Digital Phase-Locked Loops Tracked by a Relay Sensor

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***Abstract*—An optimal algorithm is presented for tracking the phase of a slowly modulating signal by means of digital sampling of its sign. Error bounds and a numerical illustration are given.**

***Index Terms*— Demodulation, digital control, optimal algorithms, phase-locked loop, sampling.**

I. Introduction

DIGITAL phase-locked loops have been much studied since the 1970's [3]–[7], [10], since they are used in communications and system design in order to track the phase of an incoming signal. In the classical theory, the given data consist of actual (possibly corrupted) measurements of the signal at appropriate sampling intervals. However, in some recent applications (e.g., [1] and [2]), it has been found necessary to consider approximately sinusoidal signals whose values are only accessible after passing through a relay sensor—that is, the given data are now the sign (± 1) of the incoming signal. In this letter, we shall provide an algorithm for tracking the phase changes of such a signal which, being central in the sense of Information-Based Complexity theory [8], [9], possesses certain optimality properties. What this means is that at the kth sampling instant, we know that the set of possible phase errors is an interval I_k; to minimize the worst-case error, we must choose the midpoint of that interval as our new estimate of the phase error.

II. A Phase-Tracking Algorithm

We shall suppose that an approximately sinusoidal signal is given by

$$u(t) = A\sin(\omega t + \theta(t))$$

where A is its amplitude, ω its nominal frequency, and θ its slowly-varying phase carrying the information. We shall suppose that θ is continuous as in the case of phase-coherent or frequency modulation. Demodulation will be possible under the additional condition

$$\|\dot{\theta}\|_\infty \leq \delta\omega \tag{1}$$

where $\delta > 0$ is sufficiently small. The measurements of u are detected by a relay sensor, which provides us with a sequence (y_k) of values, given by

$$y_k = \operatorname{sign}(u(t(k)) + n(t(k))) \tag{2}$$

where $(t(k))$ is a sequence of sampling times, and n a disturbance that is small in magnitude, that is,

$$\|n\|_\infty \leq \eta A, \qquad \text{with } 0 < \eta < 1.$$

As usual,

$$\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0. \end{cases}$$

We make no assumption about sign(0), other than that it returns a value in the interval $[-1, 1]$.

We shall present an algorithm that is optimal when we assume a bounded-error model of noise; it should be remarked here that it will also have almost-optimality properties in the case when the noise has a symmetrical distribution (e.g., Gaussian), because it will provide an interpolatory algorithm in the sense of [8] and [9]. Moreover, if there is some absolute bound that is satisfied with high probability by the noise, then this value can be used to tune the algorithm. The performance in the Gaussian case will be illustrated later by means of a simulation.

Further, we set $t(0) = 0$, and $T(k) = t(k) - t(k-1)$, the *sampling interval*. $T(k)$ is the control input of the sampling process. It will be determined in feedback-law form using the *a priori* informations ω, η, δ, and measurements $y_0, \cdots, y_{k-1}$. To maximize the amount of information obtained from the sign of u, we should sample near its zeros. As in [6], we attempt to track positive-going zero crossings of u. Such a closed-loop system is called a digital phased-locked loop. Note that condition (1) ensures that the positive-going zero crossings are well defined. In fact, the kth zero, $\bar{t}_k$, is the unique solution of

$$\omega\bar{t}_k + \theta(\bar{t}_k) = 2k\pi.$$

Let

$$\phi(k) = \omega t(k) + \theta(t(k)) - 2k\pi \tag{3}$$

denote the true, unknown, phase error at the kth sample. The phase error $\phi(k)$ can be interpreted as the controlled state, and (3) and (2) as the state-space equations:

$$\begin{aligned} \phi(k+1) &= \phi(k) + \omega T(k+1) - 2\pi + v(k+1) \\ y_k &= \operatorname{sign}(A\sin\phi(k) + n(t(k))) \end{aligned} \tag{4}$$

where $v(k+1) = \theta(t(k+1)) - \theta(t(k))$ and $n(t(k))$ are the input and output errors.

The amplitude of the *a posteriori* regulation error is given by $\kappa_\phi(k) = |\hat{\phi}(k)|$, where $\hat{\phi}(k)$ is the central minimax estimate of $\phi(k)$ given $y_k, y_{k-1}\cdots$.

