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On the existence and uniqueness of first best tolls in networks with multiple user classes and elastic demand

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Abstract. System optimal or first best pricing is examined in networks with multiple user classes and elastic demand, where different user classes have a different average value of VOT (value of time). Different flows (and first best tolls) are obtained depending on whether the system optimal characterisation is in units of generalised time or money. The standard first best tolls for time unit system optimum are unsatisfactory, due to the fact that link tolls are differentiated across users. The standard first best tolls for the money unit system optimum may seem to be practicable, but the objective function of the money unit system optimum is nonconvex, leading to possible multiple optima (and nonunique first best tolls). Since these standard first best tolls are unsatisfactory, we look to finding common money tolls which drive user equilibrium flows to time unit system optimal flows. Such tolls are known to exist in the fixed demand case, but we prove that such tolls do not exist in the elastic demand case. Although common money tolls do not exist which drive the solution to the *exact* time system optimal flows, tolls do exist which can push the system close to time system optimal flows.

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1. Introduction

As more road pricing schemes become reality in places such as London, there is increasing interest in how to design efficient schemes. Part of the design process involves a comparison of benefits against a benchmark based on system optimal or first best pricing, whereby all links in the network may be tolled. In the single user class case, the system optimal or first best tolls are easily calculated and the aggregate link flows are unique. However, there has been some discussion in the literature about the formulation of the system optimum with multiple user classes used to represent heterogeneity in value of time. In particular for system optimal formulations, it is found that differing flows are obtained depending on whether the system is described in time units or in money units. This result is well known in the fixed demand case (Yang and Huang, 2004; Engelson and Lindberg, 2006).

It has long been recognised that one of the major deficiencies highlighted in the original traffic assignment formulation is the homogeneous description of the users in the system. This problem was originally addressed by Dafermos (1971, 1972) and Netter (1972). There are two major ways of introducing heterogeneity into an assignment model. The first is used to represent multiple vehicle types, where class distinctions are based on the physical characteristics of the vehicles in the system; the second represents multiple user classes (MUC) where the distinction between different groups is based on their perception of the value of time (VOT). Furthermore, there are two approaches to dealing with variations in values of time. The first uses several discrete classes of users, each one with a VOT belonging to some interval (Dafermos (1972); Van Vliet *et al.* (1986); Yang and Huang (2004, 2005); Engelson and Lindberg (2006)); the second approach is to assume a continuously distributed VOT across the users (Dial, 1999a,b; Leurent, 1993; Mayet and Hansen, 2000).

In this paper, we consider the multiple user class distinction with an average VOT for each UC, since most large assignment models used in the design of road pricing schemes still use this approach to represent segmentation of demand by income and/or trip purpose. We extend the work by Yang and Huang (2004) and Yang and Huang (2005): we look at the properties of the system optimal solution (and first best tolls) formulated with generalised time and generalised money, but with elastic demand, and discuss the implications for use as a benchmark. We argue that the time system optimal solution - despite having desirable properties - should not be used, as the resulting common time tolls imply differentiated money tolls which could not be applied in practice. We then show that the money system optimal objective function is non-convex, and as such, there may be more than one solution, and finding the global optimum cannot be guaranteed. Given this, we consider an alternative benchmark based on common money tolls which result in the time system optimal solution. In the fixed demand case, a method has been provided to find these tolls (Yang and Huang, 2004). In the elastic demand case, not only does the method provided for the fixed demand case not give the required tolls, but we prove that it is *impossible* to find common money tolls which result in time system optimum. This then leads us to a further possible benchmark solution whereby we seek common money tolls which drive the flows as close as possible to the time system optimal solution (although without actually achieving it).

Section 2 sets out the MUC problem, considering standard first best tolling and the properties of the user equilibrium (UE) and system optimal (SO) formulations in both generalised time and money units. Section 3 discusses the alternative benchmarks, and shows that, in the elastic demand case, common money tolls do not exist which drive UE flows to the time system optimal solution. Section 4 draws conclusions and sets out directions for future research.

2. The Multiple User Class Network Equilibrium Problem

Let $G = (N, A)$ be a graph, where N is the set of nodes, and A the set of arcs. This graph can be taken to represent a transportation network where the nodes are intersections and the arcs are links between intersections. On this network, there will be a set of origin-destination (OD) pairs W and a set of paths P which connect OD pairs. The network is used by different user classes $k \in K$. The cost, c_a^k, of travelling on a particular link, a, for a particular user class k is given as a sum of travel time ($t_a(v_a)$) and a monetary cost or toll τ_a. The travel time is a function of the total flow (v_a) on the link a, and is assumed to be strictly monotonically increasing. The demand q_w from an origin to a destination is a variable, and is a function of the cost between an origin and destination with different demand functions for different user classes, and q_w^k refers to the demand for OD pair w and user class k. The demand function is assumed to be invertible, and the inverse demand function assumed to be strictly monotonically decreasing. The following constraints apply over the system:

$$\sum_p f_{pw}^k = q_w^k \qquad \forall k \in K, w \in W \tag{1a}$$

$$f_{pw}^k \geq 0 \qquad \forall k \in K, p \in P, w \in W \tag{1b}$$

$$q_w^k \geq 0 \qquad \forall k \in K, w \in W \tag{1c}$$

with definitional constraints $v_a^k = \sum_w \sum_p f_{pw}^k \delta_{ap}^w$ and $\sum_k v_a^k = v_a$. Here f_{pw}^k is the path flow on path p between OD pair w for user class (UC) k, v_a^k is the flow on link a of user class k and δ_{ap}^w is an indicator equal to 1 if link a is on path p between OD pair w, and zero otherwise. User classes differ by their value of time (VOT), β_k (with $\beta_k > 0 \; \forall k \in K$). Equations (1a)-(1c), can be written more succinctly in matrix form as $\Omega = \{(\mathbf{f}, \mathbf{q}) : B\mathbf{f} = \mathbf{q}, \mathbf{f} \geq 0, \mathbf{q} \geq 0\}$, with $\mathbf{f}$ the vector of path flows, B the OD pair path incidence matrix and $\mathbf{q}$ the vector of demand. A pair $(\mathbf{f}, \mathbf{q})$ will be said to be feasible if $(\mathbf{f}, \mathbf{q}) \in \Omega$, and the definitional constraints apply.

