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A Network Equilibrium Model with Travellers' Perception of Stochastic Travel Times

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Abstract

In this paper we consider a network whose route travel times are considered to be random variables. In this scenario travellers choose their route, uncertain of the travel time they will experience on any of the available alternative routes. The attractiveness of a given route involves evaluation of both the possible travel time outcomes, and their perceived probability of occurring. We consider a modelling framework where the perceived value and perceived probabilities of travel time outcomes are obtained via nonlinear transformations of the actual travel times and their probabilities. In this context, we present the analysis required to formulate an equilibrium condition analogous to that of User Equilibrium, wherein travellers choose the routes that maximises their *perceived value* in the face of uncertain travel times. Existence and uniqueness conditions for this equilibrium are established.

Cumulative prospect theory (CPT) provides a well supported paradigm for choices made under uncertainty, where each choice alternative presents a discrete probability distribution of a finite number of possible outcomes. Our analysis admits the particular transformations associated with CPT as a special case, and holds for a more general class of transformations and for the case of a continuous distribution of outcomes.

1 Introduction

A transportation network is a system with many sources of variability that influence network performance and hence affect travellers' experiences and their subsequent choices. Sources of variability occur in both demand and supply: variations in individuals' activity schedules and in the flexibility of their desired trips result in flows that vary from day to day, meanwhile the capacity of the road network is sporadically degraded by accidents and both planned and unplanned incidents, among other causes.

The impact of this variability on the traveller occurs through the induced variability of travel times. To represent this uncertainty, various stochastic network models have been proposed (see for example Lo and Tung 2003; Watling 2002). Such models assume demand and/or supply uncertainty and result in a continuous travel time distribution for each route. Including this travel time uncertainty into transport network modelling has received considerable attention in the literature; Noland and Small (1995), Bates et al. (2001), Noland and Polak (2002), de Jong et al. (2004), and Watling (2006) provide summaries of contributions to date. A central component of these researches concerns the way travellers decide between routes having different travel time distributions; in particular an unreliable route with low mean travel time and a reliable route that on average takes longer. Typically an additional element of route cost is associated with unreliability, relating directly to the variance of travel times (Noland and Small 1995), to the probability of being late (Watling 2006), or to the safety margin required to arrive on time (Lam et al. 2008; Lo et al. 2006). Other works concerning risk averse behaviour (Bell and Cassir 2002; Szeto et al. 2006) have ascribed a cost to the travel time uncertainty according to travellers' risk attitudes, yet still assuming that travellers accurately perceive the travel time probability distribution.

In all cases cited above, travellers continue to follow the paradigm of expected utility maximisation (EUmax) according to the actual travel time distribution. Mirchandani and Soroush (1987) propose a model that includes subjective perception of the travel time distribution. Given a continuous distribution of actual travel times, they assume different travellers each perceive the mean and variance of travel time differently, giving rise to a distribution of *perceived* travel time means and variances. On the basis of these perceived travel time distributions they compute an equilibrium assignment. The authors note that a formulation including perceived disutilities arising from nonlinear transformations of the travel time distribution remains an open problem.

In the area of choice modelling, transformations of both the distribution of outcomes' utilities and probabilities are proposed by cumulative prospect theory (CPT) (Tversky and Kahneman 1992). The tenets of CPT require specific characteristics of these transformations. CPT as a choice model has previously been applied in the transport field (for example, see Avineri and Prashker 2003; Michea and Polak 2006). Senbil and Kitamura (2004) and Jou et al (2008) applied prospect theory to commuters departure time choice and verified its explanatory power against survey data. As a model for route choice Avineri (2006) considered CPT for the case of a two-link network with stochastic travel times comprising a finite discrete set of outcomes.

This paper considers a general approach to represent travellers' route choice in the face of uncertain travel times; not only allowing for subjective perceived costs associated with the distribution of travel times, but also allowing for subjective perception of their uncertainty. A general equilibrium condition is derived for a network whose travel times arise from arbitrary probability distributions, with travellers' choosing their route based on subjective perceptions of the route travel time distributions. This formulation accommodates monotonic nonlinear transformations for the (i) perceived disutilities in terms of actual travel times and (ii) perceived risk in terms of the actual travel time probabilities. Existence and uniqueness of the resulting equilibrium is established.

In reality, and in stochastic network models, the distribution of travel times is continuous. Meanwhile travellers may in fact *perceive* a continuous or a discrete distribution of travel times; the formulation in this paper allows for both. As such the analysis presented subsumes the case of CPT with discrete outcomes, and the extension of CPT to the case of continuously distributed outcomes.

Section 2 introduces notation necessary for the formulation of network equilibrium and the two transformations of the travel time distributions. Motivated by CPT, the *perceived value* of an alternative is defined in Section 3. Section 4 then presents the equilibrium formulation and shows existence and uniqueness. Numerical examples are presented in Section 5 before the concluding remarks in Section 6.

