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ALTERNATIVE CRITERIA FOR THE BOUNDEDNESS OF VOLTERRA INTEGRAL OPERATORS IN LEBESGUE SPACES

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. Three different criteria for $L_p - L_q$ boundedness of Volterra integral operator (1.1) with locally integrable weight functions w, v and a non-negative kernel $k(x,y)$ satisfying Oinarov's condition for each case $1 < p \leqslant q < \infty$ and $1 < q < p < \infty$ are given. Relations between components of the boundedness constants are described.

1. Introduction

Let $-\infty \leqslant a < b \leqslant \infty$ and let v and w be locally integrable non-negative weight functions on (a,b). In the theory of integral operators a progress of the last three decades is related with the study of Volterra operators

$$\mathbf{K}f(x) := w(x)\int_a^x k(x,y)f(y)v(y)dy, \qquad x \in (a,b), \tag{1.1}$$

in Lebesgue spaces. Except an independent interest such transforms play an important role in applications to the spectral theory, integral and differential equations, embeddings of Sobolev spaces (see, for instance the monographs [2], [8], [12] and the papers [1], [3], [10], [11], [13], [14], [15], [16], [17], [19], [20], [21], [22]).

The first step in the study of (1.1) is the boundedness and compactness criteria which quality plays a crucial role for a further estimate of characteristic numbers and other applications. As an example we mention the operator (1.1) with $k(x,y) = \rho(x) \geqslant 0$, which was studied in the frame of the Sturm-Liouville equation theory [4]. This and other results rooted from the Hardy inequality [6], were later generalized by many authors and reached a kind of a final form on the class of Oinarov's kernels $k(x,y) \geqslant 0$ such that

$$D^{-1}k(x,y) \leqslant k(x,z) + k(z,y) \leqslant Dk(x,y), \qquad b \geqslant x \geqslant z \geqslant y \geqslant a, \tag{1.2}$$

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with a constant $D \geqslant 1$ independent on $x,\ y,\ z$. Typical examples of such kernels are the Riemann-Liouville operator with the kernel $k(x,y)=(x-y)_+^{\alpha-1}$ for $\alpha \geqslant 1$, integral kernels $k(x,y)=\left(\int_y^x h(z)dz\right)^\gamma,\ \gamma \geqslant 0,\ h(z)\geqslant 0$, and their combinations.

Let $0<p\leqslant\infty,\ -\infty\leqslant a<b\leqslant\infty$,

$$\|f\|_p=\begin{cases}\left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}}, & 0<p<\infty,\\ \operatorname*{ess\,sup}\limits_{x\in[a,b]} |f(x)|, & p=\infty.\end{cases}$$

Denote $p':=p/(p-1)$ for $0<p<\infty$, $p\neq 1$ and define the Lebesgue space $L_p[a,b]$ as a set of all measurable function f on $[a,b]$ such that $\|f\|_p<\infty$. If $a=0$ and $b=\infty$ we denote $L_p:=L_p[0,\infty)$. Without a loss of generality we assume functions f and weights $w,\ v$ to be non-negative throughout the paper.

Let the kernel $k(x,y)\geqslant 0$ of Volterra integral operator (1.1) be satisfying the condition (1.2) and the constant C in the inequality

$$\|\mathbf{K}f\|_q \leqslant C\|f\|_p \tag{1.3}$$

is the least possible, that is equal to the norm $\|\mathbf{K}\|_{L_p\to L_q}$. It is known [18], that if $0<p<1$ and $\mathbf{K}\colon L_p[a,b]\to L_q[a,b]$ then $w(x)k(x,y)v(y)=0$ for almost all (x,y), therefore $\mathbf{K}$ is the null operator. For $p=1<q<\infty$ and $1<p<q=\infty$ the boundedness of $\mathbf{K}$ from $L_p[a,b]$ to $L_q[a,b]$ is characterized by the following known criteria.

THEOREM 1.1. [7, Chapter XI, § 1.5, Theorem 4] *Let the operator* $\mathbf{K}$ *be given by* (1.1)*. Then, if* $1\leqslant q<\infty$, *we have*

$$\|\mathbf{K}\|_{L_1[a,b]\to L_q[a,b]}=\operatorname*{ess\,sup}_{t>0}\left\|\chi_{[a,\cdot]}(t)k(\cdot,t)w(\cdot)v(t)\right\|_q.$$

If $1<p\leqslant\infty$ *and* $1/p+1/p'=1$, *then*

$$\|\mathbf{K}\|_{L_p[a,b]\to L_\infty[a,b]}=\operatorname*{ess\,sup}_{t>0}\left\|\chi_{[a,\cdot]}(t)k(t,\cdot)w(\cdot)v(t)\right\|_{p'}.$$

Denote $r:=pq/(p-q)$ for $0<q<p<\infty$ and put

$$I_v f(x):=\int_a^x f(y)v(y)dy, \qquad I_w^* g(y):=\int_y^b g(x)w(x)ds,$$

$$V(x):=\int_a^x [v(y)]^{p'}dy, \qquad W(y):=\int_y^b [w(x)]^q dx,$$

$$V_1(x):=\int_a^x k(x,y)[v(y)]^{p'}dy, \qquad V_p(x):=\int_a^x [k(x,y)]^{p'}[v(y)]^{p'}dy,$$

$$W_1(y):=\int_y^b k(x,y)[w(x)]^q dx, \qquad W_q(y):=\int_y^b [k(x,y)]^q[w(x)]^q dx$$

To characterize the $L_p - L_q$ boundedness of K we need the following constants

$$\mathbf{A} := \max(\mathbf{A}_0, \mathbf{A}_1), \quad \mathbf{A}_0 := \sup_{t\in(a,b)} \mathbf{A}_0(t) = \sup_{t\in(a,b)} [W_q(t)]^{\frac{1}{q}} [V(t)]^{\frac{1}{p'}},$$

$$\mathbf{A}_1 := \sup_{t\in(a,b)} \mathbf{A}_1(t) = \sup_{t\in(a,b)} [W(t)]^{\frac{1}{q}} [V_p(t)]^{\frac{1}{p'}};$$

$$\mathbb{A} := \max(\mathbb{A}_0, \mathbb{A}_1), \quad \mathbb{A}_0 := \sup_{t\in(a,b)} \mathbb{A}_0(t) = \sup_{t\in(a,b)} [V(t)]^{-\frac{1}{p}} \left(\int_a^t [V_1(x)]^q [w(x)]^q dx\right)^{\frac{1}{q}},$$

$$\mathbb{A}_1 := \sup_{t\in(a,b)} \mathbb{A}_1(t) = \sup_{t\in(a,b)} [V_p(t)]^{-\frac{1}{p}} \left(\int_a^t [V_p(x)]^q [w(x)]^q dx\right)^{\frac{1}{q}};$$

$$\mathscr{A} := \max(\mathscr{A}_0, \mathscr{A}_1), \quad \mathscr{A}_0 := \sup_{t\in(a,b)} \mathscr{A}_0(t) = \sup_{t\in(a,b)} [W_q(t)]^{-\frac{1}{q'}} \left(\int_t^b [W_q(y)]^{p'} [v(y)]^{p'} dy\right)^{\frac{1}{p'}},$$

$$\mathscr{A}_1 := \sup_{t\in(a,b)} \mathscr{A}_1(t) = \sup_{t\in(a,b)} [W(t)]^{-\frac{1}{q'}} \left(\int_t^b [W_1(y)]^{p'} [v(y)]^{p'} dy\right)^{\frac{1}{p'}};$$

