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MINIMAL LAGRANGIAN SURFACES IN $\mathbb{CH}^2$ AND REPRESENTATIONS OF SURFACE GROUPS INTO $SU(2,1)$

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Abstract. We use an elliptic differential equation of Ţiţeica (or Toda) type to construct a minimal Lagrangian surface in $\mathbb{CH}^2$ from the data of a compact hyperbolic Riemann surface and a cubic holomorphic differential. The minimal Lagrangian surface is equivariant for an $SU(2,1)$ representation of the fundamental group. We use this data to construct a diffeomorphism between a neighbourhood of the zero section in a holomorphic vector bundle over Teichmuller space (whose fibres parameterise cubic holomorphic differentials) and a neighborhood of the $\mathbb{R}$-Fuchsian representations in the $SU(2,1)$ representation space. We show that all the representations in this neighbourhood are complex-hyperbolic quasi-Fuchsian by constructing for each a fundamental domain using an $SU(2,1)$ frame for the minimal Lagrangian immersion: the Maurer-Cartan equation for this frame is the Ţiţeica-type equation. A very similar equation to ours governs minimal surfaces in hyperbolic 3-space, and our paper can be interpreted as an analog of the theory of minimal surfaces in quasi-Fuchsian manifolds, as first studied by Uhlenbeck.

1. Introduction.

The equation

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log(s^2) = |Q|^2 s^{-4} + s^2 \tag{1.1}$$

is satisfied by the conformally flat metric $s^2|dz|^2$ of a minimal Lagrangian surface in the complex hyperbolic plane $\mathbb{CH}^2$, where $z = x + iy$ is a local conformal coordinate and $Q\,dz^3$ is a holomorphic cubic differential. We can treat this equation either as a local form or as an expression for the equations on the universal cover of a compact surface Σ. In fact, this equation is an integrability condition: satisfying it is a necessary condition for the existence of a minimal Lagrangian immersion of Σ into $\mathbb{CH}^2$.

There is also a coordinate invariant version. Fix a background metric h on a surface Σ. Then the universal cover of Σ admits a minimal Lagrangian immersion into $\mathbb{CH}^2$ with metric $e^u h$ if it admits a smooth function $u : \Sigma \to \mathbb{R}$ and a holomorphic cubic differential U for which

$$\Delta_h u - 4\|U\|_h^2 e^{-2u} - 4e^u - 2\kappa_h = 0 \tag{1.2}$$

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where Δ_h, κ_h are respectively the Laplacian and curvature, and $\|U\|_h$ is the norm on cubic differentials, all with respect to h.

We prove the existence of global solutions to this equation (1.2) on any compact hyperbolic surface Σ provided U is sufficiently small. These equations are actually necessary and sufficient conditions for the existence of a special frame $F : \tilde{\Sigma} \to SU(2,1)$ (called a *Legendrian frame*) for a minimal Lagrangian immersion $\varphi : \tilde{\Sigma} \to \mathbb{CH}^2$, where $\tilde{\Sigma}$ denotes the universal cover if Σ. This frame determines a flat $SU(2,1)$-bundle over Σ whose holonomy provides a representation of the fundamental group $\pi_1\Sigma$ into $SU(2,1)$ for which the map φ is equivariant.

The latter part of the paper concerns properties of the representations we produce. All of the representations we produce have zero Toledo invariant, as they arise from Lagrangian surfaces (the Toledo invariant characterises the connected components of the representation space of surface groups into $SU(2,1)$ [38]). When $U = 0$, the minimal Lagrangian surface is simply the canonical totally geodesic Lagrangian embedding of $\mathbb{RH}^2$ in $\mathbb{CH}^2$. The corresponding representation takes values in $SO(2,1) \simeq PSL(2,\mathbb{R})$ and is said to be $\mathbb{R}$-Fuchsian (it is a Fuchsian representation). For U small, we prove the induced minimal Lagrangian surface is properly embedded into $\mathbb{CH}^2$, and that the exponential map of the normal bundle of the surface covers all of $\mathbb{CH}^2$. This allows us to construct a locally finite fundamental domain for the $\pi_1\Sigma$ action on $\mathbb{CH}^2$, simply by taking a bundle of Lagrangian planes normal to the immersion over a fundamental domain on the minimal Lagrangian surface. As a consequence, each representation we produce is *complex-hyperbolic quasi-Fuchsian*, i.e., discrete, faithful, geometrically finite and totally loxodromic. To be precise, we prove the following theorem (this is a restatement of Theorem 9.3 below).

Theorem 1.1. *Let S be a closed oriented surface of genus $g \geq 2$. In the representation space* $\mathrm{Hom}(\pi_1 S, SU(2,1))/SU(2,1)$ *there is a neighborhood $\mathcal{P}$ of the locus of $\mathbb{R}$-Fuchsian representations so that for all $\rho \in \mathcal{P}$,*

- *ρ is complex-hyperbolic quasi-Fuchsian.*
- *There is a natural identification of ρ with a pair (Σ, U) consisting of Σ a marked conformal structure on S and U a small holomorphic cubic differential on Σ. In particular, there is submersion of $\mathcal{P}$ onto the Teichmüller space of $\mathbb{R}$-Fuchsian representations, and a complex structure on $\mathcal{P}$.*
- *There is a canonical ρ-invariant minimal Lagrangian embedding $\mathcal{D} \subset \mathbb{CH}^2$ of the Poincaré disc, and an invariant normal projection of $\mathbb{CH}^2 \to \mathcal{D}$.*

There are several aspects to this construction which we consider to be valuable and deserve further study.

First, it gives a holomorphic parameterisation for an open set of complex-hyperbolic quasi-Fuchsian representations (a neighbourhood of the locus of all $\mathbb{R}$-Fuchsian representations) as a neighbourhood of the zero section in a holomorphic vector bundle over Teichmüller space of rank $5g - 5$. It was already known from the work of Guichard [7] and Parker-Platis [25] that the $\mathbb{R}$-Fuchsian locus possesses an open neighbourhood of complex-hyperbolic quasi-Fuchsian representations (in fact, Guichard's result says that the space of complex hyperbolic quasi-Fuchsian representations is open[1]) but at this point in time not much is known about how big this set is within the Toledo invariant zero component. The fundamental domains we produce are in the end similar to those of Parker-Platis, as both consist of unions of Lagrangian planes, but with our data we get some measure, through the norm of U, of how far we are away from the $\mathbb{R}$-Fuchsian locus. Moreover, this parameterisation has an intriguing interpretation in terms of the Yang-Mills-Higgs bundle description of representation space (see Remark 1 below) which could help explain how far this parameterisation can extend.

Second, our approach is analogous to the study of minimal surfaces in quasi-Fuchsian hyperbolic 3-manifolds initiated by Uhlenbeck in [35] (and continued in [31, 17, 36, 10]). Indeed, one of the main goals of the study of surface group representations into $SU(2,1)$ is to find the extent to which the theory of quasi-Fuchsian representations of surface groups extends to the complex hyperbolic case (for a recent survey, see [26]). The conformal factor of a minimal surface in $\mathbb{RH}^3$ solves an equation analogous to (1.2),

$$\Delta_h u - 4\|V\|_h^2 e^{-u} - 4e^u - 2\kappa_h = 0, \tag{1.3}$$

where V a holomorphic quadratic differential. It is known that for quasi-Fuchsian representations near enough to Fuchsian (called almost Fuchsian) there is a unique invariant minimal surface in $\mathbb{RH}^3$. On the other hand, there are quasi-Fuchsian representations for which there are many minimal surfaces. Presumably, the complex-hyperbolic representations we produce here are analogous to the almost Fuchsian case. The solutions to (1.2) we use are what we call *small* solutions (which means, when $\kappa_h = -1$, that the metric $g = e^u h$ has curvature bounds $-3 \leq \kappa_g \leq -2$). Provided U is sufficiently small, (1.2) has exactly one small solution determined by (Σ, h, U). But it is possible that there are complex-hyperbolic quasi-Fuchsian representations far enough away from $\mathbb{R}$-Fuchsian to admit multiple invariant minimal Lagrangian surfaces.

It is also worth noting that equation (1.2) is one of several formally similar equations which arise from surface geometries corresponding to different real forms of $SL(3,\mathbb{C})$, most of which have attracted attention in the recent literature. These are all variations on the theme of *Ţiţeica's equation*,

$$u_{xy} + e^{-2u} - e^u = 0.$$

This hyperbolic equation corresponds to nonconvex proper affine spheres in $\mathbb{R}^3$, and the symmetry group is $SL(3,\mathbb{R})$. The techniques of integrating surfaces given solutions to equations of this type originated with Ţiţeica's papers [33, 34]. The

[1] Misha Kapovich has also informed us that this result was known earlier, and follows from the techniques used to address the case of $SO(n,1)$; see e.g. Izeki [12].

more modern variants, which are distinct from (1.2), are

$$\Delta_h u - 4\|U\|_h^2 e^{-2u} + 4e^u - 2\kappa_h = 0, \tag{1.4}$$
$$\Delta_h u + 4\|U\|_h^2 e^{-2u} - 4e^u - 2\kappa_h = 0, \tag{1.5}$$
$$\Delta_h u + 4\|U\|_h^2 e^{-2u} + 4e^u - 2\kappa_h = 0. \tag{1.6}$$

In each case U is a cubic holomorphic differential and $e^u h$ is a metric for, respectively: a minimal Lagrangian surface in $\mathbb{CP}^2$ (1.4), where the isometry group is $SU(3)$ (see, for example, [23, 9]); hyperbolic (1.5) and elliptic (1.6) affine spheres in $\mathbb{R}^3$, where the symmetry group is $SL(3,\mathbb{R})$ (see [37, 30]). The latter two equations were also recently studied in order to construct solutions to the Monge-Ampère equation $\det(\partial^2 u/\partial x^i \partial x^j) = 1$ on affine manifolds diffeomorphic to $\mathbb{R}^3$ minus the "Y" vertex of a graph [21, 22]. Equation (1.5) can also be used to parameterise the Hitchin component of the representation space of surface groups into $SL(3,\mathbb{R})$ [18, 19, 20].

Given Theorem 1.1, the next challenge is to understand all the representations which can be obtained from equivariant minimal Lagrangian immersions of the Poincaré disc. All will have zero Toledo invariant and therefore lie in the same connected component of representation space. This will require a greater understanding of the solutions to equation (1.2) in the case where $\|U\|_h$ is not "small." Schoen-Wolfson's theory of mean curvature flow of Lagrangian submanfolds in Kähler-Einstein surfaces [27] might be useful here. Analogous theories of surfaces which realise representations have been worked out for some of the equations mentioned above. For example, in the case of equation (1.3) each quasi-Fuchsian hyperbolic manifold admits at least one immersed minimal surface (see Uhlenbeck [35]). Cheng-Yau provide a similar theory for equation (1.5) by showing that each nondegenerate convex cone in $\mathbb{R}^3$ contains a hyperbolic affine sphere invariant under any unimodular affine automorphisms of the cone [2, 3].

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Notation. For $u, w \in \mathbb{C}^n$ we use $u \cdot v$ to denote the standard (complex bilinear) dot product, and set $\|u\| = \sqrt{u \cdot \bar{u}}$. We use $e_1, \ldots, e_n$ to denote the standard basis for $\mathbb{C}^n$. For any non-zero $u \in \mathbb{C}^{n+1}$ we use $[u] \in \mathbb{CP}^n$ to denote the complex line it generates.