2 Network Representation and Definitions

We represent the road network by a directed graph consisting of *N* links labelled $a = 1, 2, ..., N$; a demand vector $\mathbf{q}$, with entries q^r representing the travel demand on the r^{th} origin-destination (OD) movement; and a set of paths K^r connects the r^{th} OD movement. An assignment of flows to all paths is denoted by the vector $\mathbf{f}$, with f_k^r the flow on the k^{th} path connecting the r^{th} OD. The assignment is feasible for demand vector $\mathbf{q}$ if and only if

$$\sum_{k \in K^r} f_k^r = q^r \ \forall r \text{ and } f_k^r \geq 0 \ \forall k, r \ . \tag{1}$$

The (closed, convex) set of feasible path flows thus defined is denoted *F*. The link travel times are stochastic variables, T_a , whose means are single-valued, monotonic, continuous functions, $t_a(\mathbf{x})$ of the vector of link flows, $\mathbf{x} = \Delta \cdot \mathbf{f}$, where the link-path incidence matrix Δ has elements denoting the links *a* that are part of path *k* ordered by OD movement *r*. The joint distribution of the link travel times, $\mathbf{T} = \{T_1, \ldots, T_N\}$, is assumed known and independent of $\mathbf{x}$. The mapping between link flows and mean path costs arises from the link cost-flow relationships according to the standard link-additive model:

$$\mathbf{c}(\mathbf{f}) = \Delta^T \mathbf{t}(\Delta \cdot \mathbf{f}) \ . \tag{2}$$

The experienced path costs, $\mathbf{C}$ (upper case), are stochastic and have a joint distribution arising (via (2)) from that of the links.

The foundation text for deterministic network equilibrium is Wardrop (1952), who proposed that travellers will change route in order to reduce their travel cost, $\mathbf{c}(.)$. The consequence is that, at equilibrium, all used routes have equal minimum cost for each OD movement; this condition is called User Equilibrium (UE). The link flow vector $\mathbf{x}^* = \Delta \cdot \mathbf{f}^*$ is a solution to the UE if $\mathbf{f}^*$ satisfies (1) and

$$\mathbf{c}\left(\mathbf{f}^*\right)^T \left(\mathbf{f} - \mathbf{f}^*\right) \geq 0 \ \ \forall \mathbf{f} \in F \ . \tag{3}$$

This formulation ignores any travel time variability and presupposes that travellers consider only the deterministic mean route costs.

Since travel time is associated with disutility (an increase in travel time corresponds to a loss), for convenience we formulate route choice in terms of travellers' utilities, so that an increase in utility will correspond to a gain. Assuming that travel is motivated by utility gained at the destination, U_{dest} , the net utility gained by a traveller on route *k* is

$$U_k = U_{dest} - C_k = U_{dest} - c_k(\mathbf{x}) - \varepsilon_k \ . \tag{4}$$

The utilities $\{U_k\}$ are random variables; superscripts designating the OD have been dropped for brevity.

2.1 Perceived Value of a Travel Time Distribution

Consider a decision maker choosing from a set of available *alternatives*: in the context relevant here, this will be the set of routes connecting the decision maker's origin and destination. Each alternative in the choice set comprises a (continuous or discrete) set of possible *outcomes* with utilities u , and with $f_U(u)$ the probability density function (PDF) giving the probability that any outcome occurs. The expected utility of a given alternative,

$E[u] = \int_{-\infty}^{\infty} u \cdot f_U(u)\, du$, is a single value providing a summary of the overall attractiveness of this alternative's PDF to the decision maker.

Motivated by the approach taken in CPT we assume that the decision maker compares some such summary statistic for each alternative in the choice set in order to choose one; we call this the *perceived value* (PV). The perceived value of an alternative is evaluated by the decision maker from the actual outcomes and their probabilities of occurring via two transformations

(i) a value function $g(.)$ describes the payoff level experienced by the decision maker for each possible outcome (travel time)

(ii) a probability weighting function $w(.)$ that transforms the probability scale into the perceived likelihood of occurrence.

To make the formulation as general as possible, in particular to accommodate CPT, we include the notion of a reference point. This splits outcomes into gains (having utility greater than the reference point) and losses (conversely).

2.1.1 Value Function

In prospect theory the payoff level is considered to be a gain or loss from the reference point, u_0; the canonical example of a value function is

$$g(u) = \begin{cases} (u-u_0)^\alpha & u \geq u_0 \\ -\lambda(-u+u_0)^\beta & u < u_0 \end{cases}. \tag{5}$$

The parameters $0 < \alpha, \beta \leq 1$ give concavity for gains and convexity for losses, and determine the degree of diminishing sensitivity to gains/losses away from the reference point. The degree of loss aversion is governed by λ, with $\lambda \geq 1$ corresponding to the tenet of prospect theory that the value function is steeper for losses than for gains. Our analysis allows other value functions to be considered, with or without reference points, that do not necessarily conform to the specifications of prospect theory.

2.1.2 Probability Weighting Function

While decision makers could be assumed to act rationally with perfect information about the probabilities of all possible outcomes, instead we relax this assumption and consider travellers base their decisions on perceived probabilities. Based on experimental evidence, prospect theory proposes that small probabilities are typically over-weighted and moderate and high probabilities are under-weighted. However, this transformation could also be proposed as a synthesis of experienced and preconceived probabilities in the mind of the decision maker, or an internal representation based on full knowledge of the true PDF of outcomes.

The choice modelling literature provides examples of transformations of the outcome probabilities that have been validated experimentally (though not for the case of route choice in transport networks). In particular, Prelec (1998) derived the probability weight function

$$w(p) = \exp\left(-\left[-\log(p)\right]^\gamma\right), \tag{6}$$

from a behavioural invariance condition, where $w(p)$ and p denote the perceived probability and actual probability of an event respectively and $0 < \gamma \leq 1$. This probability weighting function is guaranteed to be monotonic for all parameter settings $0 < \gamma < 1$ (Rieger and Wang 2006).

