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TRANSPORT NETWORK CAPACITY EVALUATION AND DESIGN UNDER DEMAND UNCERTAINTY

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Abstract: This paper proposes a flexible transport network capacity evaluation and design problem (FNDP) under demand variability. The future stochastic demand is assumed to follow a normal distribution. Travellers' path choice behaviour is assumed to follow the Probit Stochastic User Equilibrium (SUE). The network reserve capacity is used to evaluate the performance of the network. Since the future demand is stochastic, the reserve capacity is measured by possible increases in both mean and standard deviation (SD) of the base demand distribution. The proposed model therefore represents the flexibility of the network in its robustness to OD demand variation (i.e. high SD). The proposed model can also determine an optimal network design to maximize the reserve capacity of the network in terms of both the mean and SD of the increased demand distribution. The paper applies the implicit programming approach to solve the FNDP. Sensitivity analysis is adopted to provide all necessary derivatives. The model and algorithm are tested with a hypothetical network to illustrate the merits of the proposed model.

Keywords: Network capacity, stochastic network design, network design problem, bilevel optimization

INTRODUCTION

Recently, governments and local authorities in many countries have shifted their focus to designing and developing a transportation network that can cope with future uncertainties in demand and supply (*1*). On the demand side the network should have sufficient *network reserve capacity* to deal with unexpected increases or changes in the demand pattern. The concept of network reserve capacity was originally developed for analyzing an isolated signal-controlled intersection (*2*); Wong and Yang (*3*) extended this concept to evaluate the optimal signal setting for a road network. The network reserve capacity is defined as the maximum multiplier $\theta > 1$ applied to a given Origin Destination (OD) demand matrix such that the equilibrium link flows satisfy the link capacity constraints, i.e. $\max \theta \; s.t. \, v_a(\theta \mathbf{q}) \le c_a \; \forall a \in A$ where θ is a scalar and the base OD demands are $\mathbf{q}; v_a(\theta \mathbf{q})$ is the equilibrium flow on link *a* that has capacity c_a. This problem can be considered as a deterministic network design problem (NDP).

Chen *et al.* (*4*) originally proposed capacity reliability as a new network performance index based on the concept of reserve capacity. The capacity reliability index was developed to evaluate the probability that the network capacity can accommodate a certain traffic demand at a required level of service under random link degradation (*5*). Sumalee and Kurauchi (*6*) utilized this concept to evaluate the network capacity after a major disaster. Lo and Tung (*7*) defined the capacity reliability as the maximum flow that the network can carry subject to link capacity and travel time reliabilities with random link capacities.

Most of the network capacity models mainly focussed on the reserve capacity with the deterministic demand. In reality, the travel demand forecast is uncertain from economic factor, demand concentration, energy situation, etc. and can be defined by a statistical distribution. Under stochastic demand, the definition of the network capacity can be analyzed from two perspectives. On one hand, the network should be able to handle a certain level of growth in the average OD demands, i.e. increase in the mean. On the other hand, this network may not necessarily be able to cope with increases in the variability of the OD demand, i.e. increase in the standard deviation (SD) of the demand distribution. Thus, it is important to evaluate the ability of the network to absorb increases in the variability or uncertainty of future OD demands. Patil and Ukkusuri (*8*) proposed a flexible network design problem (FNDP) to consider uncertain demand by generating a number of possible future scenarios.

In this paper, FNDP is defined for strategic policy design, allowing planners to accommodate different levels of variation in future stochastic OD demands based on a desired level of link capacity reliability (risk preference measure). The stochastic demand is assumed to follow a normal distribution. Both the mean and SD of the base OD demand can be scaled up by some multipliers. The increase in the mean demand can be interpreted as the original network reserve capacity. The increase in the SD of demand, on the other hand, can be viewed as the flexibility of the network to cope with an increase in the level of demand variability.

A mathematical program is proposed to find the maximum multipliers for both the mean and SD of the existing OD demands, subject to the link capacity chance-constraints (due to the stochastic link flows). The chance-constraint could be also referred to as the Value-at-Risk in Chen et al. (*9*). The objective function of this problem is defined as a weighted sum of the mean and SD of the increased demands. This objective function is referred to as the mean-standard deviation network capacity (M-SD). The model is then incorporated into FNDP by optimizing the link capacity investments in the future so as to maximize the M-SD network capacity. An analytical approach to solve the proposed model, following the implicit programming approach (6, *10*, *11*), is developed. The method of sensitivity analysis (SA) (*12*, *13*) is employed to calculate both the gradient of the objective function, and Jacobian of the constraints with respect to changes in the design parameters, namely the mean and SD demand multipliers, and link capacity investments. This paper is organized into four further sections. The next section defines notation and assumptions of the stochastic model. The model formulation and solution algorithm is then presented in the third section. The fourth section presents an application of the proposed model to a test network and discusses the results. The conclusion and discussion for future research are given in the last section.

STOCHASTIC NETWORK FRAMEWORK

Traffic Flow Distribution

The network is represented by a graph with a set of directed links *A* and nodes *N*. Let *R* as the set of OD pairs with $|R|$ total number of OD pairs. Due to demand uncertainty, the day-to-day travel demand for the OD pair from node *r* to *s* ($r \neq s; r, s \in N$), denoted by $Q_{rs}, rs \in R$, is assumed to follow the Normal distribution (*14*-*16*). This can be expressed as $Q_{rs} \sim N\left(q_{rs}, \left(\sigma_q^{rs}\right)^2\right)$, where q_{rs} and σ_q^{rs} are mean and SD, respectively. Define **p** having size $|K|$, where *K* is the set of possible paths, as the vector of path choice proportions in which each entry p_k^{rs} ($k \in K_{rs}$) serves an OD pair *rs*. The random path flow can then be expressed as $F_k^{rs} = p_k^{rs} Q_{rs}$. Since the OD demand is a normally distributed random variable, the path flow, which is the product of deterministic path choice proportion and stochastic OD demand, then follows a normal distribution. The mean and variance of random path flow, denoted by f_k^{rs} and $\left(\sigma_f^{k,rs}\right)^2$, respectively, can be defined as:

$$\begin{aligned} f_k^{rs} &= E\left[F_k^{rs}\right] \\ &= E\left[p_k^{rs} Q_{rs}\right] \\ &= q_{rs} p_k^{rs} \qquad \forall k \in K_{rs}; \forall rs \in R, \end{aligned} \tag{1}$$

$$\begin{aligned} \left(\sigma_f^{k,rs}\right)^2 &= Var\left(F_k^{rs}\right) \\ &= Var\left(p_k^{rs} Q_{rs}\right) \\ &= \left(\sigma_q^{rs} p_k^{rs}\right)^2 \qquad \forall k \in K_{rs}; \forall rs \in R. \end{aligned} \tag{2}$$

The covariance between two arbitrary path flows (say F_k^{rs} and F_j^{rs}) joining the same OD pair *rs* is formulated, following (*16*), as:

$$Cov\left(F_k^{rs}, F_j^{rs}\right) = \left(\sigma_q^{rs}\right)^2 p_k^{rs} p_j^{rs} \quad k \neq j; \forall k, j \in K_{rs}; \forall rs \in R, \tag{3}$$

while the covariance of the path flows from paths connecting different OD pairs is zero.

Let $\mathbf{f}$ be a $|K|$ column vector of the mean path flows from Eq. (1) and $\mathbf{\Sigma}^f$, of size $|K|\times|K|$, is the variance-covariance matrix of path flows from Eqs. (2) and (3). The link flow V_a, which is a sum of stochastic path flows (i.e. $F_k^{rs} \sim N\left(f_k^{rs}, \left(\sigma_f^{k,rs}\right)^2 \right)$), follows MVN, i.e. $\mathbf{V} \sim \text{MVN}\left(\mathbf{v}, \mathbf{\Sigma}^v\right)$ where $\mathbf{v}$ is the vector of mean link flows with size $|A|$, and $\mathbf{\Sigma}^v$ is the $|A|\times|A|$ variance-covariance matrix of link flows. $\mathbf{v}$ and $\mathbf{\Sigma}^v$ are defined as:

$$\begin{aligned} \mathbf{v} &= \mathbf{\Delta}\cdot\mathbf{f} \\ &= \sum_{rs=1}^{R} \mathbf{\Delta}_{rs}\cdot q_{rs}\cdot\mathbf{p}^{rs}, \end{aligned} \tag{4}$$

$$\begin{aligned} \mathbf{\Sigma}^v &= \mathbf{\Delta}\cdot\mathbf{\Sigma}^f\cdot\mathbf{\Delta}^T \\ &= \sum_{rs=1}^{R} \mathbf{\Delta}_{rs}\cdot\mathbf{\Sigma}_{rs}^f\cdot\left(\mathbf{\Delta}_{rs}\right)^T, \end{aligned} \tag{5}$$

where $\mathbf{\Delta} = \left(\ldots, \mathbf{\Delta}_{rs}, \ldots\right)$ is the link-path incident matrix of the network, and $\mathbf{\Delta}_{rs}$ is the link-path incident matrix associated with OD pair rs.

Link and Path Travel Times

The link travel time function is assumed to follow a standard Bureau of Public Roads (BPR) function, which is defined as:

$$T_a\left(V_a\right) = t_a^0 + \frac{b_a}{\left(c_a + s_a\right)^n} V_a^n \quad \forall a \in A, \tag{6}$$

where t_a^0 is the free-flow travel time on link a; b_a and n are parameters for the link travel time function; c_a is the existing link capacity; and s_a is the additional link capacity to be determined.

Since link flows are stochastic, link and path travel times are random variables. Let $\mathbf{t}_a$ be a $|A|$ vector of mean link travel times in which each element, denoted by t_a, is:

$$\begin{aligned} t_a &= E\left[T_a\left(V_a\right)\right] \\ &= t_a^0 + \frac{b_a}{\left(c_a + s_a\right)^n} E\left[V_a^n\right] \quad \forall a \in A. \end{aligned} \tag{7}$$

In Eq. (7), $E\left[V_a^n\right]$ is the n^{th} raw moment of the normally distributed link flow. The method of moment generating function (MGF) applied in (11) and (*17*) is adopted to calculate $E\left[V_a^n\right]$. Note that previous studies (*11*) and (*17*) only applied the MGF method to derive the expected link travel time from Poisson and Binomial randomly distributed link flows, respectively.

