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# Odd Hole Recognition in Graphs of Bounded Clique Size

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### **Abstract**

In a graph  $G$ , an odd hole is an induced odd cycle of length at least five. A clique of  $G$  is a set of pairwise adjacent vertices. In this paper we consider the class  $\mathcal{C}_k$  of graphs whose cliques have a size bounded by a constant  $k$ . Given a graph  $G$  in  $\mathcal{C}_k$ , we show how to recognize in polynomial time whether  $G$  contains an odd hole.

**Keywords:** odd hole, recognition algorithm, cleaning, decomposition

# 1 Introduction

A *hole* is a graph induced by a cycle of length at least four. A hole is *odd* if it contains an odd number of vertices. Otherwise, it is even. Graph  $G$  *contains* graph  $H$  if  $H$  is isomorphic to an induced subgraph of  $G$ . Chudnovsky, Cornuéjols, Liu, Seymour and Vušković recently proved that it is polynomial to test whether a graph contains an odd hole or its complement [2]. However, it is still an open problem to test whether a graph contains an odd hole. Bienstock [1] proved that it is  $NP$ -complete to test whether a graph contains an odd hole passing through a specific vertex. A *clique* is a set of pairwise adjacent vertices. The *clique number* of a graph is the size of its largest clique. In this paper, we show that it is polynomial to test whether a graph of bounded clique number contains an odd hole.

We use the same general strategy as in [2]. Let  $H$  be an odd hole in a graph  $G$ . We say that  $u \in V(G) \setminus V(H)$  is  *$H$ -minor* if its neighbors in  $H$  lie in some 2-edge path of  $H$ . In particular,  $u$  is  *$H$ -minor* if  $u$  has no neighbor in  $H$ . A vertex  $u \in V(G) \setminus V(H)$  is  *$H$ -major* if it is not  *$H$ -minor*. We say that  $H$  is *clean* if  $G$  contains no  *$H$ -major* vertex. A graph  $G$  is *clean* if either it is odd-hole-free or it contains a clean shortest odd hole. As in [2] our approach for testing whether a graph  $G$  of bounded clique number contains an odd hole consists of two steps:

- (i) constructing in polynomial time a clean graph  $G'$  that contains an odd hole if and only if  $G$  does, or in some cases identifying an odd hole of  $G$ , and
- (ii) checking whether the clean graph  $G'$  contains an odd hole.

For step (ii), we can use the polynomial algorithms in [2]. The main result of this paper is a polynomial algorithm for step (i). Step (i) is called *cleaning* the graph  $G$ .

## 1.1 Notation

For a graph  $G$  and a set  $B$  of vertices of  $G$ , we denote by  $G(B)$  the subgraph of  $G$  induced by the vertex set  $B$ . For a vertex  $v$ ,  $N(v)$  denotes the set of vertices adjacent to  $v$ .

A *pyramid*  $\Pi(xyz; u)$  is a graph induced by three paths  $P_1 = x, \dots, u$ ,  $P_2 = y, \dots, u$  and  $P_3 = z, \dots, u$  having no common or adjacent intermediate vertices, such that at most one of the paths is of length 1 and the vertex set  $\{x, y, z\}$  induces a clique of size 3. Note that every two of the paths  $P_1, P_2, P_3$  induce a hole. Since two of the three paths must have the same parity, one of these holes is odd. Therefore, every pyramid contains an odd hole.

A *wheel*, denoted by  $(H, x)$ , is a graph induced by a hole  $H$  and a vertex  $x \notin V(H)$  having at least three neighbors in  $H$ , say  $x_1, \dots, x_n$ . Vertex  $x$  is the *center* of the wheel. A subpath of  $H$  connecting  $x_i$  and  $x_j$  is a *sector* if it contains no intermediate vertex  $x_l$ ,  $l \in \{1, \dots, n\}$ . A *short sector* is a sector of length 1, and a *long sector* is a sector of length at least 2. A wheel is *odd* if it contains an odd number of short sectors, and *even* otherwise. Each of the long sectors together with vertex  $x$  induces a hole. If each of these holes is even and the wheel  $(H, v)$  is odd then  $H$  is an odd hole, since the wheel  $(H, x)$  contains an odd number of short sectors. Therefore, every odd wheel contains an odd hole.

