and $N(x_k) \cap H_1 = A_1$, or $N(x_i) \cap H_2 = B_2$ and $N(x_k) \cap H_1 = B_1$.*

Lemma 12.10 *If x_j is the node of lowest index adjacent to a node in H_2, then $x_1, \ldots, x_j$ is a chordless path. Similarly, if x_j is the node of highest index adjacent to a node in H_1, then $x_j, \ldots, x_n$ is a chordless path.*

Theorem 12.11 *Let G be a graph and H an induced subgraph of G with 2-join $H_1|H_2$ and special sets A_1, B_1, A_2, B_2. Let H' be an induced subgraph of G with 2-join $H_1'|H_2$ and special sets A_1', B_1', A_2, B_2 such that $A_1' \cap A_1 \neq \emptyset$ and $B_1' \cap B_1 \neq \emptyset$. If S is a blocking sequence for $H_1|H_2$ and $H_1' \cap S \neq \emptyset$, then a proper subset of S is a blocking sequence for $H_1'|H_2$.*

12.2 Decomposable $3PC(\Delta, \Delta)$ and 2-Joins

Lemma 12.12 *Let Σ be a decomposable $3PC(a_1a_2a_3, b_1b_2b_3)$ and d a Type t2 or t2p node adjacent to a_2 and a_3, or to b_2 and b_3. Let $H_1|H_2$ be the 2-join of $H = \Sigma, d$ and let A_1, B_1, A_2, B_2 be its special sets. If u is of Type p3, t2 or t2p w.r.t. Σ, of Type t3p w.r.t. Σ being a sibling of a_2, a_3, b_2 or b_3, or of Type t1 w.r.t. Σ adjacent to a node in $\{a_2, a_3, b_2, b_3\}$ and being attached to Σ, then there exists a decomposable $\Sigma' = 3PC(\Delta, \Delta)$ and a Type t2 or t2p node d' such that $H' = \Sigma', d'$ satisfies the following properties: $H_i \subseteq H'$, for some $i \in \{1, 2\}$, $u \in H_j' = H' \setminus H_i$ where $j = 3 - i$, and $H_i|H_j'$ is a 2-join of H' with special sets A_i, B_i, A_j' and B_j', where $A_j \cap A_j' \neq \emptyset$ and $B_j \cap B_j' \neq \emptyset$.*

Proof: We consider the following cases.

Case 1: u is of Type p3 w.r.t. Σ.

Let Σ' be obtained by substituting u into Σ. By Lemma 6.1, d is of Type t2 or t2p w.r.t. Σ'.

Suppose u has a neighbor in P^1. Let P' be the a_1b_1-path of Σ'. Suppose $P = x_1, \ldots, x_n$ is a P'-crosspath w.r.t. Σ'. W.l.o.g. x_1 has a neighbor in P' and x_n in P^3. Then x_1 has a neighbor in P^1. Let x_i be the node of P with highest index that has a neighbor in P^1. By Theorem 8.3, $P_{x_ix_n}$ is a P^1-crosspath w.r.t. Σ, a contradiction. Therefore, there is no P'-crosspath w.r.t. Σ'.

If u has a neighbor in $P^2 \cup P^3$, then by analogous argument, there is no P^1-crosspath w.r.t. Σ'. Hence, Σ' is the desired decomposable $3PC(\Delta, \Delta)$.

Case 2: u is of Type t2 or t2p w.r.t. Σ, or of Type t3p w.r.t. Σ being a sibling of a_2, a_3, b_2 or b_3.

If u is of Type t2 or t2p w.r.t. Σ adjacent to a_2 and a_3 or to b_2 and b_3, then $(H \setminus d) \cup u$ is the desired decomposable $3PC(\Delta, \Delta)$. So by symmetry, it is enough to consider the case when u is adjacent to a_1 and a_2, and it is of Type t2, t2p or t3p w.r.t. Σ. If u is of Type t2p or t3p w.r.t. Σ, let Σ' be obtained from Σ by substituting u for its sibling. If u is of Type t2 w.r.t. Σ, then by Lemma 10.5, there is an attachment $Q = y_1, \ldots, y_m$ of u to Σ. In this case, let Σ' be obtained by substituting u and Q into Σ. Note that $P^1 \cup P^2 \subseteq \Sigma'$. Let P^3_u be the path of $\Sigma' \setminus (P^1 \cup P^2)$.