The cost of travelling on link a is given by:

$$c_a^{k\,\text{time}} = t_a(v_a) + \beta_k^{-1} \tau_a \tag{2}$$

Note that the costs from equation (2) are in time units but with tolls (τ_a on link a) in money units. The tolls are common money tolls across all users on each link. Note that since the users differ only by their VOT, a common toll in time units would result in the same link cost for all user classes, rendering the MUC distinction redundant.

The same cost function (2) written in money units gives:

$$c_a^{k\,\text{money}} = \beta_k t_a(v_a) + \tau_a \tag{3}$$

These generalised cost functions give the mechanism by which different user classes will make different trade-offs between time and money: a class with a high VOT will not be as affected by a monetary toll as a low VOT group. The MUC user equilibrium (UE) (with tolls) is also known as bicriteria/multicriteria UE (Nagurney, 2002; Dial, 1999a,b) or cost versus time UE (Leurent, 1993).

Flows satisfying the multiple user class user equilibrium (UE) with elastic demand (ED) can be obtained by solving the following optimisation:

$$\min_{\mathbf{f},\mathbf{q}} Z = \sum_a \int_0^{v_a} t_a(x)dx + \sum_a \sum_k \beta_k^{-1} \tau_a^k v_a^k - \sum_w \sum_k \int_0^{q_w^k} g_w^k(\omega)d\omega \quad (4)$$
$$\text{subject to } (\mathbf{f},\mathbf{q}) \in \Omega$$

where $g_w^k(q)$ is the inverse demand function for UC k on OD pair w, and is in time units (although can be converted into money units by multiplying by the user class VOT). The first order conditions of the minimisation program (4) give the MUC ED UE conditions (in time units), where the analysis follows that in the fixed demand (FD) case (Yang and Huang, 2004):

$$f_{pw}^k \left(\sum_a \left(t_a(v_a) + \frac{\tau_a}{\beta_k} \right) \delta_{ap}^w - \mu_w^k \right) = 0 \qquad \forall k \in K, p \in P, w \in W \quad (5a)$$

$$f_{pw}^k \geq 0 \qquad \forall k \in K, p \in P, w \in W \quad (5b)$$

$$\sum_a \left(t_a(v_a) + \frac{\tau_a}{\beta_k} \right) \delta_{ap}^w - \mu_w^k \geq 0 \qquad \forall k \in K, p \in P, w \in W \quad (5c)$$

$$q_w^k(\mu_w^k - g_w^k(q_w^k)) = 0 \qquad \forall k \in K, w \in W \quad (5d)$$

$$q_w^k \geq 0 \qquad \forall k \in K, w \in W \quad (5e)$$

$$\mu_w^k - g_w^k(q_w^k) \geq 0 \qquad \forall k \in K, w \in W \quad (5f)$$

Multiplying equations (5) by β_k, gives the MUC ED UE conditions in money units:

$$f_{pw}^k \left(\sum_a ((\beta_k t_a(v_a) + \tau_a))\delta_{ap}^w - \mu_w^{k\prime} \right) = 0 \qquad \forall k \in K, p \in P, w \in W \quad (6a)$$

$$f_{pw}^k \geq 0 \qquad \forall k \in K, p \in P, w \in W \quad (6b)$$

$$\sum_a ((\beta_k t_a(v_a) + \tau_a))\delta_{ap}^w - \mu_w^{k\prime} \geq 0 \qquad \forall k \in K, p \in P, w \in W \quad (6c)$$

$$q_w^k(\mu_w^{k\prime} - \beta_k g_w^k(q_w^k)) = 0 \qquad \forall k \in K, w \in W \quad (6d)$$

$$q_w^k \geq 0 \qquad \forall k \in K, w \in W \quad (6e)$$

$$\mu_w^{k\prime} - \beta_k g_w^k(q_w^k) \geq 0 \qquad \forall k \in K, w \in W \quad (6f)$$

Therefore (4) gives the MUC ED UE conditions in both time and money units.

We now consider the system optimal (SO) solutions in time and money units. In the case where demand is elastic, the objective of the SO problem is to maximise economic benefit (EB) (or minimise negative EB) where EB = User Benefit - Social Cost. Time system optimal (TSO) flows are therefore solutions to

$$\min_{\mathbf{f},\mathbf{q}} Z^{\text{time}} = \sum_a v_a t_a(v_a) - \sum_w \sum_k \int_0^{q_w^k} g_w^k(\omega)d\omega \quad (7)$$
$$\text{subject to } (\mathbf{f},\mathbf{q}) \in \Omega$$

Similarly, money system optimal (MSO) flows are solutions to

$$\min_{\mathbf{f},\mathbf{q}} Z^{\text{money}} = \sum_a \sum_k \beta_k v_a^k t_a(v_a) - \sum_w \sum_k \int_0^{q_w^k} \beta_k g_w^k(\omega) d\omega \tag{8}$$
$$\text{subject to } (\mathbf{f},\mathbf{q}) \in \Omega$$

TSO (7) maximises the EB measured in time units, whereas MSO (8) maximises EB in money units. Although this appears to be an innocuous distinction, these two formulations give rise to different solution flows.