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Our procedure is to choose $T(k+1)$ in order to minimize the *a priori* worst-case phase error. Doing so, we shall obtain as a byproduct $\hat{\theta}(k)$, the central (minimax) estimate of $\theta(t(k))$ knowing $y_k, y_{k-1}\cdots$.

Theorem II.1: Suppose that $\delta < (\pi - 4\arcsin(\eta))/(5\pi)$. Then, under the conditions above, the worst-case optimal phase estimates and the control $T(k+1)$ are given by

$$\kappa_\phi(k+1) = \frac{1-\delta y_k}{2}\,\kappa_\phi(k) + \delta\pi, \qquad \kappa_\phi(0) = \frac{\pi}{2} \tag{5}$$

$$\begin{aligned}\hat{\phi}(k) &= y_k\kappa_\phi(k),\\ \hat{\theta}(k) &= \hat{\theta}(k-1) + \hat{\phi}(k)\end{aligned} \tag{6}$$

$$T(k+1) = \frac{2\pi}{\omega} - \frac{1}{\omega}\,\hat{\phi}(k) \tag{7}$$

and the worst-case bounds on $\phi(k)$ and $\hat{\theta}(k) - \theta(t(k))$ are

$$|\phi(k) - \hat{\phi}(k)| \le \rho_\phi(k) \quad \text{and} \quad |\theta(t(k)) - \hat{\theta}(k)| \le \rho_\phi(k) \tag{8}$$

where

$$\rho_\phi(k) = \kappa_\phi(k) + \arcsin(\eta), \qquad k \ge 0. \tag{9}$$

Asymptotically, the estimation and phase errors satisfy

$$\begin{aligned}&\limsup|\hat{\theta}(k) - \theta(t(k))|\\ &\quad\le \arcsin(\eta) + \frac{2\pi\delta}{1-\delta} = O(\eta) + O(\delta)\end{aligned} \tag{10}$$

$$\begin{aligned}&\limsup|\phi(k)|\\ &\quad\le \arcsin(\eta) + \frac{4\pi\delta}{1-\delta} = O(\eta) + O(\delta).\end{aligned} \tag{11}$$

Proof: We suppose that, for $k \ge 1$, $\phi(k)$ lies within an interval $[-\alpha(k), \alpha(k)]$; indeed, we shall see that we can take $0 \le \alpha(k) \le \pi - \arcsin(\eta)$ for each k. Assuming this for the moment, then given the additional information y_k, we know that

$$y_k(A\sin(\phi(k)) + n(t(k))) \ge 0$$

from which we conclude that

$$\sin(y_k\phi(k)) + \eta \ge 0$$

or $y_k\phi(k) \in I_k = [-\arcsin(\eta), \alpha(k)]$ for $k \ge 1$. The case $k = 0$ is slightly different, and our initial data y_0 allow us to conclude only that we may assume that $y_0\phi(0) \in I_0 = [-\arcsin(\eta), \pi + \arcsin(\eta)]$. We denote by $\rho_\phi(k)$ the radius of the interval I_k. All points of this interval are possible. Thus our central (minimax) estimate of $\phi(k)$ is now

$$\hat{\phi}(k) = y_k(\alpha(k) - \arcsin(\eta))/2, \qquad \text{for } k \ge 1 \tag{12}$$

and $\hat{\phi}(0) = y_k\pi/2$. To compensate for this, the optimal choice for $T(k+1)$ is given by

$$T(k+1) = \frac{2\pi}{\omega} - \frac{1}{\omega}\,\hat{\phi}(k). \tag{13}$$

Therefore, knowing that $\phi(k)$ lies in an interval of length $\alpha(k) + \arcsin(\eta)$, $\phi(k+1)$ will lie in a similar interval, symmetrized about 0, and expanded by the output error v. The error $v(k)$ is bounded in modulus by $\delta\omega T(k+1)$, and thus we can take

$$\alpha(k+1) = \frac{\alpha(k) + \arcsin(\eta)}{2} + \delta\omega T(k+1) \tag{14}$$

for $k > 1$, whereas

$$\begin{aligned}\alpha(1) &= \{\pi/2\} + \arcsin(\eta) + \delta\omega T(1)\\ &\le \{\pi/2\} + \arcsin(\eta) + \{5\pi/2\}\delta < \pi - \arcsin(\eta).\end{aligned} \tag{15}$$