We first show that there is no P^1-crosspath w.r.t. Σ'. Suppose not and let $P = x_1, \dots, x_n$ be a P^1-crosspath w.r.t. Σ'. W.l.o.g. x_1 has a neighbor in P^1. If a node of P^3 has a neighbor in $P \setminus x_n$, then by Lemma 6.1 and Theorem 8.3, a subpath of P is a P^1-crosspath w.r.t. Σ, a contradiction. So no node of P^3 is adjacent to or coincident with a node of $P \setminus x_n$. Suppose that x_n has a neighbor in P^2. Then by Lemma 6.1, x_n is of Type p2 or p4 w.r.t. Σ. Since P cannot be a P^1-crosspath w.r.t. Σ, $n > 1$, x_n is of Type p4 w.r.t. Σ, and $N(x_n) \cap \Sigma \subseteq P^2 \cup P^3$. But then $(\Sigma \setminus \{a_1, a_2, a_3\}) \cup P$ contains a $3PC(b_1b_2b_3, x_n)$. So x_n does not have a neighbor in P^2, and hence it has a neighbor in P^3_u. If x_n has a neighbor in P^3, then by Lemma 6.1 and Theorem 8.3, P is a P^1-crosspath w.r.t. Σ. So x_n does not have a neighbor in P^3, and hence the neighbors of x_n in P^3_u are contained in $P^3_u \setminus P^3$. Since x_n is of Type p2 or p4 w.r.t. Σ', x_n has a neighbor in Q. Let y_i be such a neighbor with highest index. Node a_3 does not have a neighbor in $Q_{y_iy_{m-1}}$, since otherwise $P, Q_{y_iy_j}$ (where y_j is the neighbor of a_3 in $Q_{y_iy_{m-1}}$ with lowest index) contradicts Theorem 8.3 applied to Σ. If y_m is of Type t1, p1, p2 or p3 w.r.t. Σ, then by Theorem 8.3, $P, Q_{y_iy_m}$ is a P^1-crosspath w.r.t. Σ. So y_m is of Type t2, t2p or t3p w.r.t. Σ, adjacent to b_1, b_2 and no node of $(P^1 \cup P^2) \setminus \{b_1, b_2\}$. If y_m is of Type t2 w.r.t. Σ, then $P, Q_{y_iy_m}$ contradicts Lemma 8.5 applied to Σ. So y_m is of Type t2p or t3p w.r.t. Σ. Let Σ'' be obtained by substituting y_m into Σ. But then $P, Q_{y_iy_{m-1}}$ contradicts Theorem 8.3 applied to Σ''. Therefore, there is no P^1-crosspath w.r.t. Σ'.

If d is adjacent to a_2, a_3 then, by Lemma 10.6(ii), d is adjacent to u. Therefore, by Lemma 6.1, d is of Type t2 or t2p w.r.t. Σ', and hence Σ' is the desired decomposable $3PC(\Delta, \Delta)$. Now consider the case where d is adjacent to b_2, b_3. If y_m is of Type t1, p1, p2 or p3, Σ' is the desired decomposable $3PC(\Delta, \Delta)$. If y_m is of Type t2, t2p or t3p then y_m is adjacent to d by Lemma 10.6(ii), and therefore d is of Type t2 or t2p w.r.t. Σ'. Hence Σ' is the desired decomposable $3PC(\Delta, \Delta)$.

Case 3: u is of Type t1 w.r.t. Σ adjacent to a node in $\{a_2, a_3, b_2, b_3\}$ and it is attached to Σ.

W.l.o.g. u is adjacent to a_3. Let $Q = y_1, \dots, y_m$ be an attachment of u to Σ, and let Σ' be obtained by substituting u and Q into Σ. Note that $P^1 \cup P^2 \subseteq \Sigma'$. Let P^3_u be the path of $\Sigma' \setminus (P^1 \cup P^2)$.

We first show that there is no P^1-crosspath w.r.t. Σ'. Suppose not and let $P = x_1, \dots, x_n$ be a P^1-crosspath w.r.t. Σ'. W.l.o.g. x_1 has a neighbor in P^1. By the same argument as in Case 2, no node of P^3 is adjacent to or coincident with a node of $P \setminus x_n$ and x_n does not have a neighbor in $P^2 \cup P^3$. So x_n has a neighbor in P^3_u, and its neighbors in P^3_u are contained in u, Q. Let y_i be the neighbor of x_n in Q with highest index. If a'_3 has a neighbor in $Q_{y_iy_{m-1}}$ then $Q_{y_iy_j}, P$ (where y_j is its neighbor in $Q_{y_iy_{m-1}}$ with lowest index) contradicts Theorem 8.3 applied to Σ. So a'_3 does not have a neighbor in $Q_{y_iy_{m-1}}$. If y_m is of Type t1, p1, p2 or p3 w.r.t. Σ, then by Theorem 8.3 applied to Σ, $P, Q_{y_iy_m}$ is a P^1-crosspath w.r.t. Σ. If y_m is of Type t2 w.r.t. Σ, then $P, Q_{y_iy_m}$ contradicts Lemma 8.5 applied to Σ. So y_m is of Type t2p or t3p w.r.t. Σ. Let Σ'' be obtained by substituting y_m into Σ. Then $P, Q_{y_iy_{m-1}}$ contradicts Lemma 8.4 applied to Σ''. Therefore, there is no P^1-crosspath w.r.t. Σ'.

Assume d is adjacent to a_2, a_3. By Lemma 6.1, d is of Type t2 or t2p w.r.t. Σ' and hence Σ' is the desired decomposable $3PC(\Delta, \Delta)$. Assume d is adjacent to b_2, b_3. If y_m is of Type t1, p1, p2 or p3, then d is of Type t2 or t2p w.r.t. Σ', and hence Σ' is the desired decomposable $3PC(\Delta,\Delta)$. If y_m is of Type t2, t2p or t3p, then, by Lemma 10.6(ii), y_m is adjacent to d and therefore d is of Type t2 or t2p w.r.t. Σ' and hence Σ' is the desired decomposable $3PC(\Delta,\Delta)$. □

Theorem 12.13 *Let G be a square-free even-signable graph. If G contains a $3PC(\Delta,\Delta)$ with a Type t2 or t2p node, then G has a star cutset or a 2-join.*

Proof: Assume G has no star cutset. By Theorems 4.6 and 5.3, G contains neither a proper wheel that is not a beetle nor an L-parachute. By Theorem 9.1, there is no node of Type t4, t5 or t6b w.r.t. a $3PC(\Delta,\Delta)$.