Practitioners may wish to use system optimal flows as benchmarks against which to compare possible tolling schemes: in the single user class case, the distinction between time and money is irrelevant; both TSO and MSO formulations give the same solutions. In the MUC case, however, it is crucial to know whether analysis is to be done in time units or in money units, since different flows and hence different benchmark figures will be obtained in each case. The next section considers these issues and the implications for finding the associated system optimal and first best tolls.

2.1. **Standard First Best Tolling.** In this paper a set of tolls is defined to be first best if the UE flows under this tolling regime coincide with the SO flows. A first best toll can easily be derived from the first order optimality conditions. Yang and Huang (2004) show this in the fixed demand (FD) case, and their results hold under elastic demand. We will refer to this as the *standard first best toll*. If SO is in time units (equation (7)), the standard first best toll τ_a for user class k on link a is:

$$\tau_a^k = \beta_k \sum_k v_a^k \frac{dt_a(v_a)}{dv_a} = \beta_k v_a \frac{dt_a(v_a)}{dv_a} \tag{9}$$

Note that this standard first best toll (for TSO) is different for each user class, since β_k is different for each class. The VOT is only an inferred quantity and is not known for any specific user or class of users, and so such tolls could not be applied in practice. Furthermore, as will be shown in section 3, no *common* money toll exists that can drive UE to TSO in the elastic demand case.

Similar analysis in the MSO case (8) gives a common toll in money units:

$$\tau_a = \sum_k \beta_k v_a^k \frac{dt_a(v_a)}{dv_a} \tag{10}$$

This would suggest that the MSO with common money tolls would be a useful and practical benchmark. However, as we will show in the next section, the MSO solution and thus the tolls from equation (10) are not necessarily unique.

2.2. **Uniqueness of UE and SO flows with MUC.** In general the UE MUC link flows are not unique (Dafermos, 1972; Netter, 1972; Van Vliet *et al.*, 1986). The derivation of the uniqueness results has been relegated to the appendix. These results essentially follow the fixed demand case (Engelson and Lindberg, 2006; Yang and Huang, 2005) and a summary of uniqueness results is given in table 1.

The analysis for the MSO optimisation shows that the objective function is nonconvex with respect to individual user class link flows (this is shown in detail in the appendix), so there

TABLE 1. Summary of uniqueness results for MUC ED flows.

	UE	SO
Time	A non-unique set of user class link flows specified by the **unique aggregate link flow** and **unique demand**	A non-unique set of user class link flows specified by the **unique aggregate link flow** and **unique demand** as long as $t_a''(v_a) \geq 0$
Money	Same as UE time	**Unique aggregate link flow** as long as $t_a''(v_a) \geq 0$ and **unique demand** but possible **multiple optima** for individual user class link flows

may be multiple solutions for these individual user class link flows. However, for the MSO, the *aggregate* link flow and user class demand are both unique. Although these parameters are unique at MSO, it is impossible to find them without first calculating the individual user class link flow. Since these latter are not necessarily unique, the uniqueness of the aggregate link flow and user class demand can not be used in any practical way.

It is noteworthy that the unique optimal aggregate TSO link flows are not equal to the unique optimal aggregate MSO link flows. This follows from the first order KKT conditions. It may seem trivial to state that two different optimisation problems do not have the same solution, but it is important to highlight this point, particularly when in practice the choice of units may appear to be arbitrary. In the single user class case, there will be no difference whether the aim is to maximise economic benefit in time or in money units.

2.3. **An Example.** To take a simple example, consider a two link network connecting one OD pair with two user classes. Assume the cost functions are given as:

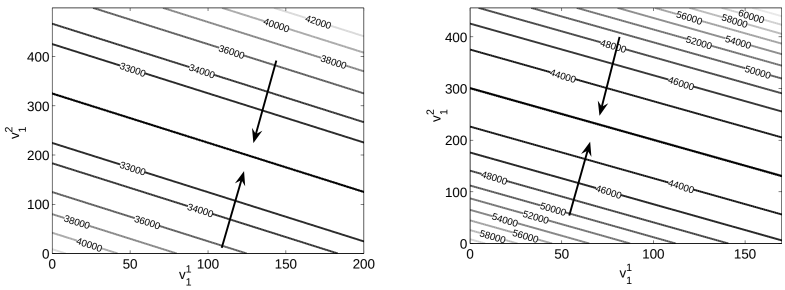
$$t_1 = 15 + 0.1v_1 \quad \text{and} \quad t_2 = 10 + 0.1v_2$$

Assume there are two user classes: the first with VOT of 3.0 (in pence per minute (PPM)), and the second with VOT of 1.5 PPM and inverse demand functions:

$$D_1{}^{-1} = 47.5\left(\frac{q^1}{200}\right)^{-\frac{1}{0.35}} \qquad D_2{}^{-1} = 47.5\left(\frac{q^2}{500}\right)^{-\frac{1}{0.2}}$$

where the elasticity for user class 1 is -0.35, and that for user class 2 is -0.2. Possible UE flows are: $v_1^1 = 12.5, v_1^2 = 312.5, v_2^1 = 187.5$ and $v_2^2 = 187.5$ (or, since the solution is not unique: $v_1^1 = 162.5, v_1^2 = 162.5, v_2^1 = 37.5, v_2^2 = 337.5$). Note that the aggregate link flows, and the UC specific demands are unique, giving $v_1 = 325$ and $v_2 = 375$ and $q^1 = 200, q^2 = 500$ (the subscripts refer to links, and the superscripts to user classes). A contour plot of the objective function for varying flows on link 1 is given in Figure 1(a). This plot shows the value of the objective function over the possible range for v_1^1 and v_1^2. The plot shows that there is a 'basin' of solutions across the plot, where the objective function (given by negative economic benefit) finds its minimum. It shows the line along which the UC specific link flow solutions lie, and thus that there are infinitely many solutions.