3 Definition of Perceived Value from Cumulative Prospect Theory

For the case of stochastic travel times we require a formulation that does not exclude a continuous distribution of actual outcomes. In deriving CPT, Tversky and Kahneman (1992) defined the PV of an alternative only for the case of a finite discrete set of outcomes, though explicitly noted CPT "can be extended to continuous distributions". Several researches have made this step, applying prospect theory or CPT to continuous outcome distributions (Barberis and Huang 2005; Davies and Satchell 2004; De Giorgi et al. 2004). In particular, Levy and Levy (2004) compare a prospect theory based approach with the mean-variance rule in the case of portfolio investments, where outcomes (asset returns) are continuous variables following normal or lognormal distributions. This is extended to a model for portfolio selection under CPT in continuous-time (including continuous outcome distributions) by Jin and Zhou (2008). Rieger and Wang (2006) determine analytical properties of CPT in the case of arbitrary (i.e. including continuous) outcome distributions, and have formulated standard prospect theory in the continuous cases (Rieger and Wang 2008). Despite these works, the existing literature does not explicitly present a derivation of the continuous formulation of CPT, nor verify that the continuous formulation recovers the original discrete theory. In this section we present the necessary formulation, with the supporting analysis included in the appendix.

Consider an alternative with finite discrete outcomes, $\{u_i\}$, ordered by their utility; the greatest loss being u_{-m} and the greatest gain u_n. If these outcome have probabilities $\{p_i\}$, Tversky and Kahneman (1992) define

$$\pi_i^+ = w^+\left(p_i + \cdots + p_n\right) - w^+\left(p_{i+1} + \cdots + p_n\right) \qquad \text{for } 0 \le i < n \text{ and } \pi_n^+ = w^+\left(p_n\right) \tag{7}$$

$$\pi_i^- = w^-\left(p_{-m} + \cdots + p_i\right) - w^-\left(p_{-m} + \cdots + p_{i-1}\right) \qquad \text{for } -m < i \le 0 \text{ and } \pi_{-m}^- = w^-\left(p_{-m}\right). \tag{8}$$

Here different weighting functions are permitted for gains (w^+) and losses (w^-). According to CPT, the PV of an alternative having a finite discrete set of outcomes is defined to be

$$v = v^+ + v^- \quad \text{with} \quad v^+ = \sum_i \pi_i^+ g\left(u_i\right) \text{ and } v^- = \sum_i \pi_i^- g\left(u_i\right) \tag{9}$$

with v^+ being the contribution from those outcomes designated as gains by choice of reference point, and v^- is the contribution from losses. This original discrete formulation can be extended to give the PV for an alternative with a continuous distribution of outcomes. For an alternative with (continuous or discrete) outcomes described by the CDF $F_U\left(u\right)$, and reference point u_0, the PV is defined to be:

$$v = v^+ + v^- = \int_{u_0}^{\infty} -\frac{dw^+\left(1 - F_U\left(u\right)\right)}{du} g\left(u\right) du + \int_{-\infty}^{u_0} \frac{dw^-\left(F_U\left(u\right)\right)}{du} g\left(u\right) du . \tag{10}$$

The appendix includes the derivation of (10) and illustration that it reproduces (7)-(9) when the probability density function represents a discrete distribution of outcomes. It is also shown that the PV (10) recovers the expected utility when both $w(.)$ and $g(.)$ are the identity mappings.

The original version of prospect theory simply mapped outcome probabilities and utilities, giving the *prospect value* to be $\sum_i w\left(p_i\right) g\left(u_i\right)$. This clearly reduces to the expected utility under the identity value and decision weight mappings. Note that in (10) the argument of the weight function is the outcomes' CDF; for the continuous case, the probability of any particular outcome is zero so mapping the probabilities of 'individual' outcomes under $w(.)$, as in original prospect theory, would be meaningless.

4 Network Equilibrium Based On Perceived Value

The PV equilibrium condition analogous to UE is that, at equilibrium, all used routes have equal (maximum) perceived value, as defined in (10). With v_k the flow dependent perceived value of route k, and $\mathbf{v(f)}$ the corresponding vector of perceived values, we have that $\mathbf{f}^* \in F$ is a PV-UE if and only if

$$\mathbf{v}\left(\mathbf{f}^*\right)^T \left(\mathbf{f} - \mathbf{f}^*\right) \le 0 \;\; \forall \mathbf{f} \in F . \tag{11}$$

We assume that (i) the link costs functions, (ii) the value function, (iii) the probability weighting function, (iv) the cumulative density function of the outcomes, are all continuous and strictly monotonic. Under these assumptions, the PV is a sum of integrals of continuous functions (of link flow); the derivatives appearing in (10) are therefore integrable. The PV is a continuous function of network flow.

The path-based proofs of existence and uniqueness follow those of Watling (2006), originating from Smith (1979) and are only briefly described here.

4.1 Existence

The feasible region of path flows, F, is closed and convex, so for any vector of path flows, $\mathbf{f}$, there exists a unique nearest feasible point $\mathbf{p}\left(\mathbf{f}\right) \in F$, the projection of $\mathbf{f}$ onto F. We define a mapping $\theta : F \to F$, $\theta\left(\mathbf{f}\right) = \mathbf{p}\left(\mathbf{f} + \mathbf{v}\left(\mathbf{f}\right)\right)$, so that $\mathbf{f}$ is a PV-UE if and only if $\theta\left(\mathbf{f}\right) = \mathbf{f}$ (see Smith 1979). Since the constituent functions are continuous as stated above, so $\theta(.)$ is continuous, and by Brouwer's fixed point theorem an equilibrium solution exists.