Assume G contains a $3PC(\Delta,\Delta)$ with a Type t2 or t2p node. Then by Theorem 12.2, G contains a decomposable $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ together with a node d of Type t2 or t2p adjacent to a_2, a_3 or to b_2, b_3. Suppose that the 2-join $H_1|H_2$ of $H = \Sigma, d$ does not extend to a 2-join of G. By Theorem 12.7, there is a blocking sequence $S = x_1, \ldots, x_n$. W.l.o.g. assume that H and S are chosen so that the size of S is minimized. Since Σ has no P^1-crosspath, no node is of Type t6a w.r.t. Σ by Theorem 11.1. By Lemma 12.12 and Theorem 12.11, no node of S is of Type p3, t2 or t2p w.r.t. Σ, of Type t3p w.r.t. Σ being a sibling of a_2, a_3, b_2 or b_3, or of Type t1 w.r.t. Σ adjacent to a node in $\{a_2, a_3, b_2, b_3\}$ and being attached to Σ.

Claim 1: *If x_i is of Type p4 w.r.t. Σ, then $N(x_i) \cap H \subseteq P^2 \cup P^3$. If x_i is of Type t1, p1 or p2 w.r.t. Σ and $N(x_i) \cap \Sigma \subseteq P^2 \cup P^3$, then $N(x_i) \cap H \subseteq P^2 \cup P^3$.*

Proof of Claim 1: W.l.o.g. assume that d is adjacent to a_2, a_3. Suppose x_i is of Type p4 w.r.t. Σ. Since there is no P^1-crosspath w.r.t. Σ, $N(x_i) \cap \Sigma \subseteq P^2 \cup P^3$. Suppose x_i is adjacent to d. If d is of Type t2 w.r.t. Σ, then d, x_i contradicts Lemma 8.5. So d is of Type t2p w.r.t. Σ, and hence $(\Sigma \setminus \{a_1, a_2, a_3\}) \cup \{d, x_i\}$ contains a $3PC(b_1b_2b_3, x_i)$. So x_i is not adjacent to d.

Now suppose that x_i is of Type t1, p1 or p2 w.r.t. Σ, with neighbors in Σ contained in say P^3. It is enough to show that x_i is not adjacent to d. Suppose x_i is adjacent to d. Suppose x_i has a neighbor in $P^3 \setminus a_3$. If d is of Type t2 w.r.t. Σ, then d, x_i contradicts Lemma 8.5. If d is of Type t2p w.r.t. Σ, then $(\Sigma \setminus \{a_1, a_3\}) \cup \{d, x_i\}$ contains a $3PC(b_1b_2b_3, d)$. So x_i is of Type t1 w.r.t. Σ adjacent to a_3. We have seen above that x_i cannot be attached to Σ. So, by Lemma 10.4, there is a bad connection $Q = y_1, \ldots, y_m$ of x_i to Σ, where y_m is of Type t2 or t2p w.r.t. Σ, adjacent to a_1 and a_2. If d has a neighbor in Q, let y_j be such a neighbor with highest index. Then $Q_{y_jy_m} \cup \{a_1, a_2, a_3, d\}$ either contains a square or induces an odd wheel with center a_2. So d does not have a neighbor in Q. If d is of Type t2p w.r.t. Σ, then $Q \cup P^1 \cup \{x_i, a_1, d\}$ contains a $3PC(x_ia_3d, a_1)$. So d is of Type t2 w.r.t. Σ. By Lemma 10.5 there is an attachment $P = u_1, \ldots, u_l$ of d to Σ. Let $\Sigma' = 3PC(\Delta,\Delta)$ obtained by substituting d and its attachment P into Σ. By Lemma 6.1 applied to Σ', x_i does not have a neighbor in P. If $(P \setminus u_1) \cup (Q \setminus y_m) \cup P^1 \cup \{x_i, b_3\}$ contains a path from x_i to a_1, then a shortest such path together with a_2, a_3 and d induces a proper wheel with center a_3 that is not a beetle. Therefore no such path exists and hence no node of $Q \setminus y_m$ is adjacent to or coincident with a node of $P \setminus u_1$. By Lemma 6.1 applied to Σ', y_m does not have a neighbor in P. Suppose u_1 has a neigbor in Q and let y_j be such a neighbor with lowest index. If a_3' has a neighbor in $Q_{y_1y_j}$ then a subpath of $Q_{y_1y_j}$ contradicts Lemma 6.1 or Theorem 8.3 applied to Σ'. So a_3' does not have a neighbor in $Q_{y_1y_j}$ and hence $x_i, Q_{y_1y_j}$

contradicts Lemma 8.5 applied to Σ'. Therefore u_1 does not have a neighbor in Q and so no node of Q is adjacent to or coincident with a node of P. Let R be a shortest path from d to a_1 in $P \cup P^1 \cup \{d, b_3\}$. Then $R \cup Q \cup \{x_i, a_3\}$ induces a proper wheel with center a_3 that is not a beetle. This completes the proof of Claim 1.