$$\mathbf{B} := \max(\mathbf{B}_0, \mathbf{B}_1), \quad \mathbf{B}_0 := \left(\int_a^b [W_q(t)]^{\frac{r}{q}} d[V(t)]^{\frac{r}{p'}}\right)^{\frac{1}{r}},$$

$$\mathbf{B}_1 := \left(\int_a^b [V_p(t)]^{\frac{r}{p'}} d\left(-[W(t)]^{\frac{r}{q}}\right)\right)^{\frac{1}{r}};$$

$$\mathbb{B} := \max(\mathbb{B}_0, \mathbb{B}_1), \quad \mathbb{B}_0 := \left(\int_a^b [V(t)]^{-\frac{r}{p}} \, d\left(\int_a^t [V_1(x)]^q [w(x)]^q dx\right)^{\frac{r}{q}}\right)^{\frac{1}{r}},$$

$$\mathbb{B}_1 := \left(\int_a^b [V_p(t)]^{-\frac{r}{p}} \, d\left(\int_a^t [V_p(x)]^q [w(x)]^q dx\right)^{\frac{r}{q}}\right)^{\frac{1}{r}};$$

$$\mathscr{B} := \max(\mathscr{B}_0, \mathscr{B}_1), \quad \mathscr{B}_0 := \left(\int_a^b [W_q(t)]^{-\frac{r}{q'}} \, d\left(-\left(\int_t^b [W_q(y)]^{p'} [v(y)]^{p'} dy\right)^{\frac{r}{p'}}\right)\right)^{\frac{1}{r}},$$

$$\mathscr{B}_1 := \left(\int_a^b [W(t)]^{-\frac{r}{q'}} \, d\left(-\left(\int_t^b [W_1(y)]^{p'} [v(y)]^{p'} dy\right)^{\frac{r}{p'}}\right)\right)^{\frac{1}{r}}.$$

For $1 < p \leqslant q < \infty$ the characterization of (1.3) by the condition $\mathscr{A} < \infty$ was obtained in [1], by $\mathbf{A} < \infty$ in [13] and by $\mathbf{A} < \infty$ or by $\mathbb{A} < \infty$ in [20] provided the kernel $k(x,y)$ of $\mathbf{K}$ satisfies (1.2) and some monotonicity or continuity conditions, which were later removed in [10]. Moreover, some relations between $\mathbf{A}_i$, $\mathbb{A}_i$, $\mathscr{A}_i$, $i = 0,1$ have been noted in [20]. The opposite case $1 < q < p < \infty$ was studied in [13] and [20] by giving the only criterion $\mathbf{B} < \infty$. An implicit criterion for (1.3) to hold in the case $0 < q < 1 < p < \infty$ was found in [9]. Explicit, but separate necessary or sufficient conditions for the case $0 < q < 1 < p < \infty$ under some monotonicity requirements on the kernel $k(x,y)$ were obtained in [20]. An extention of the above criteria $\mathbf{A} < \infty$ and $\mathbf{B} < \infty$ from [20] for the case, when the weight functions replaced by the arbitrary Borel measures is given in [17]. The compactness problem for the operator (1.1) with $1 < p,q < \infty$ was solved in [20].

In this work we analyse the relations between the components of the boundedness critera for (1.3) hold in both cases $1 < p \leqslant q < \infty$ (Theorem 2.1) and $1 < q < p < \infty$ (Theorem 2.2) providing alternative proofs.

Without a loss of generality we assume that $k(x,y)$ is non-decreasing with respect to the variable x and non-increasing in y. Otherwise we replace the kernel $k(x,y)$ of the operator (1.1) by the kernel $k_0(x,y) := \sup_{y\leqslant z\leqslant x} \overline{k}(x,z)$, where $\overline{k}(x,y) := \sup_{y\leqslant t\leqslant x} k(t,y)$. Then $k_0(x,y)$ has the both monotonicities, satisfies Oinarov's condition and $k(x,y) \leqslant k_0(x,y) \leqslant D^2 k(x,y)$ (see [10, Lemma 3] for details).

Throughout the paper the expressions of the type $0\cdot\infty$ are taken to be equal to 0. Relations $A \ll B$ mean $A \leqslant cB$ with some constants c depending only on parameters of summations and, possibly, on the constants of equivalence in the inequalities of the type (1.2). We write $A \approx B$ instead of $A \ll B \ll A$ or $A = cB$. $\mathbb{Z}$ denotes the set of all integers and χ_E stands for a characteristic function (indicator) of a subset $E \subset \mathbb{R}^+$. Also we make use of marks $:=$ and $=:$ for introducing new quantities and suppose $p' := p/(p-1)$ for $1 < p < \infty$ and $r := pq/(p-q)$ for $1 < q < p < \infty$.

2. Relations between components of boundedness constants

We need the following definition and technical proposition from [5].

DEFINITION 2.1. ([5, Definition 2.2(a)]) A nonnegative sequence $\{a_k\}_{k\in\mathbb{Z}}$ is said to be *strongly increasing* (*strongly decreasing*) if

$$\inf_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} > 1 \qquad \left(\sup_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} < 1\right),$$

and we write $a_k \upuparrows$ ($a_k \downdownarrows$).

PROPOSITION 2.1. ([5, Proposition 2.1]) *Let* $\{a_k\}_{k\in\mathbb{Z}}$, $\{\sigma_k\}_{k\in\mathbb{Z}}$ *and* $\{\tau_k\}_{k\in\mathbb{Z}}$ *be nonnegative sequences and* $0 < p < \infty$.

(a) *If* $\sigma_k \upuparrows$, *then* $\left(\sum_{k\in\mathbb{Z}} \left[\sum_{m\geqslant k} a_m\right]^p \sigma_k^p\right)^{\frac{1}{p}} \ll \left(\sum_{m\in\mathbb{Z}} [a_m\sigma_m]^p\right)^{\frac{1}{p}}$.

(b) *If* $\tau_k \downdownarrows$, *then* $\left(\sum_{k\in\mathbb{Z}} \left[\sum_{m\leqslant k} a_m\right]^p \tau_k^p\right)^{\frac{1}{p}} \ll \left(\sum_{m\in\mathbb{Z}} [a_m\tau_m]^p\right)^{\frac{1}{p}}$.

Boundedness of the operator $\mathbf{K}\colon L_p \to L_q$ with the Oinarov kernel (1.2) can be characterized by three alternative criteria (see Theorem 3.1 and also [1], [20]). Thus, there are tree pairs of conditions such that, for instance in the case $1 < p \leqslant q < \infty$

$$\mathbf{A}_0 + \mathbf{A}_1 < \infty \iff \mathbb{A}_0 + \mathbb{A}_1 < \infty \iff \mathscr{A}_0 + \mathscr{A}_1 < \infty.$$

It is known that the components in the pairs are independent on each other in general and finiteness of the only one of them do not guarantee the boundedness of $\mathbf{K}$. By the following theorem we describe relations between the above components.

THEOREM 2.1. *Let* $1 < p \leqslant q < \infty$ *and let a function* $k(x,y) \geqslant 0$ *on* $\{(x,y)\colon x > y > 0\}$ *be non-decreasing in* x *and non-increasing in* y. *Then*

(i) $\mathbf{A}_0 < \infty \Longleftrightarrow \mathscr{A}_0 < \infty$, (ii) $\mathbf{A}_1 < \infty \Longleftrightarrow \mathbb{A}_1 < \infty$,

(iii) $\mathbf{A}_0 < \infty \Longrightarrow \mathbb{A}_0 < \infty$, (iv) $\mathbf{A}_1 < \infty \Longrightarrow \mathscr{A}_1 < \infty$,

(v) $\mathscr{A}_0 < \infty \Longrightarrow \mathbb{A}_0 < \infty$, (vi) $\mathbb{A}_1 < \infty \Longrightarrow \mathscr{A}_1 < \infty$.