2. Complex hyperbolic geometry.

2.1. **Complex hyperbolic n-space.** Recall that complex hyperbolic n-space is the complex manifold

$$\mathbb{CH}^n = \{w = (w_1, \ldots, w_n) \in \mathbb{C}^n : \|w\|^2 < 1\}$$

equipped with the Hermitian metric $\sum_{j,k=1}^{n} h_{j\bar{k}} dw_j \otimes d\bar{w}_k$ with components

$$h_{j\bar{k}} = \frac{1}{1-\|w\|^2}(\delta_{jk} + \frac{\bar{w}_j w_k}{1-\|w\|^2}). \tag{2.1}$$

and Kähler form

$$\omega = \frac{i}{2}\partial\bar{\partial}\log(1-\|w\|^2).$$

With this metric $\mathbb{CH}^n$ has constant holomorphic sectional curvature -4. We can embed $\mathbb{CH}^n$ in $\mathbb{CP}^n$ by

$$\mathbb{CH}^n \to \mathbb{CP}^n; \quad w \mapsto [w,1] = [w_1,\ldots,w_n,1]. \tag{2.2}$$

Let $\langle\ ,\ \rangle$ denote the indefinite Hermitian form on $\mathbb{C}^{n+1}$ given by

$$\langle u, v\rangle = u_1\bar{v}_1 + \ldots + u_n\bar{v}_n - u_{n+1}\bar{v}_{n+1}.$$

We see that (2.2) identifies $\mathbb{CH}^n$, as a manifold, with $\mathbb{P}W_-$, the space of complex lines in $W_- = \{u \in \mathbb{C}^{n+1} : \langle u,u\rangle < 0\}$.

Let $\pi : L \to \mathbb{P}W_-$ denote the tautological line bundle. We note that W_- can be identified with L with its zero section removed. Using the standard identification $T_zW_- \simeq \mathbb{C}^{n+1}$ we obtain a splitting

$$TW_- = V + H$$

where $V = \ker(d\pi)$ and the horizontal subspace at z is

$$H_z = \{u \in \mathbb{C}^{n+1} : \langle u, z\rangle = 0\}. \tag{2.3}$$

On H the form $\langle\ ,\ \rangle$ is positive definite. Further, this splitting is invariant for the $\mathbb{C}^\times$ action along fibres of π, since these fibres are the $\mathbb{C}^\times$ orbits, and the metric on H is also invariant. Therefore we obtain an identification of $T\mathbb{P}W_-$ with π_*H, by assigning to each tangent vector its horizontal lift. This equips $\mathbb{P}W_-$ with a Kähler structure whose Levi-Civita connexion is the horizontal projection of flat differentiation in $\pi_*H \subset \mathbb{P}W_- \times \mathbb{C}^3$. One can easily show that the Kähler structure $\mathbb{P}W_-$ inherits from $\mathbb{CH}^n$ agrees with that obtained from π_*H, hence π is a pseudo-Riemannian submersion.

2.2. $\mathbb{CH}^n$ **as a symmetric space.** As a symmetric space, $\mathbb{CH}^n$ is the non-compact dual to $\mathbb{CP}^n$. Since we will make use of this for deriving the equations let us summarise the relevant facts. The Lie group $G = U(n,1)$ of isometries for $\langle\ ,\ \rangle$ acts transitively on the pseudo-sphere

$$S_- = \{v \in \mathbb{C}^{n+1} : \langle v, v\rangle = -1\} \subset W_-,$$

and we will consider S_- to be the G-orbit of e_{n+1}. This action descends to a transitive action of $U(n,1)$ on $\mathbb{CH}^n$ by holomorphic isometries. The isotropy group K for this action is isomorphic to $U(n)\times S^1$, and $\mathbb{CH}^n \simeq G/K$ as manifolds.

Let q be the diagonal matrix $\mathrm{diag}(1,\dots,1,-1)$ which represents the form $\langle\ ,\ \rangle$ (i.e., $\bar{v}^t q u = \langle u, v\rangle$). Define a involution $\sigma \in \mathrm{Aut}(\mathfrak{gl}(n+1,\mathbb{C}))$ by $\sigma(A) = qAq$. Then the Lie algebra of G is

$$\mathfrak{g} = \mathfrak{u}(n,1) = \{A \in \mathfrak{gl}(n+1,\mathbb{C}) : \sigma(A) = -\bar{A}^t\}.$$

This involution σ restricts to $\mathfrak{g}$ and provides the symmetric space decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ into ± 1 eigenspaces of σ, with $\mathfrak{k} \simeq \mathfrak{u}(n) \times i\mathbb{R}$ and $\mathfrak{m} \simeq \mathbb{C}^n$, the latter via

$$\mathbb{C}^n \to \mathfrak{m}; \quad u \mapsto \begin{pmatrix} O_n & u \\ \bar{u}^t & 0 \end{pmatrix}$$

where O_n in the $n \times n$ zero matrix.

The coset map $G \to G/K$ is a principal K-bundle. Using the right adjoint action of K on $\mathfrak{m}$ we obtain as associated vector bundle $[\mathfrak{m}] = G \times_K \mathfrak{m}$ which can be identified with $T(G/K)$ (see, for example, [1, p6]). The Hermitian metric on G/K is obtained from the (unique up to scale) $\mathrm{Ad}\,K$-invariant inner product on $\mathfrak{m}$ and the map $G/K \to \mathbb{P}W_-$; $gK \mapsto [ge_{n+1}]$ provides an isomorphism of Kähler manifolds, $G/K \simeq \mathbb{CH}^n$.

3. Lagrangian immersions in $\mathbb{CH}^2$.

On $\mathbb{C}^3$, let $\langle\ ,\ \rangle$, W_- and S_- be as above. For $u \in S_-$, the tangent space

$$T_u S_- = \{v \in \mathbb{C}^3 : \mathrm{Re}\langle v, u\rangle = 0\}$$

contains the horizontal subspace $H_u = \{v \in T_u S_- : \langle v, u\rangle = 0\}$. The form $\langle\ ,\ \rangle$ is a Hermitian inner product on H_u with real and imaginary components

$$\langle\ ,\ \rangle = g(\ ,\ ) + i\omega(\ ,\ ).$$

These provide the Riemannian metric and the symplectic structure on the horizontal bundle. The map $S_- \to \mathbb{CH}^2$ is a horizontal isometry of Hermitian structures.

For $\mathcal{D}$ the Poincaré disc, let $\varphi : \mathcal{D} \to \mathbb{CH}^2$ be a Lagrangian immersion. It lifts horizontally to a map $f : \mathcal{D} \to S_-$ which is Legendrian. This has a natural $U(2,1)$-frame for f, which we describe below. We will show that the Maurer-Cartan equations for this frame depend only on the induced metric, the mean curvature and a cubic differential. In section 4 we will that see that when φ is minimal the cubic differential is holomorphic, when φ is also equivariant for a representation of a surface group into $PU(2,1)$, the metric and cubic differential live on the quotient surface.

Using the conformal parameter z on $\mathcal{D}$ we can characterise f as Legendrian by the equations

$$\langle f_z, f\rangle = 0 = \langle f_{\bar{z}}, f\rangle. \tag{3.1}$$

A priori this only seems to force f to be horizontal, but by differentiating the first equation with respect to $\bar{z}$, and the second with respect to z, we find $\langle f_z, f_z\rangle =$

$\langle f_{\bar z}, f_{\bar z}\rangle$, which implies the Legendrian condition $\omega(f_x, f_y) = 0$. Since $\langle\ ,\ \rangle$ is positive definite on the horizontal subspace we can write $|f_z| = \sqrt{\langle f_z, f_z\rangle} = |f_{\bar z}|$.

We will also assume that φ is conformal, and hence f is horizontally conformal, i.e.,

$$\langle f_z, f_{\bar z}\rangle = 0. \tag{3.2}$$

Thus we obtain a global frame $F : \mathcal{D} \to U(2,1)$ with columns

$$F = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}, \quad f_1 = \frac{1}{|f_z|} f_z,\ f_2 = \frac{1}{|f_{\bar z}|} f_{\bar z},\ f_3 = f. \tag{3.3}$$

Now set $\alpha = F^{-1}dF$. We want to calculate the Maurer-Cartan equations

$$d\alpha + \alpha \wedge \alpha = 0. \tag{3.4}$$

Let $A = F^{-1}F_z$ and $B = F^{-1}F_{\bar z}$, then

$$\alpha = Adz + Bd\bar z \tag{3.5}$$

This is a $\mathfrak{u}(2,1)$-valued 1-form, i.e, $q\alpha q = -\bar\alpha^t$, where $q = \mathrm{diag}(1,1,-1)$. It follows that $B = -q\bar A^t q$ and the entries of the matrix qA are $\langle (f_j)_z, f_k\rangle$ for the j^{th} column and k^{th} row.

Set $s = |f_z| = |f_{\bar z}|$ so that metric induced on $\mathcal{D}$ by φ is $s^2|dz|^2$. We calculate

$$(f_1)_z = s^{-1}(f_{zz} - s_z f_1), \quad (f_2)_z = s^{-1}(f_{z\bar z} - s_z f_2), \quad (f_3)_z = f_z. \tag{3.6}$$

Now it is useful to have an expression for the second fundamental form and the mean curvature of φ. Write $z = x + iy$ and define an orthonormal basis for $T\mathcal{D} \subset f^{-1}TS_-$ by

$$E_1 = \frac{f_x}{|f_x|}, \quad E_2 = \frac{f_y}{|f_y|}.$$

Then iE_1, iE_2 span the normal bundle $T\mathcal{D}^\perp$. Since $S_- \to \mathbb{CH}^2$ is a horizontal isometry, the second fundamental form of φ is given by

$$\mathrm{I\!I}(X,Y) = \sum_{j=1}^{2} g(XYf, iE_j)iE_j, \quad X, Y \in \Gamma(T\mathcal{D}).$$

(This follows from the fact mentioned above that the Levi-Civita connection on $\mathbb{CH}^2$ is the projection of the flat connection on $W_- \subset \mathbb{C}^3$.) Therefore, the mean curvature is

$$H = \frac{1}{2}\sum_{j=1}^{2} g(E_1^2 f + E_2^2 f, iE_j)iE_j = \frac{1}{2}(E_1^2 f + E_2^2 f)^\perp.$$

Now $|f_x|^2 = |f_y|^2 = 2s^2$ and it is simple to check that

$$g(f_{xx} + f_{yy}, f_x) = 0 = g(f_{xx} + f_{yy}, f_y)$$

and therefore

$$(E_1^2 f + E_2^2 f)^\perp = \frac{2}{s^2}(f_{z\bar z} + \langle f_{z\bar z}, f\rangle f),$$

taking care to observe that $\langle f, f\rangle = -1$. But $\langle f_{z\bar z}, f\rangle = -|f_z|^2$, therefore

$$H = \frac{1}{s^2} f_{z\bar z} - f = \frac{1}{4}\Delta_g f - f,$$

where we have abused notation by letting g stand for $\varphi^* g$ as well.

Using (3.6) we write the quantities $\langle (f_j)_z, f_k\rangle$ as they would appear in the matrix qA.

$$\begin{matrix} \langle f_{zz}, f_z\rangle s^{-2} - s_z s^{-1} & \langle f_{z\bar z}, f_z\rangle s^{-2} & s \\ \langle f_{zz}, f_{\bar z}\rangle & \langle f_{z\bar z}, f_{\bar z}\rangle s^{-2} - s_z s^{-1} & 0 \\ \langle f_{zz}, f\rangle s^{-1} & \langle f_{z\bar z}, f\rangle s^{-1} & 0 \end{matrix} \tag{3.7}$$

Many of these terms simplify. We have

$$\langle f_{zz}, f_z\rangle s^{-2} = (\langle f_z, f_z\rangle_z - \langle f_z, f_{z\bar z}\rangle) s^{-2} = 2 s_z s^{-1} - \langle f_z, H\rangle.$$

Together, $\langle f_z, f\rangle = 0$ and $\langle f_z, f_{\bar z}\rangle = 0$ imply $\langle f_{zz}, f\rangle = 0$, and therefore $\langle f_{zz}, f_{\bar z}\rangle = -\langle f_{zzz}, f\rangle$. Those identities also show that for $Q = \langle f_{zzz}, f\rangle$ the quantity $Q\,dz^3$ is a cubic differential.