In a graph  $G$ , a *jewel* is a sequence  $v_1, \dots, v_5, P$  such that  $v_1, \dots, v_5$  are distinct vertices,  $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1$  are edges,  $v_1v_3, v_2v_4, v_1v_4$  are nonedges, and  $P$  is a path of  $G$  between  $v_1$

and  $v_4$  such that  $v_2, v_3, v_5$  have no neighbors in  $V(P) \setminus \{v_1, v_4\}$ . Clearly a jewel either contains an odd wheel or a 5-hole, so if there is a jewel in a graph  $G$  then there is an odd hole in  $G$ .

Chudnovsky and Seymour found an  $O(|V(G)|^9)$  algorithm to test whether a graph  $G$  contains a pyramid and an  $O(|V(G)|^6)$  algorithm to test whether a graph  $G$  contains a jewel (see [2]).

## 2 Cleaning

In this section, we show how to clean a graph  $G$  of bounded clique number. That is, we perform step (i) above. The cleaning algorithm produces a polynomial family of induced subgraphs of  $G$  such that if  $G$  contains a shortest odd hole  $H^*$ , then one of the graphs produced by the cleaning algorithm, say  $G'$ , contains  $H^*$  and  $H^*$  is clean in  $G'$ .

Roughly speaking, this is accomplished by showing that there exists a set  $X$  of vertices of  $H^*$ , whose size depends only on the clique number, such that every major vertex for  $H^*$  has a neighbor in  $X$ . Since the set  $Y$  of vertices of  $H^*$  with neighbors in  $X$  has at most  $2|X|$  elements, we may enumerate all possible choices for  $X$  and  $Y$ , and for each choice of  $X$  and  $Y$  add to the family the graph obtained by removing the vertices of  $V(G) \setminus Y$  that have a neighbor in  $X$ .

### 2.1 Vertices with At Most Three Neighbors in $H^*$

**Lemma 1** *Let  $H^*$  be a shortest odd hole in  $G$ . Suppose that  $G$  does not contain a pyramid. If a vertex  $u \notin V(H^*)$  has a neighbor but no more than three neighbors in  $H^*$  then  $u$  is  $H^*$ -minor.*

*Proof:* If  $u$  has one neighbor in  $H^*$  then  $u$  is  $H^*$ -minor. Now suppose that  $u$  has two neighbors in  $H^*$ , say  $u_1$  and  $u_2$ . Let  $P_1$  and  $P_2$  be the two  $u_1u_2$ -subpaths of  $H^*$ . Since  $H^*$  is odd,  $P_1$  and  $P_2$  have different parity, say  $P_1$  is odd. If  $P_1$  is of length 1 then  $u$  is  $H^*$ -minor. Otherwise,  $V(P_1) \cup \{u\}$  induces an odd hole. Since this hole cannot be shorter than  $H^*$ ,  $P_2$  is of length 2, and hence  $u$  is  $H^*$ -minor.

Now assume that  $u$  has three neighbors in  $H^*$ , and let  $P_1, P_2$  and  $P_3$  be the three sectors of the wheel  $(H^*, u)$ . If exactly one of the sectors is short then  $V(H^*) \cup \{u\}$  induces a pyramid. If two of the sectors are short then  $u$  is  $H^*$ -minor. Finally suppose that all three sectors are long. Since  $H^*$  is odd, at least one of the sectors, say  $P_1$ , is odd. Then  $V(P_1) \cup \{u\}$  induces an odd hole shorter than  $H^*$ , a contradiction.  $\square$

### 2.2 Vertices with More Than Three Neighbors in $H^*$

Let  $H^*$  be a shortest odd hole in  $G$ . Let  $S(H^*)$  be the set of  $H^*$ -major vertices that have four or more neighbors in  $H^*$ . Note that, for any  $u \in S(H^*)$ , every long sector of the wheel  $(H^*, u)$  is of even length since  $H^*$  is a shortest odd hole of  $G$ ; hence,  $(H^*, u)$  contains an odd number of short sectors.