4.2 Uniqueness

We begin by showing that the PV is a decreasing function of flow. The location (mean) of the distribution of route travel times (hence utilities) is determined by the deterministic, flow-dependent travel times The dependence of PV on the flow therefore arises via the CDF of utility. Consider the CDF $F_U(u)$, as determined by the mean and variance of its underlying PDF, $f_U(u;\mu,\sigma^2)$. The effect on $F_U(u)$ of a small increase in the mean is exactly that obtained by a small decrease in *u*.

$$\frac{\partial F_U(u)}{\partial \mu} = -\frac{\partial F_U(u)}{\partial u} = -f_U(u)$$

For a single link with mean cost function $c(x)$ and a distribution of utilities (as above) centred on $\mu = U_{dest} - c(x)$ this gives

$$\frac{\partial F_U(u,x)}{\partial x} = \frac{\partial F_U(u,x)}{\partial \mu}\frac{\partial\left[U_{dest} - c(x)\right]}{\partial x} = f_U(u)c'(x).$$

Therefore the PV is affected by the link flow as follows:

$$\frac{\partial v(x)}{\partial x} = -c'(x)\left\{\int_{u_0}^{\infty}\left.\frac{dw^+(s)}{ds}\right|_{1-F_U(u)} f_U(u)\frac{dg(u)}{du}du + \int_{-\infty}^{u_0}\left.\frac{dw^-(s)}{ds}\right|_{F_U(u)} f_U(u)\frac{dg(u)}{du}du\right\}.$$

Following the assumption that the link travel time functions are strictly increasing functions of flow, $c'(x) > 0$. If both the value function and decision weight function are also strictly increasing, all terms in the integrands are positive and hence PV is a monotonic and strictly decreasing function of flow.

Uniqueness is now established by contradiction. Given that solutions exist, consider two PV-UE path flow solutions **f** and **g** with distinct link flows $\Delta\mathbf{f} \neq \Delta\mathbf{g}$. The mean path travel times give

$$\left(\mathbf{c}(\mathbf{f})-\mathbf{c}(\mathbf{g})\right)^T(\mathbf{f}-\mathbf{g}) = \left[\Delta^T\left(\mathbf{t}(\Delta\mathbf{f})-\mathbf{t}(\Delta\mathbf{g})\right)\right]^T(\mathbf{f}-\mathbf{g}) = \left(\mathbf{t}(\Delta\mathbf{f})-\mathbf{t}(\Delta\mathbf{g})\right)^T(\Delta\mathbf{f}-\Delta\mathbf{g}) > 0.$$

This is positive by the definition of **t**(.) being strictly monotone. Above we assumed that the link travel time functions, value and decision weight functions are all strictly monotonic, hence (from above) $\mathbf{v}(.)$ is strictly monotonic decreasing:

$$\left(\mathbf{v}(\mathbf{f})-\mathbf{v}(\mathbf{g})\right)^T(\mathbf{f}-\mathbf{g}) < 0 \quad \mathbf{f},\mathbf{g} \in F, \Delta\mathbf{f} \neq \Delta\mathbf{g}. \qquad (12)$$

We can then write

$$\mathbf{v}(\mathbf{f})^T(\mathbf{f}-\mathbf{g}) = \mathbf{v}(\mathbf{g})^T(\mathbf{f}-\mathbf{g}) + \left(\mathbf{v}(\mathbf{f})-\mathbf{v}(\mathbf{g})\right)^T(\mathbf{f}-\mathbf{g}) < 0$$

The first term is non-negative since **g** is a PV-UE solution, and the second term is strictly positive by (12). Since **f** is also a PV-UE solution, by (11),

$$\mathbf{v}(\mathbf{f})^T(\mathbf{f}-\mathbf{g}) = -\mathbf{v}(\mathbf{f})^T(\mathbf{g}-\mathbf{f}) \geq 0$$

Giving a contradiction. Hence two distinct PV-UE link flow solutions cannot exist and the PV-UE solution is unique (in the link flows).

The PV-UE defined in this section is deterministic: given travel time distributions for each route, and the relevant CPT model parameters, (11) defines a vector of PV-UE path flows. However, since the travel time variances are assumed independent of the mean (and hence independent of the flow) we may assume that this variability persists and that the equilibrium predicted is one where the mean flows are given by (11) and the variability is that prescribed in the network definition.