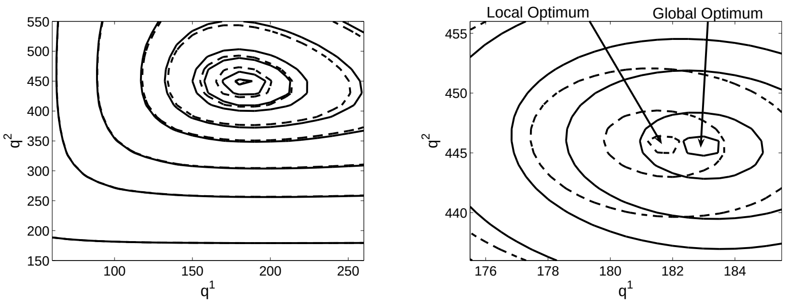
In the TSO case, there are multiple individual user class flow solutions, whereas the demands and aggregate link flows are unique, $q^1 = 170.33, q^2 = 456.17, v_1 = 300.75$ and $v_2 = 325.75$. A



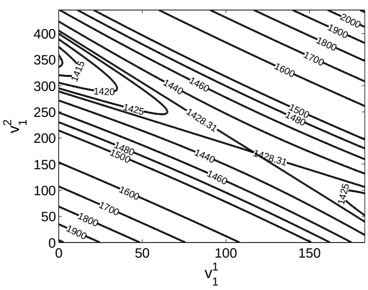
(a) Objective Function of ED UE in time over v_1^1 and v_1^2.
(b) Objective Function of ED TSO over v_1^1 and v_1^2.

FIGURE 1. Objective Functions of TSO and UE.

contour plot of the objective function (for flows on link 1) is given in Figure 1(b). This shows a similar picture as in Figure 1(a).



(a) Contour map of objective function over UC demands. Dashed line shows local min., and continuous line the global min.
(b) Contour map of objective function for different UC demands showing two local optima.



(c) Contour Plot of ED Money case over v_1^1 and v_1^2 for optimal FD slice with $q^1 = 182.92$ and $q^2 = 445.6$.

FIGURE 2. Contour plots for MSO.

As discussed in the previous section, the MSO case is different. It is more difficult to provide a visual representation in this case, as it is difficult to ensure that we have arrived at the optimal solution (due to the nonconvexity of the objective function). Using different initial conditions for the optimisation, two solutions may be found. Using two different starting points for the algorithm, we can track these two optima under various fixed demands for each UC. Figure 2(a) shows this done over a large area: the non-dashed contours track the global optimal solution, and the dashed ones the local optimal solution. Although it looks on this scale that there is only one optimum, we can see when we zoom in on the relevant area (figure 2(b)), that there are in fact two optima: one global and one local. Figure 2(c) then shows the contour plot of the objective function at the fixed demand slice for the global optimal solution indicated in figure 2(b). The solution flows are given in table 2 for both the local and global minima. Note that the first best tolls (given by equation (10)) and aggregate link flows differ considerably between the local and global optimal solutions.

TABLE 2. MSO flows for global and local optima. The objective function value at the solution for global (local) minimum is 1411.3 (1422.7).

Optimum	v_1^1	v_1^2	v_2^1	v_2^2	v_1	v_2	q^1	q^2	τ_1	τ_2
Global	0	347.49	182.92	98.11	347.49	281.03	182.92	445.60	0.87	1.16
Local	181.72	74.09	0	371.67	255.81	371.67	181.72	445.76	1.09	0.93

3. NON-EXISTENCE OF A COMMON MONEY TOLL FOR TSO

With the first best toll defined as any toll set that drives UE to SO, then there may be more than one first best toll (Bergendorff *et al.*, 1997; Hearn and Yildirim, 2002). First best tolls provide useful benchmark figures for the best possible welfare gain available, even if they cannot be implemented in practice.

We have seen that there are two system optimal characterisations based on whether the objective function is measured in generalised time or money. We have argued that the TSO solution, although theoretically appealing, does not transfer well to practice since the common time tolls imply different money tolls per user class on each link. We have also shown that the MSO, despite having the desired property of common money tolls per link, has a nonconvex objective function implying that a global optimum cannot be guaranteed. In this section, we investigate an alternative benchmark based on trying to drive flows to the TSO solution but with a common money toll on each link. This builds on the work by Yang and Huang (2004) for the fixed demand case.

3.1. Time SO with common money tolls.

In the preceding section, we argued that the standard first best tolls do not provide a useful benchmark. An obvious step is to try to find common money tolls that will drive UE to a TSO. This has been done in the fixed demand (FD) case for both UE (Yang and Huang, 2004; Yin and Yang, 2004), and mixed equilibrium (Zhang *et al.*, 2008; Yang and Zhang, 2008), but the extension to the elastic demand (ED) case is not straightforward. We now show that in a general network, it is *impossible* to derive common money tolls that will push UE to TSO. Although it has been noted that there are problems with finding common money tolls in the elastic demand case (Karakostas and Kolliopoulos, 2006), the proof of the nonexistence of these tolls is new.

