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Gibbs cluster measures on configuration spaces

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Abstract

The distribution $g_{\rm cl}$ of a Gibbs cluster point process in $X=\mathbb{R}^d$ (with i.i.d. random clusters attached to points of a Gibbs configuration with distribution g) is studied via the projection of an auxiliary Gibbs measure $\hat{g}$ in the space of configurations $\hat{\gamma}=\{(x,\bar{y})\}\subset X\times\mathfrak{X}$, where $x\in X$ indicates a cluster "center" and $\bar{y}\in\mathfrak{X}:=\bigsqcup_n X^n$ represents a corresponding cluster relative to x. We show that the measure $g_{\rm cl}$ is quasi-invariant with respect to the group $\mathrm{Diff}_0(X)$ of compactly supported diffeomorphisms of X, and prove an integration-by-parts formula for $g_{\rm cl}$. The associated equilibrium stochastic dynamics is then constructed using the method of Dirichlet forms. These results are quite general; in particular, the uniqueness of the background Gibbs measure g is not required. The paper is an extension of the earlier results for Poisson cluster measures [J. Funct. Analysis 256 (2009) 432–478], where a different projection construction was utilized specific to this "exactly soluble" case.

Keywords: Cluster point process, Gibbs measure, Poisson measure, Interaction potential, Configuration space, Quasi-invariance, Integration by parts, Dirichlet form, Stochastic dynamics
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1. Introduction

The concept of particle configurations is instrumental in mathematical modelling of multi-component stochastic systems. Rooted in statistical mechanics and theory of point processes, the development of the general mathematical framework for suitable classes of configurations has been a recurrent research theme fostered by widespread applications across the board, including quantum physics, astrophysics, chemical physics, biology, ecology, computer science, economics, finance, etc. (see an extensive bibliography in [9]).

In the past 15 years or so, there has been a more specific interest in the *analysis* on configuration spaces. To fix basic notation, let X be a topological space (e.g., a Euclidean space $X = \mathbb{R}^d$), and let $\Gamma_X = \{\gamma\}$ be the configuration space over X, that is, the space of countable subsets (called *configurations*) $\gamma \subset X$ without accumulation points. Albeverio, Kondratiev and Röckner [2, 3] have proposed an approach to configuration spaces Γ_X as *infinite-dimensional manifolds*, based on the choice of a suitable probability measure μ on Γ_X which is quasi-invariant with respect to $\mathrm{Diff}_0(X)$, the group of compactly supported diffeomorphisms of X. Providing that the measure μ can be shown to satisfy an integration-by-parts formula, one can construct, using the theory of Dirichlet forms, an associated equilibrium dynamics (stochastic process) on Γ_X such that μ is its invariant measure [2, 3, 22] (see [1, 3, 4, 26] and references therein for further discussion of various theoretical aspects and applications).

This general programme has been first implemented in [2] for the *Poisson* measure μ on Γ_X, and then extended in [3] to a wider class of *Gibbs* measures, which appear in statistical mechanics of classical continuous gases. In the Poisson case, the canonical equilibrium dynamics is given by the well-known independent particle process, that is, an infinite family of independent (distorted) Brownian motions started at the points of a random Poisson configuration. In the Gibbsian case, the equilibrium dynamics is much more complex due to interaction between the particles.

In our earlier papers [6, 7], a similar analysis was developed for a different class of random spatial structures, namely *Poisson cluster point processes*, featured by spatial grouping (“clustering”) of points around the background random (Poisson) configuration of invisible “centers”. Cluster models are well known in the general theory of random point processes [8, 9] and are widely used in numerous applications ranging from neurophysiology (nerve impulses) and ecology (spatial aggregation of species) to seismology (earthquakes) and cosmology (constellations and galaxies); see [7, 8, 9] for some references to original papers.

Our technique in [6, 7] was based on the representation of a given Poisson cluster measure on the configuration space $\varGamma_X$ as the projection image of an auxiliary Poisson measure on a more complex configuration space $\varGamma_{\mathfrak{X}}$ over the disjoint-union space $\mathfrak{X} := \bigsqcup_n X^n$, with "droplet" points $\bar{y} \in \mathfrak{X}$ representing individual clusters (of variable size). The principal advantage of this construction is that it allows one to apply the well-developed apparatus of Poisson measures to the study of the Poisson cluster measure.