5 Illustrative Examples

5.1 Two Link Network

Consider the two link network with travel time distributions

$$T_i \sim N\left(t_i(x_i), \sigma_i^2\right)$$

so that the utility of travelling on route *i* will be

$$U_i \sim N\left(U_{dest} - c_i\left(f_i\right), \Sigma_i^2\right),$$

where the path costs and the path utility variances Σ_i^2 derive from the constituent link cost distributions. For each OD movement, the equilibrium path flows give equal maximum PV for each used path. The equilibrium flows are the solution to the following gap-function minimisation programme (dependence on other parameters in calculating the PV omitted here for brevity):

$$\min_{\mathbf{f}} \left\{ \sum_i \max_j \left\{ v_j\left(\mathbf{f}\right) \right\} - v_i\left(\mathbf{f}\right) \right\}.$$

Consider the two link network with deterministic mean travel times depending on flow as follows

$$t_1\left(x_1\right) = 10 + \left(\frac{x_1}{10}\right)^2 \text{ and } \quad t_2\left(x_2\right) = 15 + \left(\frac{x_2}{25}\right)^2.$$

With fixed total travel demand = 100, the mean costs on the two links are uniquely determined by the flow on link 1, since the remainder will travel on link 2. We set the utility gain at the destination to be $U_{dest} = 200$. The UE solution is $f_1 = 34.45, c_1\left(f_1\right) = 21.87 = c_2\left(f_2\right)$ corresponding to an expected utility of 178.13.

With $\sigma_2 = 2\sigma_1$ we compute the PV for both links, under each feasible flow pattern for different values of the reference point. The other parameters are given as before by alpha = 0.52 = beta, lambda = 2.25, gamma = 0.74. The equilibrium flows, defined as those giving equal PV on all used routes, occur at the intersection of the surfaces in. The absolute difference in link-PV is shown in Figure 1; equilibrium flows occur at the minimum. The influence of the reference point on the equilibrium solution can be seen. Note that as the reference point becomes increasingly distant from the expected utility at UE, so the PV-UE flows return to the UE solution. In practice, the reference point can be calibrated from survey data, see e.g. Sebil and Kitamura (2004) and Jou et al (2008).

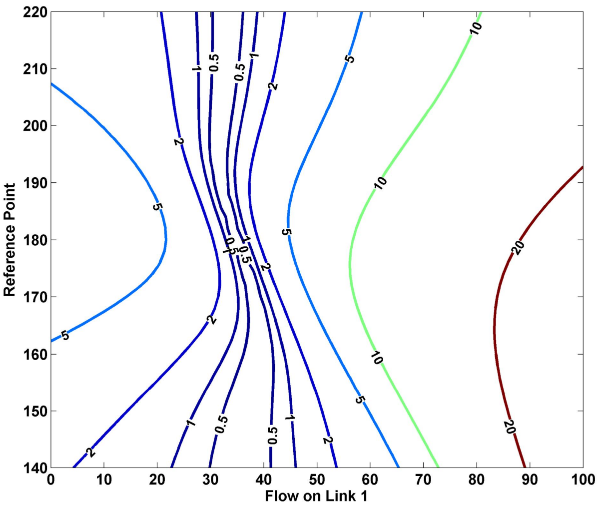

Figure 1: PV disequilibrium: │PV1-PV2│

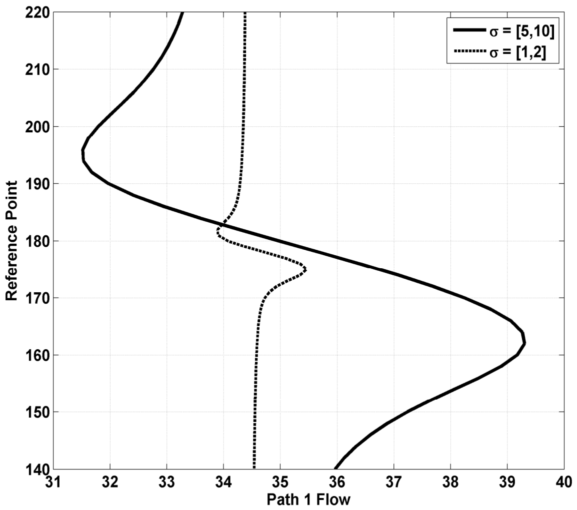

Figure 2: PV equilibrium as reference point and travel time variance change

Different route cost variances were deliberately chosen for this illustration. The PV includes the route travel time variance **(**TTV) through its definition that depends on the entire CDF of the travel time outcomes and hence the comparison of PV-UE with UE cannot be fully illustrated unless there is a choice between alternatives with different variabilities; with $\sigma_2 = \sigma_1$ we find that PV-UE reproduces the UE flows.

The above figure shows how the magnitude of TTV impacts upon the way PV-UE departs from the UE solution. Recall that for all values of the reference point the UE path 1 flow is 34.45, this is where the σ = [1,2] line intersects the x-axis. Both lines on this graph shows the link 1 flow PV-UE solution for different values of the reference point. The solid line for utility distributions having standard deviations [1,2], and the broken line for the case with standard deviations [5,10]. Increasing the TTV on each link while maintaining the same ratio of variability shows the same type of behaviour around the underlying UE solution, which PV-UE recovers as the reference point becomes more distant from the UE expected utility.

5.2 Five Link Network

Consider the five link network with two OD movements: a demand of 3 from node 1 to node 3 (OD1) and a demand of 7 from node 1 to node 4 (OD2). The path covariance matrix is assumed diagonal with elements [7.5, 2.5, 5, 7.5, 10] corresponding to paths using links [(1,3), 2, (2,5), (1,4), (1,3,5)]. The link cost functions (LCFs) are given in Figure 3.