By Claim 1, $n > 1$. Since $H_1|H_2 \cup x_1$ is not a 2-join of $H \cup x_1$, x_1 has a neighbor in $P^1 \cup \{d\}$ and either (i) $N(x_1) \cap H \subseteq P^1 \cup \{d\}$, or (ii) x_1 is of Type t3p w.r.t. Σ being a sibling of a_1 or b_1, or (iii) x_1 is of Type t3 w.r.t. Σ adjacent to, say, a_1, a_2 and a_3, x_1 is not adjacent to d, and d is adjacent to a_2, a_3. Note that the case where x_1 is of Type t3 adjacent to a_1, a_2, a_3 and d where d is adjacent to b_2, b_3 cannot occur since, in this case, there is a $3PC(x_1a_1a_3, b_3)$. Since $H_1 \cup x_n|H_2$ is not a 2-join of $H \cup x_n$, x_n has a neighbor in $P^2 \cup P^3$, and it is of Type t1, p1, p2 or p4 w.r.t. Σ. By Lemma 12.5, for $i \in \{2, \dots, n-1\}$, x_i either has no neighbor in H or $N(x_i) \cap \Sigma = \{a_1, a_2, a_3\}$ or $\{b_1, b_2, b_3\}$ and, furthermore, if say $N(x_i) \cap \Sigma = \{a_1, a_2, a_3\}$ then x_i is adjacent to d if d is adjacent to a_2, a_3, and x_i is not adjacent to d if d is adjacent to b_2, b_3. Let x_j be the node of S with highest index adjacent to a node of H_1. By Lemma 12.10, $x_j, \dots, x_n$ is a chordless path. Note that nodes $x_{j+1}, \dots, x_{n-1}$ have no neighbors in H.

Claim 2: *Let Σ be a $3PC(a_1a_2a_3, b_1b_2b_3)$ with no P^1-crosspath. Suppose that x_j is of Type t3 w.r.t. Σ, say adjacent to b_1, b_2 and b_3, and there is a $\Sigma' = 3PC(a_1a_2t, b_1b_2x_j)$ that contains $P^1 \cup P^2$ and such that t is not of Type t3 w.r.t. Σ. Then there is no P^1-crosspath w.r.t. Σ'.*

Proof of Claim 2: Let P' be the path of $\Sigma' \setminus (P^1 \cup P^2)$. Suppose $P = y_1, \dots, y_m$ is a P^1-crosspath w.r.t. Σ'. W.l.o.g. y_1 has a neighbor in P^1. Suppose y_m has a neighbor in P^2. Since P cannot be a P^1-crosspath w.r.t. Σ, a node of P has a neighbor in P^3. Let y_i be such a node with lowest index. If $i \neq m$ then by Theorem 8.3, $P_{y_1y_i}$ is a P^1-crosspath w.r.t. Σ. So $i = m$ and hence y_m is of Type p4 w.r.t. Σ. But then $(\Sigma \setminus \{a_1, a_2, a_3\}) \cup P$ contains a $3PC(b_1b_2b_3, y_m)$. So y_m has a neighbor in P'. Suppose that $P \cup P^3 \cup P' \setminus \{x_j, t\}$ contains a path from y_1 to P^3. Then by Theorem 8.3 applied to the shortest such path, there is a P^1-crosspath w.r.t. Σ. So no such path exists and hence $t \neq a_3$ and no node of P^3 is adjacent to or coincident with a node of $P \cup P' \setminus \{x_j, t\}$. So t is of Type t2, t2p or t3p w.r.t. Σ. Note that $P \cup P' \setminus x_j$ contains a chordless path T from y_1 to t. If t is of Type t2 w.r.t. Σ, then T contradicts Lemma 8.5 applied to Σ. So t is of Type t2p or t3p w.r.t. Σ. Let Σ'' be obtained by substituting t into Σ. Then $T \setminus t$ contradicts Theorem 8.3 applied to Σ''. This completes the proof of Claim 2.

We now consider the following cases.

Case 1: x_j is of Type t3 w.r.t. Σ.

If x_n is of Type p1 or p4 w.r.t. Σ, then $x_j, \dots, x_n$ contradicts Lemma 8.6.

Case 1.1: x_n is of Type p2 w.r.t. Σ.

W.l.o.g. x_n has a neighbor in P^3 and d is adjacent to a_2, a_3. Suppose x_j is adjacent to b_1, b_2 and b_3. Then there is a $\Sigma' = 3PC(a_1a_2a_3, b_1b_2x_j)$ contained in $(\Sigma \setminus b_3) \cup \{x_j, \dots, x_n\}$. By Claim 2, there is no P^1-crosspath w.r.t. Σ'. By Lemma 6.1, d is of Type t2 or t2p w.r.t. Σ' and hence Σ', d is a decomposable $3PC(\Delta, \Delta)$. But then, by Theorem 12.11, the minimality of S is contradicted. So x_j is adjacent to a_1, a_2 and a_3. Let $\Sigma' = 3PC(a_1a_2x_j, b_1b_2b_3)$ be contained in $(\Sigma \setminus a_3) \cup \{x_j, \dots, x_n\}$. By Claim 2, there is no P^1-crosspath w.r.t. Σ'. If d is adjacent to x_j, then by Lemma 6.1 applied to Σ', x_j is of Type t2 or t2p w.r.t. Σ', and hence Σ', d is a decomposable $3PC(\Delta, \Delta)$ and the minimality of S is contradicted.