3.1.1. *Common Money Tolls in a General Network.*

It is perhaps useful to return to the look at the FD case before venturing into the ED case, to show the differences that arise when ED is introduced. Looking at the toll set in the FD case, we get conditions so that (Yang and Huang, 2004):

$$f_{pw}^{k\ *}((\sum_a (t_a(v_a^*) + \frac{\tau_a}{\beta_k}))\delta_{ap}^w - \mu_w^k) = 0 \tag{11a}$$

$$f_{pw}^{k\ *} \geq 0 \tag{11b}$$

$$\sum_a (t_a(v_a^*) + \frac{\tau_a}{\beta_k})\delta_{ap}^w - \mu_w^k \geq 0 \tag{11c}$$

where the asterisk indicates that the flows are at TSO. Note that in the FD case, the minimum cost parameters, μ_w^k are unconstrained. In the ED case, however, there are extra conditions to be added, so that along with constraints (11a-11c) the following also hold:

$$q_w^{k\ *}(\mu_w^k - g_w^k(q_w^{k\ *})) = 0 \tag{12a}$$

$$q_w^{k\ *} \geq 0 \tag{12b}$$

$$(\mu_w^k - g_w^k(q_w^{k\ *})) \geq 0 \tag{12c}$$

Conditions (12a-12c) mean that the minimum OD cost for a particular user class (μ_w^k) will be equal to the inverse demand function for that user class evaluated at the TSO solution, as long as that user class travels on the OD pair.

At the TSO solution (v^*, q^*), the aggregate link flow and the user class specific demands are unique; this means that the minimum OD cost for a user class must also be unique. Not only this, but if we want equations (11a) - (12c) to hold (so we want to find tolls τ that push UE to TSO flows), then the minimum OD costs for two different user classes on the same OD will be equal if the user classes travel on the same path. This is shown in the following lemma.

Lemma 1. *If equations (11a) - (12c) hold at TSO flow (v^*, q^*), and if some user classes travel on OD pair w (so $q_w^k > 0$ for some $k \in K$), then the minimum OD cost, μ_w is equal for all of these user classes if they travel on the same path in the network, and this minimum cost is unique.*

Proof. Since the flows are at UE, it follows that if $q_w^{k\ *} > 0$ then $g_w^k(q_w^{k\ *}) = \mu_w^k$, by the complementarity condition. Since $q_w^{k\ *}$ is unique at time SO flow, and the inverse demand function is strictly monotonic, then $g_w^k(q_w^{k\ *}) = \mu_w^k$ is unique also.

Now for the second part of the lemma, to show that $\mu_w^k = \mu_w^j$ for any two user classes travelling on the same path. Assume for contradiction that $\mu_w^k \neq \mu_w^j$ when user classes k and j travel on path p. Notice that if this holds, then at SO flow:

$$g_w^k = \mu_w^k = \sum_a (t_a(v^*) + v_a^* t_a{}'(v^*))\delta_{ap}^w \neq \sum_a (t_a(v^*) + v_a^* t_a{}'(v^*))\delta_{ap}^w = \mu_w^j = g_w^j$$

But of course this gives rise to a contradiction, so then $\mu_w^k = \mu_w^j = \mu_w$ for any two user classes travelling on path p on OD pair w. □

The fact that the minimum OD cost will be the same for all user classes at TSO flows means that only one user class can travel on any path in the network if the tolled UE is to hold. This is shown in the following lemma:

Lemma 2. *If equations (11a) - (12c) hold at TSO flow (v^*, q^*) then there is only one user class using any particular path in the network. All other user classes must have zero flow on that path.*

Proof. Assume for contradiction that there are two user classes (k and j) with non-zero flow on a path p between OD pair w. This means that (by the complementarity conditions):

$$\sum_a (t_a(v_a^*) + \frac{\tau_a}{\beta_k})\delta_{ap}^w = \mu_w^k$$

$$\sum_a (t_a(v_a^*) + \frac{\tau_a}{\beta_j})\delta_{ap}^w = \mu_w^j$$

These equations can be rewritten as:

$$\sum_a \frac{\tau_a}{\beta_k}\delta_{ap}^w = \mu_w^k - \sum_a t_a(v_a^*)\delta_{ap}^w$$

$$\sum_a \frac{\tau_a}{\beta_j}\delta_{ap}^w = \mu_w^j - \sum_a t_a(v_a^*)\delta_{ap}^w$$

As shown in lemma (1), if user classes k and j travel on the same path (at TSO flow), it holds that $\mu_w^k = \mu_w^j = \mu_w$ and this minimum OD cost is a unique constant. Since v_a^* is unique at SO flow, and the travel time function is strictly monotonic, then $\sum_a t_a(v_a^*)\delta_{ap}^w$ is unique also. Now let $K = \mu_w - \sum_a t_a(v_a^*)\delta_{ap}^w$ where K is a unique constant defined by the TSO flows. This implies that

$$\sum_a \tau_a \delta_{ap}^w = \beta_k K = \beta_j K$$

Which will only be possible if $\beta_k = \beta_j$, so the users belong to the same class. This contradiction shows that only one user class can travel on a particular path, and the proof is complete. □

The question of course arises, as to which user class travels on the paths in the network. First we will show that the minimum path cost on any OD pair will be equal for all user classes, and this will enable us to show that only the highest VOT class can travel on a path in the network.