We set the utility of arriving at the destination as [OD1, OD2] = [30,50]. The UE solution has OD costs [15.89, 29.04] and hence expected OD utilities [14.11, 20.96]. The PV-UE solution at different combinations of reference point is shown in Figure 11

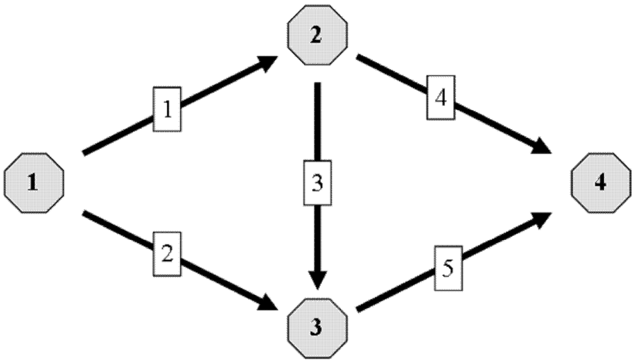

Figure 3: Five Link Network

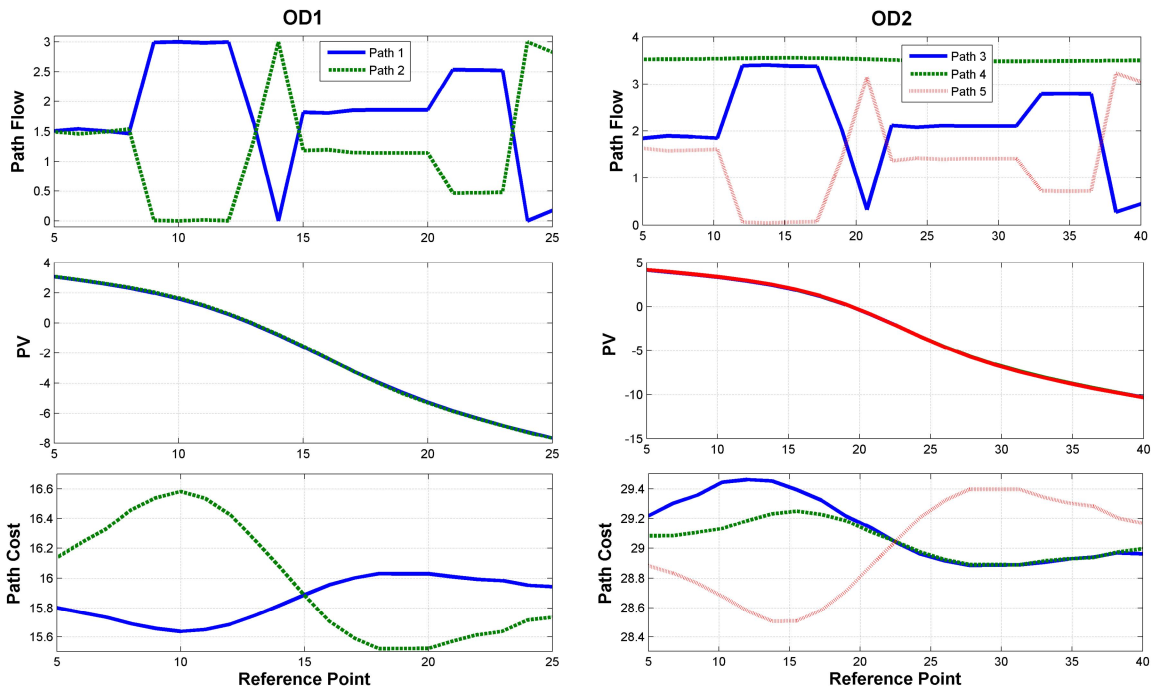

Figure 4: PV-UE for the five link network with 2 OD movements

The central row of figures shows the PV for each used route and confirms that equilibrium is attained, since the plotted lines for each route coincide (2 lines for OD1, 3 lines for OD2). The range of reference points was chosen to cover a range around the UE solution for each OD, the reference points tested spanned the range and were changed simultaneously, other combinations (e.g. large reference point for OD1, small for OD2) were not tested. Although it appears at this level of detail that all three path costs intersect at a point for OD2, on closer inspection this is not the case and its appearance is coincidental.

In the first row of figures we see the path-flow switching phenomena that is a familiar artefact in UE, due to the non-uniqueness of path flows; the equilibrium algorithm converges to an arbitrary path flow solution while satisfying the equilibrium condition, the flow conservation constraints, and corresponding to a unique (aggregate) link flow solution. For this test network, flow on paths 1 and 2 (OD1 flow) can be interchanged with OD2 flows on paths 3 and 5, while maintaining the same total link flows. Such non-unique path flows can be avoided in UE with non-additive route costs

(Gabriel and Bernstein 1997). However, the non-additive nature of the PV route choice model arises from the reference point being defined for each OD separately and hence the PVs for all paths connecting a given OD are evaluated with respect to the same (non-additive) reference point. The switching phenomena displayed above are exactly those familiar in UE analysis: a unique link flow PV-UE gives unique link costs, path costs and hence PV for each OD. Interchanging path flows can be done due to linear dependence in the path flow specification:

$$\begin{bmatrix} \Delta \\ \Psi \end{bmatrix} \cdot \mathbf{f} = \begin{bmatrix} \mathbf{x}^* \\ \mathbf{q} \end{bmatrix}$$

Here $\mathbf{x}^*$ is the unique equilibrium link flow, $\mathbf{q}$ the demand vector, Δ the link path incidence matrix and the matrix Ψ represents the demand conservation constraint. For this network, the matrix pre-multiplying $\mathbf{f}$ is of rank 4, leaving one degree of freedom that allows the path flow switching.