Let  $S \subseteq V(G)$ . We say that vertex  $x \in V(G) \setminus S$  is  $S$ -complete if  $x$  is adjacent to every vertex in  $S$ . We say that an edge  $xy$  is  $S$ -complete if both vertices  $x$  and  $y$  are  $S$ -complete.

**Lemma 2** *Let  $H^*$  be a shortest odd hole in  $G$ . Suppose that  $G$  does not contain a jewel. If  $u, v \in S(H^*)$  are not adjacent then an odd number of edges of  $H^*$  are  $\{u, v\}$ -complete.*

*Proof:* Let  $u$  and  $v$  be nonadjacent vertices of  $S(H^*)$ . Suppose that an even number of edges of  $H^*$  are  $\{u, v\}$ -complete. Then some long sector  $P$  of the wheel  $(H^*, u)$  contains an odd number of short sectors of  $(H^*, v)$ . Let  $u_1$  and  $u_2$  be the endvertices of  $P$ .  $P$  has even length. Let  $P'$  be the subpath of  $H^*$  induced by  $(V(H^*) \setminus V(P)) \cup \{u_1, u_2\}$ .  $P'$  has odd length. Note that  $P'$  must be of length at least four, since otherwise  $(H^*, u)$  is a jewel, a contradiction. If  $P$  contains three or more neighbors of  $v$ , then the vertex set  $V(P) \cup \{u, v\}$  induces an odd wheel with center  $v$ , and hence contains an odd hole shorter than  $H^*$ , contradicting our choice of  $H^*$ . Otherwise, let  $v_1$  and  $v_2$  be the two neighbors of  $v$  in  $P$ . Vertex  $v$  cannot have exactly four neighbors in  $H^*$ , say  $v_1, v_2, v_3, v_4$ , such that both  $v_3u_1$  and  $v_4u_2$  are edges, because otherwise the vertex set  $(V(H^*) \setminus V(P)) \cup \{v\}$  induces a shorter odd hole than  $H^*$ , since  $P$  is even and  $P'$  is of length at least four. Therefore, there exist vertices  $u_3, v_3 \in V(H^*) \setminus V(P)$ , the neighbors of  $u$  and  $v$  respectively, such that  $u$  and  $v$  have no other neighbors on  $u_3v_3$ -subpath of  $H^*$ , call it  $Q$ , and vertices  $u_3$  and  $v_3$  are not adjacent to  $u_1$  or  $u_2$ . But now the vertex set  $V(Q) \cup V(P) \cup \{u, v\}$  induces a pyramid  $\Pi(v_1v_2v; u)$ , and hence contains an odd hole shorter than  $H^*$ , contradicting our choice of  $H^*$ .  $\square$

The following, which is an easy consequence of Lemma 2, will be used in several places.

**Lemma 3** *Let  $H^*$  be a shortest odd hole in  $G$ ,  $P$  be a subpath of  $H^*$  such that  $|V(H^*) \setminus V(P)| \geq 3$ , and  $x, y$  be two nonadjacent vertices in  $S(H^*)$ . Assume that no ends of  $P$  are  $\{x, y\}$ -complete and there is no  $\{x, y\}$ -complete edge in  $P$ . Then there exists an  $\{x, y\}$ -complete vertex in  $H^*$  with no neighbor in  $P$ .*

*Proof:* By Lemma 2, there exists an  $\{x, y\}$  complete edge  $e$  in  $H^*$ . One of the two endvertices of  $e$  has the desired property.  $\square$

**Lemma 4** *Suppose that  $G$  does not contain a jewel. If  $A \subseteq S(H^*)$  is a stable set, then an odd number of edges of  $H^*$  are  $A$ -complete.*