From (*18*), the MGF of the normal distribution can be defined as $M_{V_a}\left(\alpha\right) = \exp\left(v_a\alpha + \frac{1}{2}\left(\sigma_t^a\alpha\right)^2\right)$. Since this paper aims to apply the BPR function with $n = 4$, the *4*th raw moment of V_a can be derived as:

$$\begin{aligned} \mathrm{E}\left[V_a^4\right] &\equiv M_{V_a}^4(0) \\ &= \left.\frac{d^4 M_{V_a}(\alpha)}{d\alpha^4}\right|_{\alpha=0} \\ &= v_a^4 + 6v_a^2\left(\sigma_v^a\right)^2 + 3\left(\sigma_v^a\right)^4. \end{aligned} \tag{8}$$

Thus, mean link travel time can be explicitly expressed in the closed form as a function of the mean and variance of the random link flow:

$$t_a = t_a^0 + \frac{b_a}{\left(c_a + s_a\right)^4}\left\{v_a^4 + 6v_a^2\left(\sigma_v^a\right)^2 + 3\left(\sigma_v^a\right)^4\right\} \quad \forall a \in A. \tag{9}$$

Finally, the $|K|$ mean path travel time vector $\mathbf{t}_k$, with entry t_k^{rs}, can be calculated from:

$$\mathbf{t}_k = \mathbf{\Delta}^T.\mathbf{t}_a. \tag{10}$$

Path Choice Model

This paper assumes travellers only consider personal travel time in the dis-utility of their trips. Since travel times are stochastic, the perceived expected travel time is considered to be the deterministic dis-utility term of each path, following (*11*). The perceived expected travel cost for path k is defined as $\tilde{C}_k^{rs} = E\left[C_k^{rs}\right] + \varepsilon_k^{rs}\ \forall k \in K_{rs}; \forall rs \in R$ where $E\left[C_k^{rs}\right]$ is the mean travel cost (i.e. mean route travel time t_k^{rs}); and ε_k^{rs} is the travel time perception error on path k of OD pair rs. ε is assumed to be MVN with zero mean and variance-covariance matrix $\mathbf{\Sigma}^{\varepsilon}$, which can be determined from link travel time perception errors and the link path incident matrix, see e.g. (*13*).

Following the Probit SUE model (*19*), the fixed-point (FP) condition for the assignment problem can be defined as:

$$\begin{aligned} p_k^{rs} &= \Pr\left[\tilde{C}_k^{rs} \le \tilde{C}_j^{rs}\right] \\ &= \Pr\left[\begin{aligned} &t_k^{rs}\left(t_a\left(v_a\left(p_k^{rs}, q_{rs}\right), \sigma_v^a\left(p_k^{rs}, \sigma_q^{rs}\right)\right)\right) + \varepsilon_k^{rs} \le \\ &t_j^{rs}\left(t_a\left(v_a\left(p_j^{rs}, q_{rs}\right), \sigma_v^a\left(p_j^{rs}, \sigma_q^{rs}\right)\right)\right) + \varepsilon_j^{rs} \quad \forall j \in K_{rs} \end{aligned}\right], \end{aligned} \tag{11}$$

where $\Pr[.]$ denotes probability.

In Eq. (11), $t_k^{rs} \equiv E\left[C_k^{rs}\right]$ can be calculated from t_a which can be further expressed as a function of the mean and SD of random link flow. These two parameters of link flow can be determined from the mean and SD of stochastic OD demands and path choice proportion. Eq. (11) is thus the FP condition. To solve this FP problem, the method of successive average (MSA) (*19*) is adopted, with the simple step size of 1/i where i is the current iteration of the solution. The Probit path choice probabilities (Pr) are analytically calculated following (*13*). The MSA algorithm can be summarized as follows:

Step 0 *Initialization*: Set i = 1; define sets of possible paths for each OD pair (denoted as K_{rs}). Find a feasible path choice proportion vector $\mathbf{g}^i$ and set $\mathbf{p}^i = \mathbf{g}^i$.

Step 1 *Travel time calculation*: Assign $\mathbf{p}^i$ to the stochastic network according to Eqs. (1)-(5) and then calculate new mean link and path travel time vectors following Eqs. (9) and (10).
Step 2 *Descent direction finding*: For each OD pair, evaluate the path choice proportions $g_k^{rs,i} = \Pr\left[t_k^{rs,i} + \varepsilon_k^{rs,i} \le t_j^{rs,i} + \varepsilon_j^{rs,i}\ \forall j \in K_{rs}\right]$.
Step 3 *Convergence test*: If $\left\|\mathbf{g}^i - \mathbf{p}^i\right\| / \left\|\mathbf{p}^i\right\| \le \delta$ or $i \ge i_{\max}$ then terminate the algorithm where δ is the convergence criteria and $i_{\max}$ is the predefined maximum number of iterations.
Step 4 *Move*: $\mathbf{p}^{i+1} = \mathbf{p}^i + \alpha^i.\left(\mathbf{g}^i - \mathbf{p}^i\right)$ where $\alpha^i = 1/i$; set $i := i+1$ and return to step 1.

Link Capacity Chance-Constraint

Let θ_1^{rs} and θ_2^{rs} represent the multipliers of the r to s OD demand distribution's mean and SD; $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ denote the vectors comprising θ_1^{rs} and θ_2^{rs} $\forall rs \in R$ and $\mathbf{s}$ is the vector of s_a $\forall a \in A$. After multiplying the existing OD demand distribution with θ_1^{rs} and θ_2^{rs}, the mean and SD of the new OD demand distribution are $\tilde{q}_{rs} = \theta_1^{rs} q_{rs}$ and $\tilde{\sigma}_q^{rs} = \theta_2^{rs} \sigma_q^{rs}$, respectively. The equilibrated stochastic link flow V_a^* can be reformulated as a function of the vectors of mean demand $\tilde{\mathbf{q}}\left(\boldsymbol{\theta}_1\right)$, SD of demand $\tilde{\boldsymbol{\sigma}}_q\left(\boldsymbol{\theta}_2\right)$, and Probit SUE route choice proportion $\mathbf{p}^*\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \mathbf{s}\right)$ satisfying FP condition in Eq. (11).

Adopting the chance-constraint concept in (*7*, *9*), the chance-constraint of the equilibrated stochastic link flow, based on the desired level of link capacity reliability, is shown in Figure 1.

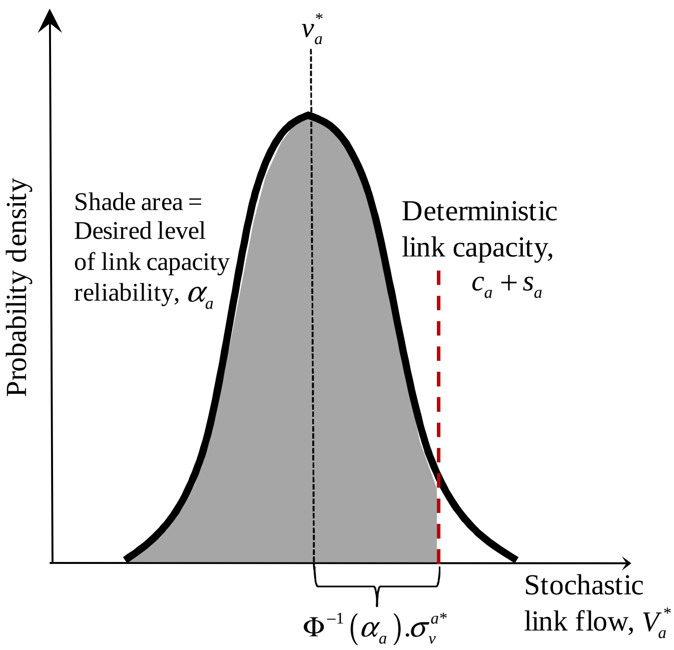


FIGURE 1 Chance-constraint of stochastic link flow.

In Figure 1, the link capacity reliability can be evaluated from the frequency that the equilibrated stochastic link flow is less than or equal to total deterministic link capacity. This capacity reliability must not be less than the desired level α_a (shaded area). The relation can be written as:

$$\Pr\left[V_a^*\left(\tilde{\mathbf{q}}(\boldsymbol{\theta}_1),\tilde{\boldsymbol{\sigma}}_q(\boldsymbol{\theta}_2),\mathbf{p}^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})\right)\le c_a+s_a\right]\ge\alpha_a \quad \forall a\in A. \tag{12}$$

The level α_a allows planners to easily specify a risk preference. Increasing α_a increases the risk aversion level (meaning more reliability is desired). Eq. (12) is explicitly reformulated as the chance-constrained equation:

$$v_a^*\left(\tilde{\mathbf{q}}(\boldsymbol{\theta}_1),\mathbf{p}^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})\right)+\Phi^{-1}(\alpha_a)\sigma_v^{a*}\left(\tilde{\boldsymbol{\sigma}}_q(\boldsymbol{\theta}_2),\mathbf{p}^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})\right)\le c_a+s_a \quad \forall a\in A, \tag{13}$$

where $\Phi^{-1}(.)$ is the inverse cumulative distribution function (CDF) of the standard normal distribution. Note that the left hand side of Eq. (13) could be referred to as VaR of the equilibrated stochastic link flow with $\Phi^{-1}(\alpha_a)\sigma_v^{a*}$ above the mean value.