6 Conclusion & Discussion

In this paper we proposed cumulative prospect theory as a model for individual route choice behaviour in a network where link travel times are random variables. Since the existing literature does not provides a clear derivation of the continuous analogue of Tversky and Kahneman's original discrete formulation of cumulative prospect value, we provide a derivation and show that the continuous formulation obtained generalises both the discrete definition, and the expected utility. We then propose an equilibrium criterion based on UE and show that the PV-UE exists and is unique in the link flows under assumptions of continuity and monotonicity that are no more onerous than the corresponding requirements for UE. Finally we present numerical examples to show that the PV includes information about the variability of travel times, and we find PV-UE for two simple networks and illustrate the dependence of PV-UE on the underlying TTV. The parameters needed to compute perceived values are taken from the choice modelling literature and are not necessarily appropriate in the context of route choice.

It remains to find meaningful parameters for the value and decision weight functions that comprise the formulation of CPT. The reference point in particular is shown to have a strong influence on the equilibrium achieved. One intuitively appealing approach would be to allow determination of the reference point to be endogenous: travellers modify their reference point based on experienced (stochastic) travel times via a learning model. Conditions for an equilibrium to be attained under a stochastic process model such as this is left for future research.

It is worth making some remarks on the equilibrium formulation under this framework that would be analogous to that of SUE. At any given set of flows, for each path in the network the perceived value is deterministic. Following the case of stochastic user equilibrium (SUE) we can instead propose that there is a distribution of PVs for each path and, analogously to the case of SUE (Daganzo and Sheffi 1977), define a random variable of path PVs

$$\mathbf{V}(\mathbf{f}) = \mathbf{v}(\mathbf{f}) + \zeta \quad \text{where } \zeta \sim D(0, \Xi),$$

with ζ a zero-mean random variable, having variance-covariance matrix Ξ and distribution function $D(.)$. Assuming that travellers choose the route with the highest random PV, the flow on route k will be that proportion of the OD flow, q^r, determined by the probability that route k is more attractive than all other OD routes.

$$f_k^r = q^r \Pr\left(V_k^r(\mathbf{f}) \geq V_j^r(\mathbf{f})\ \forall j\right).$$

The equilibrium flows therefore solve the following fixed point problem:

$$\mathbf{f}^* = \mathbf{q} \cdot \mathbf{P}\left(\mathbf{V}\left(\mathbf{f}^*\right)\right),$$

where $\mathbf{P}(.)$ is the function mapping the vector of random PVs to flow proportions. As for the case of PV-UE, since the mappings composed to form the fixed point condition are all continuous, Brouwer's fixed point theorem guarantees that an equilibrium solution exists. Note that, as for SUE, flow conservation is automatically satisfied at equilibrium and is not required as an additional constraint. Uniqueness can also be established following the analysis (see Sheffi 1985) establishing this results for SUE, replacing the cost function, $\mathbf{c}(.)$ used in SUE with the PV. The necessary properties of the cost functions (continuity and monotonicity) are attributes of the PV.

However, this formulation is implausible since the stochasticity introduced in this way is not attributed to any identifiable behavioural cause; the underlying PV formulation captures subjective perception of value and probability, and introducing variation in these perceptions between individuals should obey the model structure. A more plausible explanatory model could therefore be proposed from the assumption that the reference point (or other parameters in the value function) have a distribution across the population of travellers, resulting in a continuous distribution of PVs for each route, as written above. However, due to the nonlinear transformation of utility into PV, the induced stochastic behaviour of the PV (its PDF) will not follow a standard distribution therefore making analysis difficult. A consistent model incorporating a continuous distribution of travellers is left for further research.

Appendix: Derivation of the Perceived Value for a Continuous Distribution of Outcomes

Ordered Discrete Alternatives

The perceived value of an alternative, taking into account the probabilities of all its possible outcomes is

$$v = v^+ + v^- \text{ with } v^+ = \sum_i \pi_i^+ g(u_i) \text{ and } v^- = \sum_i \pi_i^- g(u_i)$$

With (an ordered) discrete set of outcomes

Outcomes	u_{-m}	$\cdots$	u_{-1}	u_0	u_1	$\cdots$	u_n
Probability	p_{-m}	$\cdots$	p_{-1}	p_0	p_1	$\cdots$	p_n
Values	$g(u_{-m})$	$\cdots$	$g(u_{-1})$	$g(u_0)=0$	$g(u_1)$	$\cdots$	$g(u_n)$
Decision Weight	$\pi_{-m}^- = w^-(p_{-m})$	$\cdots$	π_{-1}^-	$\pi_0^- \left(=\pi_0^+\right)$	π_1^+	$\cdots$	π_n^+

Tversky and Kahneman (1992) define

$$\pi_i^+ = w^+(p_i + \cdots + p_n) - w^+(p_{i+1} + \cdots + p_n) \qquad \text{for } 0 \le i < n \text{ and } \pi_n^+ = w^+(p_n)$$

$$\pi_i^- = w^-(p_{-m} + \cdots + p_i) - w^-(p_{-m} + \cdots + p_{i-1}) \qquad \text{for } -m < i \le 0 \text{ and } \pi_{-m}^- = w^-(p_{-m}).$$