In the present paper,[3] our aim is to extend this approach to a more general class of *Gibbs cluster measures* on the configuration space $\varGamma_X$, where the distribution of cluster centers is given by a Gibbs (grand canonical) measure $g \in \mathscr{G}(\theta, \varPhi)$ on $\varGamma_X$, with a reference measure θ on X and an interaction potential $\varPhi$. We focus on Gibbs cluster processes in $X = \mathbb{R}^d$ with i.i.d. random clusters of random size. Let us point out that we do not require the uniqueness of the Gibbs measure, so our results are not affected by possible "phase transitions" (i.e., non-uniqueness of $g \in \mathscr{G}(\theta, \varPhi)$). Under some natural smoothness conditions on the reference measure θ and the distribution η of the generic cluster, we prove the $\mathrm{Diff}_0(X)$-quasi-invariance of the corresponding Gibbs cluster measure g_{cl} (Section 3.2), establish the integration-by-parts formula (Section 3.3) and construct the associated Dirichlet operator, which leads to the existence of the equilibrium stochastic dynamics on the configuration space $\varGamma_X$ (Section 4).

Unlike the Poisson cluster case, it is now impossible to work with the measure arising in the space $\varGamma_{\mathfrak{X}}$ of droplet configurations $\bar{\gamma} = \{\bar{y}\}$, which is hard to characterize for Gibbs cluster measures. Instead, in order to be able to pursue our projection approach while still having a tractable pre-projection measure, we choose the configuration space $\varGamma_{\mathcal{Z}}$ over the set $\mathcal{Z} := X \times \mathfrak{X}$, where each configuration $\hat{\gamma} \in \varGamma_{\mathcal{Z}}$ is a (countable) set of pairs $z = (x, \bar{y})$ with $x \in X$ indicating a cluster center and $\bar{y} \in \mathfrak{X}$ representing a cluster attached to x. A crucial step is to show that the corresponding measure $\hat{g}$ on $\varGamma_{\mathcal{Z}}$ is again Gibbsian, with the reference measure $\sigma = \theta \otimes \eta$ and a "cylinder" interaction potential $\hat{\varPhi}(\hat{\gamma}) := \varPhi(\mathfrak{p}(\hat{\gamma}))$, where $\varPhi$ is the original interaction potential associated with the background Gibbs measure g and $\mathfrak{p}$ is the operator on the configuration space $\varGamma_{\mathcal{Z}}$ projecting a configuration $\hat{\gamma} = \{(x, \bar{y})\}$ to the configuration of cluster centers, $\gamma = \{x\}$. We then project the Gibbs measure $\hat{g}$ from the "higher floor" $\varGamma_{\mathcal{Z}}$ directly to the configuration space $\varGamma_X$ (thus skipping the "intermediate floor" $\varGamma_{\mathfrak{X}}$), and show that the resulting measure coincides with the original Gibbs cluster measure g_{cl} (Section 2).

[3] Some of our results have been announced in [5] (in the case of clusters of fixed size).

In fact, it can be we shown (Section 2.3) that *any* cluster measure $\mu_{\rm cl}$ on $\varGamma_X$ can be obtained by a similar projection from $\varGamma_{\mathcal{Z}}$. Even though it may not always be possible to find an intrinsic characterization of the corresponding lifted measure $\hat{\mu}$ on the configuration space $\varGamma_{\mathcal{Z}}$ (unlike the Poisson and Gibbs cases), we expect that the projection approach can be instrumental in the study of more general cluster point processes by a reduction to point processes in more complex phase spaces but with a simpler correlation structure. We intend to develop these ideas elsewhere.

2. Gibbs cluster measures via projections

In this section, we start by recalling some basic concepts and notations for random point processes and associated probability measures in configuration spaces (Section 2.1), followed in Section 2.2 by a definition of a general cluster point process (CPP). In Section 2.3, we explain our main "projection" construction allowing one to represent CPPs in the phase space X in terms of auxiliary measures on a more complex configuration space involving Cartesian powers of X. The implications of such a description are discussed in greater detail for the particular case of Gibbs CPPs (Sections 2.4, 2.5).