MODEL FORMULATION AND SOLUTION ALGORITHM

Flexible Transport Network Capacity Evaluation and Design Problem (FNDP)

This paper proposes a model to solve a flexible transport network capacity evaluation and design problem (FNDP). The model can also be employed to evaluate the M-SD reserve capacity. The model assesses the reserve capacity of the existing network from the maximum multipliers, $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, of the base OD demand distributions' mean and SD respectively, subject to the link capacity chance-constraints (due to the stochastic link flows). To consider the preferred trade-off level of mean and SD reserve capacity, the objective function can be defined as a weighted sum of mean and SD of the increased demands with a predefined weight factor τ. This model is then used within the FNDP, to optimize the link capacity investments $\mathbf{s}$, so as to maximize the M-SD reserve capacity (under different variations of OD demand distributions) subject to link capacity chance-constraints under the Probit SUE condition, budgetary constraint, and design parameter constraints (on $\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s}$). The FNDP can be formulated as a mathematical program with equilibrium constraint (MPEC) as:

$$\max_{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})} Z\equiv\tau.\boldsymbol{\theta}_1^T.\mathbf{q}+(1-\tau).\left\{\left(\boldsymbol{\theta}_2^T\right)^2.\boldsymbol{\sigma}_q^2\right\}^{\frac{1}{2}} \tag{14}$$

subject to:

$$\mathbf{v}^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})+\Phi^{-1}(\boldsymbol{\alpha}).\boldsymbol{\sigma}_v^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})\le\mathbf{c}+\mathbf{s}, \tag{15}$$

$$\gamma.\mathbf{l}^T.\mathbf{s}\le\beta, \tag{16}$$

$$\text{CV }\boldsymbol{\theta}_1\le\delta_{\theta_1}, \tag{17}$$

$$\text{CV }\boldsymbol{\theta}_2\le\delta_{\theta_2}, \tag{18}$$

$$0\le\tau\le 1, \tag{19}$$

$$1\le\theta_1^{rs}\le\theta_1^{\max},\ 1\le\theta_2^{rs}\le\theta_2^{\max}, \tag{20}$$

$$0\le s_a\le s_{\max}, \tag{21}$$

where γ is the cost per unit link length per unit capacity investment; $\mathbf{l}$ is the vector of link lengths; β is the total budget; $\text{CV}\boldsymbol{\theta}_1 = \sqrt{\frac{1}{R-1}\sum_{rs=1}^{R}\left(\theta_1^{rs} - \frac{1}{R}\sum_{rs=1}^{R}\theta_1^{rs}\right)^2} \Big/ \frac{1}{R}\sum_{rs=1}^{R}\theta_1^{rs}$ and $\text{CV}\boldsymbol{\theta}_2 = \sqrt{\frac{1}{R-1}\sum_{rs=1}^{R}\left(\theta_2^{rs} - \frac{1}{R}\sum_{rs=1}^{R}\theta_2^{rs}\right)^2} \Big/ \frac{1}{R}\sum_{rs=1}^{R}\theta_2^{rs}$; δ_{θ_1} and δ_{θ_2} are the allowable coefficients of variation (CV) of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, respectively; $\theta_1^{\max}$ and $\theta_2^{\max}$ are the mean and SD demand multiplier upper bounds; and $s_{\max}$ are the link capacity investment upper bounds.

The objective of the optimisation (14) is to maximize the M-SD reserve capacity using the predefined weight factor. Constraint (15) is the link capacity chance-constraint due to the stochastic link flow under the Probit SUE condition. This chance-constraint is written as an implicit function of design parameters ($\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, and $\mathbf{s}$). The budgetary constraint is written in Eq. (16). The OD demand multiplier constraints (17) and (18) allow the model to deal with the FNDP by controlling the variation of the increasing demands on different OD pairs. For example, if both mean and SD of demands from all OD pairs equally increase, δ_{θ_1} and δ_{θ_2} are set to be 0. In addition, $\delta_{\theta_1}, \delta_{\theta_2} > 0$ can be adjusted to control different variations of the increasing OD demands; higher $\delta_{\theta_1}, \delta_{\theta_2}$ allows higher variation between the increasing OD demands. The former is referred to as a whole area based control scheme when the latter is referred to as an OD based control scheme. Finally, the bounds for possible τ; θ_1^{rs} and θ_2^{rs}; and s_a are expressed in constraints (19)-(21), respectively.

Solution Algorithm

An analytical approach to solve the FNDP follows the implicit programming approach (*6, 9, 10*). The fmincon solver in MATLAB, which implements Sequential Quadratic Programming (SQP) algorithm, is used to find an optimal solution. This method requires the gradient of the objective function (14) and the Jacobians of link capacity chance-constraint (15) and OD demand multiplier constraints (17)-(18) with respect to the design parameters ($\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, and $\mathbf{s}$). The chain rule and SA of the equilibrium link flow under the Probit SUE (*12, 13*) are used to attain all derivative expressions. The details of gradient of the objective function and Jacobians of two constraints are shown in APPENDIX A, B and C, respectively.

ILLUSTRATIVE TESTS

The test network with 18 directed links, six OD pairs, and 34 paths from (*10*) is adopted as shown in FIGURE 2. The mean and CV of OD demands (CV in brackets) are listed in Table 1. The BPR link cost function, i.e. $T_a\left(V_a\right) = t_a^0 + \frac{b_a}{\left(c_a + s_a\right)^n} V_a^n$, is used with $n_a = 4, \forall a \in A$. The other parameters of the BPR function and the length for each link are given in Table 2.

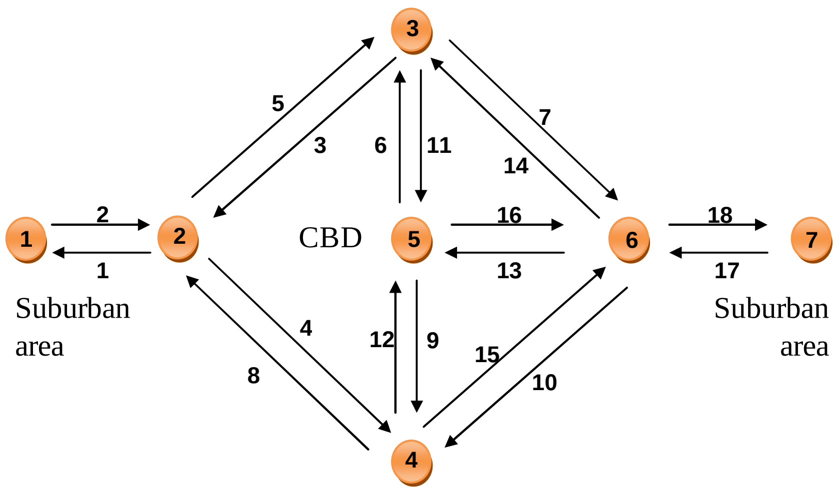


FIGURE 2 Hypothetical network.

TABLE 1 Mean and CV of OD Normal Distribution Demand

Origin	Destination		
	1	*5*	*7*
1	-	600 (0.25)	400 (0.20)
5	500 (0.50)	-	600 (0.60)
7	375 (0.28)	800 (0.20)	-

TABLE 2 Link Cost Parameters and Lengths

Link	t_a^0	b_a	c_a	l_a
1	0.0125	0.0025	1800	1.0
2	0.0125	0.0025	1800	1.0
3	0.0917	0.6261	1100	5.5
4	0.0917	0.6261	1100	5.5
5	0.0917	0.6261	1100	5.5
6	0.0250	0.1708	1100	1.5
7	0.0750	0.1087	1100	6.0
8	0.0917	0.6261	1100	5.5
9	0.0250	0.1708	1100	1.5
10	0.0750	0.1087	1100	6.0
11	0.0250	0.1708	1100	1.5
12	0.0250	0.1708	1100	1.5
13	0.0200	0.1366	1100	1.2
14	0.0750	0.1087	1100	6.0
15	0.0750	0.1087	1100	6.0
16	0.0200	0.1366	1100	1.2
17	0.0125	0.0025	1800	1.0
18	0.0125	0.0025	1800	1.0

To illustrate the applications of the FNDP model and solution algorithm, two sets of tests are conducted. Without any improvement, the first set is to evaluate the M-SD reserve capacity of the existing network. The second set is to enhance the M-SD reserve capacity of the network by optimizing the link capacity investment while satisfying the link capacity chance-constraints under the Probit SUE condition and budgetary constraint. The details are explained as follows.

Network Capacity Evaluation

To evaluate the existing M-SD network reserve capacity, Eqs. (16) and (21) in the FNDP are excluded. The first test is to determine the maximum θ_1^* and θ_2^* without violating the link capacity chance-constraints (15). The desired level of link capacity reliability is set as $\alpha_a = 0.9\ \forall a \in A$. To solve Probit SUE, the independent link travel time perception error is assumed as $\varepsilon_a \sim N\left(0, 0.3\left(t_a^0\right)^2\right) \forall a \in A$, and the convergence criteria and maximum iteration number are set to be $\delta = 1e^{-6}$ and $i_{\max} = 500$. This test is conducted on the basis of the whole area based control scheme ($\delta_{\theta_1}, \delta_{\theta_2}$ = 0) when $\theta_1^{\max} = 10$ and $\theta_2^{\max} = 10$. The results of maximum demand multipliers and M-SD network capacities under different values of τ with the increasing step of 0.2 are shown in Figure 3.

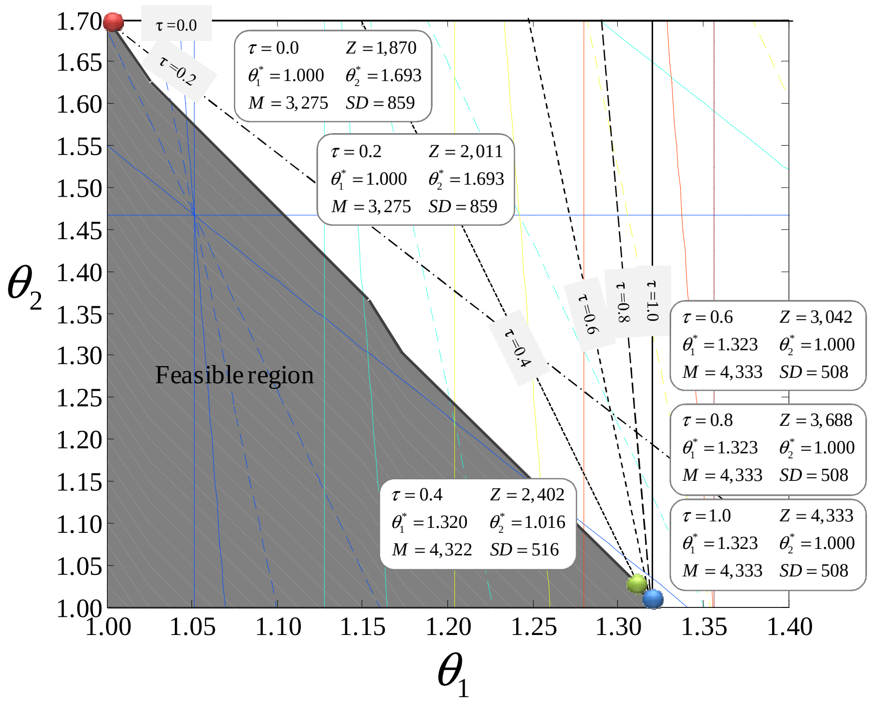
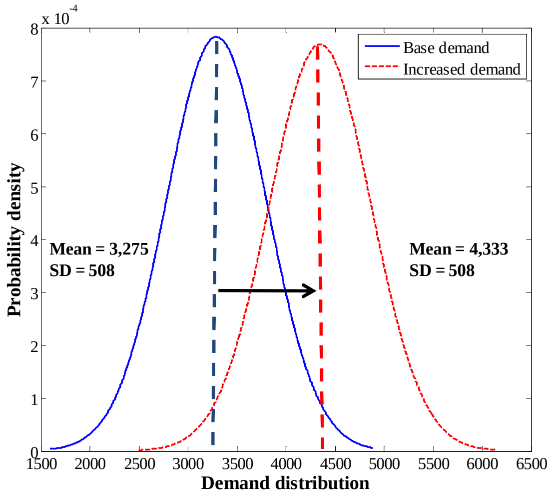


FIGURE 3 Maximum demand multipliers and M-SD network capacities.