Moreover, the opposite relations to (iii) – (vi) *are not true in general.*

Proof. We start with (i). For any $t \in (a,b)$ we define an increasing integer-valued function $k\colon (a,b) \to \mathbb{Z}$

$$k(t) := \max\left\{k \in \mathbb{Z}\colon W_q(t) \leqslant 2^{-k}\right\} \tag{2.1}$$

and let $\{k_i\}_{i\in\mathbb{Z}_1}, \mathbb{Z}_1 \subseteq \mathbb{Z}$ be the values of $k(t)$. Then, it corresponds to each $i \in \mathbb{Z}_1$ either the interval $\Delta_i := [t_i, t_{i+1})$ or the interval $\Delta_i := (t_i, t_{i+1})$, when

$$2^{-k_i} \geqslant W_q(t) > 2^{-k_i-1}, \qquad t \in \Delta_i, \qquad i \in \mathbb{Z}_{1,1} \subseteq \mathbb{Z}_1 \subseteq \mathbb{Z}, \tag{2.2}$$

or the points $\{t_i\}$ for which

$$2^{-k_i} \geqslant W_q(t_i) > 2^{-k_i-1}, \qquad i \in \mathbb{Z}_{1,2} \subseteq \mathbb{Z}_1 \subseteq \mathbb{Z}, \tag{2.3}$$

and $\mathbb{Z}_1 = \mathbb{Z}_{1,1} \sqcup \mathbb{Z}_{1,2}$. Observe that the function $W_q(t)$ is non-increasing and almost everywhere equal to the left continuous function $\widetilde{W}_q(t) := W_q(t-0)$

$$2^{-k_m} \geqslant \widetilde{W}_q(t) \geqslant 2^{-k_m-1}, \qquad t \in (t_m, t_{m+1}], \qquad m \in \mathbb{Z}_{1,1} \tag{2.4}$$

and to the right continuous function $\overline{W}_q(t) := W_q(t+0)$

$$2^{-k_m} \geqslant \overline{W}_q(t) > 2^{-k_m-1}, \qquad t \in [t_m, t_{m+1}), \qquad m \in \mathbb{Z}_{1,1}. \tag{2.5}$$

Let $t \in \Delta_i,\ i \in \mathbb{Z}_{1,1}$. Then we have

$$\mathbf{A}_0(t) \overset{(2.2)}{\ll} 2^{-\frac{k_i}{q}} \left(\sum_{m\leqslant i} \int_{\Delta_m} [v(y)]^{p'} dy\right)^{\frac{1}{p'}}.$$

Since for any $t_m < t < t_{m+1},\ m \in \mathbb{Z}_{1,1}$,

$$\begin{aligned} 2^{\frac{k_m\cdot p'}{q'}} \left(\int_t^{t_{m+1}} [v(y)]^{p'} dy\right) &= 2^{\frac{k_m\cdot p'}{q'}} \left(\int_t^{t_{m+1}} [W_q(y)]^{p'} [v(y)]^{p'} [\widetilde{W}_q(y)]^{-p'} dy\right) \\ &\overset{(2.4)}{\ll} 2^{k_m\cdot p'} \cdot 2^{\frac{k_m\cdot p'}{q'}} \left(\int_t^{t_{m+1}} [W_q(y)]^{p'} [v(y)]^{p'} dy\right) \\ &\overset{(2.2)}{\ll} 2^{k_m\cdot p'} \cdot [W_q(t)]^{-\frac{p'}{q'}} \left(\int_t^{t_{m+1}} [W_q(y)]^{p'} [v(y)]^{p'} dy\right) \\ &\leqslant 2^{k_m\cdot p'} \cdot \mathscr{A}_0^{p'}, \end{aligned} \tag{2.6}$$

then

$$\int_{\Delta_m} [v(y)]^{p'} dy \ll \mathscr{A}_0^{p'} \cdot 2^{\frac{k_m \cdot p'}{q}}. \tag{2.7}$$

Therefore,

$$\mathbf{A}_0(t) \ll \mathscr{A}_0 \cdot 2^{-\frac{k_i}{q}} \left(\sum_{m \leqslant i} 2^{\frac{k_m \cdot p'}{q}} \right)^{\frac{1}{p'}} \leqslant \left(1 - 2^{-\frac{p'}{q}} \right)^{-\frac{1}{p'}} \mathscr{A}_0. \tag{2.8}$$

Hence,

$$\sup_{t \in \cup_{i \in \mathbb{Z}_{1,1}} \Delta_i} \mathbf{A}_0(t) \ll \mathscr{A}_0. \tag{2.9}$$

In the case $t = t_i,\ i \in \mathbb{Z}_{1,2},$ we have

$$\mathbf{A}_0(t_i) \overset{(2.3)}{\ll} 2^{-\frac{k_i}{q}} \left(\sum_{m \leqslant i-1} \int_{\Delta_m} [v(y)]^{p'} dy \right)^{\frac{1}{p'}}$$

and using (2.7) we obtain (2.8) for $t = t_i$. Thus,

$$\sup_{i \in \mathbb{Z}_{1,2}} \mathbf{A}_0(t_i) \ll \mathscr{A}_0$$

and combining this with (2.9) we see that $\mathbf{A}_0 \ll \mathscr{A}_0$ holds.

For the converse direction, using (2.2) we write for $t \in \Delta_i,\ i \in \mathbb{Z}_{1,1}$

$$\begin{aligned} \mathscr{A}_0(t) \ &\ll\ 2^{\frac{k_i}{q'}} \left(\sum_{m \geqslant i} \int_{\Delta_m} [W_q(y)]^{p'} [v(y)]^{p'} dy \right)^{\frac{1}{p'}} \\ &\overset{(2.2)}{\leqslant}\ 2^{\frac{k_i}{q'}} \left(\sum_{m \geqslant i} 2^{-k_m \cdot p'} \int_{\Delta_m} [v(y)]^{p'} dy \right)^{\frac{1}{p'}}. \end{aligned} \tag{2.10}$$

Since for any $t_m < t < t_{m+1},\ m \in \mathbb{Z}_{1,1},$

$$2^{-k_m \cdot p'} \int_{t_m}^{t} [v(y)]^{p'} dy \overset{(2.2)}{\ll} 2^{-\frac{k_m \cdot p'}{q'}} \cdot [W_q(t)]^{\frac{p'}{q}} V(t) \leqslant \mathbf{A}_0^{p'} \cdot 2^{-\frac{k_m \cdot p'}{q'}}, \tag{2.11}$$

then

$$2^{-k_m \cdot p'} \int_{\Delta_m} [v(y)]^{p'} dy \ll \mathbf{A}_0^{p'} \cdot 2^{-\frac{k_m \cdot p'}{q'}},$$

and

$$\mathscr{A}_0(t) \ll \mathbf{A}_0 \cdot 2^{\frac{k_i}{q'}} \left(\sum_{m \geqslant i} 2^{-\frac{k_m \cdot p'}{q'}} \right)^{\frac{1}{p'}} \leqslant \left(1 - 2^{-\frac{p'}{q'}} \right)^{-\frac{1}{p'}} \mathbf{A}_0. \tag{2.12}$$

The argument is analogous in the proof of (2.12) for $t = t_i$. It implies $\mathscr{A}_0 \ll \mathbf{A}_0$ and the equivalence $\mathbf{A}_0 < \infty \Longleftrightarrow \mathscr{A}_0 < \infty$ follows.