2.1. Probability measures on configuration spaces

Let X be a Polish space equipped with the Borel σ-algebra $\mathcal{B}(X)$ generated by the open sets. Denote $\overline{\mathbb{Z}}_+ := \mathbb{Z}_+ \cup \{\infty\}$, where $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, and consider a space $\mathfrak{X}$ built from all Cartesian powers of X, that is, the disjoint union

$$\mathfrak{X} := \bigsqcup_{n\in\overline{\mathbb{Z}}_+} X^n, \tag{2.1}$$

including $X^0 = \{\emptyset\}$ and the space X^∞ of infinite sequences $(x_1, x_2, \dots)$. That is, $\bar{x} = (x_1, x_2, \dots) \in \mathfrak{X}$ if and only if $\bar{x} \in X^n$ for some $n \in \overline{\mathbb{Z}}_+$. We take the liberty to write $x_i \in \bar{x}$ if x_i is a coordinate of the "vector" $\bar{x}$. The space $\mathfrak{X}$ is endowed with the natural disjoint union topology induced by the topology in X.

Remark 2.1. Note that a set $K \subset \mathfrak{X}$ is compact if and only if $K = \bigsqcup_{n=0}^{N} K_n$, where $N < \infty$ and K_n are compact subsets of X^n, respectively.

Remark 2.2. $\mathfrak{X}$ is a Polish space as a disjoint union of Polish spaces.

Denote by $\mathcal{N}(X)$ the space of $\overline{\mathbb{Z}}_+$-valued measures $N(\cdot)$ on $\mathcal{B}(X)$ with countable (i.e., finite or countably infinite) support. Consider the natural projection

$$\mathfrak{X} \ni \bar{x} \mapsto \mathfrak{p}(\bar{x}) := \sum_{x_i\in\bar{x}} \delta_{x_i} \in \mathcal{N}(X), \tag{2.2}$$

where δ_x is the Dirac measure at point $x \in X$. That is to say, under the map $\mathfrak{p}$ each vector from $\mathfrak{X}$ is "unpacked" into its components to yield a countable aggregate of (possibly multiple) points in X, which can be interpreted as a generalized configuration γ,

$$\mathfrak{p}(\bar{x}) \leftrightarrow \gamma := \bigsqcup_{x_i \in \bar{x}} \{x_i\}, \qquad \bar{x} = (x_1, x_2, \dots) \in \mathfrak{X}. \tag{2.3}$$

In what follows, we interpret the notation γ either as an aggregate of points in X or as a $\overline{\mathbb{Z}}_+$-valued measure or both, depending on the context. Even though generalized configurations are not, strictly speaking, subsets of X (because of possible multiplicities), it is convenient to use set-theoretic notations, which should not cause any confusion. For instance, we write $\gamma \cap B$ for the restriction of configuration γ to a subset $B \in \mathcal{B}(X)$. For a function $f : X \to \mathbb{R}$ we denote

$$\langle f, \gamma \rangle := \sum_{x_i \in \gamma} f(x_i) \equiv \int_X f(x)\, \gamma(\mathrm{d}x). \tag{2.4}$$

In particular, if $\mathbf{1}_B(x)$ is the indicator function of a set $B \in \mathcal{B}(X)$ then $\langle \mathbf{1}_B, \gamma \rangle = \gamma(B)$ is the total number of points (counted with their multiplicities) in $\gamma \cap B$.

Definition 2.1. A *configuration space* $\mathit{\Gamma}_X^\sharp$ is the set of generalized configurations γ in X, endowed with the *cylinder σ-algebra* $\mathcal{B}(\mathit{\Gamma}_X^\sharp)$ generated by the class of cylinder sets $C_B^n := \{\gamma \in \mathit{\Gamma}_X^\sharp : \gamma(B) = n\},\ B \in \mathcal{B}(X),\ n \in \mathbb{Z}_+$.

Remark 2.3. It is easy to see that the map $\mathfrak{p} : \mathfrak{X} \to \mathit{\Gamma}_X^\sharp$ defined by formula (2.3) is measurable.

In fact, conventional theory of point processes (and their distributions as probability measures on configuration spaces) usually rules out the possibility of accumulation points or multiple points (see, e.g., [9]).

Definition 2.2. A configuration $\gamma \in \mathit{\Gamma}_X^\sharp$ is said to be *locally finite* if $\gamma(B) < \infty$ for any compact set $B \subset X$. A configuration $\gamma \in \mathit{\Gamma}_X^\sharp$ is called *simple* if $\gamma(\{x\}) \le 1$ for each $x \in X$. A configuration $\gamma \in \mathit{\Gamma}_X^\sharp$ is called *proper* if it is both locally finite and simple. The set of proper configurations will be denoted by $\mathit{\Gamma}_X$ and called the *proper configuration space* over X. The corresponding σ-algebra $\mathcal{B}(\mathit{\Gamma}_X)$ is generated by the cylinder sets $\{\gamma \in \mathit{\Gamma}_X : \gamma(B) = n\}$ ($B \in \mathcal{B}(X),\ n \in \mathbb{Z}_+$).