Lemma 3. *If equations (11a) - (12c) hold, then the minimum OD cost will be equal for all user classes on a particular OD pair.*

Proof. Assume for contradiction that there are two user classes k and j with differing OD costs on OD pair w: $\mu_w^k \neq \mu_w^j$. From lemma 2, we know that these two user classes must travel on different paths between the origin and destination. Let user class k travel on path p, and user class j travel on path p'. At TSO flows, if user class k travels on path p, it follows, by the complementarity condition that:

$$\sum_a (t_a(v_a^*) + v_a^* t_a{}'(v_a^*))\delta_{ap}^w = \mu_w^k \tag{13}$$

Similarly, since user class j travels on path p':

$$\sum_a (t_a(v_a^*) + v_a^* t_a{}'(v_a^*))\delta_{ap'}^w = \mu_w^j \tag{14}$$

From lemma 2, only one user class can travel on any path thus, by the complementarity conditions, the cost of travel for user class k on path p' must be greater than the minimum OD cost for user class k:

$$\sum_a (t_a(v_a^*) + v_a^* t_a{}'(v_a^*))\delta_{ap'}^w \geq \mu_w^k \tag{15}$$

Similarly, user class j does not travel on path p, so:

$$\sum_a (t_a(v_a^*) + v_a^* t_a{}'(v_a^*))\delta_{ap}^w \geq \mu_w^j \tag{16}$$

From equations (13) and (16) we see that $\mu_w^k \geq \mu_w^j$, and from equations (14) and (15) we know that $\mu_w^j \geq \mu_w^k$. This means that $\mu_w^k = \mu_w^j$, against our assumption. Thus the minimum OD cost for all user classes on an OD is equal for all user classes. □

Lemma 4. *It is only possible for the user class with the highest VOT to travel on any path in the network if MUC UE conditions prevail at SO flows (so equations (11a) - (12c) hold).*

Proof. For contradiction, assume that a user class k which does not have the highest VOT travels on some path p in the network. So $\beta_j > \beta_k$, and $f_{pw}^k > 0$. From lemma (3), we know that the minimum OD cost is the same for all user classes on an OD, so $\mu_w^k = \mu_w^j = \mu_w$. By the complementarity constraint at UE:

$$\sum_a (t_a(v_a{}^*) + \frac{\tau_a}{\beta_k})\delta_{ap}^w = \mu_w$$

remembering of course, that at TSO flow, μ_w and $\sum_a t_a(v_a{}^*)\delta_{ap}^w$ are unique. We know from lemma (2) that if user class k is travelling on the path, then user class j can not travel on the path, thus path p for user class j will be unused, so it must hold that (again by the complementarity condition):

$$\sum_a (t_a(v_a{}^*) + \frac{\tau_a}{\beta_j})\delta_{ap}^w \geq \mu_w \tag{17}$$

Now, we know that

$$\sum_a \frac{\tau_a}{\beta_k}\delta_{ap}^w = \mu_w - \sum_a t_a(v_a)\delta_{ap}^w$$

(because user class k is using the path). But because we assumed that $\beta_k < \beta_j$ then it holds that:

$$\sum_a \frac{\tau_a}{\beta_j}\delta_{ap}^w < \sum_a \frac{\tau_a}{\beta_k}\delta_{ap}^w = \mu_w - \sum_a t_a(v_a{}^*)\delta_{ap}^w$$

But this implies that:

$$\sum_a (t_a(v_a{}^*) + \frac{\tau_a}{\beta_j})\delta_{ap}^w < \mu_w$$

And this provides a contradiction to the assumption that a class which does not have the highest VOT can travel on any path in the network. This concludes the proof that only the highest VOT group can travel on any path in the network. □

This then leads us to our main proposition, that there are no common money tolls that will push UE flows to TSO flows in a general network.

Proposition 1. *Consider a general network with separable link cost functions given in generalised time units by equation (2), and with different user classes identical except for their values of time. The travel time functions are assumed to be strictly monotonically increasing, and the demand functions strictly monotonically decreasing. In this case the TSO optimum flows can not also be at user equilibrium with common money tolls on the links.*

Proof. From the description of proposition 1, we need equations (11a) - (12c) to hold. From lemma (4), we see that only the highest VOT class can travel on any path in the network. This means that, if there is flow on some path in the network, then this flow constitutes of only users in the highest VOT class. But then the TSO solution must only include demand for the highest VOT group. Except in the special case where the TSO solution only has demand for the highest VOT user class, there will be more than one user class travelling on the network at TSO flow. Thus no common money tolls exist on the network that can support MUC UE at TSO flows. □

Note that proposition (1) allows there to be common money tolls in the extreme case where only the highest VOT user class is on the network at TSO.

3.2. **Some small examples.** Although it is impossible to have common money tolls that drive ED MUC UE to TSO, it may be possible to get tolls which achieve flows which are close to TSO. In order to find these tolls, it is required to solve a bi-level programming problem which is formulated as:

$$\min_{v,q} \quad \sum_a v_a t_a(v_a) - \sum_k \sum_w \int_0^{q_w^k} g(\omega) d\omega \tag{18}$$

subject to $(v,q) \in \Omega$ and (v,q) are at tolled MUC UE (with common money tolls).

It is instructive to look at some examples in order to have an idea of how close it might be possible to get to the TSO while still being at tolled UE with common money tolls. For small networks it is practical to find the optimal tolls by a grid search method, looping over all possible tolls, or through direct optimisation.