To prove (ii) we define the integer-valued function $l\colon (a,b) \to \mathbb{Z}$ such that

$$l(t) = \max\left\{ l \in \mathbb{Z}\colon V_p(t) \geqslant 2^l \right\}. \tag{2.13}$$

Denote values of $l(t)$ by $\{l_j\}_{j\in\mathbb{Z}_2\subseteq\mathbb{Z}}$. Here again each number j corresponds either to the interval

$$\Delta_j := [t_j, t_{j+1}) \qquad \text{or} \qquad \Delta_j := (t_j, t_{j+1}),$$

where

$$2^{l_j} \leqslant V_p(t) < 2^{l_j+1}, \qquad t \in \Delta_j, \qquad j \in \mathbb{Z}_{2,1} \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}, \tag{2.14}$$

or to the points $\{t_j\}$, where

$$2^{l_j} \leqslant V_p(t_j) < 2^{l_j+1}, \qquad j \in \mathbb{Z}_{2,2} \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}, \tag{2.15}$$

and $\mathbb{Z}_2 = \mathbb{Z}_{2,1} \sqcup \mathbb{Z}_{2,2}$. Note that the function $V_p(t)$ is non-decreasing and almost everywhere equal to the left continuous function $\widetilde{V}_p(t) := V_p(t-0)$

$$2^{l_m} \leqslant \widetilde{V}_p(t) \leqslant 2^{l_m+1}, \qquad t \in (t_m, t_{m+1}], \qquad m \in \mathbb{Z}_{2,1}, \tag{2.16}$$

as well as to the right continuous function $\overline{V}_p(t) := V_p(t+0)$

$$2^{l_m} \leqslant \overline{V}_p(t) < 2^{l_m+1}, \qquad t \in [t_m, t_{m+1}), \qquad m \in \mathbb{Z}_{2,1}. \tag{2.17}$$

For $t \in \Delta_j,\ j \in \mathbb{Z}_{2,1}$, we have

$$\begin{aligned} \mathbb{A}_1(t) &\overset{(2.14)}{\ll} 2^{-\frac{l_j}{p}} \left(\sum_{m\leqslant j} \int_{\Delta_m} [V_p(x)]^q [w(x)]^q dx \right)^{\frac{1}{q}} \\ &\overset{(2.14)}{\leqslant} 2^{-\frac{l_j}{p}} \left(\sum_{m\leqslant j} 2^{(l_m+1)\cdot q} \int_{\Delta_m} [w(x)]^q dx \right)^{\frac{1}{q}}. \end{aligned} \tag{2.18}$$

The inequality

$$2^{\frac{l_m\cdot q}{p'}} \int_t^{t_{m+1}} [w(x)]^q dx \overset{(2.14)}{\ll} [V_p(t)]^{\frac{q}{p'}} W(t) \leqslant \mathbf{A}_1^q, \tag{2.19}$$

where $t_m < t < t_{m+1},\ m \in \mathbb{Z}_{2,1}$, yields

$$2^{(l_m+1)\cdot q} \int_{\Delta_m} [w(x)]^q dx \ll \mathbf{A}_1^q \cdot 2^{\frac{l_m\cdot q}{p}}.$$

Therefore,

$$\mathbb{A}_1(t) \ll \mathbf{A}_1 \cdot 2^{-\frac{l_j}{p}} \left(\sum_{m\leqslant j} 2^{\frac{l_m\cdot q}{p}} \right)^{\frac{1}{q}} \leqslant \left(1 - 2^{-\frac{q}{p}}\right)^{-\frac{1}{q}} \mathbf{A}_1. \tag{2.20}$$

For $t=t_j,\ j\in\mathbb{Z}_{2,2}$, by (2.15) we have similar to (2.18) inequality

$$\mathbb{A}_1(t_j)\ll 2^{-\frac{l_j}{p}}\left(\sum_{m\leqslant j-1}2^{(l_m+1)\cdot q}\int_{\Delta_m}[w(x)]^q dx\right)^{\frac{1}{q}}.$$

Now by (2.19) the estimate (2.20) is true for $t=t_j$ too. Hence, $\mathbb{A}_1\ll\mathbf{A}_1$.

For the opposite estimate we note that for $t\in\Delta_j,\ j\in\mathbb{Z}_{2,1}$, it holds by (2.14) that

$$\mathbf{A}_1(t)\ll 2^{\frac{l_j}{p'}}\left(\sum_{m\geqslant j}\int_{\Delta_m}[w(x)]^q dx\right)^{\frac{1}{q}}.$$

We have for any $t_m<t<t_{m+1},\ m\in\mathbb{Z}_{2,1}$,

$$\begin{aligned}
2^{-\frac{l_m\cdot q}{p}}\left(\int_{t_m}^t[w(x)]^q dx\right) &= 2^{-\frac{l_m\cdot q}{p}}\left(\int_{t_m}^t[V_p(x)]^q[w(x)]^q[\overline{V}_p(x)]^{-q}dx\right)\\
&\overset{(2.14)}{\leqslant} 2^{-l_m\cdot q}\cdot 2^{-\frac{l_m\cdot q}{p}}\left(\int_{t_m}^t[V_p(x)]^q[w(x)]^q dx\right)\\
&\overset{(2.14)}{\ll} 2^{-l_m\cdot q}\cdot[V_p(t)]^{-\frac{q}{p}}\int_{t_m}^t[V_p(x)]^q[w(x)]^q dx\\
&\leqslant 2^{-l_m\cdot q}\cdot\mathbb{A}_1^q. \qquad (2.21)
\end{aligned}$$

Therefore,

$$\int_{\Delta_m}[w(x)]^q dx\ll\mathbb{A}_1^q\cdot 2^{-\frac{l_m\cdot q}{p'}}. \qquad (2.22)$$

Then,

$$\mathbf{A}_1(t)\ll\mathbb{A}_1\cdot 2^{\frac{l_j}{p'}}\left(\sum_{m\geqslant j}2^{-\frac{l_m\cdot q}{p'}}\right)^{\frac{1}{q}}\leqslant\left(1-2^{-\frac{p'}{q}}\right)^{-\frac{1}{p'}}\mathbb{A}_1, \qquad (2.23)$$

and, thus,

$$\sup_{t\in\cup_{j\in\mathbb{Z}_{2,1}}\Delta_j}\mathbf{A}_1(t)\ll\mathbb{A}_1. \qquad (2.24)$$

If $t=t_j,\ j\in\mathbb{Z}_{2,2}$, then

$$\mathbf{A}_1(t_j)\overset{(2.15)}{\ll}2^{\frac{l_j}{p'}}\left(\sum_{m\geqslant j}\int_{\Delta_m}[w(x)]^q dx\right)^{\frac{1}{q}},$$

and now by (2.21) and (2.22) the estimate (2.23) holds too and the assertion (ii) follows.

The implications (iii) and (iv) follow by Minkowski's inequality (see [20, Proposition]). The relation (v) follows from (i) and (iii), while (vi) – from (ii) and (iv). The assertion about the implications reverse to (iii) – (vi) is proved in [20, Proposition].

Analogously, in the case $1<q<p<\infty$ there are tree pairs of conditions such that

$$\mathbf{B}_0+\mathbf{B}_1<\infty \Longleftrightarrow \mathbb{B}_0+\mathbb{B}_1<\infty \Longleftrightarrow \mathscr{B}_0+\mathscr{B}_1<\infty,$$

and the components are related to each other by the following way.