So d is not adjacent to x_j, and hence by Lemma 6.1, d is of Type t1 w.r.t. Σ' and of Type t2 w.r.t. Σ. By Lemma 10.5, let $Q = y_1, \dots, y_m$ be an attachment of d to Σ.

First we show that no node of Q is adjacent to or coincident with a node of $\{x_j, \dots, x_n\}$. Suppose not and let y_k be the node of Q with highest index that has a neighbor in $\{x_j, \dots, x_n\}$. Let x_i be the neighbor of y_k in $\{x_j, \dots, x_n\}$ with highest index.

Suppose $i \neq j$. If a_1 has a neighbor in $Q_{y_k y_{m-1}}$ then let y_l be such a neighbor with lowest index. Then $Q_{y_k y_l}, x_i, \dots, x_n$ contradicts Lemma 8.4 applied to Σ. So a_1 has no neighbor in $Q_{y_k y_{m-1}}$. If y_m is of Type t1, p1, p2 or p3 w.r.t. Σ, then by Theorem 8.3 applied to Σ, $Q_{y_k y_m}, x_i, \dots, x_n$ is a P^1-crosspath w.r.t. Σ. If y_m is of Type t2 w.r.t. Σ, then $Q_{y_k y_m}, x_i, \dots, x_n$ contradicts Lemma 8.5 applied to Σ. So y_m is of Type t2p or t3p w.r.t. Σ. Let Σ'' be obtained by substituting y_m into Σ. Then either $Q_{y_{k+1} y_{m-1}}, x_i, \dots, x_n$ (if $k \neq m$) or $x_i, \dots, x_n$ (otherwise) contradicts Lemma 8.4 applied to Σ''. Therefore, $i = j$.

If x_j is adjacent to y_m, then y_m and Σ' contradict Lemma 6.1 (since by our assumption y_m cannot be of Type t4 or t5 w.r.t. Σ'). So x_j is not adjacent to y_m, i.e. $k < m$. If a_1 does not have a neighbor in $Q_{y_k y_{m-1}}$, then y_m is not of Type p2 w.r.t. Σ and hence $x_j, Q_{y_k y_m}$ contradicts Lemma 8.6 applied to Σ. So a_1 has a neighbor in $Q_{y_k y_{m-1}}$. If y_m is of Type t2, t2p or t3p w.r.t. Σ, then let H' be the hole induced by $P^2 \cup Q_{y_k y_m} \cup x_j$, and H'' the hole contained in $(P^3 \setminus a_3) \cup Q_{y_k y_m} \cup \{x_j, \dots x_n\}$. If y_m is of Type t1, p1 ,p2 or p3 w.r.t. Σ, then let v be the neighbor of y_m in P^1 that is closest to b_1, let H' be the hole induced by $P^1_{vb_1} \cup P^2 \cup Q_{y_k y_m} \cup x_j$ and let H'' be the hole contained in $P^1_{vb_1} \cup (P^3 \setminus a_3) \cup Q_{y_k y_m} \cup \{x_j, \dots, x_n\}$. Since neither (H', a_1) nor (H'', a_1) can be an odd wheel, y_k is the unique neighbor of a_1 in $Q_{y_k y_m}$. So y_k is of Type t2 w.r.t. Σ' and hence $Q_{y_k y_m}$ contradicts Lemma 8.5 applied to Σ.

Therefore, no node of Q is adjacent to or coincident with a node of $\{x_j, \dots, x_n\}$. Let $\Sigma'' = 3PC(a_2 a_3 d, \Delta)$ be obtained by substituting d and Q into Σ. Then $x_j, \dots, x_n$ contradicts Lemma 8.5 applied to Σ''.

Case 1.2: x_n is of Type t1 w.r.t. Σ.

Assume w.l.o.g. that x_j is adjacent to b_1, b_2 and b_3. If x_n is adjacent to a_3, then $x_j, \dots, x_n$ contradicts Lemma 8.6. So x_n is adjacent to b_3.

Claim 3: *If there exists a $\Sigma' = 3PC(a_1 a_2 t, b_1 b_2 x_j)$ that contains $P^1 \cup P^2$, then t is of Type t3 w.r.t. Σ.*

Proof of Claim 3: Suppose that there exists a $\Sigma' = 3PC(a_1 a_2 t, b_1 b_2 x_j)$ that contains $P^1 \cup P^2$ and such that t is not of Type t3 w.r.t. Σ. Then by Claim 2, there is no P^1-crosspath w.r.t. Σ'. Suppose d is of Type t2 or t2p w.r.t. Σ'. Then Σ', d is a decomposable $3PC(\Delta, \Delta)$. Now, by Theorem 12.11 applied to $H = \Sigma, d$ and $H' = \Sigma', d$, the minimality of S is contradicted. Therefore d cannot be of Type t2 or t2p w.r.t. Σ'.