Finally, $\langle f_{z\bar z}, f_{\bar z}\rangle = s^2\langle H, f_{\bar z}\rangle$, and therefore we deduce that

$$A = \begin{pmatrix} s_z s^{-1} - \langle f_z, H\rangle & \langle H, f_z\rangle & s \\ -Q s^{-2} & \langle H, f_{\bar z}\rangle - s_z s^{-1} & 0 \\ 0 & s & 0 \end{pmatrix} \tag{3.8}$$

It follows that

$$B = \begin{pmatrix} -s_{\bar z} s^{-1} + \langle H, f_z\rangle & \bar Q s^{-2} & 0 \\ -\langle f_z, H\rangle & -\langle f_{\bar z}, H\rangle + s_{\bar z} s^{-1} & s \\ s & 0 & 0 \end{pmatrix} \tag{3.9}$$

At this point the following observation is useful.

Lemma 3.1. *Let $\sigma_H = \omega(H, df)$, which is the mean curvature 1-form for φ (pulled back by the lift f). Then $i\sigma_H = \langle H, df\rangle = \frac{1}{2}\operatorname{tr}\alpha$.*

Proof. First observe that since $g(H, df) = 0$, we clearly have $i\sigma_H = \langle H, df\rangle$. Now observe that

$$\begin{aligned} \langle H, f_{\bar z}\rangle &= \tfrac{1}{2}\langle H, f_x + i f_y\rangle \\ &= \tfrac{1}{2}\langle H, f_x\rangle - \tfrac{i}{2}\langle H, f_y\rangle \\ &= \tfrac{i}{2}\omega(H, f_x) + \tfrac{1}{2}\omega(H, f_y) \end{aligned}$$

and therefore $\langle H, f_z\rangle = -\langle f_{\bar z}, H\rangle$ (and, equally, $\langle f_z, H\rangle = -\langle H, f_{\bar z}\rangle$). Now

$$\begin{aligned} \operatorname{tr}\alpha &= (\langle H, f_{\bar z}\rangle - \langle f_z, H\rangle) dz + (\langle H, f_z\rangle - \langle f_{\bar z}, H\rangle) d\bar z \\ &= 2\langle H, f_{\bar z} d\bar z + f_z dz\rangle. \end{aligned}$$

□

It follows that the frame F has $\det(F) = 1$ if and only if φ is a minimal immersion, i.e., $H = 0$. Otherwise to make the frame an $SU(2,1)$ frame we must divide F by $\det(F)^{1/3}$, which is the cube root of the Lagrangian angle function $\exp(2i\int \sigma_H) : \mathcal{D} \to S^1$. Notice that $d\,\mathrm{tr}\,\alpha = 0$ (which follows from the Maurer-Cartan equations) implies that σ_H is a closed 1-form: this is to be expected since $\mathbb{CH}^2$ is a Kähler-Einstein manifold.

Now that we have the form of α, we can state the Maurer-Cartan equations.

Proposition 3.2. *The form α satisfies the Maurer-Cartan equations if and only if all three of the following equations hold*

$$\frac{\partial^2}{\partial z \partial \bar z}\log(s^2) = s^2 + |Q|^2 s^{-4} - \frac{s^2}{2}|H|^2 \tag{3.10}$$

$$d\sigma_H = 0 \tag{3.11}$$

$$Q_{\bar z} s^{-4} = -(\langle H, f_{\bar z}\rangle s^{-2})_z. \tag{3.12}$$

Here we have used

$$\langle H, f_z\rangle = \frac{1}{2}(\omega(H, f_x) - i\omega(H, f_y)) = \frac{s}{\sqrt{2}}(g(H, iE_1) - ig(H, iE_2)),$$

to deduce that

$$|\langle H, f_z\rangle|^2 = \frac{s^2}{2}|H|^2.$$

Note that, via the isomorphism $T\mathbb{CH}^2 \simeq G \times_K \mathfrak{m}$, the differential $d\varphi$ corresponds to $\mathrm{Ad}\,F \cdot \alpha_{\mathfrak{m}}$ and the first equation in (3.10) is essentially the Gauss equation for φ.

4. Minimal Lagrangian immersions in $\mathbb{CH}^2$.

For a minimal Lagrangian surface $H = 0$, and so for the Maurer-Cartan form $\alpha = A\,dz + B\,d\bar z$,

$$A = \begin{pmatrix} s^{-1}s_z & 0 & s \\ -Qs^{-2} & -s^{-1}s_z & 0 \\ 0 & s & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -s^{-1}s_{\bar z} & \bar Q s^{-2} & 0 \\ 0 & s^{-1}s_{\bar z} & s \\ s & 0 & 0 \end{pmatrix} \tag{4.1}$$

Moreover, the integrability conditions (3.10) and (3.12) become (1.1) and $Q_{\bar z} = 0$, so that Q is a holomorphic cubic differential.

Now consider the global theory on a Riemann surface. Let (Σ, h) be a closed Riemann surface of genus at least 2 with metric h. Fix a uniformisation $\mathcal{D} \to \Sigma$, and express the metric h over $\mathcal{D}$ as $\gamma|dz|^2$.

Suppose $\varphi : \mathcal{D} \to \mathbb{CH}^2$ is minimal Lagrangian and $\bar\rho$-equivariant for a representation $\bar\rho : \pi_1\Sigma \to PSU(2,1)$. The circle bundle $S_- \to \mathbb{CH}^2$ is the unit subbundle of the tautological bundle L and as a bundle with connexion $L^3 \simeq K_{\mathbb{CH}^2}$ (viewing $\mathbb{CH}^2 \subset \mathbb{CP}^2$). Let f be a global horizontal section of the flat S^1-bundle $\varphi^{-1}S_-$. For the same reasons as the case of $\mathbb{CP}^2$ [11] the mean curvature 1-form σ_H is the connexion 1-form for this flat connexion on $\varphi^{-1}K_{\mathbb{CH}^2}$. Since φ is minimal this connexion has trivial holonomy, so holonomy group for the contact structure connexion on $\varphi^{-1}S_-$ is either trivial or $\mathbb{Z}_3$. Hence f is equivariant for a representation $\rho : \pi_1\Sigma \to SU(2,1)$ which lies over $\bar{\rho}$. It induces a metric $|df|^2 = e^u h = s^2|dz|^2$ and a cubic holomorphic differential $U = Qdz^3 = \langle f_{zzz}, f\rangle dz^3$ on $\mathcal{D}$ which are both ρ-invariant. According to the previous section, they satisfy

$$\frac{\partial^2}{\partial z\partial\bar{z}}\log(\gamma) + \frac{\partial^2 u}{\partial z\partial\bar{z}} = e^u\gamma + |Q|^2 e^{-2u}\gamma^{-2}.$$

The Laplacian and curvature with respect to h are given by

$$\Delta_h = \frac{4}{\gamma}\frac{\partial^2}{\partial z\partial\bar{z}}, \quad \kappa_h = -\frac{2}{\gamma}\frac{\partial^2}{\partial z\partial\bar{z}}\log(\gamma),$$

so that the equation becomes

$$\Delta_h u - 2\kappa_h = 4e^u + 4\frac{|Q|^2}{\gamma^3}e^{-2u}.$$

But in these coordinates $\|U\|_h^2 = |Q|^2/\gamma^3$ and therefore we obtain (1.2).

Conversely, suppose we have a triple (Σ, U, u), where $u : \Sigma \to \mathbb{R}$ is a global solution of (1.2). Let α be the Maurer-Cartan form over $\mathcal{D}$ given by (4.1). Fix any base point $z_0 \in \mathcal{D}$ and let F be the unique $SU(2,1)$ frame for which $F^{-1}dF = \alpha$ and $F(z_0) = I$. It is easy to see that a holomorphic change of coordinates $w(z)$ results in the change of frame

$$F \mapsto Fc_{zw}, \quad c_{zw} = \begin{pmatrix} z_w/|z_w| & 0 & 0 \\ 0 & \overline{z_w}/|z_w| & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.2}$$

The quantities $z_w/|z_w|$ and $\overline{z_w}/|z_w|$ are, respectively, the transition functions for the unit circle subbundle in $T^{1,0}\Sigma$ and and its inverse. Hence α determines a principal $SU(2,1)$-bundle $P \to \Sigma$ equipped with a flat connexion θ whose expression in the local frame F is the Maurer-Cartan form α above. The holonomy of this flat connexion determines a representation ρ up to conjugacy in $SU(2,1)$, hence we obtain an well-defined element of $\mathrm{Hom}(\pi_1\Sigma, SU(2,1))/SU(2,1)$. Moreover, by (4.2) the last column of F is independent of the local coordinate and determines a ρ-equivariant map $f : \mathcal{D} \to S_-$, which is minimal Legendrian and the horizontal lift of a ρ-equivariant minimal Lagrangian map $\varphi : \mathcal{D} \to \mathbb{CH}^2$ with induced metric $e^u h$ and cubic holomorphic differential U.

In summary, we have proven the following theorem. As above, we assume we have fixed a uniformisation $\mathcal{D} \to \Sigma$ and holomorphic coordinate z on $\mathcal{D}$.

Theorem 4.1. *Let (Σ, h) be a compact Riemannian surface of genus at least 2 and U a globally holomorphic cubic differential on Σ for which there exists a solution $u : \Sigma \to \mathbb{R}$ to (1.2). Let α be given by (4.1), with $e^u h = s^2|dz|^2$ and $U = Qdz^3$. Then we obtain a minimal Lagrangian immersion $\varphi : \mathcal{D} \to \mathbb{CH}^2$ by integration of the equations $F^{-1}dF = \alpha$, $F : \mathcal{D} \to SU(2,1)$. The map φ is uniquely determined, up to isometries of $\mathbb{CH}^2$, by the data $(\Sigma, e^u h, U)$, and is* *equivariant with respect to a holonomy representation $\rho : \pi_1\Sigma \to SU(2,1)$ lying in the conjugacy class corresponding to the flat $SU(2,1)$-bundle (Σ, P, θ) determined by α.*

Conversely, a minimal Lagrangian immersion $\varphi : \mathcal{D} \to \mathbb{CH}^2$ equivariant with respect to a representation $\bar{\rho} : \pi_1\Sigma \to PSU(2,1)$ determines a metric $e^u h$ and a holomorphic cubic differential U on Σ which satisfy (1.2). Up to conjugacy, the representation ρ this data determines lies over $\bar{\rho}$.