THEOREM 2.2. *Let* $1<q<p<\infty$ *and let a function* $k(x,y)\geqslant 0$ *on* $\{(x,y)\colon x>y>0\}$ *be non-decreasing in* x *and non-increasing in* y*. Then*

$$\begin{array}{llll} \text{(i)} & \mathbf{B}_0<\infty \Longleftrightarrow \mathscr{B}_0<\infty, & \text{(ii)} & \mathbf{B}_1<\infty \Longleftrightarrow \mathbb{B}_1<\infty,\\ \text{(iii)} & \mathbf{B}_0<\infty \Longrightarrow \mathbb{B}_0<\infty, & \text{(iv)} & \mathbf{B}_1<\infty \Longrightarrow \mathscr{B}_1<\infty,\\ \text{(v)} & \mathscr{B}_0<\infty \Longrightarrow \mathbb{B}_0<\infty, & \text{(vi)} & \mathbb{B}_1<\infty \Longrightarrow \mathscr{B}_1<\infty, \end{array}$$

and the relations opposite to (iii) – (vi) *are not true in general.*

Proof. We start with (i). Using the definitions (2.1) – (2.5) we write

$$\begin{aligned}\mathbf{B}_0^r &= \sum_{i\in\mathbb{Z}_{1,1}}\int_{\Delta_i}[W_q(t)]^{\frac{r}{q}}d[V(t)]^{\frac{r}{p'}} \overset{(2.2)}{\leqslant} \sum_{i\in\mathbb{Z}_{1,1}} 2^{\frac{-k_i\cdot r}{q}}\int_{\Delta_i}d[V(t)]^{\frac{r}{p'}}\\ &\leqslant \sum_{i\in\mathbb{Z}_{1,1}} 2^{\frac{-k_i\cdot r}{q}}[V(t_{i+1})]^{\frac{r}{p'}} \leqslant \sum_{i\in\mathbb{Z}} 2^{-\frac{k_i\cdot r}{q}}\left(\sum_{m\leqslant i}\int_{\Delta_m}[v(y)]^{p'}dy\right)^{\frac{r}{p'}}.\end{aligned}$$

Let $\Delta_m=\varnothing$ if $m\in\mathbb{Z}\setminus\mathbb{Z}_{1,1}$. By Proposition 2.1(b)

$$\begin{aligned}\mathbf{B}_0^r &\ll \sum_{i\in\mathbb{Z}} 2^{-\frac{k_i\cdot r}{q}}\left(\int_{\Delta_i}[v(y)]^{p'}dy\right)^{\frac{r}{p'}}\\ &\overset{(2.4)}{\ll} \sum_{i\in\mathbb{Z}_{1,1}} 2^{\frac{k_i\cdot r}{q'}}\left(\int_{t_i}^{t_{i+1}}[v(y)]^{p'}dy\right)^{\frac{r}{p'}}[\widetilde{W}_q(t_{i+1})]^r\\ &\leqslant \sum_{i\in\mathbb{Z}_{1,1}} 2^{\frac{k_i\cdot r}{q'}}\left(\int_{t_i}^{t_{i+1}}[W_q(y)]^{p'}[v(y)]^{p'}dy\right)^{\frac{r}{p'}}\\ &\overset{(2.5)}{\ll} \sum_{i\in\mathbb{Z}_{1,1}} [\overline{W}_q(t_i)]^{-\frac{r}{q'}}\int_{t_i}^{t_{i+1}}d\left(-\left(\int_t^{t_{i+1}}[W_q(y)]^{p'}[v(y)]^{p'}dy\right)^{\frac{r}{p'}}\right)\\ &\leqslant \sum_{i\in\mathbb{Z}_{1,1}}\int_{t_i}^{t_{i+1}}[W_q(t)]^{-\frac{r}{q'}}d\left(-\left(\int_t^{b}[W_q(y)]^{p'}[v(y)]^{p'}dy\right)^{\frac{r}{p'}}\right) = \mathscr{B}_0^r.\end{aligned}$$

By Proposition 2.1(a) for the opposite estimate we write

$$\begin{aligned}
\mathscr{B}_0^r &\overset{(2.2)}{\ll} \sum_{i\in\mathbb{Z}_{1,1}} 2^{\frac{k_i\cdot r}{q'}} \int_{\Delta_i} d\left(-\left(\int_t^b [W_q(y)]^{p'}[v(y)]^{p'}dy\right)^{\frac{r}{p'}}\right)\\
&\leqslant \sum_{i\in\mathbb{Z}_{1,1}} 2^{\frac{k_i\cdot r}{q'}} \left(\int_{t_i}^b [W_q(y)]^{p'}[v(y)]^{p'}dy\right)^{\frac{r}{p'}}\\
&= \sum_{i\in\mathbb{Z}_{1,1}} 2^{\frac{k_i\cdot r}{q'}} \left(\sum_{m\geqslant i}\int_{\Delta_m} [W_q(y)]^{p'}[v(y)]^{p'}dy\right)^{\frac{r}{p'}}\\
&\ll \sum_{m\in\mathbb{Z}_{1,1}} 2^{\frac{k_m\cdot r}{q'}} \left(\int_{\Delta_m} [W_q(y)]^{p'}[v(y)]^{p'}dy\right)^{\frac{r}{p'}}\\
&\overset{(2.2)}{\ll} \sum_{m\in\mathbb{Z}_{1,1}} 2^{\frac{-k_m\cdot r}{q}} \left(\int_{\Delta_m} [v(y)]^{p'}dy\right)^{\frac{r}{p'}}\\
&\approx \sum_{m\in\mathbb{Z}_{1,1}} 2^{\frac{-k_m\cdot r}{q}} \int_{t_m}^{t_{m+1}} \left(\int_{t_m}^t [v(y)]^{p'}dy\right)^{\frac{r}{q'}} [v(t)]^{p'}dt\\
&\overset{(2.4)}{\ll} \sum_{m\in\mathbb{Z}_{1,1}} [\widetilde{W}_q(t_{m+1})]^{\frac{r}{q}} \int_{t_m}^{t_{m+1}} d[V(t)]^{\frac{r}{p'}}\\
&\leqslant \sum_{m\in\mathbb{Z}_{1,1}} \int_{t_m}^{t_{m+1}} [W_q(t)]^{\frac{r}{q}} d[V(t)]^{\frac{r}{p'}} = \mathbf{B}_0^r.
\end{aligned}$$

To prove (ii) we use the sequence $\{l_j\}_{j\in\mathbb{Z}_2\subseteq\mathbb{Z}}$ with the properties (2.13) – (2.17). Then

$$\begin{aligned}
\mathbf{B}_1^r &= \sum_{j\in\mathbb{Z}_{2,1}} \int_{\Delta_j} [V_p(t)]^{\frac{r}{p'}} d\left(-[W(t)]^{\frac{r}{q}}\right) \overset{(2.14)}{\leqslant} \sum_{j\in\mathbb{Z}_{2,1}} 2^{\frac{(l_j+1)\cdot r}{p'}} \int_{\Delta_j} d\left(-[W(t)]^{\frac{r}{q}}\right)\\
&\leqslant \sum_{j\in\mathbb{Z}_{2,1}} 2^{\frac{(l_j+1)\cdot r}{p'}} [W(t_j)]^{\frac{r}{q}} \leqslant 2^{\frac{r}{p'}} \sum_{j\in\mathbb{Z}} 2^{\frac{l_j\cdot r}{p'}} \left(\sum_{m\geqslant j}\int_{\Delta_m}[w(x)]^q dx\right)^{\frac{r}{q}}.
\end{aligned}$$