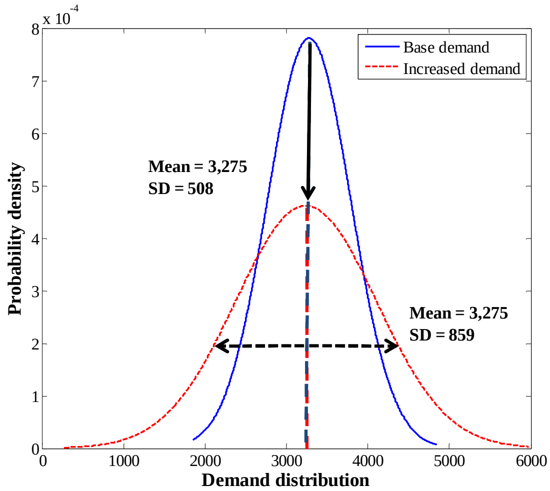
From Figure 3, the pairs of θ_1^* and θ_2^* can be classified into three categories. When τ = 0.6-1.0, the first case focuses on the higher expected network capacity, θ_1^* = 1.323 (θ_2^* = 1.000), as optimal M-SD capacities are M = 4,333 (SD = 508). This is referred to as the mean capacity approach. On the other hand, the solutions change to θ_1^* = 1.000 and θ_2^* = 1.693 for τ = 0.0 and 0.2. This allows the existing network to cope with the higher variation or uncertainty of the demand, SD = 859 (M = 3,275), which is referred to as the SD capacity approach. Lastly, the trade-off between M-SD multipliers can be observed at τ = 0.4. The solutions are θ_1^* = 1.320 and θ_2^* = 1.016 as M = 4,322 and SD = 516. The changes of demand distributions from these three evaluation approaches can be clearly explained as depicted in Figure 4.

Figure 4a) shows the mean capacity approach which shifts only the mean of the initial demand distribution to be 4,333 (same SD). In contrast, Figure 4b) presents the SD capacity approach which changes the shape of the base demand distribution to be flattened with the new SD = 859 (same mean). A joint result, trade-off approach, is illustrated in Figure 4c) when both mean and SD of the increased demand distribution are shifted and flattened to M = 4,322 and SD = 516. From the results, these three evaluation approaches can be adopted to the network design problem to enhance the reserve capacity of the network.

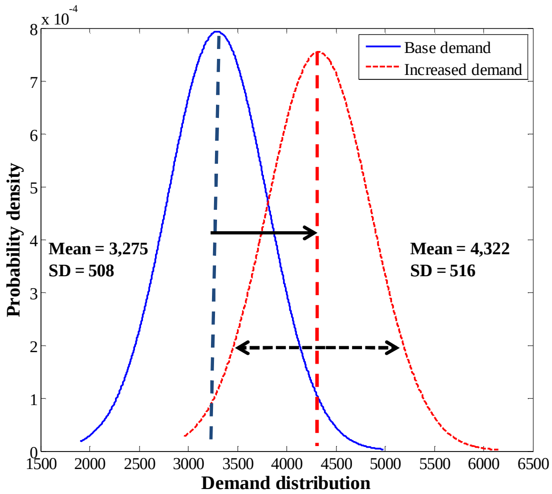
The second test assesses the vulnerability of each link to increasing OD demands in terms of the link capacity reliability. The results based on three evaluation approaches are shown in Figure 5. Figure 5a) shows that Links 17 and 18 are the first two critical links that violate the minimum desirable value (α_a = 0.90) at θ_1^* = 1.323 (θ_2 = 1.0). As θ_1 increases (mean demand increases) without any improvement on each link, the link capacity reliability gradually decreases. The next four critical links are found to be Links 2, 13, 1, and 16, respectively. Figure 5b) indicates that Link 18 is the first critical link based on the SD capacity approach (θ_1 = 1.0, θ_2^* = 1.693) following by Links 17, 16, 1 in that order. The order of critical links from the trade-off approach is similar to that of SD approach. Figure 5c) only shows Link 18 which is the first critical link at (θ_1^* = 1.32, θ_2^* = 1.016). Here the series of critical links from different evaluation approaches are identified. The next section will investigate link capacity investments to enhance the reserve capacity of the network.



a) Mean capacity approach ($\theta_1^* = 1.323$, $\theta_2^* = 1.000$)

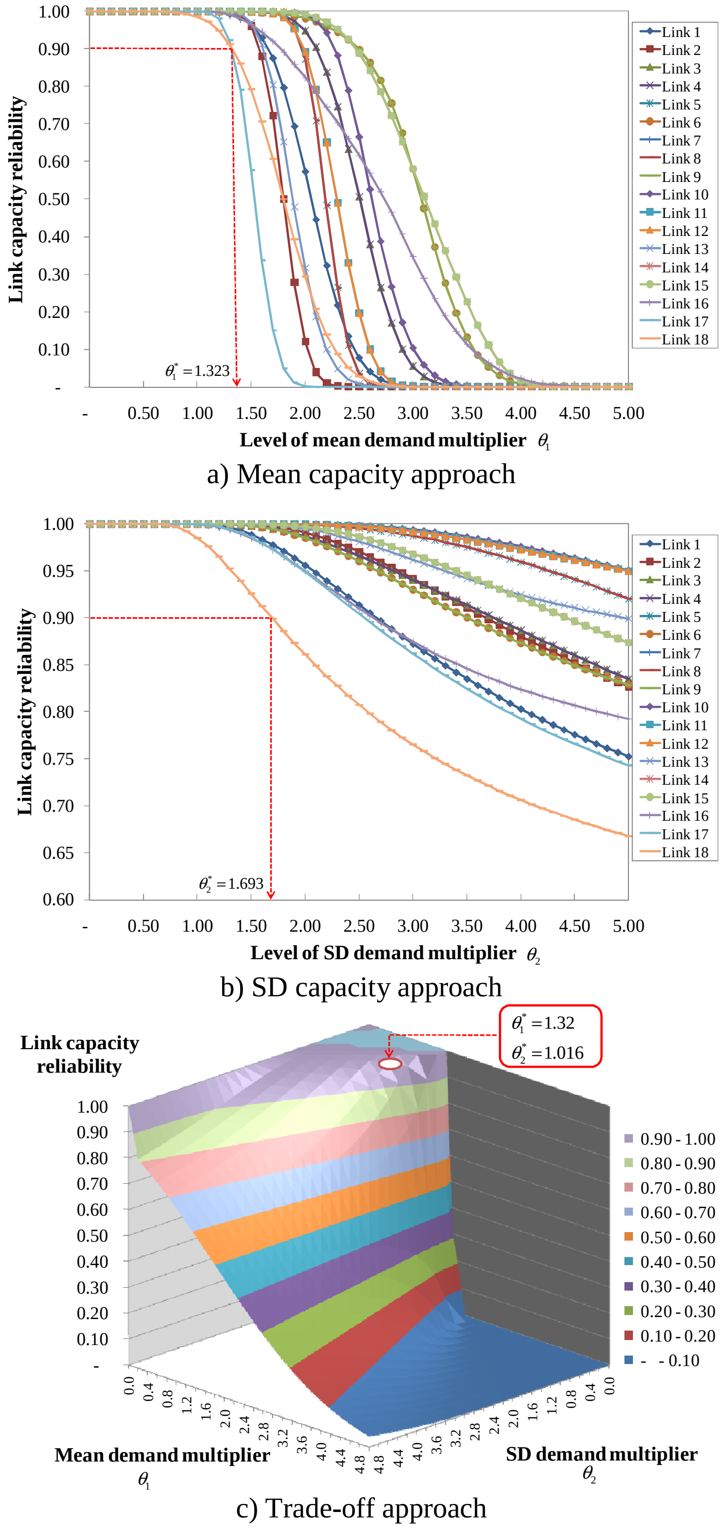


b) SD capacity approach ($\theta_1^* = 1.000$, $\theta_2^* = 1.693$)



c) Trade-off approach ($\theta_1^* = 1.32$, $\theta_2^* = 1.016$)

FIGURE 4 The changes of demand distributions based on three evaluation approaches.