3.2.1. *1 Link Example.* Take a very simple one link example with two user classes. The travel time and inverse demand functions on this link are given as:

$$t_1 = 12 + 0.1 v_1 \quad g_1^1 = 15 \left(\frac{q^1}{10}\right)^{-\left(\frac{1}{0.2}\right)} ; \quad g_1^2 = 15 \left(\frac{q^2}{20}\right)^{-\left(\frac{1}{0.35}\right)}$$

With the generalised cost function for UC k given by $c_1^k = t_1 + \tau_1/\beta_k$, and the VOT given as $\beta_1 = 1, \beta_2 = 2$. The TSO flows of this system are given as $v_1^1 = 9.67, v_1^2 = 18.87$, and these flows are unique. The welfare gain calculated is the difference between the welfare of the system when it is at UE flows with no toll employed ($\tau = 0$) and the welfare of the system when it is at UE flow with a toll employed ($\tau > 0$) The total possible welfare achieved is at TSO. This is a standard way of calculating change in benefits for a new toll scheme (Rouwendal and Verhoef, 2006). Plotting this welfare gain against possible tolls gives figure (3). We see that the welfare gain increases up to a maximum, and then decreases. This maximum occurs when $\tau_1 = 4.27$, with corresponding UE flows $v_1^1 = 9.52, v_1^2 = 19.14$ and at this point, 91.00% of the maximum possible welfare gain is achieved.

3.2.2. *5 Link Example.* Optimal tolls are also calculated for the 5 link network in figure 4. There are two OD pairs, and two user classes (with VOT of 1 and 2). The inverse demand

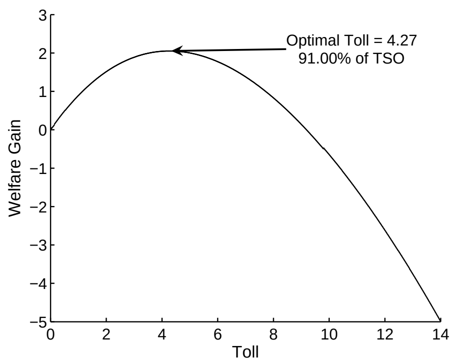


FIGURE 3. Plot showing the welfare gain compared to no toll UE for different tolls on a 1 link network with ED.

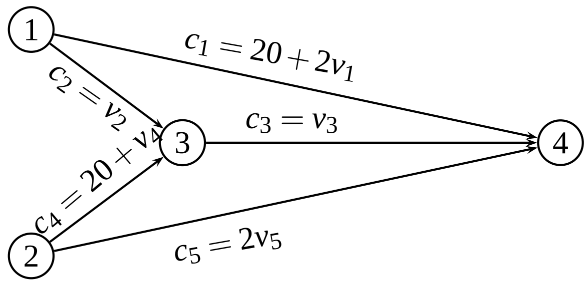


FIGURE 4. Diagram for five link network as seen in Yang and Huang (2004).

functions are:

$$g_1^1 = 33.33\left(\frac{q_1^1}{10}\right)^{-\frac{1}{0.2}} \qquad g_1^2 = 33.33\left(\frac{q_1^2}{10}\right)^{-\frac{1}{0.35}}$$

$$g_2^1 = 46.67\left(\frac{q_2^1}{20}\right)^{-\frac{1}{0.2}} \qquad g_2^2 = 46.67\left(\frac{q_2^2}{10}\right)^{-\frac{1}{0.35}}$$

where superscripts refer to the UC, and subscripts to the OD pair. It is of course impossible to provide a visual representation of the welfare gain in this case, since there are 5 tolls over which we vary. However, we can still find the optimal tolls which are $\tau_1 = 22.15, \tau_2 = 27.33, \tau_3 = 18.01, \tau_4 = 19.60$ and $\tau_5 = 47.25$. These tolls correspond to UE flows shown in table 3, and achieve 94.53% of the total possible welfare gain.

These examples show that, in these small cases, it is possible to achieve close to TSO flows and welfare, but the method used here can not be used in large cases, as it cannot be readily applied to large networks. Methods of bi-level programming should be used instead, although it should be noted that these methods do not guarantee finding the global optimal solution.

TABLE 3. Tolled MUC UE flows with optimal tolls, and TSO flows for 5 link network.

	v_1	v_2	v_3	v_4	v_5	q_1^1	q_1^2	q_2^1	q_2^2
Tolled UE	8.89	8.88	17.42	8.54	18.64	8.82	8.95	18.08	9.13
TSO	8.44	9.11	17.76	8.64	18.21	9.09	8.46	18.30	8.56

4. CONCLUSIONS AND FURTHER WORK

First best tolls are used as benchmarks against which other tolling schemes can be compared. First best tolls in the MUC case are tolls which, when added to the cost function, cause users to reach SO flows where SO flows maximise economic benefit either in time or money units. This paper has shown that the flows (and therefore the first best tolls) are different depending on whether the system is in time or money units. Although this result is known, it is important to highlight this, as this differs from the single user class case where the unit of analysis does not affect the flows or tolls.

Furthermore, it is straightforward to derive standard first best tolls in the ED case, as the analysis follows that done in the FD case. We suggest that the most appropriate benchmark should have tolls in money units which are common across links (so that the same money toll is applied to all user classes across a link). We dismiss the TSO solution since these tolls are differentiated by user class which is unacceptable since the users only differ by their VOT which is an invisible economic concept rather than a physical characteristic.

However, having selected the money-based benchmark, we then showed that the MSO flows cannot be guaranteed to be found, since the optimisation problem has a nonconvex objective function. Although this is known for some cases (Engelson and Lindberg, 2006; Yang and Huang, 2005), the proof provided here that shows non-convexity (see Appendix), allows for a more general class of network, including allowance for the elastic demand case. The standard first best tolls under MSO depend on individual user class link flows, and thus are unsatisfactory, since there may be multiple solutions. This has important implications since current guidance in the UK (DfT, 2007) proposes a money-based benchmark, and furthermore suggests an iterative approach may be used to obtain the so-called marginal cost tolls. As with any optimisation problem with multiple solutions, such an iterative approach cannot guarantee that a global solution will be obtained.