If $m\in\mathbb{Z}\setminus\mathbb{Z}_{2,1}$ we assume $\Delta_m=\varnothing$. By Proposition 2.1(a)

$$\begin{aligned}
\mathbf{B}_1^r &\ll \sum_{j\in\mathbb{Z}} 2^{\frac{l_j\cdot r}{p'}} \left(\int_{\Delta_j}[w(x)]^q dx\right)^{\frac{r}{q}}\\
&\overset{(2.17)}{\ll} \sum_{j\in\mathbb{Z}_{2,1}} 2^{\frac{-l_j\cdot r}{p}} \left(\int_{t_j}^{t_{j+1}}[w(x)]^q dx\right)^{\frac{r}{q}} [\overline{V}_p(t_j)]^r\\
&\leqslant \sum_{j\in\mathbb{Z}_{2,1}} 2^{\frac{-l_j\cdot r}{p}} \left(\int_{t_j}^{t_{j+1}}[V_p(x)]^q[w(x)]^q dx\right)^{\frac{r}{q}}\\
&\overset{(2.16)}{\ll} \sum_{j\in\mathbb{Z}_{2,1}} [\widetilde{V}_p(t_{j+1})]^{-\frac{r}{p}} \int_{t_j}^{t_{j+1}} d\left(\int_{t_j}^t [V_p(x)]^q[w(x)]^q dx\right)^{\frac{r}{q}}
\end{aligned}$$

$$\leqslant \sum_{j\in\mathbb{Z}_{2,1}} \int_{t_j}^{t_{j+1}} [V_p(t)]^{-\frac{r}{p}} \, d\left(\int_a^t [V_p(x)]^q [w(x)]^q dx\right)^{\frac{r}{q}} = \mathbb{B}_1^r.$$

By Proposition 2.1(b) for the reverse estimate we write

$$\begin{aligned}
\mathbb{B}_1^r &= \sum_{j\in\mathbb{Z}_{2,1}} \int_{t_j}^{t_{j+1}} [V_p(t)]^{-\frac{r}{p}} \, d\left(\int_a^t [V_p(x)]^q [w(x)]^q dx\right)^{\frac{r}{q}} \\
&\overset{(2.14)}{\leqslant} \sum_{j\in\mathbb{Z}_{2,1}} 2^{\frac{-l_j\cdot r}{p}} \left(\int_a^{t_{j+1}} [V_p(x)]^q [w(x)]^q dx\right)^{\frac{r}{q}} \\
&= \sum_{j\in\mathbb{Z}_{2,1}} 2^{\frac{-l_j\cdot r}{p}} \left(\sum_{m\leqslant j} \int_{\Delta_m} [V_p(x)]^q [w(x)]^q dx\right)^{\frac{r}{q}} \\
&\ll \sum_{m\in\mathbb{Z}_{2,1}} 2^{\frac{-l_m\cdot r}{p}} \left(\int_{\Delta_m} [V_p(x)]^q [w(x)]^q dx\right)^{\frac{r}{q}} \\
&\overset{(2.14)}{\leqslant} \sum_{m\in\mathbb{Z}_{2,1}} 2^{\frac{l_m\cdot r}{p'}} \left(\int_{\Delta_m} [w(x)]^q dx\right)^{\frac{r}{q}} \\
&\leqslant \sum_{m\in\mathbb{Z}_{2,1}} 2^{\frac{l_m\cdot r}{p'}} \int_{\Delta_m} d\left(-[W(x)]^{\frac{r}{q}}\right) \\
&\overset{(2.14)}{\leqslant} \sum_{m\in\mathbb{Z}_{2,1}} \int_{\Delta_m} [V_p(x)]^{\frac{r}{p'}} d\left(-[W(x)]^{\frac{r}{q}}\right) = \mathbf{B}_1^r.
\end{aligned}$$

For (iii) we obtain by Minkowski's inequality

$$\left(\int_a^t [V_1(x)]^q [w(x)]^q dx\right)^{\frac{1}{q}} \leqslant \int_a^t \left(\int_y^t [k(x,y)]^q [w(x)]^q dx\right)^{\frac{1}{q}} dV(y) \leqslant \int_a^t [W_q(y)]^{\frac{1}{q}} dV(y). \tag{2.25}$$

Integrating by parts we have

$$\mathbb{B}_0^r = [V(b)]^{-\frac{r}{p}} \left(\int_a^b [V_1(x)]^q [w(x)]^q dx\right)^{\frac{r}{q}} + \frac{r}{p} \bar{\mathbb{B}}_0^r, \tag{2.26}$$

provided $V(b) < \infty$, where

$$\bar{\mathbb{B}}_0^r := \int_a^b \left(\int_a^t [V_1(x)]^q [w(x)]^q dy\right)^{\frac{r}{q}} [V(t)]^{-\frac{r}{q}} dV(t)$$

and $\mathbb{B}_0^r = \bar{\mathbb{B}}_0^r$ if $V(b) = \infty$. Let $\alpha \in \left(\frac{1}{q'}, \frac{1}{r'}\right)$. By applying Hölder's inequality with the powers r and $r' = \frac{r}{r-1}$ we have

$$\bar{\mathbb{B}}_0^r \ll \int_a^b \left(\int_a^t [W_q(y)]^{\frac{1}{q}}[V(y)]^{\alpha}[V(y)]^{-\alpha}dV(y)\right)^r [V(t)]^{-\frac{r}{q}}dV(t)$$
$$\leqslant \int_a^b \int_a^t [W_q(y)]^{\frac{r}{q}}[V(y)]^{\alpha r}dV(y)\left(\int_a^t [V(z)]^{-\alpha r'}dV(z)\right)^{r-1}[V(t)]^{-\frac{r}{q}}dV(t)$$
$$\approx \int_a^b [W_q(y)]^{\frac{r}{q}}[V(y)]^{\alpha r}\left(\int_y^b [V(y)]^{r-1-\alpha r-\frac{r}{q}}dV(y)\right)dV(y) \ll \mathbf{B}_0^r. \qquad (2.27)$$

To estimate the first term in (2.26) we write by Minkowski's and Hölder's inequalities

$$\left(\int_a^b [V_1(x)]^q[w(x)]^q dx\right)^{\frac{r}{q}} \leqslant \left(\int_a^b \left(\int_y^b [k(x,y)]^q[w(x)]^q dx\right)^{\frac{1}{q}} dV(y)\right)^r$$
$$= \left(\int_a^b [W_q(y)]^{\frac{1}{q}}[V(y)]^{\frac{1}{q'}}[V(y)]^{-\frac{1}{q'}}dV(y)\right)^r$$
$$\leqslant \int_a^b [W_q(y)]^{\frac{r}{q}}[V(y)]^{\frac{r}{q'}}dV(y)\left(\int_a^b [V(z)]^{-\frac{r'}{q'}}dV(z)\right)^{r-1}$$
$$\approx \mathbf{B}_0^r[V(b)]^{\frac{r}{p}}. \qquad (2.28)$$

From this and (2.26) – (2.27) the estimate $\mathbb{B}_0 \ll \mathbf{B}_0$ follows.