While these definitions are not necessarily consistent for $i = 0$

$$\pi_0^- = w^-(p_{-m} + \cdots + p_0) - w^-(p_{-m} + \cdots + p_{-1}) \ne w^+(p_0 + \cdots + p_n) - w^+(p_1 + \cdots + p_n) = \pi_0^+$$

the contribution of the reference outcome is zero since $g(u_0) = 0$. The arguments of the decision weight functions used here can be written in terms of the cumulative distribution function of the outcomes

$$p_i + \cdots + p_n = 1 - [p_{-m} + \cdots + p_{i-1}] = 1 - F_U(u_{i-1}),$$

$$p_{-m} + \cdots + p_i = F_U(u_i).$$

So that

$$\pi_i^+ = w^+\left(1 - F_U(u_{i-1})\right) - w^+\left(1 - F_U(u_i)\right) = -\frac{w^+\left(1 - F_U(u_i)\right) - w^+\left(1 - F_U(u_{i-1})\right)}{u_i - u_{i-1}}(u_i - u_{i-1})$$

Similarly

$$\pi_i^- = w^-\left(F_U(u_i)\right) - w^-\left(F_U(u_{i-1})\right) = \frac{w^-\left(F_U(u_i)\right) - w^-\left(F_U(u_{i-1})\right)}{u_i - u_{i-1}}(u_i - u_{i-1}).$$

Continuous Distribution of Alternatives

In the case of a continuous probability density function, begin by sampling the PDF to give an approximate discrete distribution of outcomes. If we consider a symmetric (about the reference point) and regularly spaced sampling, the probability associated with outcome u_i reflects continuous outcomes in the interval $[u_i - du/2, u_i + du/2]$, which occur with probability approximated by $f_U(u_i)\,du$:

Outcome Intervals	$\cdots$	u_{-m}	$\cdots$	u_{-1}	u_0	u_1	$\cdots$	u_n	$\cdots$
Probabilities	$\cdots$	$f_U(u_{-m})\ du$	$\cdots$	$f_U(u_{-1})\ du$	$f_U(u_0)\ du$	$f_U(u_1)\ du$	$\cdots$	$f_U(u_n)\ du$	$\cdots$

Values	$\cdots$	$g(u_{-m})$	$\cdots$	$g(u_{-1})$	$g(u_0)=0$	$g(u_1)$	$\cdots$	$g(u_n)$	$\cdots$
Decision Weight	$\cdots$	π_{-m}^{-}	$\cdots$	π_{-1}^{-}	π_0	π_1^{+}	$\cdots$	π_n^{+}	$\cdots$

Note that $\pi_0 = \pi_0^- = \pi_0^+$. Writing

$$p_i + \cdots + p_n + \cdots = 1 - [\cdots + p_{i-1}] = 1 - \sum_{i=-\infty}^{i-1} f_U(u_j)\,du \approx 1 - \int_{-\infty}^{u_{i-1}} f_U(w)\,dw = 1 - F_U(u_{i-1}),$$

where the discrepancy between the sum and integral vanishes in the limit $du \to 0$. We find that

$$\pi_i^+ = -\left[w^+(1-F_U(u_i)) - w^+(1-F_U(u_{i-1}))\right] = -\frac{w^+(1-F_U(u_i)) - w^+(1-F_U(u_{i-1}))}{du}du$$

Hence we define the function

$$\pi^+(u) := -\frac{dw^+(1-F_U(u))}{du}du. \tag{13}$$

Note that the negative sign is appropriate since it gives a positive decision weight (event probabilities are mapped to positive decision weights)

$$\pi^+(u) = -\frac{dw^+(1-F_U(u))}{du}du = -\left.\frac{dw^+(s)}{ds}\right|_{s=1-F_U(u)}\left(-\frac{dF_U(u)}{du}\right)du = \left.\frac{dw^+(s)}{ds}\right|_{s=1-F_U(u)} f_U(u)\,du \geq 0$$

Similarly

$$\pi_i^- = w^-(F_U(u_i)) - w^-(F_U(u_{i-1})) = \frac{w^-(F_U(u_i)) - w^-(F_U(u_{i-1}))}{du}du \tag{14}$$

and we define the function

$$\pi^-(u) := \frac{dw^-(F_U(u))}{du}du. \tag{15}$$

Again this gives decision weights that are positive. Hence the perceived value for the gains, with arbitrary reference point u_0, is

$$v^+ = \int_{u_0}^{\infty} -\frac{dw^+(1-F_U(u))}{du}\cdot g(u)\,du = \int_{u_0}^{\infty} \left.\frac{dw^+(s)}{ds}\right|_{1-F_U(u)} f_U(u)\cdot g(u)\,du \tag{16}$$

Similarly, for the losses

$$v^- = \int_{-\infty}^{u_0} \frac{dw^-(F_U(u))}{du}\cdot g(u)\,du. \tag{17}$$

The perceived value for the distribution of utilities, $f_U(u)$, with reference point u_0 is then:

$$v = v^+ + v^- = \int_{u_0}^{\infty} -\frac{dw^+(1-F_U(u))}{du}g(u)du + \int_{-\infty}^{u_0}\frac{dw^-(F_U(u))}{du}g(u)du \tag{18}$$

Notice that when both w^+, w^- and g are the identity mappings, integrating by parts gives

$$v = \int_{u_0}^{\infty} 1 - F_U(u)du - \int_{-\infty}^{u_0} F_U(u)du = \int_{-\infty}^{\infty} u\, f_T(u)du - u_0$$

the expected value of the r.v. U (when the reference point is zero).

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