a) Mean capacity approach

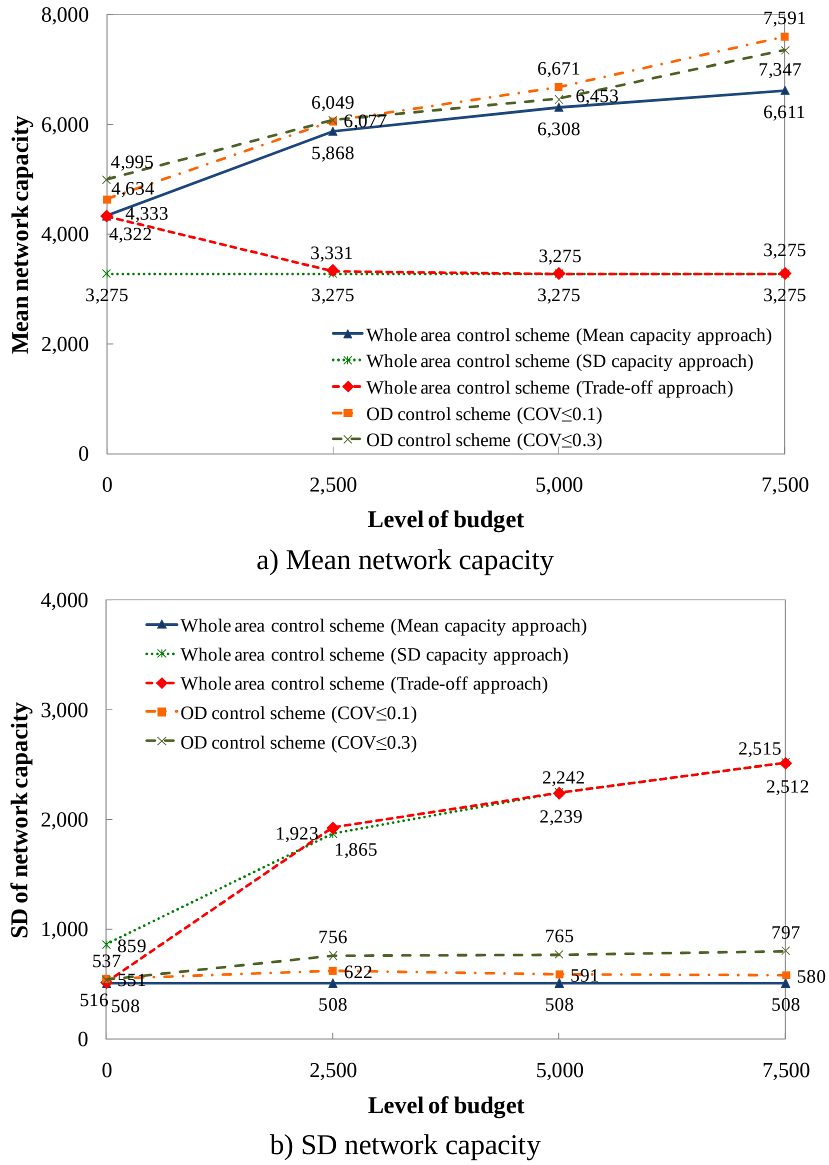
b) SD capacity approach

c) Trade-off approach

FIGURE 5 Link capacity reliability based on three evaluation approaches.

Network Capacity Enhancement

This test is to optimize the link capacity investments so as to maximize the M-SD reserve capacity subject to Eq. (16). The demands on different OD pairs can increase independently, with some constraints in (17) and (18). This is referred to as the OD based control scheme. For the test, we assume $\gamma = 1$ and $s_{\max} = 1,800$. The M-SD network capacities are measured based on the OD control scheme with two scenarios, i.e. (i) $\delta_{\theta_1}, \delta_{\theta_2} \leq 0.1$ and (ii) $\delta_{\theta_1}, \delta_{\theta_2} \leq 0.3$. Note that $\delta_{\theta_1}, \delta_{\theta_2} \leq 0.1$ and $\delta_{\theta_1}, \delta_{\theta_2} \leq 0.3$ represent low and high variations of the increasing demands on different OD pairs. The results at different budgets are shown in Figure 6.



a) Mean network capacity

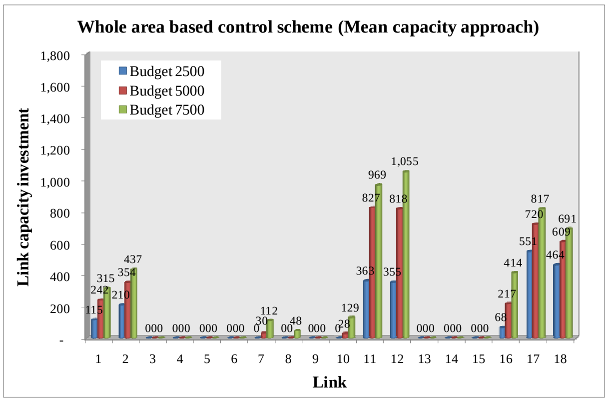
b) SD network capacity

FIGURE 6 M-SD network capacities based on whole area and OD control schemes at different budgets.

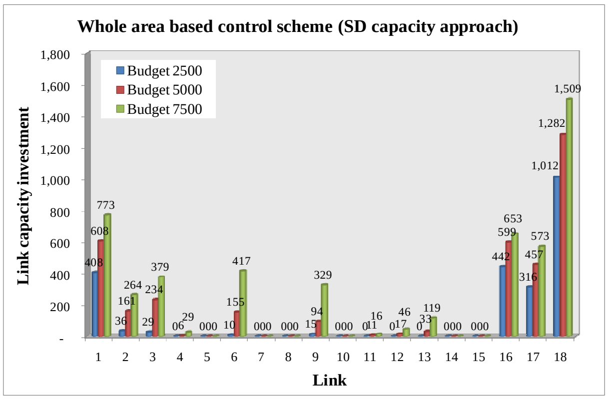
From Figure 6a the mean network capacities based on mean design approach and two OD control schemes increase in a similar way as the budget increases. In contrast, the results from the trade-off design approach decrease from 4,322 (budget = 0) to 3,275 (budget = 5,000). However, the mean capacities under the SD design approach are constant. On the other hand, Figure 6b shows that the SD network capacities under the SD and trade-off design approaches increase as the budget increases. The results from the OD control scheme ($\delta_{\theta_1}, \delta_{\theta_2} \leq 0.3$) slightly increases when that from $\delta_{\theta_1}, \delta_{\theta_2} \leq 0.1$ initially increase to 622 (budget = 2,500) and then decreases as the budget increases. The SD capacities under the mean design approach are constant.

Figure 7a, 7b, and 7c show the link capacity investments from three network design approaches under the whole area based control scheme. The amount of additional link capacity investment occurs in the same order as mentioned in the vulnerable link evaluation test (see Figure 5). However, under the mean approach, the first two investment links change to Links 12 and 11 at high budgets (5,000 and 7,500) instead of Links 17 and 18 at budget = 2,500. From Figure 7d and 7e, as OD demands are allowed to increase independently, the patterns of link capacity investment are different. However, Link 18 which is related to the highest volume of traffic is the most significant link to be improved. Figure 7 implies that different approaches provide different link capacity investment forms.

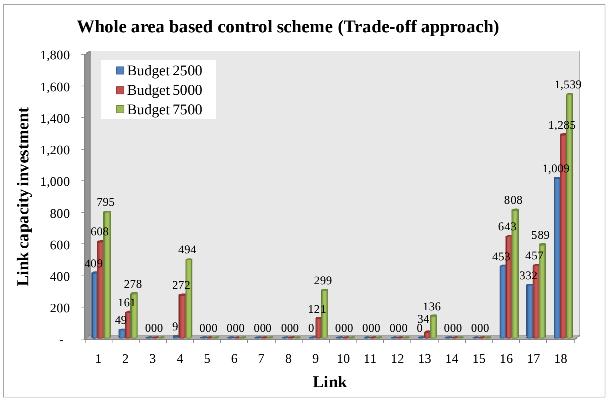
Note that the computational times for network design (at budget = 2,500) under the whole area based control scheme are 22, 57, and 25 seconds for mean-, SD-, trade-off design approach, respectively; whereas the computational times under the OD based control scheme are around 3 minutes for $\delta_{\theta_1}, \delta_{\theta_2} \leq 0.3$, and 18 minutes for $\delta_{\theta_1}, \delta_{\theta_2} \leq 0.1$. The tests were carried out with a computer with Pentium Dual Core 1.86 GHz. and 2 GB RAM. In these tests, computational time increases as higher variations are considered in the increasing OD demands. This result requires exploration with large networks in future research.



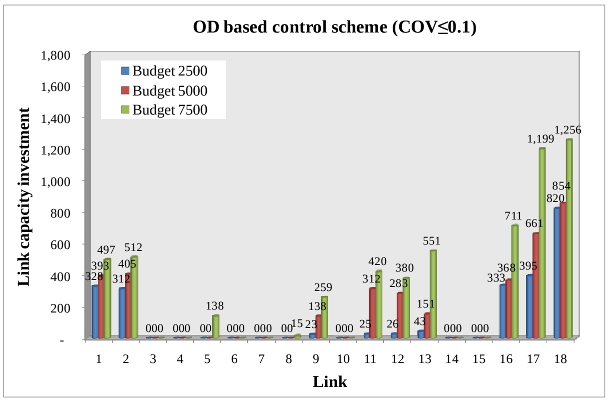
a) Mean capacity approach



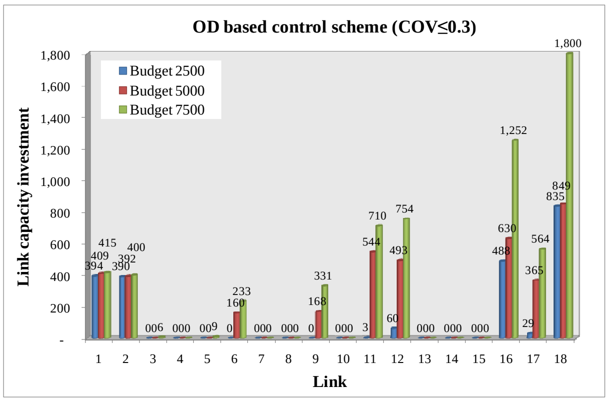
b) SD capacity approach



c) Trade-off approach



d) $\delta_{\theta_1}, \delta_{\theta_2} \leq 0.1$



e) $\delta_{\theta_1}, \delta_{\theta_2} \leq 0.3$

FIGURE 7 Link capacity investments based on the whole area and OD control schemes at different levels of budget.