*Proof:* Let  $A \subseteq S(H^*)$  be a stable set and suppose that an even number of edges of  $H^*$  are  $A$ -complete. Let  $A'$  be a smallest subset of  $A$  with the property that an even number of edges of  $H^*$  are  $A'$ -complete. Note that by Lemma 2,  $|A'| \geq 3$ . Let  $s_1, \dots, s_m$  be the vertices of  $H^*$  adjacent to at least one vertex in  $A'$ , encountered in that order when traversing  $H^*$  clockwise. For  $i \in [m]$ , let  $S_i$  be the  $s_i s_{i+1}$ -subpath of  $H^*$  (indices taken modulo  $m$ ), that does not contain any intermediate vertex  $s_j$ ,  $j \in [m]$ .

**Claim** For every  $i \in [m]$ ,  $S_i$  is either an edge whose endvertices are both adjacent to some vertex  $x \in A$ , or  $S_i$  has even length.

*Proof of Claim:* If there is a vertex  $x \in A'$  adjacent to both  $s_i$  and  $s_{i+1}$ , then  $S_i$  is a sector of the wheel  $(H^*, x)$  and hence the result holds. Otherwise, let  $x_1$  and  $x_2$  be vertices of  $A'$  such that  $x_1$  is adjacent to  $s_i$  and  $x_2$  is adjacent to  $s_{i+1}$ . By Lemma 3 there exists an  $\{x_1, x_2\}$ -complete vertex  $u$  in  $H^*$  with no neighbor in  $S_i$ . Then the vertex set  $V(S_i) \cup \{x_1, x_2, u\}$  induces a hole. Since both  $x_1$  and  $x_2$  have at least four neighbors in  $H^*$ , this hole is shorter than  $H^*$ , so it must be even, hence  $S_i$  is of even length. This completes the proof of the claim.  $\diamond$

For  $C \subseteq A'$ , let  $\delta_C$  denote the number of edges of  $H^*$  that are  $C$ -complete. Let  $\delta$  be the number of paths in  $S_1, \dots, S_m$  of length one. Then

$$\delta = \sum_{i=1}^{|A'|} (-1)^{i+1} \sum_{C \subseteq A', |C|=i} \delta_C$$

By the choice of  $A'$ , for every  $C \subseteq A'$  such that  $C \neq A'$ ,  $\delta_C$  is odd. Hence the parity of  $\delta$  is equal to the parity of

$$\sum_{i=1}^{|A'|-1} \binom{|A'|}{i} + \delta_{A'}$$

which is itself equal to the parity of  $\delta_{A'}$  since

$$\sum_{i=1}^{|A'|-1} \binom{|A'|}{i} = 2^{|A'|} - 2$$

By the Claim and because  $H^*$  is odd,  $\delta$  is odd. Hence  $\delta_{A'}$  must be odd as well, contradicting the choice of  $A'$ .  $\square$

**Theorem 5** *Suppose that  $G$  does not contain a jewel. Let  $A$  be a stable set of  $S(H^*)$  and let  $x_1x_2$  be an edge of  $H^*$  such that every vertex of  $A$  is adjacent to both  $x_1$  and  $x_2$  (such an edge exists by Lemma 4). Let  $B$  be the set of vertices of  $S(H^*)$  that have no neighbor in  $\{x_1, x_2\}$ , and have both a neighbor and a nonneighbor in  $A$ . Then there exists an edge  $y_1y_2$  of  $H^*$  such that  $y_1$  is  $A$ -complete and every vertex of  $B$  has a neighbor in  $\{y_1, y_2\}$ .*

*Proof:* If  $B = \emptyset$  then the result is trivially true, so we may assume that  $B \neq \emptyset$ . Since every vertex of  $B$  is major, this implies that  $H^*$  is of length greater than 5.