Given the problem with MSO, we then proposed using common money tolls to drive flows toward the TSO solution. However, it has been proved that there do not exist any common money tolls that will push UE to TSO. This differs from the FD case, and although it was known that finding these tolls was problematic, their non-existence has been proved for the first time in this paper.

Although it is impossible to find common money tolls that will push UE to TSO flows, we did demonstrate via simple examples that near optimal common money tolls can be found. The comparisons in terms of welfare gain were obviously specific to those examples and an obvious area for further research is to look at the bounds on how close to TSO we can get with common money tolls.

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APPENDIX: UNIQUENESS RESULTS

A minimisation program gives a unique solution if the objective function is strictly convex, and the constraint set is convex. The UE MUC ED objective function (in time) is given in (4). The Hessian with respect to user class link flows is positive semi-definite, thus all local minima are global minima, but are not necessarily unique. Note that the Hessian with respect to *aggregate* link flows is positive definite, and thus the optimal aggregate link flows are unique. The UC specific OD demands (q_w^k) are also unique since the Hessian is positive definite with respect to the demand parameters.

The Hessian of the TSO objective function (equation (7)) is positive semi-definite if the second derivatives of the link cost functions are non-negative ($t_a^{''}(v_a) \geq 0$). In this case the uniqueness results are the same as in MUC UE ED case above.

The MSO objective function is defined in (8). The Hessian reveals that this objective function is not convex. The Hessian has elements:

$$\frac{\partial^2 Z_2}{\partial v_a^k \partial v_b^j} = \begin{cases} \sum_k \beta_k v_a^k t_a^{''}(v_a) + 2\beta_k t_a^{'}(v_a), & \text{if } b = a, j = k \\ \sum_k \beta_k v_a^k t_a^{''}(v_a) + \beta_k t_a^{'}(v_a) + \beta_j t_a^{'}(v_a), & \text{if } b = a, j \neq k \\ 0, & \text{otherwise} \end{cases}$$

Consider a single link with two user classes. The determinant of the Hessian is given by:

$$|H| = \left| \begin{pmatrix} \beta_1 v_1^1 t_1^{''}(v_1) + \beta_2 v_1^2 t_1^{''}(v_1) + 2\beta_1 t_1^{'}(v_1) & \beta_1 v_1^1 t_1^{''}(v_1) + \beta_2 v_1^2 t_1^{''}(v_1) + (\beta_1+\beta_2) t_1^{'}(v_1) \\ \beta_1 v_1^1 t_1^{''}(v_1) + \beta_2 v_1^2 t_1^{''}(v_1) + (\beta_1+\beta_2) t_1^{'}(v_1) & \beta_1 v_1^1 t_1^{''}(v_1) + \beta_2 v_1^2 t_1^{''}(v_1) + 2\beta_2 t_1^{'}(v_1) \end{pmatrix} \right| \tag{A-1}$$

We decompose this as $H = A + M$ where

$$A = \begin{pmatrix} \beta_1 v_1^1 t_1^{''}(v_1) + \beta_2 v_1^2 t_1^{''}(v_1) & \beta_1 v_1^1 t_1^{''}(v_1) + \beta_2 v_1^2 t_1^{''}(v_1) \\ \beta_1 v_1^1 t_1^{''}(v_1) + \beta_2 v_1^2 t_1^{''}(v_1) & \beta_1 v_1^1 t_1^{''}(v_1) + \beta_2 v_1^2 t_1^{''}(v_1) \end{pmatrix} \tag{A-2}$$

$$M = \begin{pmatrix} 2\beta_1 t_1^{'}(v_1) & (\beta_1+\beta_2) t_1^{'}(v_1) \\ (\beta_1+\beta_2) t_1^{'}(v_1) & 2\beta_2 t_1^{'}(v_1) \end{pmatrix} \tag{A-3}$$

For a matrix $B = \begin{pmatrix} a+\alpha & b+\alpha \\ c+\alpha & d+\alpha \end{pmatrix}$, $\det(B) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ as long as $a+d-b-c=0$, which in this case holds. So then

$$\det(H) = \det(M) = t_1^{'}(v_1)^2 (4\beta_1\beta_2 - (\beta_1+\beta_2)^2) \tag{A-4}$$

without loss of generality, set $\beta_1 = 1$, to give

$$\det(H) = t_1^{'}(v_1)^2 (-(1-\beta_2)^2) < 0 \tag{A-5}$$

for all $\beta_2 \neq 1$. The first principal minor is non-negative (as long as $t_1^{''}(v_1) \geq 0$), and the second is negative, no matter what values we use (unless $\beta_1 = \beta_2$ wherein the problem reduces to the single user class case).

Extending this to a larger network with n links, and m user classes, the vector of individual user class link flows will be

$$\mathbf{v} = (v_1^1, v_1^2, \ldots, v_1^m, v_2^1, \ldots, v_2^m, \ldots, v_n^1, \ldots, v_m^n)^{\mathrm{T}}$$

so the first two principal minors of the Hessian will always look like those in equation (A-1) with first principal minor unrestricted in sign, while the second always negative. This means that the objective function is not convex for any values of VOT, and any network, as long as the restrictions on the cost function apply. This strengthens the result in Engelson and Lindberg (2006) which says that the money unit objective function will be nonconvex *in general* in the fixed demand case, along with the analysis from Yang and Huang (2005) which points out the possibility of this nonconvexity, again in the fixed demand case.

The UC demands follows the analysis for the TSO case; the MSO formulation leads to unique user class specific OD demands.