We shall now show inductively that $\pi - \arcsin(\eta) \ge \alpha(k) \ge \arcsin(\eta)$ for all $k \ge 1$. However, it is not necessary for our purposes that $(\alpha(k))$ decrease monotonically. Note that

$$\begin{aligned}&\alpha(k+1) - \arcsin(\eta)\\ &\quad= \frac{\alpha(k) - \arcsin(\eta)}{2} + \delta\omega T(k+1)\\ &\quad= (1-\delta y_k)\,\frac{\alpha(k) - \arcsin(\eta)}{2} + \delta 2\pi.\end{aligned}$$

As $\alpha(1) - \arcsin(\eta) > 0$ we have inductively that $\alpha(k) > \arcsin(\eta)$ for each $k \ge 1$. Further, we have inductively that

$$\begin{aligned}&\alpha(k+1) - \arcsin(\eta)\\ &\quad< \frac{(1+\delta)(\pi - 2\arcsin(\eta))}{2} + 2\pi\delta < \pi - 2\arcsin(\eta)\end{aligned} \tag{16}$$

by virtue of the hypothesis on δ. We deduce that $\kappa_\phi(k) = (\alpha(k) - \arcsin(\eta))/2$, so that $\hat{\phi}(k) = y_k\kappa_\phi(k)$, $\rho_\phi(k) = \kappa_\phi(k) + \arcsin(\eta)$, and (14) can be rewritten

$$\kappa_\phi(k+1) = \frac{1-\delta y_k}{2}\kappa_\phi(k) + \delta\pi.$$

At this point, note that $T(k+1) \ge (1/\omega)(2\pi - \pi/2) > 0$, that is, the control is admissible. Due to (3), the central estimate of $\theta(t(k))$ at time $t(k)$ satisfies

$$\hat{\theta}(k) - \theta(t(k)) = \hat{\phi}(k) - \phi(k), \tag{17}$$

leading to (8). Now to obtain the second formula in (6), we use (17) to obtain

$$\begin{aligned}\hat{\theta}(k) - \hat{\theta}(k-1) = \theta(t(k)) + \hat{\phi}(k) - \phi(k) - \theta(t(k-1))\\ - \hat{\phi}(k-1) + \phi(k-1)\end{aligned} \tag{18}$$

which simplifies, using (4) and the definition of v, to give

$$\hat{\theta}(k) - \hat{\theta}(k-1) = \hat{\phi}(k) - \hat{\phi}(k-1) - \omega T(k) + 2\pi \tag{19}$$

and then the formula follows using (13).

Let $\beta_i(k), i = 0, 1$ be given by $\beta_i(0) = \pi/2$ and for $k \ge 0$,

$$\beta_i(k+1) = \frac{1-(-1)^i\delta}{2}\,\beta_i(k) + \delta\pi.$$

Then clearly $0 \le \beta_0(k) \le \kappa_\phi(k) \le \beta_1(k)$ for each k.

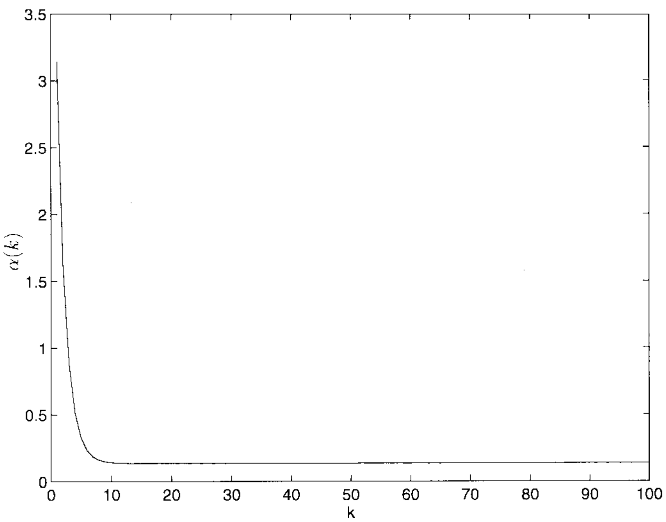

Fig. 1.