**Claim 1** For every  $u \in B$ , an edge of  $H^*$  is  $(A \cup \{u\})$ -complete.

*Proof of Claim 1:* Let  $A_1$  be the neighbors of  $u$  in  $A$  and  $A_2 = A \setminus A_1$ . By Lemma 4, there is an edge  $u_1u_2$  of  $H^*$  such that every vertex of  $A_2 \cup \{u\}$  is adjacent to both  $u_1$  and  $u_2$ . Since  $u$  has no neighbor in  $\{x_1, x_2\}$ , every vertex of  $A_1$  must be adjacent to both  $u_1$  and  $u_2$ , else there is a 5-hole. This completes the proof of Claim 1.  $\diamond$

**Claim 2** If  $X$  is a stable set of  $B$ , then there exists an edge  $z_1z_2$  of  $H^*$  such that  $z_1$  is  $A$ -complete and every vertex of  $X$  has a neighbor in  $\{z_1, z_2\}$ .

*Proof of Claim 2:* We consider the following two cases.

*Case 1* There is a vertex in  $A$  that is not adjacent to any vertex in  $X$ .

Let  $A_1 \subseteq A$  be such that  $A_1 \cup X$  is a maximal stable set. By Lemma 4, an edge of  $H^*$  is  $(A_1 \cup X)$ -complete, say  $u_1u_2$ . Let  $w \in A \setminus A_1$ . Note that  $w$  is adjacent to some  $x \in X$ . If  $w$  is not adjacent to  $u_1$  or  $u_2$ , then there is a 5-hole in the graph induced by  $\{x, y, w, u_1, u_2, x_1, x_2\}$ , where  $y \in A_1$ . So every vertex of  $A \setminus A_1$  is adjacent to both  $u_1$  and  $u_2$ .

*Case 2* Every vertex of  $A$  is adjacent to some vertex in  $X$ .

By Claim 1 and Case 1, we may assume w.l.o.g. that  $|X| > 1$  and for every proper subset of  $X$  the result holds. Let  $w \in A$  be such that  $|N(w) \cap X|$  is minimum. Let  $Z = N(w) \cap X$ . Since every vertex of  $X$  has a non-neighbor in  $A$  and  $|Z|$  is minimum,  $|Z| < |X|$ . By our assumption, there exists an edge  $y_1y_2$  of  $H^*$  such that  $y_1$  is  $A$ -complete and every vertex of  $X \setminus Z$  has a neighbor in  $\{y_1, y_2\}$ . By Lemma 4 an edge of  $H^*$  is  $X$ -complete, say edge  $y_3y_4$ .

We may assume that vertices  $y_1, y_2, y_3, y_4$  are all distinct and  $y_1y_3$  and  $y_1y_4$  are not edges, since otherwise the result trivially holds. Also w.l.o.g.  $y_2y_4$  is not an edge.

Suppose that  $wy_4$  is not an edge. We may assume that some  $z \in Z$  is not adjacent to  $y_1$ , since otherwise the edge  $y_1y_2$  satisfies the claim. If some  $v \in X \setminus Z$  is adjacent to  $y_1$ , then  $\{y_1, v, w, z, y_4\}$  induces a 5-hole. So for every  $v \in X \setminus Z$ ,  $vy_1$  is not an edge, and hence  $vy_2$  is an edge. If  $w$  is adjacent to  $y_2$ , then  $\{y_2, w, v, z, y_4\}$  induces a 5-hole. So  $w$  is not adjacent to  $y_2$ . By Lemma 3, there is a vertex  $u$  of  $H^*$  adjacent to both  $v$  and  $w$ , but with no neighbor in  $\{y_1, y_2\}$ . Then  $\{y_1, y_2, u, v, w\}$  induces a 5-hole.

Therefore  $wy_4$  is an edge. We now show that  $y_4$  is  $A$ -complete. Let  $w' \in A$  and assume  $w'y_4$  is not an edge. By the choice of  $w$  and by the above argument, there is a vertex  $v \in X \setminus Z$  adjacent to  $w'$ . But then the graph induced by  $\{w, w', x_1, x_2, v, y_4\}$  contains a 5-hole. This completes the proof of Claim 2.  $\diamond$

**Claim 3** For every edge  $v_1v_2$  in  $G(B)$ , there exists  $v \in A$  that is adjacent to neither  $v_1$  nor  $v_2$ .