For (iv) we find analogously to (2.25)

$$\left(\int_t^b [W_1(y)]^{p'}v^{p'}(y)dy\right)^{\frac{1}{p'}} \leqslant \int_t^b [V_p(y)]^{\frac{1}{p'}}d\left(-W(y)\right).$$

Further, analogously to (2.27) for $\alpha \in \left(\frac{1}{p},\frac{1}{r'}\right)$ and (2.28) we obtain the required estimate. The relation (v) follows from (i) and (iii), while (vi) goes from ((ii) and (iv). Proofs of the assertion about the implications reverse to (iii) – (vi) are analogous to [20, Proposition].

3. Boundedness criteria

Let the operator $\mathbf{K}$ be given by (1.1) with a kernel $k(x,y)$ satisfying Oinarov's condition (1.2). We consider the cases $1 < p \leqslant q < \infty$ and $1 < q < p < \infty$ separately but both basing on the following auxiliary lemmas.

LEMMA 3.1. [20, Lemma 1] *Let* $1 < q < \infty$ *and the operator* $\mathbf{K}$ *be given by* (1.1) *with the kernel* $k(x,y)$ *satisfying* (1.2). *Denote* $Kf(x) := \int_a^x k(x,y)f(y)v(y)dy$ *and suppose that*

$$\|\mathbf{K}f\|_q^q < \infty.$$

Then

$$\|\mathbf{K}f\|_q^q \approx \int_a^b f(t)v(t)[I_v f(t)]^{q-1}W_q(t)dt + \int_a^b f(t)v(t)[Kf(t)]^{q-1}W_1(t)dt. \qquad (3.1)$$

LEMMA 3.2. [20, Lemma 2] *Let* $1 < p' < \infty$ *and* $\mathbf{K}^*$ *be an adjoint to* $\mathbf{K}$ *operator of the form*

$$\mathbf{K}^*g(y) := v(y)\int_y^b k(x,y)w(x)g(x)dy, \qquad y \in (a,b), \tag{3.2}$$

with $k(x,y)$ *satisfying* (1.2). *Denote* $K^*g(y) := \int_y^b k(x,y)g(x)w(x)dx$ *and suppose that*

$$\|\mathbf{K}^*g\|_{p'}^{p'} < \infty.$$

Then

$$\|\mathbf{K}^*g\|_{p'}^{p'} \approx \int_a^b g(t)w(t)[I_w^*g(t)]^{p'-1}V_p(t)dt + \int_a^b g(t)w(t)[K^*g(t)]^{p'-1}V_1(t)dt. \tag{3.3}$$

Denote $\|\mathbf{K}\| := \|\mathbf{K}\|_{L_p[a,b]\to L_q[a,b]}$. Our main result reads

THEOREM 3.1. *Let the operator* $\mathbf{K}$ *be defined by the formula* (1.1) *with the kernel satisfying the condition* (1.2). *If* $1 < p \leqslant q < \infty$, *then*

$$(\mathbf{a})\ \|\mathbf{K}\| \approx \mathbf{A} \qquad (\mathbf{b})\ \|\mathbf{K}\| \approx \mathbb{A} \qquad (\mathbf{c})\ \|\mathbf{K}\| \approx \mathscr{A}. \tag{3.4}$$

For $1 < q < p < \infty$ *we have*

$$(\mathbf{a})\ \|\mathbf{K}\| \approx \mathbf{B} \qquad (\mathbf{b})\ \|\mathbf{K}\| \approx \mathbb{B} \qquad (\mathbf{c})\ \|\mathbf{K}\| \approx \mathscr{B}. \tag{3.5}$$

Proof. The lower estimates in (3.4)(a), (3.5)(a) and (3.4)(c) follow by inserting in (1.1) the test functions (see [20, Proofs of Theorems 1 and 2] and [1, Proof of Theorem 2.1]). Moreover, the lower estimate in (3.4)(c) can be obtained from $\|\mathbf{K}\| \gg \mathbf{A}$ by Theorem 2.1 (i), (ii) and (vi) as well as the lower estimate in (3.4)(b) by Theorem 2.2 (i), (ii) and (vi). Similar assertions of Theorem 2.2 works for the lower estimates in (3.5)(b) and (3.5)(c) from $\|\mathbf{K}\| \gg \mathbf{B}$ to obtain.

For *the upper estimates* in (3.4)(a), (3.5)(a) and (3.4)(c) we give the proofs different from [20] and [1]. For the simplicity we suppose that $a = 0$, $b = \infty$. Remind that we suppose functions f to be non-negative. We start from (a) and (c) both basing on Lemma 3.1. We have

$$\|\mathbf{K}f\|_q^q \approx \int_0^\infty f(t)v(t)[I_vf(t)]^{q-1}W_q(t)dt + \int_0^\infty f(t)v(t)[Kf(t)]^{q-1}W_1(t)dt =: J_1 + J_2. \tag{3.6}$$

To estimate J_1 we use the sequence $\{k_i\}_{i\in\mathbb{Z}_1\subseteq\mathbb{Z}}$ with the properties (2.1) – (2.4). Then, by Proposition 2.1(b) and Hölder's inequality

$$J_1 = \sum_{i\in\mathbb{Z}_{1,1}} \int_{\Delta_i} f(t)v(t)[I_vf(t)]^{q-1}W_q(t)dt \overset{(2.2)}{\leqslant} \sum_{i\in\mathbb{Z}_{1,1}} 2^{-k_i}\int_{\Delta_i} f(t)v(t)[I_vf(t)]^{q-1}dt$$

$$\ll \sum_{i\in\mathbb{Z}_{1,1}} 2^{-k_i}[I_vf(t_{i+1})]^q \leqslant \sum_{i\in\mathbb{Z}_{1,1}} 2^{-k_i}\left(\sum_{m\leqslant i}\int_{\Delta_m} f(y)v(y)dy\right)^q$$

$$\ll \sum_{m\in\mathbb{Z}_{1,1}} 2^{-k_m}\left(\int_{\Delta_m} f(y)v(y)dy\right)^q = \sum_{m\in\mathbb{Z}_{1,1}} 2^{-k_m}\left(\int_{\Delta_m} f(y)v(y)dy\right)^q$$
$$\leqslant \sum_{m\in\mathbb{Z}_{1,1}} 2^{-k_m}\left(\int_{\Delta_m}[f(y)]^p dy\right)^{\frac{q}{p}}\left(\int_{\Delta_m}[v(y)]^{p'}dy\right)^{\frac{q}{p'}} =: J_{1,1}. \tag{3.7}$$

If $t_m < t < t_{m+1}$, then by (2.2)

$$2^{-k_m}\left(\int_{t_m}^t [v(y)]^{p'}dy\right)^{\frac{q}{p'}} \leqslant 2\cdot W_q(t)\left(\int_{t_m}^t [v(y)]^{p'}dy\right)^{\frac{q}{p'}} \leqslant 2\cdot \mathbf{A}_0^q.$$

Consequently,

$$2^{-k_m}\left(\int_{\Delta_m}[v(y)]^{p'}dy\right)^{\frac{q}{p'}} \leqslant 2\cdot \mathbf{A}_0^q.$$

If $1 < p \leqslant q < \infty$, we obtain by Jensen's inequality with the power q/p

$$J_{1,1} \leqslant 2\cdot \mathbf{A}_0^q\left(\sum_{m\in\mathbb{Z}}\int_{\Delta_m}[f(y)]^p dy\right)^{\frac{q}{p}} \leqslant 2\cdot\mathbf{A}_0^q\,\|f\|_p^q \approx \mathscr{A}_0^q\,\|f\|_p^q, \tag{3.8}$$

where the last equivalence follows from Theorem 2.1 (i).