CONCLUSIONS

This paper extended the original network reserve capacity to the case with stochastic demand. The FNDP is proposed to analyze the ability of the network to accommodate different levels of variation in future stochastic OD demands. The stochastic OD demand is assumed to follow a normal distribution with travellers' route choice behaviour following Probit SUE. By introducing multipliers to both mean and SD of the base OD demand distributions, the network capacity is assessed by the maximum weighted sum of the mean and SD of increased demands, satisfying the link capacity chance-constraints. The proposed model therefore represents the flexibility of the network in its robustness to OD demand variation. The proposed model can also be employed to determine an optimal network design to enhance the M-SD reserve capacity of the network. The optimal design also satisfied the probability link capacity and budget constraints. The paper adopted the implicit programming approach to solve this optimization problem, applying the SA method to obtain all essential derivatives.

The proposed model and algorithm were tested with a hypothetical network. Firstly, the reserve capacity of the existing network was evaluated, based on the whole area control scheme. The results were classified into three main categories: the mean capacity, SD capacity, and trade-off approaches. Secondly, the vulnerabilities of links were assessed, based on these three evaluation approaches. Next, the optimal network designs were determined at different budgets based on the three approaches under the whole area control strategy, and on two conditions under the OD based control scheme. The results suggest different investment strategies under mean and robustness perspective. The planners can focus on certain critical links (intensive investment program) to enhance the network in coping with typical growth of demand. On the other hand, the spread-investment strategy (investing on a larger set of links) is more appropriate for strengthening the network robustness against future uncertain demand pattern.

In current practice, the approach taken to evaluate the robustness of existing network or level of service (LOS) (at link and network level) is to assess the LOS of links at different assumptions of demand (e.g. low, medium, and high demand level). The proposed FNDP can provide an alternative modelling framework to evaluate the probability of different LOS of the network. Such probabilistic outcome of link-flow and travel time distributions can be compared against some required statistical criteria of LOS over a period of time (e.g. % of link flow to exceed the 70^{th} percentile of link capacity over a year). Future research should introduce the time dimension into the design process and apply the proposed model to a practical large scale case study to evaluate the computational efficiency.

ACKNOWLEDGEMENT

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APPENDIX A. Gradient of Objective Function

The gradient of objective function Eq. (14) with respect to $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, and $\mathbf{s}$ can be easily derived as $\tau.\mathbf{I}.\mathbf{q}$; $(1-\tau).\left\{\left(\boldsymbol{\theta}_2{}^T\right)^2.\boldsymbol{\sigma}_q^2\right\}^{-1/2}.\left(\boldsymbol{\theta}_2{}^T.\boldsymbol{\sigma}_q^2\right)$; and $\mathbf{0}$, respectively.

APPENDIX B. Jacobian of Link Capacity Chance-Constraints

Let $\boldsymbol{\Psi}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})$ be the vector of the capacity chance-constraints, Eq. (15), for all links. It can be rewritten as: $\mathbf{v}^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})+\Phi^{-1}(\boldsymbol{\alpha}).\boldsymbol{\sigma}_v^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})\le\mathbf{c}+\mathbf{s}$,

$$\boldsymbol{\Psi}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})=\mathbf{v}^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})+\Phi^{-1}(\boldsymbol{\alpha}).\boldsymbol{\sigma}_v^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})-\mathbf{c}-\mathbf{s}. \tag{B.1}$$

Then, the Jacobian of $\boldsymbol{\Psi}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})$ evaluated at $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, and $\mathbf{s}$ can be formulated as:

$$\nabla_{\boldsymbol{\theta}_1}\boldsymbol{\Psi}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})=\nabla_{\boldsymbol{\theta}_1}\mathbf{v}^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})+\Phi^{-1}(\boldsymbol{\alpha}).\nabla_{\boldsymbol{\theta}_1}\boldsymbol{\sigma}_v^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s}), \tag{B.2}$$

$$\nabla_{\boldsymbol{\theta}_2}\boldsymbol{\Psi}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})=\nabla_{\boldsymbol{\theta}_2}\mathbf{v}^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})+\Phi^{-1}(\boldsymbol{\alpha}).\nabla_{\boldsymbol{\theta}_2}\boldsymbol{\sigma}_v^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s}), \tag{B.3}$$

$$\nabla_{\mathbf{s}}\boldsymbol{\Psi}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})=\nabla_{\mathbf{s}}\mathbf{v}^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})+\Phi^{-1}(\boldsymbol{\alpha}).\nabla_{\mathbf{s}}\boldsymbol{\sigma}_v^*(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\mathbf{s})-\mathbf{I}_{(|A|\times|A|)}. \tag{B.4}$$

$\nabla_{\boldsymbol{\theta}_1}\mathbf{v}^*$, $\nabla_{\boldsymbol{\theta}_2}\mathbf{v}^*$, and $\nabla_{\mathbf{s}}\mathbf{v}^*$ in Eqs. (B.2)-(B.4) can be determined by using the mean link flow in Eq. (4) and the chain rule as follows:

$$\begin{aligned}\nabla_{\boldsymbol{\theta}_1}\mathbf{v}^* &=\sum_{rs=1}^{R}\boldsymbol{\Delta}_{rs}.\left(\tilde{q}_{rs}.\nabla_{\boldsymbol{\theta}_1}\mathbf{p}^{rs*}+\mathbf{p}^{rs*}.\nabla_{\boldsymbol{\theta}_1}\tilde{q}_{rs}\right)\\ &=\sum_{rs=1}^{R}\boldsymbol{\Delta}_{rs}.\left(\tilde{q}_{rs}.\nabla_{\boldsymbol{\theta}_1}\mathbf{p}^{rs*}+q_{rs}.\mathbf{p}^{rs*}\left(\mathbf{I}^{rs}_{(|R|\times|R|)}\right)^T\right),\end{aligned} \tag{B.5}$$

$$\begin{aligned}\nabla_{\boldsymbol{\theta}_2}\mathbf{v}^* &=\sum_{rs=1}^{R}\boldsymbol{\Delta}_{rs}.\left(\tilde{q}_{rs}.\nabla_{\boldsymbol{\theta}_2}\mathbf{p}^{rs*}+\mathbf{p}^{rs*}.\nabla_{\boldsymbol{\theta}_2}\tilde{q}_{rs}\right)\\ &=\sum_{rs=1}^{R}\boldsymbol{\Delta}_{rs}.\left(\tilde{q}_{rs}.\nabla_{\boldsymbol{\theta}_2}\mathbf{p}^{rs*}\right),\end{aligned} \tag{B.6}$$

$$\begin{aligned}\nabla_{\mathbf{s}}\mathbf{v}^* &=\sum_{rs=1}^{R}\boldsymbol{\Delta}_{rs}.\left(\tilde{q}_{rs}.\nabla_{\mathbf{s}}\mathbf{p}^{rs*}+\mathbf{p}^{rs*}.\nabla_{\mathbf{s}}\tilde{q}_{rs}\right)\\ &=\sum_{rs=1}^{R}\boldsymbol{\Delta}_{rs}.\left(\tilde{q}_{rs}.\nabla_{\mathbf{s}}\mathbf{p}^{rs*}\right).\end{aligned} \tag{B.7}$$

Similarly, $\nabla_{\boldsymbol{\theta}_1}\boldsymbol{\sigma}_v^*$, $\nabla_{\boldsymbol{\theta}_2}\boldsymbol{\sigma}_v^*$, and $\nabla_{\mathbf{s}}\boldsymbol{\sigma}_v^*$ can be derived from the link flow variance from variance-covariance matrix in Eq. (5) and the chain rule as follows:

$$\nabla_{\boldsymbol{\theta}_1}\boldsymbol{\sigma}_v^*=\frac{1}{2\boldsymbol{\sigma}_v^*}.diag\left[\boldsymbol{\Delta}.\left(\nabla_{\boldsymbol{\theta}_1}\boldsymbol{\Sigma}^f\right).\boldsymbol{\Delta}^T\right], \tag{B.8}$$

$$\nabla_{\boldsymbol{\theta}_2}\boldsymbol{\sigma}_v^*=\frac{1}{2\boldsymbol{\sigma}_v^*}.diag\left[\boldsymbol{\Delta}.\left(\nabla_{\boldsymbol{\theta}_2}\boldsymbol{\Sigma}^f\right).\boldsymbol{\Delta}^T\right], \tag{B.9}$$

$$\nabla_{\mathbf{s}}\boldsymbol{\sigma}_v^* = \frac{1}{2\boldsymbol{\sigma}_v^*}.diag\left[\boldsymbol{\Delta}.\left(\nabla_{\mathbf{s}}\boldsymbol{\Sigma}^f\right).\boldsymbol{\Delta}^T\right], \tag{B.10}$$

where the diagonal entries of $\nabla_{\boldsymbol{\theta}_1}\boldsymbol{\Sigma}^f$, $\nabla_{\boldsymbol{\theta}_2}\boldsymbol{\Sigma}^f$ and $\nabla_s\boldsymbol{\Sigma}^f$ can be calculated from:

$$\begin{aligned}\frac{\partial Var\left(F_k^{rs}\right)}{\partial\theta_1^{rs}} &= \frac{\partial\left(\tilde{\sigma}_q^{rs}\right)^2}{\partial\theta_1^{rs}}\left(p_k^{rs*}\right)^2 + \left(\tilde{\sigma}_q^{rs}\right)^2\frac{\partial\left(p_k^{rs*}\right)^2}{\partial\theta_1^{rs}} \\ &= \left(\tilde{\sigma}_q^{rs}\right)^2\left(2p_k^{rs*}\right)\frac{\partial p_k^{rs*}}{\partial\theta_1^{rs}},\end{aligned} \tag{B.11}$$

$$\begin{aligned}\frac{\partial Var\left(F_k^{rs}\right)}{\partial\theta_2^{rs}} &= \frac{\partial\left(\tilde{\sigma}_q^{rs}\right)^2}{\partial\theta_2^{rs}}\left(p_k^{rs*}\right)^2 + \left(\tilde{\sigma}_q^{rs}\right)^2\frac{\partial\left(p_k^{rs*}\right)^2}{\partial\theta_2^{rs}} \\ &= \left(2\theta_2^{rs}\right)\left(\sigma_q^{rs}\right)^2\left(p_k^{rs*}\right)^2 + \left(\tilde{\sigma}_q^{rs}\right)^2\left(2p_k^{rs*}\right)\frac{\partial p_k^{rs*}}{\partial\theta_2^{rs}},\end{aligned} \tag{B.12}$$