Like in the standard theory based on proper configuration spaces (see, e.g., [9, § 6.1]), every probability measure μ on the generalized configuration space $\mathit{\Gamma}_X^\sharp$

can be characterized by its Laplace functional (cf. [7])

$$L_\mu(f) := \int_{\Gamma_X^\sharp} \mathrm{e}^{-\langle f,\gamma\rangle}\,\mu(\mathrm{d}\gamma), \qquad f\in \mathrm{M}_+(X), \tag{2.5}$$

where $\mathrm{M}_+(X)$ is the class of measurable non-negative functions on X.

2.2. Cluster point processes

Let us recall the notion of a general cluster point process (CPP). Its realizations are constructed in two steps: (i) a background random configuration of (invisible) "centers" is obtained as a realization of some point process γ_c governed by a probability measure μ_c on $\Gamma_X^\sharp$, and (ii) relative to each center $x\in\gamma_\mathrm{c}$, a set of observable secondary points (referred to as a *cluster* centered at x) is generated according to a point process γ'_x with probability measure μ_x on $\Gamma_X^\sharp$ ($x\in X$).

The resulting (countable) assembly of random points, called the *cluster point process*, can be symbolically expressed as

$$\gamma = \bigsqcup_{x\in\gamma_\mathrm{c}} \gamma'_x \in \Gamma_X^\sharp,$$

where the disjoint union signifies that multiplicities of points should be taken into account. More precisely, assuming that the family of secondary processes $\gamma'_x(\cdot)$ is measurable as a function of $x\in X$, the integer-valued measure corresponding to a CPP realization γ is given by

$$\gamma(B) = \int_X \gamma'_x(B)\,\gamma_\mathrm{c}(\mathrm{d}x) = \sum_{x\in\gamma_\mathrm{c}} \gamma'_x(B), \qquad B\in\mathcal{B}(X). \tag{2.6}$$

In what follows, we assume that (i) X is a linear space (e.g., $X=\mathbb{R}^d$) so that translations $X\ni y\mapsto y+x\in X$ are defined, and (ii) random clusters are independent and identically distributed (i.i.d.), being governed by the same probability law translated to the cluster centers, so that, for any $x\in X$, we have $\mu_x(A)=\mu_0(A-x)$ ($A\in\mathcal{B}(\Gamma_X^\sharp)$).

In turn, the measure μ_0 on $\Gamma_X^\sharp$ determines a probability distribution η in $\mathfrak{X}$ which is symmetric with respect to permutations of coordinates. Conversely, μ_0 is a push-forward of the measure η under the projection map $\mathfrak{p}:\mathfrak{X}\to\Gamma_X^\sharp$ defined by (2.3), that is,

$$\mu_0 = \mathfrak{p}^*\eta \equiv \eta\circ\mathfrak{p}^{-1}. \tag{2.7}$$

Remark 2.4. Unlike the standard CPP theory when sample configurations are *presumed* to be a.s. locally finite (see, e.g., [9, Definition 6.3.I]), the description of the CPP given above only implies that its configurations γ are countable aggregates in X, but possibly with multiple and/or accumulation points, even if the background point process γ_{c} is proper. Therefore, the distribution μ of the CPP (2.6) is a probability measure defined on the space $\varGamma_X^\sharp$ of *generalized* configurations. It is a matter of interest to obtain conditions in order that μ be actually supported on the proper configuration space $\varGamma_X$, and we shall address this issue in Section 2.4 below for Gibbs CPPs (see [7] for the case of Poisson CPPs).

The following fact is well known in the case of CPPs without accumulation points (see, e.g., [9, $\S$ 6.3]); its proof in the general case is essentially the same (see [7, Proposition 2.5]).