Remark 1. The flat bundle (Σ, P, θ) has a corresponding Yang-Mills-Higgs description in the sense of Corlette [4]. For we can split α into $\alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$ according to the reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. Since the transition functions (4.2) lie in the isotropy subgroup K there is a corresponding splitting of $\operatorname{ad} P$ into $\operatorname{ad} K$-invariant subbundles $V_{\mathfrak{k}} + V_{\mathfrak{m}}$ and θ determines a K-connexion D on each subbundle together with a section Ψ of $V_{\mathfrak{m}}$. In our local frame $D = d + \operatorname{ad} \alpha_{\mathfrak{k}}$ and $\Psi = \alpha_{\mathfrak{m}}$. These satisfy the Yang-Mills-Higgs equations, which assert that $D + \operatorname{ad} \Psi$ is flat and $D^*\Psi = 0$. This data can also be treated as holomorphic Higgs bundle data using the type decomposition of 1-forms and connexions (see, for example, Xia [38]). From our point of view D is the Levi-Civita connexion on $\varphi^{-1}T\mathbb{CH}^2$ and Ψ is the differential $d\varphi$. Now recall that according to Corlette for *any* choice of conformal structure on Σ every reductive representation in $\mathrm{Hom}(\pi_1\Sigma, SU(2,1))/SU(2,1)$ corresponds to Yang-Mills-Higgs data. Indeed, what Corlette proves is that for every choice of conformal structure on the smooth surface Σ and every reductive representation $\rho : \pi_1\Sigma \to SU(2,1)$ there is a ρ-equivariant harmonic map $\varphi : \mathcal{D} \to \mathbb{CH}^2$. The data we have satisfies the extra conditions corresponding to φ being conformal harmonic and Lagrangian: in terms of (D, Ψ) conformality is the condition that $\langle \Psi^{1,0}, \Psi^{0,1} \rangle = 0$ while φ is Lagrangian when $\mathrm{Im}\langle \Psi, \Psi \rangle = 0$. Thus the conformal Lagrangian conditions tie the conformal structure of Σ to the data (D, Ψ) and impose conditions on Ψ which could be thought of as putting it into a normal form.

5. Global solutions.

In this section, we find global solutions to (1.2) on a compact hyperbolic surface, by using similar techniques to those developed in [21, 22].

Theorem 5.1. *Let Σ be a compact Riemann surface equipped with a metric h of constant Gaussian curvature -1. If U is a nonzero holomorphic cubic differential on Σ which satisfies*

$$\max_{\Sigma} \|U\|_h^2 \leq \frac{1}{54}$$

for $\|\cdot\|_h$ the metric on cubic differentials determined by h, then there is a smooth solution u to the equation of Ţiţeica type

$$\Delta u - 4\|U\|_h^2 e^{-2u} - 4e^u + 2 = 0 \tag{5.1}$$

on Σ, where Δ is Laplacian with respect to h.

Proof. Let $H(u) = \Delta u - 4\|U\|_h^2 e^{-2u} - 4e^u + 2$. The existence of a smooth solution to $H(u) = 0$ follows if we can construct sub- and super-solutions s, S on Σ satisfying

$$s \leq S, \qquad H(s) \geq 0, \qquad H(S) \leq 0$$

(see e.g. Schoen-Yau [28], Proposition V.1.1). Such a solution satisfies $S \geq u \geq s$. Let $S = -\log 2$. Then clearly $H(S) = -4\|U\|_h^2 e^{-2S} \leq 0$.

Similarly, let $s = C$ for C a negative number, and let $M = \max_\Sigma \|U\|_h^2$. Then compute

$$H(s) = -4\|U\|_h^2 e^{-2C} - 4e^C + 2 \geq -4Me^{-2C} - 4e^C + 2.$$

Consider the function $f(C) = -4Me^{-2C} - 4e^C + 2$ for $M > 0$. Then $f(-\log 2) < 0$ and $f \to -\infty$ as $C \to -\infty$. The only critical point of f occurs at

$$C = C_{\max} = \frac{1}{3}\log(2M).$$

Compute

$$f(C_{\max}) = -6 \cdot 2^{\frac{1}{3}} \cdot M^{\frac{1}{3}} + 2 \geq 0 \qquad \Longleftrightarrow \qquad M \leq \frac{1}{54}.$$

So if M satisfies this bound, $C_{\max} \leq -\log 3 < -\log 2$ and $f(C_{\max}) \geq 0$, which shows that $s = C_{\max}$ is a subsolution. $\square$

To obtain a corresponding uniqueness for these solutions, we must introduce the following constraint.

Definition 5.2. *On a hyperbolic Riemann surface* (Σ, h) *equipped with a holomorphic cubic differential* U*, we call a function* u small *when it provides the bound* $2\|U\|_h^2 e^{-3u} \leq 1$*, in other words, when*

$$u \geq \frac{1}{3}\log(2\max_\Sigma \|U\|_h^2). \tag{5.2}$$

Remark 2. The geometric significance of this constraint, and the reason for the somewhat counter-intuitive use of the word *small* to describe a function bounded *below*, is that u is a small solution of (5.1) if and only if the metric $g = e^u h$ has *small curvature*

$$-3 \leq \kappa_g \leq -2.$$

The upper bound is true for any solution to (5.1): it is the lower bound which equates to (5.2).

We will mainly be interested in small solutions to (5.1). Let

$$\theta(x, u) = -4\|U\|_h^2 e^{-2u} - 4e^u + 2,$$

so that (5.1) becomes $\Delta u + \theta(x, u) = 0$. A simple computation implies

Lemma 5.3. *For a small function* v*,*

$$\frac{\partial \theta}{\partial u}(x, v) \leq 0.$$

Proposition 5.4. *Given (Σ, h, U), there is at most one small solution u to (5.1). The solutions produced by Theorem 5.1 above are small.*

Proof. Suppose that v, w are two small solutions of (5.1). We will use the comparison principle and Lemma 5.3 show that $v = w$. By assumption, $\Delta v + \theta(x, v) = 0$, $\Delta w + \theta(x, w) = 0$. Consider the path of functions $u_t = tv + (1-t)w$. Then a standard computation shows $v - w$ satisfies

$$\Delta(v-w) = -\left(\int_0^1 \frac{\partial\theta}{\partial u}(x, u_t)\, dt\right)(v-w).$$

Therefore, $v - w$ satisfies the linear elliptic equation $L(v-w) = 0$, for

$$L\phi = \Delta\phi + \left(\int_0^1 \frac{\partial\theta}{\partial u}(x, u_t)\, dt\right)\phi \equiv \Delta\phi + c\phi. \tag{5.3}$$

Since v and w are both small, u_t is small for all $t \in [0,1]$, and Lemma 5.3 shows $c \le 0$. At this point, the strong maximum principle (Theorem 3.5 in [5]) applies, and we may conclude that $v - w$ is either constant on Σ or has no nonnegative maximum. If $v - w$ is constant, it is easy to show the constant must be 0. Therefore, we may conclude $v - w$ has no positive maximum—i.e., $v - w \le 0$ on Σ. By symmetry, $w - v \le 0$ on Σ also, and so we must have $v = w$.

That the solutions produced in Theorem 5.1 are small is evident from the proof. □

Lemma 5.5. *Assume $M = \max_\Sigma \|U\|_h^2 \le \frac{1}{16}$. Then any small solution u to (5.1) satisfies*

$$\chi_M \le u \le -\log 2,$$

where χ_M is the largest real root of $f(C) = -4Me^{-2C} - 4e^C + 2$ if $M \le \frac{1}{54}$ and $\chi_M = \log(2M)/3$ if $\frac{1}{54} < M \le \frac{1}{16}$.

Proof. If $M \le \frac{1}{16}$, we may easily check that $w = -\log 2$ is small. Set $u_t = tu + (1-t)w$ and compute as in the proof of Proposition 5.4

$$\begin{aligned} H(w) &= \Delta w + \theta(x, w) = -4\|U\|_h^2 e^{-2w} \le 0, \\ \Delta(u-w) &= -\theta(x,u) + \theta(x,w) - H(w) \\ &\ge -[\theta(x,u) + \theta(x,w)] \\ &= -\left(\int_0^1 \frac{\partial\theta}{\partial u}(x, u_t)\, dt\right)(u-w). \end{aligned}$$

Therefore, $u - w$ satisfies $L(u-w) \ge 0$ for $L\phi = \Delta\phi + c\phi$ as in (5.3), with $c \le 0$ by Lemma 5.3. Again, the strong maximum principle implies either that $u - w$ is constant (which is easily ruled out except for the case $U = 0$, $u = -\log 2$) or that $u - w$ has no nonnegative maximum. Therefore $u \le w$ on all Σ.

Similar reasoning shows that χ_M is a lower bound for any small solution u. When $M \le \frac{1}{54}$, χ_M is small. The proof of Theorem 5.1 shows that f achieves a nonnegative maximum value at its only critical point $C_{\max} < -\log 2$ when

$M \le \frac{1}{54}$. On the other hand, $f(-\log 2) < 0$. The Intermediate Value Theorem implies $\chi_M \ge C_{\max}$, which is equivalent to the definition for χ_M to be small.

Moreover,

$$H(\chi_M) = \Delta\chi_M - 4\|U\|_h^2 e^{-2\chi_m} - 4e^{\chi_M} + 2 \ge f(\chi_M) = 0.$$

This implies that for $u_t = t\chi_M + (1-t)u$ and $L\phi = \Delta\phi + c\phi$ as in (5.3), $L(\chi_M - u) \ge 0$ with $c \le 0$. Then the strong maximum principle implies $\chi_M \le u$ on all Σ. □

Corollary 5.6. *Fix (Σ, h). As $U \to 0$, the unique small solution $u = u_U$ to (5.1) approaches $-\log 2$ uniformly.*

Proof. As $M \to 0$, $\chi_M \to -\log 2$. □

Theorem 5.7. *Given a closed hyperbolic Riemann surface (Σ, h), the family of small solutions $u = u_U$ to (5.1) is smoothly varying in U for*

$$U \in \mathcal{U}_{1/54} = \left\{ U : M = \max_{\Sigma} \|U\|_h^2 \le \frac{1}{54} \right\}.$$

Proof. We use the continuity method. Consider

$$H(u, U) = \Delta u - 4\|U\|_h^2 e^{-2u} - 4e^u + 2$$

for $U \in \mathcal{U}_{1/54}$ and $u \in C^{2,\alpha}(\Sigma)$. Then it is straightforward to check that H is a Fréchet differentiable map from $C^{2,\alpha} \times \mathcal{U}_{1/54} \to C^{0,\alpha}$. In order to use the Implicit Function Theorem (for U in the interior of $\mathcal{U}_{1/54}$), we must check that the partial differential

$$\frac{\delta H}{\delta u} : \eta \mapsto \Delta\eta + 8\|U\|_h^2 e^{-2u}\eta - 4e^u\eta$$

has a continuous inverse from $C^{0,\alpha}$ to $C^{2,\alpha}$. This follows from checking the kernel of $\frac{\delta H}{\delta u}$ vanishes, which is true by the assumption that u is small and the maximum principle. Thus there is a family of solutions u_U for each U in a neighborhood of each U_0 in the interior of $\mathcal{U}_{1/54}$. These solutions are still small by continuity and the improved bound in Lemma 5.5 (since $\chi_M > \frac{1}{3}\log(2M)$). Then Proposition 5.4 allows us to identify these solutions with the ones already produced in Theorem 5.1.

For good measure, the closedness part of the continuity method follows from Lemma 5.5, which shows that u_U and Δu_U are uniformly in L^p for any $p < \infty$. Then the elliptic theory shows $u_U \in L^p_2$ uniformly. Sobolev embedding then gives uniform bounds in $C^{1,\alpha}$, and further bootstrapping implies uniform $C^{2,\alpha}$ bounds of u_U as U varies. Ascoli-Arzéla allows us to take limits to show closedness.

The variation is smooth by standard elliptic theory. □

Remark 3. We do not expect the bound $\|U\|_h^2 \le \frac{1}{54}$ to be sharp as a condition for the existence of solutions.

Proposition 5.8. *Let Σ be a closed Riemann surface of genus $g \geq 2$ equipped with a hyperbolic metric h as above. If U is a holomorphic cubic differential on Σ which is large in the sense that*

$$\int_\Sigma |U|^{\frac{2}{3}} > \frac{2\pi\sqrt[3]{4}}{3}(g-1),$$

then there is no solution to (5.1) on Σ.