$$\begin{aligned}\frac{\partial Var\left(F_k^{rs}\right)}{\partial s_a} &= \frac{\partial\left(\tilde{\sigma}_q^{rs}\right)^2}{\partial s_a}\left(p_k^{rs*}\right)^2 + \left(\tilde{\sigma}_q^{rs}\right)^2\frac{\partial\left(p_k^{rs*}\right)^2}{\partial s_a} \\ &= \left(\tilde{\sigma}_q^{rs}\right)^2\left(2p_k^{rs*}\right)\frac{\partial p_k^{rs*}}{\partial s_a},\end{aligned} \tag{B.13}$$

and off-diagonal elements of $\nabla_{\boldsymbol{\theta}_1}\boldsymbol{\Sigma}^f$, $\nabla_{\boldsymbol{\theta}_2}\boldsymbol{\Sigma}^f$ and $\nabla_s\boldsymbol{\Sigma}^f$ can be derived from:

$$\begin{aligned}\frac{\partial Cov\left(F_k^{rs},F_j^{rs}\right)}{\partial\theta_1^{rs}} &= \frac{\partial\left(\tilde{\sigma}_q^{rs}\right)^2}{\partial\theta_1^{rs}}p_k^{rs*}p_j^{rs*} + \left(\tilde{\sigma}_q^{rs}\right)^2\left(p_k^{rs*}\frac{\partial p_j^{rs*}}{\partial\theta_1^{rs}} + p_j^{rs*}\frac{\partial p_k^{rs*}}{\partial\theta_1^{rs}}\right) \\ &= \left(\tilde{\sigma}_q^{rs}\right)^2\left(p_k^{rs*}\frac{\partial p_j^{rs*}}{\partial\theta_1^{rs}} + p_j^{rs*}\frac{\partial p_k^{rs*}}{\partial\theta_1^{rs}}\right),\end{aligned} \tag{B.14}$$

$$\begin{aligned}\frac{\partial Cov\left(F_k^{rs},F_j^{rs}\right)}{\partial\theta_2^{rs}} &= \frac{\partial\left(\tilde{\sigma}_q^{rs}\right)^2}{\partial\theta_2^{rs}}p_k^{rs*}p_j^{rs*} + \left(\tilde{\sigma}_q^{rs}\right)^2\left(p_k^{rs*}\frac{\partial p_j^{rs*}}{\partial\theta_2^{rs}} + p_j^{rs*}\frac{\partial p_k^{rs*}}{\partial\theta_2^{rs}}\right) \\ &= 2\theta_2^{rs}\left(\sigma_q^{rs}\right)^2 p_k^{rs*}p_j^{rs*} + \left(\tilde{\sigma}_q^{rs}\right)^2\left(p_k^{rs*}\frac{\partial p_j^{rs*}}{\partial\theta_2^{rs}} + p_j^{rs*}\frac{\partial p_k^{rs*}}{\partial\theta_2^{rs}}\right),\end{aligned} \tag{B.15}$$

$$\begin{aligned}\frac{\partial Cov\left(F_k^{rs},F_j^{rs}\right)}{\partial s_a} &= \frac{\partial\left(\tilde{\sigma}_q^{rs}\right)^2}{\partial s_a}p_k^{rs*}p_j^{rs*} + \left(\tilde{\sigma}_q^{rs}\right)^2\left(p_k^{rs*}\frac{\partial p_j^{rs*}}{\partial s_a} + p_j^{rs*}\frac{\partial p_k^{rs*}}{\partial s_a}\right) \\ &= \left(\tilde{\sigma}_q^{rs}\right)^2\left(p_k^{rs*}\frac{\partial p_j^{rs*}}{\partial s_a} + p_j^{rs*}\frac{\partial p_k^{rs*}}{\partial s_a}\right).\end{aligned} \tag{B.16}$$

To complete Eqs. (B.5)-(B.7) and (B.11)-(B.16), the sensitivity analysis of Probit SUE path choice probability ($\nabla_{\boldsymbol{\theta}_1}\mathbf{p}^*$, $\nabla_{\boldsymbol{\theta}_2}\mathbf{p}^*$, and $\nabla_{\mathbf{s}}\mathbf{p}^*$) are required.

Sensitivity Analysis of Probit SUE Path Choice Probability

Let $\boldsymbol{\omega}$ be the vector of three design parameters ($\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, and $\mathbf{s}$). The gap function of the path choice proportion in Eq. (11) is defined as $\Theta(\mathbf{p},\boldsymbol{\omega}) \equiv \mathbf{p} - \mathbf{Pr}(\mathbf{p}(\boldsymbol{\omega}),\boldsymbol{\omega})$. Let $\mathbf{p}^*(\boldsymbol{\omega})$ be the solution of Probit SUE, for any given value of $\boldsymbol{\omega}$, $\Theta(\mathbf{p}^*(\boldsymbol{\omega}),\boldsymbol{\omega}) = 0$. Assuming that all related functions are differentiable, the linear approximation of $\Theta(\mathbf{p},\boldsymbol{\omega})$ around $\Theta(\mathbf{p}^*(\boldsymbol{\omega}_0),\boldsymbol{\omega}_0)$ is defined as $\Theta(\mathbf{p},\boldsymbol{\omega}) \approx \Theta(\mathbf{p}^*(\boldsymbol{\omega}_0),\boldsymbol{\omega}_0) + \nabla_{\mathbf{p}}\Theta\big|_{\mathbf{p}^*(\boldsymbol{\omega}_0),\boldsymbol{\omega}_0}(\mathbf{p}-\mathbf{p}^*(\boldsymbol{\omega}_0)) + \nabla_{\boldsymbol{\omega}}\Theta\big|_{\mathbf{p}^*(\boldsymbol{\omega}_0),\boldsymbol{\omega}_0}(\boldsymbol{\omega}-\boldsymbol{\omega}_0)$, where $\nabla_{\mathbf{p}}\Theta\big|_{\mathbf{p}^*(\boldsymbol{\omega}_0),\boldsymbol{\omega}_0}$ and $\nabla_{\boldsymbol{\omega}}\Theta\big|_{\mathbf{p}^*(\boldsymbol{\omega}_0),\boldsymbol{\omega}_0}$ are the Jacobian matrices ($\mathbf{J}_1$ and $\mathbf{J}_2$) of Θ with respect to $\mathbf{p}$ and $\boldsymbol{\omega}$, evaluated at the solution $\mathbf{p}^*(\boldsymbol{\omega}_0)$ and $\boldsymbol{\omega}_0$. The equilibrium condition $\Theta(\mathbf{p},\boldsymbol{\omega}) = 0$ can be approximately solved for $\mathbf{p}(\boldsymbol{\omega})$, $\boldsymbol{\omega} \neq \boldsymbol{\omega}_0$, from $0 \approx 0 + \mathbf{J}_1(\mathbf{p}-\mathbf{p}^*(\boldsymbol{\omega}_0)) + \mathbf{J}_2(\boldsymbol{\omega}-\boldsymbol{\omega}_0)$, and hence:

$$\begin{aligned}\nabla_{\boldsymbol{\omega}}\mathbf{p}^* &= \lim_{\boldsymbol{\omega}\to\boldsymbol{\omega}_0}\frac{(\mathbf{p}-\mathbf{p}^*(\boldsymbol{\omega}_0))}{(\boldsymbol{\omega}-\boldsymbol{\omega}_0)} \\ &= -\mathbf{J}_1^{-1}\mathbf{J}_2,\end{aligned} \tag{B.17}$$

where

$$\mathbf{J}_1 = \mathbf{I} - \nabla_{E[\mathbf{C}]}\mathbf{Pr}.\Delta^T.\left(\nabla_{\tilde{\mathbf{v}}}\mathbf{t_a}.\nabla_{\mathbf{p}}\tilde{\mathbf{v}} + \nabla_{\tilde{\boldsymbol{\sigma}}_v^2}\mathbf{t_a}.\nabla_{\mathbf{p}}\tilde{\boldsymbol{\sigma}}_v^2\right), \tag{B.18}$$

$$\mathbf{J}_2 = -\nabla_{E[\mathbf{C}]}\mathbf{Pr}.\Delta^T.\nabla_{\mathbf{s}}\mathbf{t_a}. \tag{B.19}$$

To complete Eqs. (B.18) and (B.19), $\nabla_{E[\mathbf{C}]}\mathbf{Pr}$ can be determined following (*13*). Due to $\partial t_a(\tilde{x}_a)/\partial\tilde{x}_b = 0$ for $\forall a \neq b$, off-diagonal entries of $\nabla_{\tilde{\mathbf{v}}}\mathbf{t_a}$ and $\nabla_{\tilde{\boldsymbol{\sigma}}_v^2}\mathbf{t_a}$ are zero. When diagonal elements can be calculated from:

$$\frac{\partial t_a}{\partial\tilde{v}_a} = \frac{b_a}{(c_a+s_a)^4}\left\{4\tilde{v}_a^3 + 12\tilde{v}_a\left(\tilde{\sigma}_v^a\right)^2\right\}, \tag{B.20}$$

$$\frac{\partial t_a}{\partial\left(\tilde{\sigma}_v^a\right)^2} = \frac{b_a}{(c_a+s_a)^4}\left\{6\tilde{v}_a^2 + 6\left(\tilde{\sigma}_v^a\right)^2\right\}, \tag{B.21}$$

where $\tilde{v}_a$ and $\tilde{\sigma}_v^a$ are computed from perturbed mean demand $\tilde{\mathbf{q}}(\boldsymbol{\theta}_1)$, SD of demand $\tilde{\boldsymbol{\sigma}}_q(\boldsymbol{\theta}_2)$ using Eqs. (1)-(5).