Assume first that d is adjacent to a_2 and a_3. If $t = a_3$ then d is of Type t2 or t2p w.r.t. Σ' by Lemma 6.1 applied to Σ', a contradiction. So we may assume that $t \neq a_3$. Since t is adjacent to a_1, a_2 and it is not of Type t3 w.r.t. Σ, it is of Type t2, t2p or t3p w.r.t. Σ. By Lemma 10.6(ii), d is adjacent to t, and hence by Lemma 6.1, d is of Type t2 or t2p w.r.t. Σ', a contradiction.

Assume now that d is adjacent to b_2 and b_3. If d is adjacent to x_j, then by Lemma 6.1, d is of Type t2 or t2p w.r.t. Σ', a contradiction. So d is not adjacent to x_j, and hence it is of Type t1 w.r.t. Σ'. So d is of Type t2 w.r.t. Σ. By Lemma 10.5, let $Q = y_1, \ldots, y_m$ be an attachment of d to Σ. Let Σ^d be obtained by substituting d and Q into Σ. Let P' be the tx_j-path of Σ'.

We now show that x_j has a neighbor in Q. Suppose it does not. Suppose that a node of $Q \setminus y_m$ is adjacent to a node of P'. Let y_i be such a node with lowest index. Let p be a neighbor of y_i in P' that is closest to x_j. If b_1 does not have a neighbor in $Q_{y_1y_i}$, then $d, Q_{y_1y_i}$ contradicts Lemma 8.4 applied to Σ'. So b_1 has a neighbor in $Q_{y_1y_i}$. Let y_b be such a neighbor with lowest index. If $y_b = y_i$ then y_b contradicts Lemma 6.1 applied to Σ'. So $y_b \neq y_i$ and hence $d, Q_{y_1y_b}$ contradicts Theorem 8.3 applied to Σ'. So no node of $Q \setminus y_m$ is adjacent to or coincident with a node of P'. By the same argument as above, b_1 does not have a neighbor in $Q \setminus y_m$. Then by Lemma 8.4 applied to d, Q and Σ', $N(y_m) \cap (\Sigma \cup \Sigma') = \{a_2, a_3\}$ and $t \neq a_3$. If a_3 has no neighbor in P', then $Q \cup P' \cup \{a_1, a_2, a_3, b_2, d\}$ induces an odd wheel with center a_2. Otherwise, let u be the neighbor of a_3 in P' that is closest to x_j. Note that since t is not of Type t3 w.r.t. Σ, $u \neq t$. But then $Q \cup P^2 \cup P'_{ux_j} \cup \{a_3, d\}$ induces a $3PC(a_2a_3y_m, b_2)$. Therefore, x_j has a neighbor in Q.

Let y_i be the neighbor of x_j in Q with highest index. We now show that b_1 is adjacent to y_i, it is not adjacent to y_{i+1} and it has at most two neighbors in $Q_{y_iy_m}$, and y_m is not of Type p2 w.r.t. Σ. Suppose that b_1 has no neighbor in $Q_{y_iy_m}$. Since y_m is not adjacent to b_1, $x_j, Q_{y_iy_m}$ and Σ violate Lemma 8.6. So b_1 has a neighbor in $Q_{y_iy_m}$. Let R be a shortest path from y_m to b_2 in $(\Sigma \cup y_m) \setminus \{b_1, b_3\}$. Let H be the hole induced by $R \cup Q_{y_iy_m} \cup x_j$. If y_m is of Type p2 w.r.t. Σ, then (H, b_1) is an odd wheel. So y_m is not of Type p2 w.r.t. Σ. If (H, b_1) is a line wheel, then $H \cup b_1 \cup P^1$ contains an L-parachute, contradicting Theorem 5.3. So (H, b_1) is a twin wheel or a beetle, and hence the result holds.

Next we show that no node of $Q_{y_iy_m}$ is adjacent to or coincident with a node of $P' \setminus x_j$. Assume not and let y_k be a node of $Q_{y_iy_m}$ with lowest index adjacent to a node of $P' \setminus x_j$. If $k \neq m$ then either $Q_{y_iy_k}$ contradicts Lemma 8.5 applied to Σ' (if y_i is the unique neighbor of b_1 in $Q_{y_iy_k}$) or a subpath of $Q_{y_{i+1}y_k}$ contradicts Theorem 8.3 applied to Σ' (otherwise). So $k = m$. By Lemma 6.1 applied to y_m and Σ', either $t = a_3$ and y_m is of Type t2, t2p or t3p w.r.t. Σ', adjacent to a_2, a_3 and possibly a node of $P^1 \setminus a_1$, or $t \neq a_3$ and $N(y_m) \cap (\Sigma \cup \Sigma') = \{a_1, t\}$. In the second case, t and Σ^d violate Lemma 6.1. So $t = a_3$. If b_1 has a neighbor in $Q_{y_{i+1}y_m}$ then $Q_{y_iy_m} \cup P' \cup b_1$ induces an odd wheel with center b_1. So b_1 has no neighbor in $Q_{y_{i+1}y_m}$. But then $Q_{y_iy_m}$ contradicts Lemma 8.5 applied to Σ'. Therefore, no node of $Q_{y_iy_m}$ is adjacent to or coincident with a node of $P' \setminus x_j$.