For $1 < q < p < \infty$ we have by (2.4), Hölder's inequality with the powers p/q, r/q and Theorem 2.2 (i)

$$J_1 \ll \left(\sum_{i\in\mathbb{Z}_{1,1}}\int_{\Delta_i}[f(y)]^p dy\right)^{\frac{q}{p}}\left(\sum_{i\in\mathbb{Z}_{1,1}}[\widetilde{W}_q(t_{i+1})]^{\frac{r}{q}}\left(\int_{\Delta_i}[v(y)]^{p'}dy\right)^{\frac{r}{p'}}\right)^{\frac{q}{r}}$$
$$\leqslant \left(\int_0^\infty [f(y)]^p dy\right)^{\frac{q}{p}}\left(\sum_{i\in\mathbb{Z}_{1,1}}\int_{\Delta_i}[W_q(t)]^{\frac{r}{q}}\, d\left(\int_{t_i}^t [v(y)]^{p'}dy\right)^{\frac{r}{p'}}\right)^{\frac{q}{r}}$$
$$\leqslant \mathbf{B}_0^q\,\|f\|_p^q \approx \mathscr{B}_0^q\,\|f\|_p^q. \tag{3.9}$$

Thus, by (3.7) – (3.9)

$$J_1 \ll \mathbf{F}^q\,\|f\|_p^q, \qquad \text{where} \qquad \mathbf{F} := \begin{cases}\mathbf{A} \text{ or } \mathscr{A}, & \text{if } 1 < p \leqslant q < \infty,\\ \mathbf{B} \text{ or } \mathscr{B}, & \text{if } 1 < q < p < \infty.\end{cases} \tag{3.10}$$

Since $W(t)$ is absolutely continuous, we can find the sequence $\{z_m\} \subset (0,\infty)$ such that

$$W(t_m) = 2^{-m}, m \in \mathbb{Z}_3 \subset \mathbb{Z}.$$

Obviously, the function $W(t)$ is non-increasing and

$$2^{-m-1} \leqslant W(t) \leqslant 2^{-m} \qquad \text{for} \qquad t \in \Delta_m := [t_m, t_{m+1}]. \tag{3.11}$$

By Hölder's inequality it holds that

$$J_2 \leqslant J_3^{\frac{1}{p'}} \left(\int_0^\infty [f(y)]^p dy \right)^{\frac{1}{p}}, \tag{3.12}$$

with

$$J_3 := \int_0^\infty [Kf(t)]^{p'(q-1)} [W_1(t)]^{p'} [v(t)]^{p'} dt,$$

where $Kf(t) := \int_0^t k(t,s) f(s) v(s) ds$. Since $Kf(t)$ is non-decreasing we have

$$\begin{aligned} J_3 &= \sum_{m \in \mathbb{Z}_3} \int_{\Delta_m} [Kf(t)]^{p'(q-1)} [W_1(t)]^{p'} [v(t)]^{p'} dt \\ &\overset{(3.11)}{\ll} \sum_{m \in \mathbb{Z}_3} 2^{-m\frac{p'}{q'}} [Kf(t_{m+1})]^{p'(q-1)} \left([W(t_m)]^{-\frac{p'}{q'}} \int_{\Delta_m} [W_1(t)]^{p'} [v(t)]^{p'} dt \right). \end{aligned} \tag{3.13}$$

Let $1 < p \leqslant q < \infty$. By Jensen's inequality and Theorem 2.1 (iv)

$$\begin{aligned} J_3 &\ll \mathscr{A}_1^{p'} \sum_{m \in \mathbb{Z}_3} \left(2^{-m} [Kf(t_{m+1})]^q \right)^{\frac{p'}{q'}} \approx \mathscr{A}_1^{p'} \sum_{m \in \mathbb{Z}_3} \left([Kf(t_{m+1})]^q \int_{\Delta_{m+1}} [w(x)]^q dx \right)^{\frac{p'}{q'}} \\ &\ll \mathscr{A}_1^{p'} \left(\sum_{m \in \mathbb{Z}_3} \int_{t_{m+1}}^{t_{m+2}} [\mathbf{K}f(x)]^q dx \right)^{\frac{p'}{q'}} \leqslant \mathscr{A}_1^{p'} \|\mathbf{K}f\|_q^{p'(q-1)} \ll \mathbf{A}_1^{p'} \|\mathbf{K}f\|_q^{p'(q-1)}. \end{aligned} \tag{3.14}$$

Now let $1 < q < p < \infty$. By applying Hölder's inequality to (3.13) with the powers q'/p' and r/p' we obtain

$$J_3 \ll \left(\sum_{m \in \mathbb{Z}_3} 2^{-m} [Kf(t_{m+1})]^q \right)^{\frac{p'}{q'}} \left(\sum_{m \in \mathbb{Z}_3} 2^{m\frac{r}{q'}} \left(\int_{\Delta_m} [W_1(t)]^{p'} [v(t)]^{p'} dt \right)^{\frac{r}{p'}} \right)^{\frac{p'}{r}}.$$

On the strength of (3.11)

$$\left(\sum_{m \in \mathbb{Z}_3} 2^{-m} [Kf(t_{m+1})]^q \right)^{\frac{p'}{q'}} \ll \left(\sum_{m \in \mathbb{Z}_3} \int_{\Delta_{m+1}} [\mathbf{K}f(x)]^q dx \right)^{\frac{p'}{q'}} \leqslant \|\mathbf{K}f\|_q^{p'(q-1)}.$$

By the same reason and Theorem 2.2 (iv)

$$\begin{aligned} &\left(\sum_{m \in \mathbb{Z}_3} 2^{m\frac{r}{q'}} \left(\int_{\Delta_m} [W_1(t)]^{p'} [v(t)]^{p'} dt \right)^{\frac{r}{p'}} \right)^{\frac{p'}{r}} \\ &\quad = \left(\sum_{m \in \mathbb{Z}_3} 2^{m\frac{r}{q'}} \int_{\Delta_m} d \left(- \left(\int_z^{t_{m+1}} [W_1(t)]^{p'} [v(t)]^{p'} dt \right)^{\frac{r}{p'}} \right) \right)^{\frac{p'}{r}} \end{aligned}$$

$$\ll \left(\sum_{m \in \mathbb{Z}_3} \int_{\Delta_m} [W(z)]^{\frac{r}{q}} d \left(- \left(\int_z^\infty [W_1(t)]^{p'} [v(t)]^{p'} dt \right)^{\frac{r}{p'}} \right) \right)^{\frac{p'}{r}} \leqslant \mathscr{B}_1^{p'} \ll \mathbf{B}_1^{p'}.$$

Now from (3.12) – (3.14) it follows

$$J_2 \ll \mathbf{F} \|f\|_p \|\mathbf{K}f\|_q^{q-1}.$$

Thus, and from (3.10) we have proved by (3.6) that

$$\|\mathbf{K}f\|_q^q \ll \mathbf{F}^q \|f\|_p^q + \mathbf{F} \|f\|_p \|\mathbf{K}f\|_q^{q-1}.$$

Therefore,

$$\left(\frac{\|\mathbf{K}\|}{\mathbf{F}} \right)^q \ll 1 + \left(\frac{\|\mathbf{K}\|}{\mathbf{F}} \right)^{q-1}.$$

From here the upper estimate $\|\mathbf{K}\| \ll \mathbf{F}$ follows evidently and now (3.4) (a), (c) and (3.5) (a), (c) are proved.

The upper estimates in (3.4) (b) and (3.5) (b) follows similarly by using Lemma 3.2 and (2.13) – (2.17).

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