In Eqs. (B.18) and (B.19), $\nabla_{\mathbf{p}}\tilde{\mathbf{v}}$ and $\nabla_{\mathbf{p}}\tilde{\boldsymbol{\sigma}}_{\mathbf{v}}$ can be derived from:

$$\begin{aligned}\nabla_{\mathbf{p}}\tilde{\mathbf{v}} &= \sum_{rs=1}^{R}\boldsymbol{\Delta}_{rs}.\left(\tilde{q}_{rs}.\nabla_{\mathbf{p}}\mathbf{p}^{rs} + \nabla_{\mathbf{p}}\tilde{q}_{rs}.\mathbf{p}^{rs}\right) \\ &= \sum_{rs=1}^{R}\boldsymbol{\Delta}_{rs}.\left(\tilde{q}_{rs}.\mathbf{I}_{(|K|\times|K|)}^{K_{rs}}\right),\end{aligned} \tag{B.22}$$

$$\nabla_{\mathbf{p}}\tilde{\boldsymbol{\sigma}}_v^2 = diag\left[\boldsymbol{\Delta}.\left(\nabla_{\mathbf{p}}\boldsymbol{\Sigma}^f\right).\boldsymbol{\Delta}^T\right], \tag{B.23}$$

where $\nabla_{\mathbf{p}}\boldsymbol{\Sigma}^f$ can be determined from:

$$\begin{aligned}\frac{\partial Var\left(F_k^{rs}\right)}{\partial p_k^{rs}} &= \frac{\partial\left(\tilde{\sigma}_q^{rs}\right)^2}{\partial p_k^{rs}}\left(p_k^{rs}\right)^2 + \left(\tilde{\sigma}_q^{rs}\right)^2 \frac{\partial\left(p_k^{rs}\right)^2}{\partial p_k^{rs}} \\ &= \left(\tilde{\sigma}_q^{rs}\right)^2\left(2p_k^{rs}\right),\end{aligned} \tag{B.24}$$

$$\begin{aligned}\frac{\partial Cov\left(F_k^{rs},F_j^{rs}\right)}{\partial p_k^{rs}} &= \frac{\partial\left(\tilde{\sigma}_q^{rs}\right)^2}{\partial p_k^{rs}} p_k^{rs} p_j^{rs} + \left(\tilde{\sigma}_q^{rs}\right)^2\left(p_k^{rs}\frac{\partial p_j^{rs}}{\partial p_k^{rs}} + p_j^{rs}\frac{\partial p_k^{rs}}{\partial p_k^{rs}}\right) \\ &= \left(\tilde{\sigma}_q^{rs}\right)^2 p_j^{rs}.\end{aligned} \tag{B.25}$$

To achieve Eq. (B.17), the second term, $\mathbf{J}_2$, can be derived from:

$$\begin{aligned}\nabla_{\boldsymbol{\theta}_1}\Theta &= -\nabla_{\mathbf{E[C]}}\mathbf{Pr}.\Delta^T.\left(\nabla_{\tilde{\mathbf{v}}}\mathbf{t_a}.\nabla_{\boldsymbol{\theta}_1}\tilde{\mathbf{v}} + \nabla_{(\tilde{\boldsymbol{\sigma}}_{\mathbf{v}})^2}\mathbf{t_a}.\nabla_{\boldsymbol{\theta}_1}\tilde{\boldsymbol{\sigma}}_v^2\right) \\ &= -\nabla_{\mathbf{E[C]}}\mathbf{Pr}.\Delta^T.\left(\nabla_{\tilde{\mathbf{v}}}\mathbf{t_a}.\nabla_{\boldsymbol{\theta}_1}\tilde{\mathbf{v}}\right),\end{aligned} \tag{B.26}$$

$$\begin{aligned}\nabla_{\boldsymbol{\theta}_2}\Theta &= -\nabla_{\mathbf{E[C]}}\mathbf{Pr}.\Delta^T.\left(\nabla_{\tilde{\mathbf{v}}}\mathbf{t_a}.\nabla_{\boldsymbol{\theta}_2}\tilde{\mathbf{v}} + \nabla_{\tilde{\boldsymbol{\sigma}}_v^2}\mathbf{t_a}.\nabla_{\boldsymbol{\theta}_2}\tilde{\boldsymbol{\sigma}}_v^2\right) \\ &= -\nabla_{\mathbf{E[C]}}\mathbf{Pr}.\Delta^T.\left(\nabla_{\tilde{\boldsymbol{\sigma}}_v^2}\mathbf{t_a}.\nabla_{\boldsymbol{\theta}_2}\tilde{\boldsymbol{\sigma}}_v^2\right),\end{aligned} \tag{B.27}$$

$$\nabla_{\mathbf{s}}\Theta = -\nabla_{\mathbf{E[C]}}\mathbf{Pr}.\Delta^T.\nabla_{\mathbf{s}}\mathbf{t_a}, \tag{B.28}$$

where $\nabla_{\tilde{\mathbf{v}}}\mathbf{t_a}$ in Eq. (B.26) and $\nabla_{\tilde{\boldsymbol{\sigma}}_v^2}\mathbf{t_a}$ in Eq. (B.27) can be obtained from Eqs. (B.20) and (B.21), respectively.

To complete Eq. (B.26), $\nabla_{\boldsymbol{\theta}_1}\tilde{\mathbf{v}}$ can be determined from:

$$\begin{aligned}\nabla_{\boldsymbol{\theta}_1}\tilde{\mathbf{v}} &= \sum_{rs=1}^{R}\boldsymbol{\Delta}_{rs}.\left(\nabla_{\boldsymbol{\theta}_1}\tilde{q}_{rs}\right).\mathbf{p}^{rs} \\ &= \sum_{rs=1}^{R}\boldsymbol{\Delta}_{rs}.q_{rs}.\mathbf{p}^{rs}.\left(\mathbf{I}_{(|R|\times|R|)}^{rs}\right)^T.\end{aligned} \tag{B.29}$$

In Eq. (B.27), $\nabla_{\boldsymbol{\theta}_2}\boldsymbol{\sigma}_v^2$ can be calculated from $diag\left[\boldsymbol{\Delta}.\left(\nabla_{\boldsymbol{\theta}_2}\boldsymbol{\Sigma}^f\right).\boldsymbol{\Delta}^T\right]$, whence the components of $\nabla_{\boldsymbol{\theta}_2}\boldsymbol{\Sigma}^f$ can be calculated from:

$$\begin{aligned}\frac{\partial Var\left(F_k^{rs}\right)}{\partial \theta_2^{rs}} &= \frac{\partial\left(\tilde{\sigma}_q^{rs}\right)^2}{\partial \theta_2^{rs}}\left(p_k^{rs}\right)^2 \\ &= 2\theta_2^{rs}\left(\sigma_q^{rs}\right)^2\left(p_k^{rs}\right)^2,\end{aligned} \tag{B.30}$$

$$\begin{aligned}\frac{\partial Cov\left(F_k^{rs},F_j^{rs}\right)}{\partial \theta_2^{rs}} &= \frac{\partial\left(\tilde{\sigma}_q^{rs}\right)^2}{\partial \theta_2^{rs}} p_k^{rs} p_j^{rs} \\ &= 2\theta_2^{rs}\left(\sigma_q^{rs}\right)^2 p_k^{rs} p_j^{rs}.\end{aligned} \tag{B.31}$$

Finally, in Eq. (B.28), each diagonal element of $\nabla_{\mathbf{s}}\mathbf{t_a}$ can be calculated from:

$$\frac{\partial t_a}{\partial s_a} = \frac{-4b_a}{\left(c_a+s_a\right)^5}\left\{\tilde{v}_a^4 + 6\tilde{v}_a^2\left(\tilde{\sigma}_v^a\right)^2 + 3\left(\tilde{\sigma}_v^a\right)^4\right\}. \tag{B.32}$$

In summary, all derivatives in Eqs. (B.18)-(B.31) can be substituted in the reverse order to obtain the Jacobian of the Probit path choice proportions. Then, these results can be substituted in Eqs. (B.5)-(B.7) and (B.11)-(B.16) to complete the Jacobian of the link capacity chance-constraints (B.2)-(B.4).

APPENDIX C. Jacobians of OD Demand Multiplier Constraints

Let $\Psi_{\mathrm{cv}\theta_1}$ and $\Psi_{\mathrm{cv}\theta_2}$ be the OD demand multiplier constraints (17) and (18), respectively. These two constraints can be rewritten as $\Psi_{\mathrm{cv}\theta_1} = \sqrt{\frac{1}{R-1}\sum_{rs=1}^{R}\left(\theta_1^{rs} - \frac{1}{R}\sum_{rs=1}^{R}\theta_1^{rs}\right)^2} - \delta_{\theta_1}\frac{1}{R}\sum_{rs=1}^{R}\theta_1^{rs} = 0$ and $\Psi_{\mathrm{cv}\theta_2} = \sqrt{\frac{1}{R-1}\sum_{rs=1}^{R}\left(\theta_2^{rs} - \frac{1}{R}\sum_{rs=1}^{R}\theta_2^{rs}\right)^2} - \delta_{\theta_2}\frac{1}{R}\sum_{rs=1}^{R}\theta_2^{rs} = 0$, respectively. The entries of Jacobians $\nabla_{\boldsymbol{\theta}_1}\Psi_{\mathrm{cv}\theta_1}$ and $\nabla_{\boldsymbol{\theta}_2}\Psi_{\mathrm{cv}\theta_2}$ can be similarly derived from:

$$\frac{\partial \Psi_{\mathrm{cv}\theta_1}}{\partial \theta_1^{rs}} = \left\{\frac{1}{R-1}\sum_{rs=1}^{R}\left(\theta_1^{rs} - \frac{1}{R}\sum_{rs=1}^{R}\theta_1^{rs}\right)^2\right\}^{-\frac{1}{2}} \frac{1}{R-1}\left(\theta_1^{rs} - \frac{1}{R}\sum_{rs=1}^{R}\theta_1^{rs}\right)\left(1-\frac{1}{R}\right) - \delta_{\theta_1}\frac{1}{R}, \tag{B.33}$$

$$\frac{\partial \Psi_{\mathrm{cv}\theta_2}}{\partial \theta_2^{rs}} = \left\{\frac{1}{R-1}\sum_{rs=1}^{R}\left(\theta_2^{rs} - \frac{1}{R}\sum_{rs=1}^{R}\theta_2^{rs}\right)^2\right\}^{-\frac{1}{2}} \frac{1}{R-1}\left(\theta_2^{rs} - \frac{1}{R}\sum_{rs=1}^{R}\theta_2^{rs}\right)\left(1-\frac{1}{R}\right) - \delta_{\theta_2}\frac{1}{R}. \tag{B.34}$$

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