*Proof of Claim 3:* Let  $A_1$  be the set of neighbors of  $v_1$  in  $A$ , and  $A_2 = A \setminus A_1$ . Suppose the claim does not hold. Then  $v_2$  is universal for  $A_2$ . Let  $w_1$  be a vertex of  $A_1$  that  $v_2$  is not adjacent to. Then  $v_1, v_2, w_2, x_2, w_1, v_1$ , where  $w_2 \in A_2$ , is a 5-hole. This completes the proof of Claim 3.  $\diamond$

By Claim 1, we may assume that for every proper subset  $B'$  of  $B$ , the statement holds. By Claim 2 we may assume that  $B$  is not a stable set. Let  $v_1v_2$  be an edge of  $G(B)$ . By Claim 3, let  $v$  be a vertex of  $A$  that is adjacent to neither  $v_1$  nor  $v_2$ . Let  $y_1y_2$  be an edge of  $H^*$  such that  $y_1$  is  $A$ -complete and all vertices of  $B \setminus v_2$  have a neighbor in  $\{y_1, y_2\}$ . Let  $y_3y_4$  be an edge of  $H^*$  such that  $y_3$  is  $A$ -complete and all vertices of  $B \setminus v_1$  have a neighbor in  $\{y_3, y_4\}$ . Then the theorem follows from the following claim.

**Claim 4**  $v_1$  has a neighbor in  $\{y_3, y_4\}$ , or  $v_2$  has a neighbor in  $\{y_1, y_2\}$ .

*Proof of Claim 4:* Suppose the claim does not hold.  $v_1$  has no neighbor in  $\{y_3, y_4\}$  and  $v_2$  has no neighbor in  $\{y_1, y_2\}$ .

If a vertex of  $\{y_1, y_2\}$  coincides with a vertex of  $\{y_3, y_4\}$ , then  $\{y_1, y_2, y_3, y_4, v_1, v_2\}$  induces a 5-hole. Therefore, vertices  $y_1, y_2, y_3, y_4$  are all distinct.

We now show that  $v$  and  $v_1$  must have a common neighbor in  $\{y_1, y_2\}$ . Assume not. Then  $vy_1$  and  $v_1y_2$  are edges, and  $vy_2$  and  $v_1y_1$  are not. By Lemma 3, there is a vertex  $u$  of  $H^*$  that is  $\{v, v_1\}$ -complete but has no neighbor in  $\{y_1, y_2\}$ . Then  $\{y_1, y_2, v, v_1, u\}$  induces a 5-hole. Therefore,  $v$  and  $v_1$  have a common neighbor  $y$  in  $\{y_1, y_2\}$ , and similarly  $v$  and  $v_2$  have a common neighbor  $y'$  in  $\{y_3, y_4\}$ . If  $yy'$  is not an edge, then  $\{y, y', v, v_1, v_2\}$  induces a 5-hole. Therefore,  $yy'$  is an edge.

Let  $a, y, y', b$  be the subpath of  $H^*$  induced by  $\{y_1, y_2, y_3, y_4\}$ . Then  $vy, vy', v_1y, v_2y'$  are edges and  $v_2a, v_2y, v_1y', v_1b$  are not.

Let  $z_2$  be the neighbor of  $v_2$  in  $H^*$  that is closest to  $a$  in  $H^* \setminus \{y, y'\}$ . Note that  $z_2 \neq b$  since  $v_2$  is a major vertex. Let  $P_2$  be the  $az_2$ -subpath of  $H^*$  that does not contain  $y$ .

Suppose  $v$  does not have a neighbor in  $P_2$ . By Lemma 3, some vertex  $u$  of  $H^*$  is  $\{v, v_2\}$ -complete and has no neighbor in  $P_2$ . Note that  $u \neq b$  since  $b$  is not  $\{v, v_2\}$ -complete. But then  $P_2 \cup \{y, y', v, v_2, u\}$  induces a pyramid  $\Pi(vyy', v_2)$ , and hence there is an odd hole shorter than  $H^*$ , a contradiction. Therefore  $v$  must have a neighbor in  $P_2$ .