Suppose y_m is of Type t1, p1 or p3 w.r.t. Σ and let y be the neighbor of y_m in P^1 that is closest to a_1. If b_1 has a neighbor in $Q_{y_{i+1}y_m}$, then either $Q_{y_iy_m} \cup P^1_{a_1y} \cup P' \cup b_1$ or $Q_{y_iy_m} \cup P^1_{a_1y} \cup P^2 \cup \{b_1, x_j\}$ induces an odd wheel with center b_1. So b_1 has no neighbor in $Q_{y_{i+1}y_m}$. But then $Q_{y_iy_m}$ contradicts Lemma 8.5 applied to Σ'. Hence y_m is of Type t2, t2p or t3p w.r.t. Σ, and since y_m is not adjacent to t, $t \neq a_3$. If b_1 has a neighbor in $Q_{y_{i+1}y_m}$, then $Q_{y_iy_m} \cup P' \cup \{a_2, b_1\}$ induces an odd wheel with center b_1. So b_1 has no neighbor in $Q_{y_{i+1}y_m}$. By Lemma 8.5 applied to $Q_{y_iy_m}$ and Σ', y_m is of Type t1 w.r.t. Σ' adjacent to a_2. So y_m must be of Type t2 w.r.t. Σ, adjacent to a_2 and a_3. If a_3 has no neighbor in P' then $Q_{y_iy_m} \cup P' \cup \{a_1, a_2, a_3\}$ induces an odd wheel with center a_2. So a_3 has a neighbor in P'. Let u be its neighbor in P' that is closest to x_j. Since t is not of Type t3 w.r.t. Σ, $u \neq t$. But then $P^1 \cup Q_{y_i y_m} \cup P'_{ux_j} \cup a_3$ induces a $3PC(y_i b_1 x_j, a_3)$. This completes the proof of Claim 3.

Let $T = (N(b_1) \cup b_1) \setminus b'_1$ and let $Q = y_1, \ldots, y_m$ be a direct connection from x_n to $\Sigma \setminus T$ in $G \setminus T$. Let $y_0 = x_n$ and $Q' = y_0, Q$. Note that b_2 and b_3 are the only nodes of Σ that can have a neighbor in $Q \setminus y_m$. We first show that b_2 cannot have a neighbor in $Q \setminus y_m$. Suppose otherwise and let y_k be the node of $Q \setminus y_m$ with lowest index adjacent to b_2. If y_k is also adjacent to b_3, then y_k is of Type t2 w.r.t. Σ and therefore, by Lemma 10.5, it is attached. Let Σ' be obtained by substituting y_k and one of its attachment into Σ. Let y_i be the node of $Q'_{y_0 y_{k-1}}$ with highest index adjacent to b_3. By Lemma 10.6, $i \neq k-1$. Let y_l be the node of $Q'_{y_i y_{k-1}}$ with lowest index that has a neighbor in $\Sigma' \setminus b_3$. By Lemma 6.1, y_i is of Type t1 w.r.t. Σ', and hence $l \neq i$. But then $Q'_{y_i y_l}$ contradicts Theorem 8.3 applied to Σ'. So y_k is not adjacent to b_3. But then Theorem 8.3 is contradicted in Σ by the path $x_n, y_1, \ldots, y_k$ when b_3 has no neighbor in $Q_{y_1 y_k}$ or a subpath starting from y_k otherwise. Therefore b_2 does not have a neighbor in $Q \setminus y_m$.

Suppose y_m is of Type p2 or p4 w.r.t. Σ adjacent to b_3. If y_m is of Type p2 w.r.t. Σ, then $(\Sigma \setminus b_3) \cup \{x_j, x_n\}$ contains a $3PC(a_1 a_2 a_3, b_1 b_2 x_j)$ contradicting Claim 3. So y_m is of Type p4 w.r.t. Σ. W.l.o.g. y_m does not have a neighbor in P^2. Let R be the shortest path from x_j to y_m in $Q \cup \{x_j, \ldots, x_n\}$. If b_3 is adjacent to a node of $R \setminus \{x_j, y_m\}$, then $P^2 \cup P^3 \cup R$ induces a proper wheel with center b_3 that is not a beetle. Otherwise, R contradicts Lemma 8.6 applied to Σ. Therefore, y_m is not of Type p2 or p4 w.r.t. Σ adjacent to b_3.

Let y_i be the node of $Q' \setminus y_m$ with highest index adjacent to b_3. By Lemma 8.4 applied to $Q'_{y_i y_m}$ and Σ, since y_m is not of Type p2 or p4 w.r.t. Σ adjacent to b_3, y_m is either of Type t1, p1 or p3 w.r.t. Σ with a neighbor in $P^3 \setminus b_3$, or of Type t2, t2p or t3p w.r.t. Σ adjacent to a_1, a_2 and no node of $(P^1 \cup P^2) \setminus \{a_1, a_2\}$. But then $(\Sigma \setminus b_3) \cup Q \cup \{x_j, \ldots, x_n\}$ contains a $3PC(a_1 a_2 t, b_1 b_2 x_j)$ that contains $P^1 \cup P^2$ and such that t is not of Type t3 w.r.t. Σ, contradicting Claim 3.