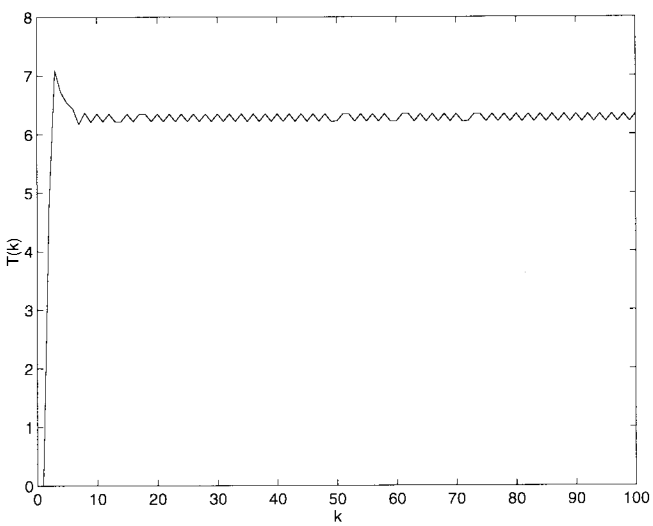

Fig. 2.

The solution to a recurrence relation of the form

$$\beta(k+1) = \lambda\beta(k) + \gamma$$

with $\lambda \neq 1$ is

$$\beta(k) = \left(\beta(0) - \frac{\gamma}{1-\lambda}\right)\lambda^k + \frac{\gamma}{1-\lambda}.$$

In this case,

$$\lambda_i = \frac{1-(-1)^i\delta}{2} \quad \text{and} \quad \gamma = \delta\pi.$$

As $\delta < 1$, we have $\lambda_i < 1$, and, asymptotically,

$$\lim_{k\to\infty} \beta_i(k) = \frac{\gamma}{1-\lambda_i} = \frac{2\pi\delta}{1+(-1)^i\delta}, \qquad i = 0, 1$$

as asserted in (10). Likewise (11) now follows from (12). We have also proved the remark below. □

Remark II.1: The feedback law from y_k to $T(k+1)$ is affine, but not time-invariant, since earlier measurements will necessarily have more effect as the sampling method locks onto the correct phase. Asymptotically, however, it is approximately time-invariant: $\lim_k \kappa_\phi(k) = 2\pi\delta + O(\delta^2)$.

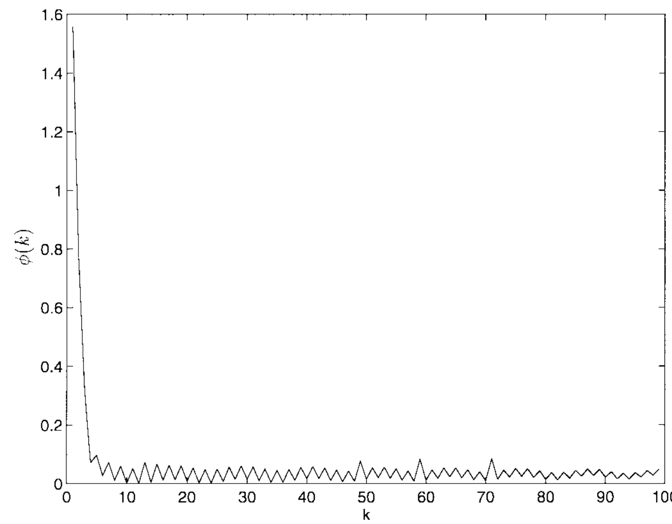

Fig. 3.

III. Example

We take $\theta(t) = (\delta/\sqrt{2})\cos(t\sqrt{2})$, and $A = 1$, $\omega = 1$, $\delta = \eta = 0.02$.

To show that the method still performs very well in the presence of Gaussian disturbances, the simulation makes use of such noise, distributed according to the law $N(0, \eta^2)$.

Fig. 1 shows the theoretical bound $\alpha(k)$ on the phase error; and Fig. 2 shows the sampling interval $T(k)$.

The phase error bound $\alpha(k)$ quickly settles down to a value of approximately 0.27, which is in accordance with the predicted worst-case value $\arcsin(0.02) + 4\pi/49 = 0.2765$. Likewise, the sampling intervals are close to the true period, 2π.

Fig. 3 shows the actual phase error, $\phi(k)$, which is not available to the algorithm, but which is in practice smaller than the predicted worst-case bound. We remark that the algorithm is very robust, in that it rapidly compensates for any outlier.

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