We now show that  $a$  is the unique neighbor of  $v$  in  $P_2$ . Let  $v'$  be the neighbor of  $v$  in  $P_2$  that is closest to  $z_2$ . Assume that  $v' \neq a$ . Let  $P'$  be the  $v'z_2$ -subpath of  $P_2$ . If  $v_1$  has no neighbor in  $P'$ , then the graph induced by  $S = P' \cup \{y, y', v, v_1, v_2\}$  is a pyramid  $\Pi(vyy', v_2)$  hence there is an odd hole shorter than  $H^*$ . If  $v_1$  has a neighbor in  $P' \setminus z_2$ , then the graph induced by  $S$  contains a pyramid  $\Pi(vyy', v_1)$  hence there is an odd hole shorter than  $H^*$ . So  $v_1$  is adjacent to  $z_2$ . If the graph induced by  $P_2 \cup \{y, y', v_1, v_2\}$  is an odd wheel with center  $v_1$ , there is an odd hole shorter than  $H^*$ . Hence  $v_1$  must have a neighbor in  $P_2 \setminus P'$ . If  $v_1$  has a neighbor  $z$  in  $P_2$  that lies strictly between  $a$  and  $v'$ , then there is a path  $Q$  from  $v$  to  $v_1$  with interior in  $z, P_2, v'$ . But then  $Q \cup \{y, y', v_2\}$  induces a pyramid  $\Pi(vyy', v_1)$ , which contains an odd hole shorter than  $H^*$ . Therefore  $a$  and  $z_2$  are the only neighbors of  $v_1$  in  $P_2$ . Then  $v$  is not adjacent to  $a$  for otherwise  $a, v, y', v_2, v_1, a$  is an odd hole. Let  $v''$  be the neighbor of  $v$  closest to  $a$  in  $P_2$ . Note that  $v'' \neq z_2$  since otherwise  $P_2 \cup \{y, y', v_2, v\}$  induces an odd wheel with center  $v$  hence there is an odd hole shorter than  $H^*$ . Let  $P''$  denote the  $av''$ -subpath of  $P_2$ . By Lemma 3, some vertex  $u$  of  $H^*$  is  $\{v, v_1\}$ -complete and has no neighbor in  $P''$ . But then the graph induced by  $P'' \cup \{y, v, v_1, u\}$  is a pyramid  $\Pi(ayv_1, v)$  hence there is an odd hole shorter than  $H^*$ . Therefore  $a$  is the unique neighbor of  $v$  in  $P_2$ .

Then  $v_1$  is not adjacent to  $a$  for otherwise  $a, v, y', v_2, v_1, a$  is an odd hole. Suppose  $v_1$  has a neighbor in  $P_2$ . By Lemma 3, there exists a vertex  $u$  of  $H^*$  adjacent to both  $v$  and  $v_1$ , but with no neighbor in  $P_2$ . Then the graph induced by  $P_2 \cup \{y, v, v_1, u\}$  contains a pyramid  $\Pi(ayv, v_1)$  hence there is an odd hole shorter than  $H^*$ . Therefore,  $v_1$  has no neighbor in  $P_2$ .

Let  $z_1$  be the neighbor of  $v_1$  in  $H^*$  that is closest to  $b$  in  $H^* \setminus \{y, y'\}$ . Let  $P_1$  be the  $bz_1$ -subpath of  $H^*$  that does not contain  $y$ . By symmetry,  $b$  is the unique neighbor of  $v$  in  $P_1$  and  $v_2$  has no neighbor in  $P_1$ . Since  $P_2, a, y, y'$  is a sector of wheel  $(H^*, v_2)$ ,  $P_2$  must be even, and similarly  $P_1$  is even. Note that  $z_1z_2$  is not an edge since  $H^*$  and the path  $a, y, y', b$  have odd length and  $P_1, P_2$  have even length. But then  $P_1 \cup P_2 \cup \{v, v_1, v_2\}$  induces an odd hole shorter than  $H^*$ , a contradiction.  $\square$

### 2.3 Cleaning Algorithm

In this section, we present our cleaning algorithm for the class of graphs of bounded clique number. The running time depends on the clique number.