Proposition 2.1. *Let μ_{cl} be a probability measure on $(\varGamma_X^\sharp, \mathcal{B}(\varGamma_X^\sharp))$ determined by the probability distribution of a CPP* (2.6)*. Then its Laplace functional is given, for all functions $f \in \mathrm{M}_+(X)$, by*

$$L_{\mu_{\mathrm{cl}}}(f) = \int_{\varGamma_X^\sharp} \prod_{x\in\gamma_{\mathrm{c}}} \left(\int_{\mathfrak{X}} \exp\Bigl(- \sum_{y_i\in\bar{y}} f(y_i + x) \Bigr) \, \eta(\mathrm{d}\bar{y}) \right) \mu_{\mathrm{c}}(\mathrm{d}\gamma_{\mathrm{c}}). \tag{2.8}$$

2.3. A projection construction of cluster measures on configurations

Denote $\mathcal{Z} := X \times \mathfrak{X}$, and consider the space $\varGamma_{\mathcal{Z}}^\sharp = \{\hat{\gamma}\}$ of (generalized) configurations in $\mathcal{Z}$. Let $p_X : \mathcal{Z} \to X$ be the natural projection to the first coordinate,

$$\mathcal{Z} \ni z = (x, \bar{y}) \mapsto p_X(z) := x \in X, \tag{2.9}$$

and consider its pointwise lifting to the configuration space $\varGamma_{\mathcal{Z}}^\sharp$ (preserving the same notation p_X), defined as follows

$$\varGamma_{\mathcal{Z}}^\sharp \ni \hat{\gamma} \mapsto p_X(\hat{\gamma}) := \bigsqcup_{z\in\hat{\gamma}} \{p_X(z)\} \in \varGamma_X^\sharp. \tag{2.10}$$

Let μ_{cl} denote the probability measure on the configuration space $\varGamma_X^\sharp$ associated with an i.i.d. cluster point process (see Section 2.2), specified by measures μ_{c} on $\varGamma_X^\sharp$ and η on $\mathfrak{X}$.

Definition 2.3. Let us define a probability measure $\hat{\mu}$ on $\varGamma_{\mathcal{Z}}^\sharp$ as the distribution of random configurations $\hat{\gamma}$ over $\mathcal{Z}$ obtained from configurations $\gamma \in \varGamma_X^\sharp$ by attaching to each point $x \in \gamma$ an i.i.d. random vector $\bar{y}_x$ with distribution η:

$$\varGamma_X^\sharp \ni \gamma_{\mathrm{c}} \mapsto \hat{\gamma} := \bigsqcup_{x\in\gamma_{\mathrm{c}}} \{(x, \bar{y}_x)\} \in \varGamma_{\mathcal{Z}}^\sharp. \tag{2.11}$$

Geometrically, the construction (2.11) may be viewed as random i.i.d. pointwise translations of configurations γ from X into the "plane" $\mathcal{Z} = X \times \mathfrak{X}$. The measure $\hat{\mu}$ so obtained may be expressed in the differential form as a skew product

$$\hat{\mu}(\mathrm{d}\hat{\gamma}) = \mu_{\mathrm{c}}(p_X(\mathrm{d}\hat{\gamma})) \bigotimes_{z\in\hat{\gamma}} \eta(p_{\mathfrak{X}}(\mathrm{d}z)), \qquad \hat{\gamma} \in \varGamma^{\sharp}_{\mathcal{Z}}. \tag{2.12}$$

Equivalently, for any function $F \in \mathrm{M}_+(\varGamma^{\sharp}_{\mathcal{Z}})$,

$$\int_{\varGamma^{\sharp}_{\mathcal{Z}}} F(\hat{\gamma})\, \hat{\mu}(\mathrm{d}\hat{\gamma}) = \int_{\varGamma^{\sharp}_{X}} \left(\int_{\mathfrak{X}^\infty} F\Bigl(\bigcup_{x\in\gamma} \{(x,\bar{y})\} \Bigr) \bigotimes_{x\in\gamma} \eta(\mathrm{d}\bar{y}) \right) \mu_{\mathrm{c}}(\mathrm{d}\gamma). \tag{2.13}$$

Remark 2.5. Note that formula (2.13) is a simple case of the general disintegration theorem, or the "total expectation formula" (see, e.g., [15, Theorem 5.4] or [24, Ch. V, §8, Theorem 8.1]).