Proof. Let u be a solution to (5.1). Integrate (5.1) and use Gauss-Bonnet to find

$$4\int_\Sigma \|U\|_h^2 e^{-2u}\, dV_h + 4\int_\Sigma e^u\, dV_h = 8\pi(g-1) \tag{5.4}$$

for dV_h the volume form of the hyperbolic metric. Hölder's inequality shows

$$\int_\Sigma |U|^{\frac{2}{3}} = \int_\Sigma \|U\|_h^{\frac{2}{3}}\, dV_h \leq \left(\int_\Sigma \|U\|_h^2 e^{-2u}\, dV_h\right)^{\frac{1}{3}} \left(\int_\Sigma e^u dV_h\right)^{\frac{2}{3}}. \tag{5.5}$$

If we denote

$$A = \int_\Sigma \|U\|_h^2 e^{-2u}\, dV_h, \qquad B = \int_\Sigma e^u\, dV_h,$$

we can maximise AB^2 for $A, B > 0$ subject to the constraint $A + B = 2\pi(g-1)$ to find that

$$AB^2 \leq \frac{32}{27}[\pi(g-1)]^3.$$

Then (5.4) and (5.5) prove the contrapositive of the proposition. □

6. Solutions with zero cubic differential.

In this section, we study minimal Lagrangian surfaces in $\mathbb{CH}^2$ corresponding to $U = Q\,dz^3 = 0$. First of all, consider solutions to (5.1) if $U = 0$. By the maximum principle, we have

Lemma 6.1. *If Σ is a closed Riemann surface equipped with hyperbolic metric h and cubic differential $U = 0$, then the unique solution to (5.1) is $u = -\log 2$.*

On the upper half-plane $\{x + iy : y > 0\}$, consider $Q = 0$. In this case, the metric corresponding to $s = \frac{1}{y\sqrt{2}}$ solves (1.1). The connexion matrices A, B

satisfy

$$\begin{aligned}
A &= \begin{pmatrix} \frac{i}{2y} & 0 & \frac{1}{y\sqrt{2}} \\ 0 & -\frac{i}{2y} & 0 \\ 0 & \frac{1}{y\sqrt{2}} & 0 \end{pmatrix}, \\
B &= \begin{pmatrix} \frac{i}{2y} & 0 & 0 \\ 0 & -\frac{i}{2y} & \frac{1}{y\sqrt{2}} \\ \frac{1}{y\sqrt{2}} & 0 & 0 \end{pmatrix}, \\
F^{-1}F_x = A + B &= \begin{pmatrix} \frac{i}{y} & 0 & \frac{1}{y\sqrt{2}} \\ 0 & -\frac{i}{y} & \frac{1}{y\sqrt{2}} \\ \frac{1}{y\sqrt{2}} & \frac{1}{y\sqrt{2}} & 0 \end{pmatrix} \equiv \frac{L}{y}, \\
F^{-1}F_y = iA - iB &= \begin{pmatrix} 0 & 0 & \frac{i}{y\sqrt{2}} \\ 0 & 0 & -\frac{i}{y\sqrt{2}} \\ -\frac{i}{y\sqrt{2}} & \frac{i}{y\sqrt{2}} & 0 \end{pmatrix} \equiv \frac{K}{y}.
\end{aligned}$$

To solve the initial value problem in y, let $y = e^t$ to find $F^{-1}F_t = K$. It is straightforward to integrate these equations to find a fundamental solution of the initial-value problem for any path from $(0,1)$ to (x,y):

$$\exp(Lx) \cdot \exp(Kt) = \exp(Lx) \cdot \exp(K \log y).$$

This formula follows from the Maurer-Cartan equations, which show the fundamental solution is independent of the choice of path. Thus we may integrate along the piecewise-linear path from $(0,1)$ to $(x,1)$ to (x,y).

It is convenient to take the initial condition

$$F_0 = (f_1 \;\; f_2 \;\; f_3) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(This $F_0 \notin SU(2,1)$, but we still use it to ensure f is real below. The factor $\det F_0 = i$ is irrelevant upon projecting $W_- \to \mathbb{CH}^2$ in any case.) So the solution is $F = F_0 \cdot \exp(Lx) \cdot \exp(K \log y)$, whose last column $f_3 = f$ is given by

$$f = \begin{pmatrix} \frac{x}{y} \\ \frac{-1+x^2+y^2}{2y} \\ \frac{1+x^2+y^2}{2y} \end{pmatrix}.$$

Note f parameterises the upper component of the real hyperboloid

$$(\operatorname{Re} u_1)^2 + (\operatorname{Re} u_2)^2 - (\operatorname{Re} u_3)^2 = -1$$

in $\mathbb{R}^3 \subset \mathbb{C}^3$, and the immersion $[f]$ into $\mathbb{CH}^2$ is the standard immersion of $\mathbb{RH}^2 \subset \mathbb{CH}^2$, which is of course minimal Lagrangian.

7. Fundamental domains in $\mathbb{CH}^2$.

We now show, at least for U near 0, that the minimal Lagrangian surface produced by a small solutions u to (5.1) determines a fundamental domain for induced action of $\pi_1\Sigma$ on $\mathbb{CH}^2$. Recall the notation of the Legendrian $SU(2,1)$ frame

$$F = (f_1\ f_2\ f_3), \qquad f_1 = \frac{f_z}{|f_z|}, \quad f_2 = \frac{f_{\bar z}}{|f_{\bar z}|}, \quad f_3 = f.$$

At a point p in $\tilde\Sigma$ the universal cover of Σ, we may choose coordinates in $\mathbb{C}^3$ so that

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad f_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{7.1}$$

We may also choose a conformal normal coordinate z so that at p, $z = 0$ and $\psi = \psi_z = \psi_{\bar z} = 0$ for the affine metric $e^\psi|dz|^2$. In terms of $s = |f_z| = |f_{\bar z}| = e^{\frac{\psi}{2}}$, this means $s = 1$, $s_z = s_{\bar z} = 0$ at $z = 0$. Moreover, we may rotate z so that at $z = 0$, $Q \in [0,\infty)$ (Note $Q = e^{-\frac{3}{2}u}\|U\|$ for $U = Q\,dz^3$ in this case). Under these assumptions, at $z = 0$,

$$F_z = A = \begin{pmatrix} 0 & 0 & 1 \\ -Q & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad F_{\bar z} = B = \begin{pmatrix} 0 & Q & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{7.2}$$

The tangent plane to $M = \pi(f(\tilde\Sigma))$ at p is spanned by $\pi_*(f_x) = \pi_*(f_z + f_{\bar z}) = \pi_*(f_1 + f_2)$ and $\pi_*(f_y) = \pi_*(if_z - if_{\bar z}) = \pi_*(if_1 - if_2)$, for $\pi : W_- \to \mathbb{CH}^2$ the projection. So the Lagrangian copy of $\mathbb{RH}^2$ tangent to M in $\mathbb{CH}^2$ can be described by

$$\mathcal{T} = \{\pi[(a+ib)f_1 + (a-ib)f_2 + cf_3] : a^2 + b^2 < \tfrac12 c^2\}.$$

This explicit description of the tangent space allows us also to describe the totally geodesic Lagrangian plane normal to $\mathcal{T}$ (the image of the normal space under the exponential map) as

$$\mathcal{N} = \{\pi[(ia-b)f_1 + (ia+b)f_2 + cf_3] : a^2 + b^2 < \tfrac12 c^2\}. \tag{7.3}$$

This is because the normal vectors to a Lagrangian tangent plane are determined by the action of the complex structure J on tangent vectors.

Theorem 7.1. *Let Σ be a compact Riemann surface equipped with a cubic differential U and a solution u to (5.1). For the disc*

$$\mathcal{D} = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 < \tfrac12\}$$

and $\tilde\Sigma$ the universal cover of Σ, consider the map $\Phi\colon \tilde\Sigma \times \mathcal{D} \to \mathbb{CH}^2$ given by

$$\Phi(z,a,b) = \pi[(ia-b)f_1(z) + (ia+b)f_2(z) + f_3(z)].$$

Then $\|U\|_m = e^{-\frac32 u}\|U\|_h \le \sqrt2$ on all of Σ if and only if Φ is an immersion.

Proof. To calculate when Φ is an immersion, we may work at a point $z=0$, and make the coordinate assumptions (7.1-7.2) above. Let $z=x+iy$, and compute to first order in x,y

$$F=(f_1\ f_2\ f_3)\sim\begin{pmatrix}1 & Qx-iQy & x+iy\\ -Qx-iQy & 1 & x-iy\\ x-iy & x+iy & 1\end{pmatrix}.$$

We choose inhomogeneous coordinates on $\mathbb{CP}^2\supset\mathbb{CH}^2$ to identify $\pi(\alpha,\beta,\gamma)=(\frac{\alpha}{\gamma},\frac{\beta}{\gamma})$. Again, to first order in x,y, compute

$$\begin{aligned}\Phi(z,a,b) &\sim \pi\left[\begin{pmatrix}(ia-b)+(ia+b)(Qx-iQy)+(x+iy)\\ (ia-b)(-Qx-iQy)+(ia+b)+(x-iy)\\ (ia-b)(x-iy)+(ia+b)(x+iy)+1\end{pmatrix}\right]\\ &\sim \begin{pmatrix}(ia-b)-(ia-b)^2(x-iy)-(ia-b)(ia+b)(x+iy)\\ +(ia+b)(Qx-iQy)+(x+iy)\\ (ia+b)-(ia+b)(ia-b)(x-iy)-(ia+b)^2(x+iy)\\ +(ia-b)(-Qx-iQy)+(x-iy)\end{pmatrix}.\end{aligned}$$

Now take real and imaginary parts to view $\mathbb{C}^2$ as $\mathbb{R}^4$ to find

$$\Phi\sim\begin{pmatrix}-b+(a^2-b^2)x+2aby+(a^2+b^2)x+bQx+aQy+x\\ a-(a^2-b^2)y+2abx+(a^2+b^2)y+aQx-bQy+y\\ b+(a^2+b^2)x+(a^2-b^2)x+2aby+bQx+aQy+x\\ a-(a^2+b^2)y-(-a^2+b^2)y-2abx-aQx+bQy-y\end{pmatrix}.$$

So the Jacobian matrix $(\Phi_a,\Phi_b,\Phi_x,\Phi_y)$ at $x=y=0$ is equal to

$$\begin{pmatrix}0 & -1 & 2a^2+bQ+1 & 2ab+aQ\\ 1 & 0 & 2ab+aQ & 2b^2-bQ+1\\ 0 & 1 & 2a^2+bQ+1 & 2ab+aQ\\ 1 & 0 & -2ab-aQ & -2b^2+bQ-1\end{pmatrix}.$$

The Jacobian determinant is then

$$\mathcal{J}=4[(a^2+b^2)Q^2+(6a^2b-2b^3)Q+(-2a^2-2b^2-1)] \tag{7.4}$$

Thus Φ is an immersion if and only if $\mathcal{J}\neq0$ for all $(a,b)\in\mathcal{D}$. We find conditions for nonnegative Q which ensure $\mathcal{J}<0$ for all $(a,b)\in\mathcal{D}$. Set

$$\ell=6a^2b-2b^3,\qquad k=a^2+b^2.$$

Then $(a,b)\in\mathcal{D}$ corresponds to

$$0\le k<\frac{1}{2},\qquad -2k^{\frac{3}{2}}\le\ell\le2k^{\frac{3}{2}}, \tag{7.5}$$

and $\mathcal{J} = 4(kQ^2 + \ell Q - 2k - 1)$. There is only one positive root to (7.4), when viewed as a quadratic polynomial in Q:

$$R = \frac{-\ell + \sqrt{\ell^2 + 8k^2 + 4k}}{2k}.$$

It is straightforward to verify that the minimum value of R on the closure of the domain in (7.5) occurs when $k = \frac{1}{2}$, $\ell = \frac{1}{\sqrt{2}}$, $R = \sqrt{2}$. This corresponds to the values $(a, b) = (0, -\frac{1}{\sqrt{2}}), (\frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}), (-\frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$.