**Input:** A graph  $G$  of bounded clique number  $k$ .

**Output:** Either an odd hole or a family  $\mathcal{F}$  of induced subgraphs of  $G$  that satisfies the following properties:

- (1)  $G$  contains an odd hole if and only if some graph of  $\mathcal{F}$  contains a clean shortest odd hole.
- (2)  $|\mathcal{F}|$  is  $O(|V(G)|^{8k})$ .

**Step 1:** Check whether  $G$  contains a jewel or a pyramid (by algorithms in [2]). If it does, output an odd hole and stop. Otherwise, set  $\mathcal{F}_1 = \{G\}$  and  $\mathcal{F}_2 = \emptyset$ .

**Step 2:** Repeat the following  $k$  times. For each graph  $F \in \mathcal{F}_1$  and every  $(P_1, P_2)$  where  $P_1 = x_0, x_1, x_2, x_3$  and  $P_2 = y_0, y_1, y_2, y_3$  are two induced paths of  $F$ , add to  $\mathcal{F}_2$  the graph obtained from  $F$  by removing the vertex set  $(N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)) \setminus (V(P_1) \cup V(P_2))$ . Set  $\mathcal{F}_1 = \mathcal{F}_2$  and  $\mathcal{F}_2 = \emptyset$ .

**Step 3:** Set  $\mathcal{F} = \mathcal{F}_1$ .

**Theorem 6** *This algorithm produces the desired output, and its running time is  $O(|V(G)|^{8k})$ .*

*Proof:* Suppose that the algorithm does not output an odd hole. Suppose  $G$  contains a shortest odd hole  $H^*$ . By Step 1  $G$  contains no jewel and no pyramid. Now we show how Step 2 generates a graph in  $\mathcal{F}_1$  that contains  $H^*$  and  $H^*$  is clean in it.

By Lemma 1,  $S(H^*)$  is the set of all  $H^*$ -major vertices. Let  $A$  be a maximal stable set of  $S(H^*)$ . We follow the notation in Theorem 5. Let  $P_1 = x_0, x_1, x_2, x_3$  and  $P_2 = y_0, y_1, y_2, y_3$  such that  $x_1x_2$  and  $y_1y_2$  satisfy the conditions stated in Theorem 5. Let  $S'(H^*)$  denote the set of vertices of  $S(H^*)$  that have no neighbor in  $\{x_1, x_2\}$ , and are  $A$ -complete. Let  $G'$  be the graph obtained from  $G$  by removing  $(N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)) \setminus (V(P_1) \cup V(P_2))$ . Then  $G'$  contains  $H^*$  and the set of major vertices for  $H^*$  in  $G'$  is contained in  $S'(H^*)$ . The clique number of the graph induced by  $S'(H^*)$  is one less than the clique number of the graph induced by  $S(H^*)$ . Hence, by the fact that the clique number of  $G$  is bounded by  $k$ , Theorem 5 implies that, when the  $k$  iterations of Step 2 are completed, some graph  $F \in \mathcal{F}_1$  contains  $H^*$  and  $H^*$  is clean in  $F$ . Hence (1) holds.

$O(|V(G)|^{8k})$  graphs are created in Step 2. Hence, (2) holds. The running time of Step 1 is  $O(|V(G)|^9)$  as discussed in [2]. The running time of Steps 2 is  $O(|V(G)|^{8k})$ . Therefore, the overall running time is  $O(|V(G)|^{8k})$ .  $\square$

In [2] a polynomial time algorithm with following specification is obtained.

**Input:** A clean graph  $G$ .

**Output:** ODD-HOLE-FREE when  $G$  is odd-hole-free, and NOT ODD-HOLE-FREE otherwise.

The above two algorithms imply that it is polynomial to test whether a graph of bounded clique number contains an odd hole.

## References

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