Recall that the "unpacking" map $\mathfrak{p} : \mathfrak{X} \to \varGamma^{\sharp}_X$ is defined in (2.3), and consider a map $\mathfrak{q} : \mathcal{Z} \to \varGamma^{\sharp}_X$ acting by the formula

$$\mathfrak{q}(x,\bar{y}) := \mathfrak{p}(\bar{y} + x) = \bigsqcup_{y_i\in\bar{y}} \{y_i + x\}, \qquad (x,\bar{y}) \in \mathcal{Z}. \tag{2.14}$$

Here and below, we use the shift notation ($x \in X$)

$$\bar{y} + x := (y_1 + x,\, y_2 + x, \dots), \qquad \bar{y} = (y_1, y_2, \dots) \in \mathfrak{X}, \tag{2.15}$$

In the usual "diagonal" way, the map $\mathfrak{q}$ can be lifted to the configuration space $\varGamma^{\sharp}_{\mathcal{Z}}$:

$$\varGamma^{\sharp}_{\mathcal{Z}} \ni \hat{\gamma} \mapsto \mathfrak{q}(\hat{\gamma}) := \bigsqcup_{z\in\hat{\gamma}} \mathfrak{q}(z) \in \varGamma^{\sharp}_X. \tag{2.16}$$

Proposition 2.2. *The map* $\mathfrak{q} : \varGamma^{\sharp}_{\mathcal{Z}} \to \varGamma^{\sharp}_X$ *defined by* (2.16) *is measurable.*

Proof. Observe that $\mathfrak{q}$ can be represented as a composition

$$\mathfrak{q} = \mathfrak{p} \circ p_{\mathfrak{X}} \circ p_+ : \varGamma^{\sharp}_{\mathcal{Z}} \xrightarrow{p_+} \varGamma^{\sharp}_{\mathcal{Z}} \xrightarrow{p_{\mathfrak{X}}} \varGamma^{\sharp}_{\mathfrak{X}} \xrightarrow{\mathfrak{p}} \varGamma^{\sharp}_X, \tag{2.17}$$

where the maps p_+, $p_{\mathfrak{X}}$ and $\mathfrak{p}$ are defined, respectively, by

$$\varGamma^{\sharp}_{\mathcal{Z}} \ni \hat{\gamma} \mapsto p_+(\hat{\gamma}) := \bigsqcup_{(x,\bar{y})\in\hat{\gamma}} \{(x, \bar{y} + x)\} \in \varGamma^{\sharp}_{\mathcal{Z}}, \tag{2.18}$$

$$\varGamma^{\sharp}_{\mathcal{Z}} \ni \hat{\gamma} \mapsto p_{\mathfrak{X}}(\hat{\gamma}) := \bigsqcup_{(x,\bar{y})\in\hat{\gamma}} \{\bar{y}\} \in \varGamma^{\sharp}_{\mathfrak{X}}, \tag{2.19}$$

$$\varGamma^{\sharp}_{\mathfrak{X}} \ni \bar{\gamma} \mapsto \mathfrak{p}(\bar{\gamma}) := \bigsqcup_{\bar{y}\in\bar{\gamma}} \mathfrak{p}(\bar{y}) \in \varGamma^{\sharp}_X. \tag{2.20}$$

To verify that the map $p_+ : \Gamma_{\mathcal{Z}}^\sharp \to \Gamma_{\mathcal{Z}}^\sharp$ is measurable, note that for a cylinder set

$$C^n_{B_1\times\bar{B}} = \{\hat{\gamma} \in \Gamma_{\mathcal{Z}} : \hat{\gamma}(B_1 \times \bar{B}) = n\} \in \mathcal{B}(\Gamma_{\mathcal{Z}}^\sharp),$$

with $B_1 \in \mathcal{B}(X)$, $\bar{B} \in \mathcal{B}(\mathfrak{X})$ and $n \in \mathbb{Z}_+$, its pre-image under p_+ is given by

$$p_+^{-1}(C^n_{B_1\times\bar{B}}) = C^n_A = \{\hat{\gamma} \in \Gamma_{\mathcal{Z}}^\sharp : \hat{\gamma}(A) = n\},$$

where

$$A := \{(x, \bar{y}) \in \mathcal{Z} : (x, \bar{y} + x) \in B_1 \times \bar{B}\} \in \mathcal{B}(\mathcal{Z}),$$

since the indicator $\mathbf{1}_A(x, \bar{y}) = \mathbf{1}_{B_1}(x) \cdot \mathbf{1}_{\bar{B}}(\bar{y} + x)$ is obviously a measurable function on $\mathcal{Z} = X \times \mathfrak{X}$. Similarly, for $p_{\mathfrak{X}} : \Gamma_{\mathcal{Z}}^\sharp \to \Gamma_{\mathfrak{X}}^\sharp$ (see (2.19)) we have

$$p_{\mathfrak{X}}^{-1}(C^n_{\bar{B}}) = C^n_{X\times\bar{B}} = \{\hat{\gamma} \in \Gamma_{\mathcal{Z}}^\sharp : \hat{\gamma}(X \times \bar{B}) = n\} \in \mathcal{B}(\Gamma_{\mathcal{Z}}^\sharp),$$

since $X \times \bar{B} \in \mathcal{B}(\mathcal{Z})$. Finally, measurability of the projection $\mathfrak{p} : \Gamma_{\mathfrak{X}}^\sharp \to \Gamma_X^\sharp$ (see (2.20)) was shown in [7, Section 3.3, p. 455]. As a result, the composition (2.17) is measurable, as claimed. □