The analysis above shows that if $\|U\|_m \leq \sqrt{2}$ on Σ then Φ is an immersion. The converse follows by noting that if $Q > \sqrt{2}$, $a = 0$ and $b = -\frac{1}{Q}$, then $\Phi_x = 0$, and so Φ cannot be an immersion. $\square$

Proposition 7.2. *There is a constant κ so that if $\|U\|_h < \kappa$ on all of Σ, then Φ is a diffeomorphism from $\tilde{\Sigma} \times \mathcal{D} \to \mathbb{CH}^2$. Moreover, the natural continuous extension $\bar{\Phi}\colon \tilde{\Sigma} \times \bar{\mathcal{D}} \to \overline{\mathbb{CH}^2}$ is injective.*

Proof. For U near 0, note that the estimates above show that $\|U\|_h$ and $\|U\|_m = e^{-\frac{3}{2}u}\|U\|_h$ are equivalent norms. Therefore, Theorem 7.1 above shows there is a bound on $\sup \|U\|_h$ which implies Φ is an immersion. On the other hand, Φ is a proper map if and only if $[f]$ is a proper map from $\tilde{\Sigma} \to \mathbb{CH}^2$. This is because Φ corresponds to the exponential map on the normal bundle in the direction transverse to the image of $[f]$, and so must be proper in that direction. Proposition 8.1 below shows there is a constant bound k so that if $\|U\|_h < k$, then $[f]$ is proper.

Therefore, there is a bound κ so that if $\|U\|_h < \kappa$, then Φ must be a proper immersion from $\tilde{\Sigma} \times \mathcal{D} \to \mathbb{CH}^2$. Thus Φ is a covering map [8], and is a diffeomorphism since its domain is simply connected.

To show $\bar{\Phi}$ is injective as well, note that the proof of Theorem 7.1 shows that if $\|U\|_m < \sqrt{2}$, then $\bar{\Phi}$ is an immersion of manifolds with boundary. The injectivity of Φ implies $\bar{\Phi}$ is injective also. $\square$

Corollary 7.3. *Corresponding to each surface produced in the previous proposition, there is a fundamental domain $\mathcal{F}$ in $\mathbb{CH}^2$ for the representation of $\pi_1\Sigma$ into $SU(2,1)$. Let $\bar{\mathcal{F}}$ be the closure of $\mathcal{F}$ in $\overline{\mathbb{CH}^2} \subset \mathbb{CP}^2$. There are only a finite number of $\gamma \in \pi_1\Sigma$ satisfying $\gamma \cdot \bar{\mathcal{F}} \cap \bar{\mathcal{F}} \neq \emptyset$.*

Proof. We discuss below in Section 9 the induced representation of $\pi_1\Sigma$ into $SU(2,1)$ from the point of view of principal bundles.

Consider a fundamental domain for the action of $\pi_1\Sigma$ on $\tilde{\Sigma}$, and then consider the portion of the total space of the normal bundle of the embedded minimal Lagrangian surface over this domain.

The last statement of the corollary follows from the injectivity of $\bar{\Phi}$ and the corresponding fact for the fundamental domain on the surface Σ. $\square$

8. Properness of the Immersion.

Proposition 8.1. *There is a constant $k > 0$ so that if $\|U\|_h < k$, then $[f]$ is proper.*

Proof. First of all, by the construction of $\mathbb{CH}^2$ above, note that $f \in S_- = \{v \in \mathbb{C}^3 : \langle v, v\rangle = -1\}$ implies that

$$[f] \to \partial\mathbb{CH}^2 \quad \Longleftrightarrow \quad \|f\|_E \to \infty$$

for $\|f\|_E$ the Euclidean norm on $\mathbb{C}^3$. Therefore, $[f]$ is proper if and only if $\|f\|_E$ is unbounded along any path to infinity in the universal cover $\tilde{\Sigma}$. In terms of suitable coordinates, we will show that $\|f\|_E$ has to grow exponentially.

The proof proceeds by treating the developing map for f as a perturbation of the developing map in the case of $U = 0$ with the background hyperbolic metric (as in Section 6 above). The key estimate involves an ODE system of form $X_t = (C + D(t))X$, where C is an explicit constant matrix and $D(t)$ is small enough and bounded in absolute value.

Identify the universal cover $\tilde{\Sigma}$ of the Riemann surface with the upper half-plane $\{z = x + iy : y > 0\}$. As above in Section 6, for our Legendrian frame F,

$$F^{-1}F_y = iA - iB = \begin{pmatrix} is^{-1}(s_z + s_{\bar z}) & -i\bar{Q}s^{-2} & is \\ -iQs^{-2} & -is^{-1}(s_z + s_{\bar z}) & -is \\ -is & is & 0 \end{pmatrix},$$

where $U = Q\,dz^3$ and $s^2|dz|^2 = e^u h$ for $h = |dz|^2/y^2$ the hyperbolic metric. Therefore,

$$F^{-1}F_y = \begin{pmatrix} \frac{i}{2}u_x & -i\bar{Q}y^2e^{-u} & ie^{\frac{u}{2}}y^{-1} \\ -iQy^2e^{-u} & -\frac{i}{2}u_x & -ie^{\frac{u}{2}}y^{-1} \\ -ie^{\frac{u}{2}}y^{-1} & ie^{\frac{u}{2}}y^{-1} & 0 \end{pmatrix}.$$

All of the terms of $F^{-1}F_y$ are on the order of y^{-1}: This is obvious for the terms in the third row and column. As for u_x, note $\|\nabla u\|_h = y\sqrt{u_x^2 + u_y^2} \ge y|u_x|$. By compactness of Σ, there is a uniform bound on $\|\nabla u\|_h$ (improved by Proposition 8.3 below), and thus there is a bound of the form $|u_x| \le Cy^{-1}$. On the other hand, $|Q|$ transforms as a section of $|K^3|$ over Σ (where K represents the canonical bundle). Since y^{-3} is an invariant section of $|K^3|$, we have have e.g. $|-iQy^2e^{-u}|$ is bounded by $C'y^{-1}$.

In fact, we have better bounds on the entries in $F^{-1}F_y$ as $U \to 0$:

$$F^{-1}F_y = \frac{1}{y}\left[\begin{pmatrix} 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} + \begin{pmatrix} iu_xy & -i\bar{Q}y^3e^{-u} & i(e^{\frac{u}{2}} - \frac{1}{\sqrt{2}}) \\ -iQy^3e^{-u} & -iu_xy & i(\frac{1}{\sqrt{2}} - e^{\frac{u}{2}}) \\ i(\frac{1}{\sqrt{2}} - e^{\frac{u}{2}}) & i(e^{\frac{u}{2}} - \frac{1}{\sqrt{2}}) & 0 \end{pmatrix}\right].$$

If we write the second matrix as $\tilde{G}$, then Corollary 5.6 above and Proposition 8.3 below show that the maximum of the entries of $\tilde{G}$ go to zero as $\sup_\Sigma \|U\|_h$ goes to 0.

We also change coordinates $t = \log y$ to show

$$F^{-1}F_t = \begin{pmatrix} 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} + \tilde{G},$$

in which the constant matrix can be diagonalised with eigenvalues $-1, 0, 1$. In fact,

$$\begin{pmatrix} i & -i & \sqrt{2} \\ 1 & 1 & 0 \\ -i & i & \sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} -\frac{i}{4} & \frac{1}{2} & \frac{i}{4} \\ \frac{i}{4} & \frac{1}{2} & -\frac{i}{4} \\ \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $Y = FP$ for P the change of frame matrix listed third above. Then

$$Y^{-1}Y_t = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + G, \tag{8.1}$$

where the conjugated matrix $G = P^{-1}\tilde{G}P$ satisfies the same sort of sup norm estimates on its entries that $\tilde{G}$ does.

For initial conditions, we follow the model case in Section 6 above by choosing at $(t, x) = (0, 0)$,

$$F_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Y_0 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \end{pmatrix}.$$

Let $X = (x_1\ x_2\ x_3) = X(t) \in \mathbb{C}^3$ represent the bottom row of Y, so that $X(0) = (\frac{1}{2\sqrt{2}}\ 0\ \frac{1}{2\sqrt{2}})$. Let $\tilde{x} = (x_1, x_2)$, and let $G = (g_{ij}(t))$ with $|g_{ij}(t)| < \delta$. Let $\epsilon > 0$. First, we use (8.1) to show that there are $\gamma > 0$ and $k > 1$ so that $|x_3(\epsilon)| - k|\tilde{x}(\epsilon)| > \gamma$ as long as δ is small, and k, γ depend only on δ, ϵ. This follows from the fact that, for a fixed $\epsilon > 0$, the solution to the linear initial value problem

$$\dot{X} = K(t)X, \qquad X(0) = \left(\tfrac{1}{2\sqrt{2}}\ \ 0\ \ \tfrac{1}{2\sqrt{2}}\right)$$

on the interval $t \in [0, \epsilon]$ varies continuously in $C^0([0, \epsilon])$ as $K(t)$ varies in $C^0([0, \epsilon])$. This in turn follows by inspection of the Picard iterates. Then note that if $K(t) = \mathrm{diag}(-1, 0, 1)$, the solution $\Phi = \left(\frac{1}{2\sqrt{2}}e^{-t}\ \ 0\ \ \frac{1}{2\sqrt{2}}e^{t}\right)$. Since our X is C^0-close to Φ, this ensures that the third component of $X(\epsilon)$ is larger than the first two components of $X(\epsilon)$.

In particular, $\hat{X}(t) = X(t-\epsilon)$ satisfies the hypotheses of Proposition 8.2 below for $k > 1$. Therefore, we can choose a $\delta > 0$ so that if $|g_{ij}(t)| < \delta$ for $G = (g_{ij})$, then

$$|x_3(t)| - k|\tilde{x}(t)| \geq (|x_3(\epsilon)| - k|\tilde{x}(\epsilon)|)e^{C(t-\epsilon)} \geq \gamma e^{C(t-\epsilon)}$$

for a constant $C = C(\delta) > 0$. The element f_{33} of F satisfies $f_{33} = x_1\sqrt{2} + x_3\sqrt{2}$, and so

$$|f_{33}(t)| \geq \sqrt{2}(|x_3(t)| - |x_1(t)|) \geq \sqrt{2}(|x_3(t)| - k|\tilde{x}(t)|) \geq \sqrt{2}\,\gamma e^{C(t-\epsilon)}.$$

For $y = e^t$, we have

$$|f_{33}(t)| \geq \sqrt{2}\,\gamma e^{-C\epsilon} y^C.$$

But f_{33} is the third component of the position vector f of the embedding, and so $\|f\|_E \to \infty$ along the path in the upper half plane $\{iy : y \to \infty\}$. In terms of the Poincaré disc model, if $w = \frac{iz+1}{z+i}$, then along the radial path $\{ir : r \to 1^-\}$,

$$\|f(ir)\|_E \geq \sqrt{2}\,\gamma e^{-C\epsilon}\left(\frac{1+r}{1-r}\right)^C.$$

But for any radial path $\{e^{i\theta}r : r \to 1^-\}$, the same estimates hold, since we may reduce to the same problem by rotating both w in the disc and f in $\mathbb{C}^3$. Therefore, for any w in the Poincaré disc, we have

$$\|f(w)\|_E \geq \sqrt{2}\,\gamma e^{-C\epsilon}\left(\frac{1+|w|}{1-|w|}\right)^C,$$

and so f is a proper map into $\mathbb{C}^3$. Therefore, $[f]$ is a proper map into $\mathbb{CH}^2$. $\square$

Proposition 8.2. *If $X = (x_1 \ \ x_2 \ \ x_3)$ is a $\mathbb{C}^3$-valued solution to the ODE system*

$$\frac{dX}{dt} = X\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + G\right],$$

where $G = (g_{ij}) = (g_{ij}(t))$ satisfies $|g_{ij}| \leq \delta < 1/(4\sqrt{2}+3)$, then there are positive constants $k = k(\delta)$ and $C = C(\delta)$ so that if the initial conditions satisfy

$$|x_3(0)| > k|\tilde{x}(0)|,$$

for $\tilde{x} = (x_1, x_2)$, then for all $t > 0$,

$$|x_3(t)| - k|\tilde{x}(t)| \geq (|x_3(0)| - k|\tilde{x}(0)|)e^{Ct}.$$

k is a continuous, decreasing function of δ, with $k(1/(4\sqrt{2}+3)) = 1$ and $k \to \infty$ as $\delta \to 0$.