Case 2: $j = 1$ and x_1 is of Type t1, p1 or p2 w.r.t. Σ.

If x_n is if Type t1, p1 or p2 w.r.t. Σ, then by Theorem 8.3, $x_1, \ldots, x_n$ is a P^1-crosspath w.r.t. Σ. If x_n is of Type p4 w.r.t. Σ, then $(\Sigma \setminus \{b_2, b_3\}) \cup \{x_1, \ldots, x_n\}$ contains a $3PC(a_1 a_2 a_3, x_n)$.

Case 3: $j = 1$ and d is the unique neighbor of x_1 in H.

W.l.o.g. assume that d is adjacent to a_2, a_3. If d is of Type t2p w.r.t. Σ, let Σ' be obtained by substituting d into Σ. If x_n is not of Type t1 adjacent to a_2 or a_3, then $x_1, \ldots, x_n$ contradicts Lemma 8.4 applied to Σ'. If x_n is of Type t1 adjacent to a_2 or a_3, then $x_1, \ldots, x_n$ contradicts Theorem 8.3 applied to Σ'.

So d is of Type t2 w.r.t. Σ. If x_n is not of Type t1 w.r.t. Σ adjacent to a_2 or a_3, then $d, x_1, \ldots, x_n$ contradicts Lemma 8.5. So x_n is of Type t1 w.r.t. Σ, w.l.o.g. adjacent to a_3. Let Σ' be a $3PC(\Delta, \Delta)$ obtained by substituting d and an attachment of d to Σ into Σ. Now $x_1, \ldots, x_n$ or a subpath starting from x_n contradicts Theorem8.3 applied to Σ'.

Case 4: $j = 1$ and x_1 is of Type t3p w.r.t. Σ.

W.l.o.g. x_1 is a sibling of b_1. Let Σ' be obtained by substituting x_1 into Σ. Let v be the neighbor of x_1 in P^1 that is closest to a_1. If x_n is not of Type t1 w.r.t. Σ adjacent to b_2 or b_3, then $x_2, \ldots, x_n$ contradicts Lemma 8.4 applied to Σ'. So x_n is of Type t1 w.r.t. Σ, w.l.o.g.

adjacent to b_3. If $n \neq 2$, then $x_1, \ldots, x_n$ contradicts Theorem 8.3 in Σ'. So $n = 2$.

Let $T = (N(b_1) \cup b_1) \setminus b'_1$ and let $Q = y_1, \ldots, y_m$ be a direct connection from x_n to $\Sigma \setminus T$ in $G \setminus T$. As in Case 1.2, b_2 does not have a neighbor in $Q \setminus y_m$. By Lemma 8.4 applied to Σ and x_n, Q (if b_3 does not have a neighbor in $Q \setminus y_m$) or a subpath of Q (otherwise), y_m is either of Type t1, p1 or p3 w.r.t. Σ with a neighbor in $P^3 \setminus b_3$, or of Type p2 or p4 w.r.t. Σ adjacent to b_3, or of Type t2, t2p or t3p w.r.t. Σ adjacent to a_1, a_2 and no node of $(P^1 \cup P^2) \setminus \{a_1, a_2\}$.

Let $x_n = y_0$ and $Q' = y_0, Q$. Let y_i be the node of Q' with highest index adjacent to x_1. By Lemma 6.1 applied to Σ' and since y_m cannot be of Type t4d w.r.t. Σ', $i \neq m$. If b_3 does not have a neighbor in $Q'_{y_i y_{m-1}}$, then $Q'_{y_i y_m}$ contradicts Lemma 8.4 applied to Σ'. So b_3 has a neighbor in $Q'_{y_i y_{m-1}}$. Let y_l be such a node with lowest index. If $l \neq i$, then $Q_{y_i y_l}$ contradicts Theorem 8.3 in Σ'. So b_3 is adjacent to y_i. If b_3 does not have a neighbor in $Q'_{y_{i+1} y_{m-1}}$, then $Q'_{y_i y_m}$ contradicts Lemma 8.5 applied to Σ'. So b_3 has a neighbor in $Q'_{y_{i+1} y_{m-1}}$. If y_m is of Type t2, t2p or t3p w.r.t. Σ, then let C be the hole induced by $P^2 \cup Q'_{y_i y_m} \cup x_1$ and C' the hole induced by $P^1_{a_1 v} \cup Q'_{y_i y_m} \cup x_1$. Then (C, b_3) or (C', b_3) is an odd wheel. If y_m is of Type t1, p1, p2 or p3 w.r.t. Σ, then let C be the hole passing through x_1 contained in $P^2 \cup (P^3 \setminus b_3) \cup Q'_{y_i y_m} \cup x_1$ and C' the hole contained in $P^1_{a_1 v} \cup (P^3 \setminus b_3) \cup Q'_{y_i y_m} \cup x_1$. Then (C, b_3) or (C', b_3) is an odd wheel. So y_m is of Type p4 w.r.t. Σ. Since there is no P^1-crosspath w.r.t. Σ, y_m does not have a neighbor in P^1. Hence, $P^1_{a_1 v} \cup P^3 \cup Q'_{y_i y_m} \cup x_1$ induces a proper wheel with center b_3 that is not a beetle. □

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