Let us define a measure on $\Gamma_X^\sharp$ as the push-forward of $\hat{\mu}$ (see Definition 2.3) under the map $\mathfrak{q}$ defined in (2.14), (2.16):

$$\mathfrak{q}^*\hat{\mu}(A) \equiv \hat{\mu}(\mathfrak{q}^{-1}(A)), \qquad A \in \mathcal{B}(\Gamma_X^\sharp), \tag{2.21}$$

or equivalently

$$\int_{\Gamma_X^\sharp} F(\gamma)\, \mathfrak{q}^*\hat{\mu}(\mathrm{d}\gamma) = \int_{\Gamma_{\mathcal{Z}}^\sharp} F(\mathfrak{q}(\hat{\gamma}))\, \hat{\mu}(\mathrm{d}\hat{\gamma}), \qquad F \in \mathrm{M}_+(\Gamma_X^\sharp). \tag{2.22}$$

The next general result shows that this measure may be identified with the original cluster measure μ_{cl}.

Theorem 2.3. *The measure* (2.21) *coincides with the cluster measure* μ_{cl},

$$\mu_{\mathrm{cl}} = \mathfrak{q}^*\hat{\mu} \equiv \hat{\mu} \circ \mathfrak{q}^{-1}. \tag{2.23}$$

Proof. Let us evaluate the Laplace transform of the measure $\mathfrak{q}^*\hat{\mu}$. For any function $f \in \mathrm{M}_+(X)$, we obtain, using (2.22), (2.16) and (2.13),

$$\begin{aligned} L_{\mathfrak{q}^*\hat{\mu}}(f) &= \int_{\Gamma_X^\sharp} \exp(-\langle f, \gamma\rangle)\, \mathfrak{q}^*\hat{\mu}(\mathrm{d}\gamma) = \int_{\Gamma_{\mathcal{Z}}^\sharp} \exp(-\langle f, \mathfrak{q}(\hat{\gamma})\rangle)\, \hat{\mu}(\mathrm{d}\hat{\gamma}) \\ &= \int_{\Gamma_X} \left(\int_{\Gamma_{\mathfrak{X}}^\sharp} \exp\biggl(-\sum_{x\in\gamma_{\mathrm{c}}} f(\mathfrak{p}(\bar{y}_x + x))\biggr) \bigotimes_{x\in\gamma_{\mathrm{c}}} \eta(\mathrm{d}\bar{y}_x)\right) \mu_{\mathrm{c}}(\mathrm{d}\gamma_{\mathrm{c}}) \\ &= \int_{\Gamma_X^\sharp} \left(\int_{\Gamma_{\mathfrak{X}}^\sharp} \prod_{x\in\gamma_{\mathrm{c}}} \exp\bigl(-f(\bar{y}_x + x)\bigr) \bigotimes_{x\in\gamma_{\mathrm{c}}} \eta(\mathrm{d}\bar{y}_x)\right) \mu_{\mathrm{c}}(\mathrm{d}\gamma_{\mathrm{c}}) \\ &= \int_{\Gamma_X^\sharp} \prod_{x\in\gamma_{\mathrm{c}}} \left(\int_{\mathfrak{X}} \exp\biggl(-\sum_{y\in\bar{y}} f(y+x)\biggr)\, \eta(\mathrm{d}\bar{y})\right) \mu_{\mathrm{c}}(\mathrm{d}\gamma_{\mathrm{c}}), \end{aligned}$$

which coincides with the Laplace transform (2.8) of the cluster measure μ_{cl}. ☐