Proof. Use the Cauchy-Schwartz Inequality to estimate

$$\begin{aligned}\frac{d}{dt}|x_3| &= \frac{\dot{x}_3\bar{x}_3 + x_3\dot{\bar{x}}_3}{2|x_3|} \\ &= \frac{2\mathrm{Re}[(1+g_3^3)x_3\bar{x}_3 + g_3^2 x_2\bar{x}_3 + g_3^1 x_1\bar{x}_3]}{2|x_3|} \\ &\geq (1-\delta)|x_3| - \delta\sqrt{2}|\tilde{x}|.\end{aligned}$$

Similarly, we may compute

$$\frac{d}{dt}|\tilde{x}| \leq \delta\sqrt{2}|x_3| + 2\delta|\tilde{x}|.$$

Therefore, we have

$$\frac{d}{dt}\left(|x_3| - k|\tilde{x}|\right) \geq (1-\delta-k\delta\sqrt{2})|x_3| - (\delta\sqrt{2}+2k\delta)|\tilde{x}|. \tag{8.2}$$

Now by our bound on δ, we may choose k to be the larger root of

$$k^2 - \left(\frac{1-3\delta}{2\delta\sqrt{2}}\right)k + 1 = 0,$$

and also choose

$$C = 1 - \delta - k\delta\sqrt{2} > 0.$$

Then we have

$$-(\delta\sqrt{2} + 2k\delta) = -Ck,$$

which, together with (8.2), implies

$$\frac{d}{dt}(|x_3| - k|\tilde{x}|) \geq C(|x_3| - k|\tilde{x}|).$$

Since the initial value of $|x_3| - k|\tilde{x}|$ is assumed to be positive, the differential inequality shows

$$|x_3(t)| - k|\tilde{x}(t)| \geq (|x_3(0)| - k|\tilde{x}(0)|)e^{Ct}.$$

It is easy to check that $\lim_{\delta\to 0} k = \infty$. □

Proposition 8.3. *Let Σ be a compact Riemann surface equipped with a conformal hyperbolic metric h and a holomorphic cubic differential U. Let u be a small solution to (5.1). Then there is a constant C depending only on $m = \sup e^{-2u}\|\nabla(\|U\|)^2\|$ so that $\|\nabla u\| \leq C$. As $U \to 0$, $m \to 0$ and $C \to 0$.*

Proof. In local coordinates, write $h = \gamma|dz|^2$. Let $v = \gamma\, u_z u_{\bar{z}} = \frac{1}{4}\|\nabla u\|^2$, and let p be a maximum point of v. Choose a local coordinate z so that at $p = \{z = 0\}$, $\gamma_z(0) = \gamma_{\bar{z}}(0) = 0$ and $\gamma(0) = 1$. The condition that h is hyperbolic is then $\gamma_{z\bar{z}}(0) = \frac{1}{2}$.

Compute

$$\begin{aligned} v_z &= \gamma_z u_z u_{\bar z} + \gamma u_{zz} u_{\bar z} + \gamma u_z u_{z\bar z}, \\ v_{\bar z} &= \gamma_{\bar z} u_z u_{\bar z} + \gamma u_{z\bar z} u_{\bar z} + \gamma u_z u_{\bar z\bar z}, \\ v_{z\bar z} &= \gamma_{z\bar z} u_z u_{\bar z} + \gamma_z u_{z\bar z} u_{\bar z} + \gamma_z u_z u_{\bar z\bar z} + \gamma_{\bar z} u_{zz} u_{\bar z} + \gamma u_{zz\bar z} u_{\bar z} + \gamma u_{zz} u_{\bar z\bar z} \\ &\quad + \gamma_{\bar z} u_z u_{z\bar z} + \gamma u_{z\bar z} u_{z\bar z} + \gamma u_z u_{z\bar z\bar z}. \end{aligned}$$

At the maximum point, $\nabla v = 0$ implies

$$u_{zz} u_{\bar z} + u_z u_{z\bar z} = 0 = u_{z\bar z} u_{\bar z} + u_z u_{\bar z\bar z}, \tag{8.3}$$

while $v_{z\bar z} \le 0$ implies

$$\tfrac{1}{2} u_z u_{\bar z} + u_{zz\bar z} u_{\bar z} + u_{zz} u_{\bar z\bar z} + u_{z\bar z} u_{z\bar z} + u_z u_{z\bar z\bar z} \le 0.$$

This becomes, by (8.3),

$$\tfrac{1}{2} u_z u_{\bar z} + 2 u_{z\bar z} u_{z\bar z} + u_{zz\bar z} u_{\bar z} + u_z u_{z\bar z\bar z} \le 0. \tag{8.4}$$

Now (5.1) implies that

$$u_{z\bar z} = Q\bar Q \gamma^{-2} e^{-2u} + \gamma e^u - \tfrac{1}{\gamma} = 0,$$

where the cubic differential $U = Q\,dz^3$. Since $\gamma = 1 + O(|z|^2)$, we compute at p

$$\begin{aligned} u_{z\bar z} &= Q\bar Q e^{-2u} + e^u - \tfrac{1}{2}, \\ u_{zz\bar z} &= Q_z \bar Q e^{-2u} - 2Q\bar Q e^{-2u} u_z + e^u u_z, \\ u_{z\bar z\bar z} &= Q \bar Q_{\bar z} e^{-2u} - 2Q\bar Q e^{-2u} u_{\bar z} + e^u u_{\bar z}. \end{aligned}$$

Therefore, (8.4) becomes

$$\begin{aligned} 0 &\ge \tfrac{1}{2} u_z u_{\bar z} + 2(Q\bar Q e^{-2u} + e^u - \tfrac{1}{2})^2 + \bar Q e^{-2u} Q_z u_{\bar z} + Q e^{-2u} \bar Q_{\bar z} u_z \\ &\quad + 2(-2Q\bar Q e^{-2u} + e^u) u_z u_{\bar z} \\ &\ge \tfrac{1}{2} u_z u_{\bar z} + \bar Q e^{-2u} Q_z u_{\bar z} + Q e^{-2u} \bar Q_{\bar z} u_z + 2(-2Q\bar Q e^{-2u} + e^u) u_z u_{\bar z} \\ &\ge \tfrac{1}{2} u_z u_{\bar z} + \bar Q e^{-2u} Q_z u_{\bar z} + Q e^{-2u} \bar Q_{\bar z} u_z \end{aligned}$$

since the assumption that u is small implies $-2Q\bar Q e^{-2u} + e^u \ge 0$. In coordinate-free notation, we see that at the maximum point of $v = \frac{1}{4}\|\nabla u\|^2$,

$$0 \ge \tfrac{1}{2}\|\nabla u\|^2 + e^{-2u}\nabla(\|U\|^2)\cdot\nabla u \ge \tfrac{1}{2}\|\nabla u\|^2 - \tfrac{\epsilon}{2}\|\nabla u\|^2 - \tfrac{1}{2\epsilon}\|e^{-2u}\nabla(\|U\|^2)\|^2$$

for any $\epsilon > 0$. For $\epsilon = \frac{1}{2}$, we see that at the maximum point p of v, that $v = \frac{1}{4}\|\nabla u\|^2 \le \|e^{-2u}\nabla(\|U\|^2)\|^2$. Thus v is bounded by the maximum value of $\|e^{-2u}\nabla(\|U\|^2)\|^2$.

That $m \to 0$ as $U \to 0$ then follows from Corollary 5.6 above. The explicit bound above shows $C \to 0$ as $m \to 0$. $\square$

9. Representations of the Fundamental Group.

Fix a smooth compact oriented surface S of genus at least two. By Theorems 5.1 and 4.1, given a marked conformal structure Σ on S and small cubic holomorphic differential on Σ we obtain, via the solution u to (5.1), a holonomy map

$$\chi \colon \mathcal{K} \to \mathrm{Hom}(\pi_1 S, SU(2,1))/SU(2,1), \quad (\Sigma, U) \mapsto [\rho],$$

into the representation space of $\pi_1 S$ in $SU(2,1)$. The domain of this map is

$$\mathcal{K} = \{(\Sigma, U) : \max_{\Sigma} \|U\|_h^2 < \tfrac{1}{54}\}$$

where Σ ranges over all marked conformal structures on S, i.e., over all points in the Teichmüller space of S. The norm $\|U\|_h$ is that induced by the hyperbolic metric h on Σ. As a manifold $\mathcal{K}$ is a fibre subbundle of the vector bundle over Teichmüller space whose fibre at Σ is the vector space $H^0(\Sigma, K^3)$ of globally holomorphic cubic differentials.

A representation of $\pi_1 S$ into $SU(2,1)$ is called $\mathbb{R}$-Fuchsian if it is discrete, faithful and conjugate to a representation into $SO(2,1)$. In this case, it preserves a Lagrangian plane. The discussion in Section 6 above shows that $\mathbb{R}$-Fuchsian representations correspond exactly to pairs (Σ, U) with cubic differential $U = 0$.

An important invariant of representations of surface groups into $SU(n,1)$ is the Toledo invariant [32]. Given an invariant surface in $\mathbb{CH}^n$, the Toledo invariant is a normalised integral of the pull-back of the Kähler form. In the present $n = 2$ case, Xia has shown that the level sets of the Toledo invariant are connected components of the representation space [38].

Proposition 9.1. *The Toledo invariant vanishes for the representations we have produced.*

Proof. For each such representation, we have constructed an equivariant Lagrangian surface. □

Theorem 9.2. *The map χ is a local diffeomorphism near the zero section $\{U = 0\}$.*

The proof uses a symplectic form on the representation space due to Goldman [7], which we pause to describe. Let G denote $SU(2,1)$, and let $\mathfrak{g}$ denote its Lie algebra. Recall that the representation space $\mathrm{Hom}(\pi_1 S, G)/G$ is diffeomorphic to the moduli space $\mathcal{M}_G$ of flat principal G-bundles over S. A point $P \in \mathcal{M}_G$ is smooth if the centraliser of the image $\rho(\pi_1 S)$ of the corresponding representation has dimension zero. At such a point the tangent space can be identified with the cohomology $H^1(S, \mathrm{ad}P)$, where $\mathrm{ad}P$ is the associated flat $\mathfrak{g}$ bundle. The cohomology is de Rham cohomology with respect to the flat connexion d_P on $\mathrm{ad}P$. Goldman's symplectic form is defined as follows. For any pair of d_P-closed 1-forms $Z, W \in \Omega^1_S(\mathrm{ad}P)$ representing cohomology classes $[Z], [W] \in H^1(S, \mathrm{ad}P)$, he shows that

$$\omega_P([Z],[W]) = \int_S \mathrm{tr}(Z \wedge W)$$

is well-defined and symplectic at each smooth point P.

In particular, suppose P is the flat principal bundle whose connexion is determined by the Maurer-Cartan 1-form α in Theorem 4.1. When $X \in T_{(\Sigma,U)}\mathcal{K}$ is tangent to a curve $\gamma(t)$ in $\mathcal{K}$ at $t=0$, its push-forward to $H^1(S,\mathrm{ad}P)$ is represented by the first variation $\delta_X\alpha$ of α along this curve, as a $\mathfrak{g}$-valued 1-form on S. Thus we have

$$\chi^*\omega(X_1,X_2) = \int_S \mathrm{tr}(\delta_{X_1}\alpha \wedge \delta_{X_2}\alpha).$$

Proof. The proof proceeds by using Goldman's symplectic form and the Inverse Function Theorem. First of all, any solution for $U=0$ is an $\mathbb{R}$-Fuchsian representation, since it corresponds to an embedding of $\mathbb{RH}^2 \subset \mathbb{CH}^2$.

Second, the representation space of $SU(2,1)$ near any $\mathbb{R}$-Fuchsian representation is smooth and has dimension $16g-16$. The smoothness follows from realizing that the holonomy representation of an $\mathbb{R}$-Fuchsian representation in $SU(2,1)$ has zero centraliser (by [6]). Moreover, the representation space is Hausdorff, since $\rho(\pi_1 S)$ is not contained in a parabolic subgroup [13]. The dimension is calculated in [6]. Note the Riemann-Roch Theorem shows that $\mathcal{K}$ has the same real dimension $16g-16$.

Third, at any point in $\mathcal{K}$ where $U=0$, we prove the tangent map of χ is a linear isomorphism by showing that the pullback $\chi^*\omega$ is nondegenerate. The tangent space $T_{(\Sigma,0)}\mathcal{K}$ can be split into a Teichmüller space part and a fibre part, and so each tangent vector can be split into a holomorphic cubic differential U plus a tangent vector to Teichmüller space, which we may represent as a harmonic Beltrami differential μ.

The nondegeneracy of $\chi^*\omega$ follows from the following three claims, where $\delta_U\alpha$ represents the variation $\frac{\partial}{\partial t}\alpha(\Sigma,tU)\big|_{t=0}$.

- For any nonzero holomorphic cubic differential U, $\omega(\delta_U\alpha,\delta_{iU}\alpha) \neq 0$.
- If U is a holomorphic cubic differential and μ is a harmonic Beltrami differential, $\omega(\delta_U\alpha,\delta_\mu\alpha)=0$.
- If μ,ν are harmonic Beltrami differentials, then $\omega(\delta_\mu\alpha,\delta_\nu\alpha)$ is a nonzero multiple of the Weil-Petersson pairing $\mathrm{Im}\int_S \mu\cdot h\cdot\bar{\nu}$, for h the hyperbolic metric.

These claims show that the above splitting of $T_{(\Sigma,0)}\mathcal{K}$ is a symplectic-orthogonal splitting of nondegenerate spaces.

To prove the first claim, note that at $U=0$, the variation of the metric $\delta_U s = 0$ (for the metric $s^2|dz|^2$ above). This follows since U appears quadratically in (5.1).

Recall $U = Q\,dz^3$.

$$\delta_U\alpha = \delta_U(A\,dz + B\,d\bar z) = \begin{pmatrix} 0 & \bar Q s^{-2}d\bar z & 0 \\ -Qs^{-2}dz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we may compute

$$\mathrm{tr}\,(\delta_U\alpha \wedge \delta_{iU}\alpha) = 2i|Q|^2 s^{-4} dz \wedge d\bar z,$$

which is equal to the nonnegative two-form $2|U|^2h^{-2}$ for U the cubic differential and h the hyperbolic metric. This shows $\omega(\delta_U\alpha, \delta_{iU}\alpha) \neq 0$.

For the second claim, note that for $U = 0$, the deformation of the connexion α in the direction of the harmonic Beltrami differential μ is of the form

$$\delta_\mu\alpha = \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ * & * & 0 \end{pmatrix}.$$

Therefore, $\mathrm{tr}\,(\delta_U\alpha \wedge \delta_\mu\alpha) = 0$, and so $\omega(\delta_U\alpha, \delta_\mu\alpha) = 0$.

The third claim follows from a result of Shimura [29] (see also Goldman [6]). If $U = 0$, the holonomy of the connexion α is contained in $SO(2,1) \subset SU(2,1)$, since the developed surface is an $\mathbb{RH}^2 \subset \mathbb{CH}^2$. Therefore, $\mathrm{tr}\,(\delta_\mu\alpha \wedge \delta_\nu\alpha)$ is the same as the trace form on $SO(2,1)$. Under the Lie algebra isomorphism $\mathfrak{so}(2,1) \sim \mathfrak{sl}(2,\mathbb{R})$, the trace forms on $SO(2,1)$ and $SL(2,\mathbb{R})$ are the same up to a nonzero constant multiple. Then Shimura's result shows this trace form is a multiple of the imaginary part of $\int_S \mu \cdot h \cdot \bar\nu$.

Therefore, $\chi^*\omega$ is nondegenerate at $(\Sigma, 0) \in \mathcal{K}$, and so the Inverse Function Theorem shows χ is a local diffeomorphism there. □

Remark 4. It is likely that this computation can be pushed further to show that χ is a local diffeomorphism away from $U = 0$. In order to do this, we must have a good model of varying both Σ and U away from $U = 0$ (and a direct verification, using connexions, of a generalisation of Shimura's result).

We say a representation $\rho\colon \pi_1 S \to SU(2,1)$ is *geometrically finite* if for $\Omega \subset \partial\mathbb{CH}^2$ the domain of discontinuity of the action, the quotient of $(\mathbb{CH}^2 \cup \Omega)/\rho(\pi_1 S)$ is a compact manifold with boundary. (This definition should be modified in situations in which cusps are allowed.)

As in Parker-Platis [25], we say a representation ρ is *complex hyperbolic quasi-Fuchsian* if it is discrete, faithful, geometrically finite and totally loxodromic. Recall that a representation ρ into $SU(2,1)$ is called *totally loxodromic* if every $\rho(\gamma)$ is loxodromic for γ not the identity. $\rho(\gamma)$ is *loxodromic* if it fixes exactly two points in $\partial\mathbb{CH}^2$.

Theorem 9.3. *There is a neighborhood $\mathcal{N}$ of the zero section $\{U = 0\}$ of the total space of the vector bundle over Teichmüller space whose fibre is the space of* *holomorphic cubic differentials so that*

$$\chi|_{\mathcal{N}} : \mathcal{N} \to \mathrm{Hom}(\pi_1 S, SU(2,1))/SU(2,1)$$

is a diffeomorphism onto its image. For each of these representations, there is a fundamental domain in $\mathbb{CH}^2$*, an equivariant minimal Lagrangian surface, and an equivariant submersion of* $\mathbb{CH}^2$ *onto the surface. Each of these representations is complex hyperbolic quasi-Fuchsian.*

Proof. Restrict to cubic differentials U so that $\sup \|U\|_h$ is small enough; then Corollary 7.3 and Theorem 9.2 provide the bulk of the theorem. All that remains is to show that the representations we produce are complex hyperbolic quasi-Fuchsian.

The existence of the fundamental domain immediately implies the representation is discrete and faithful. Geometric finiteness follows from Corollary 7.3, in particular the fact that $\bar{\Phi}$ is an injective immersion of manifolds with boundary. We show ρ is totally loxodromic below in Proposition 9.5. □

Let $\Gamma = \rho(\pi_1\Sigma)$ be the induced discrete subgroup of $SU(2,1)$. Recall the limit set $\Lambda(\Gamma)$ is the subset of $\overline{\mathbb{CH}^2}$ defined by

$$\Lambda(\Gamma) = \{y = \lim_{i\to\infty} g_i(x) \;\big|\; x \in \overline{\mathbb{CH}^2},\, g_i \in \Gamma,\, g_i \neq g_j \text{ if } i \neq j\}.$$

Our construction of the fundamental domain $\mathcal{F}$ shows the following lemma, whose proof follows immediately from Corollary 7.3.

Lemma 9.4. $\bar{\mathcal{F}} \cap \Lambda(\Gamma) = \emptyset$.

Proposition 9.5. *The representation* ρ *is totally loxodromic.*

Proof. This is a standard fact, once we have our locally finite fundamental domain $\mathcal{F}$ (see e.g. [25]), but we provide a proof for the reader's convenience. We would like to thank Bill Goldman and especially John Parker for explaining the essential ideas here to us.

We need only rule out elliptic and parabolic elements of $\Gamma \setminus \{1\}$.

Ruling out elliptic elements is straightforward. If $g \in \Gamma \setminus \{1\}$ fixes a point $p \in \mathbb{CH}^2$, then p must lie in a translate $h\bar{\mathcal{F}}$ for some $h \in \Gamma$. But since g has infinite order (as Γ is a surface group), that would imply that all $g^n p \in h\bar{\mathcal{F}}$, which violates Lemma 9.4.

The remaining case is to rule out parabolic fixed points. Let p be a fixed point of a nontrivial parabolic element of Γ. Then $p \in \partial\mathbb{CH}^2$ and $p \in \Lambda(\Gamma)$. There are analogues of the classical horoball construction, due to Kamiya and Parker [14, 15, 24, 16]. Let Γ_p denote the isotropy group of p.

The (modified) horoballs are open sets $\mathcal{B}_\ell \subset \mathbb{CH}^2$ for $\ell \geq 0$ satisfying

(a) $p \in \overline{\mathcal{B}_\ell}$ for all $\ell \geq 0$.
(b) $\overline{\mathcal{B}_\ell} \subset \mathcal{B}_k \cup \{p\}$ if $\ell > k$.
(c) $\cap_{\ell\geq 0}\overline{\mathcal{B}_\ell} = \{p\}$.

(d) For any $\ell \geq 0$, $g \in \Gamma$, $g\mathcal{B}_\ell = \mathcal{B}_\ell$ if and only if $g \in \Gamma_p$.

(e) For any $\ell \geq 0$, $g \in \Gamma$, $\overline{\mathcal{B}_\ell} \cap g\overline{\mathcal{B}_\ell} = \emptyset$ if and only if $g \notin \Gamma_p$.

Now for $i = 1, 2, \dots$, choose $z_i \in \mathcal{B}_i$ so that $z_i \to p$. Then there are elements $g_i \in \Gamma$ so that $g_i z_i \in \bar{\mathcal{F}}$. By compactness, upon passing to a subsequence, we may assume $g_i z_i \to q \in \bar{\mathcal{F}}$. Now by properties (d) and (e), we have the following two cases

Case 1: Upon passing to a subsequence, $\{g_i\mathcal{B}_1\}$ are disjoint. In this case, we may assume that the Euclidean volume (as measured in $\mathbb{R}^4 = \mathbb{C}^2 \supset \mathbb{CH}^2$) of $g_i\mathcal{B}_1$ goes to zero. This implies that the Euclidean diameter of $g_i\mathcal{B}_1$ also goes to zero. Therefore, $g_i z_i \to q$ implies $g_i p \to q$. Thus q is a limit point of Γ, which contradicts Lemma 9.4. (The assertion about the relationship between the Euclidean volume and diameter is valid for all sufficiently small domains in $\mathbb{CH}^2 \subset \mathbb{C}^2$, and may be checked infinitesimally by calculating the Jacobian matrix of the action of a general element of $SU(2,1)$ in inhomogeneous projective coordinates.)

Case 2: Upon passing to a subsequence, all $g_i\mathcal{B}_1 = g_1\mathcal{B}_1$. In this case, $g_1^{-1}g_i \in \Gamma_p$, and so $g_1^{-1}g_i\mathcal{B}_i = \mathcal{B}_i$. Therefore, $g_i z_i \in g_1\mathcal{B}_i$, and taking $i \to \infty$ shows that $q = g_1 p$. But $g_1 p$ is a limit point of Γ, which again contradicts Lemma 9.4. $\square$

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