2.4. Gibbs cluster measure via an auxiliary Gibbs measure

In this paper, we are concerned with *Gibbs cluster point processes*, for which the distribution of cluster centers is given by some Gibbs measure $g \in \mathscr{G}(\theta, \Phi)$ on the proper configuration space Γ_X (see the Appendix), specified by a *reference measure* θ on X and an *interaction potential* $\Phi : \Gamma_X^0 \to \mathbb{R} \cup \{+\infty\}$, where $\Gamma_X^0 \subset \Gamma_X$ is the subspace of finite configurations in X. We assume that the set $\mathscr{G}(\theta, \Phi)$ of all Gibbs measures on Γ_X associated with θ and Φ is non-empty.[4]

Specializing Definition 2.3 to the Gibbs case, the corresponding auxiliary measure $\hat{g}$ on the (proper) configuration space $\Gamma_{\mathcal{Z}}$ is given by (cf. (2.12), (2.13))

$$\hat{\mathsf{g}}(\mathrm{d}\hat{\gamma}) = \mathsf{g}(p_X(\mathrm{d}\hat{\gamma})) \bigotimes_{z\in\hat{\gamma}} \eta(p_{\mathfrak{X}}(\mathrm{d}z)), \qquad \hat{\gamma} \in \Gamma_{\mathcal{Z}}, \tag{2.24}$$

or equivalently, for any function $F \in \mathrm{M}_+(\Gamma_{\mathcal{Z}})$,

$$\int_{\Gamma_{\mathcal{Z}}} F(\hat{\gamma})\, \hat{\mathsf{g}}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_X} \left(\int_{\mathfrak{X}^\infty} F\Bigl(\textstyle\bigcup_{x\in\gamma} \{(x, \bar{y})\}\Bigr) \bigotimes_{x\in\gamma} \eta(\mathrm{d}\bar{y})\right) \mathsf{g}(\mathrm{d}\gamma). \tag{2.25}$$

Remark 2.6. Vector $\bar{y}$ in each pair $z = (x, \bar{y}) \in \mathcal{Z}$ may be interpreted as a *mark* attached to the point $x \in X$, so that $\hat{\gamma}$ becomes a marked configuration, with the

[4] For various sufficient conditions, consult [25, 27]; see also references in the Appendix.

mark space $\mathfrak{X}$ (see [9, 16, 19]). The corresponding *marked configuration space* is defined by

$$\Gamma_X(\mathfrak{X}) := \{\hat{\gamma} \in \Gamma_{\mathcal{Z}} : p_X(\hat{\gamma}) \in \Gamma_X\} \subset \Gamma_{\mathcal{Z}}. \tag{2.26}$$

In other words, $\Gamma_X(\mathfrak{X})$ is the set of configurations in $\Gamma_{\mathcal{Z}}$ such that the collection of their x-coordinates is a (proper) configuration in Γ_X. Clearly, $\Gamma_X(\mathfrak{X})$ is a Borel subset of $\Gamma_{\mathcal{Z}}$, that is, $\Gamma_X(\mathfrak{X}) \in \mathcal{B}(\Gamma_{\mathcal{Z}})$. Since $\mathsf{g}(\Gamma_X) = 1$, we have $\hat{\mathsf{g}}(\Gamma_X(\mathfrak{X})) = 1$.

Finally, owing to the general Theorem 2.3 (see (2.23)), the corresponding Gibbs cluster measure g_{cl} on the configuration space Γ_X is represented as a push-forward of the measure $\hat{\mathsf{g}}$ on $\Gamma_{\mathcal{Z}}$ under the map $\mathfrak{q}$ defined in (2.14), (2.16):

$$\mathsf{g}_{\text{cl}} = \mathfrak{q}^* \hat{\mathsf{g}} \equiv \hat{\mathsf{g}} \circ \mathfrak{q}^{-1}. \tag{2.27}$$

Our next goal is to show that $\hat{\mathsf{g}}$ is a *Gibbs measure* on $\Gamma_{\mathcal{Z}}$, with the reference measure σ defined as a product measure on the space $\mathcal{Z} = X \times \mathfrak{X}$,

$$\sigma := \theta \otimes \eta, \tag{2.28}$$

and with the interaction potential $\hat{\Phi} : \Gamma^0_{\mathcal{Z}} \to \mathbb{R} \cup \{+\infty\}$ given by

$$\hat{\Phi}(\hat{\gamma}) := \begin{cases} \Phi(p_X(\hat{\gamma})), & \hat{\gamma} \in \Gamma^0_{\mathcal{Z}} \cap \Gamma_X(\mathfrak{X}), \\ +\infty, & \hat{\gamma} \in \Gamma^0_{\mathcal{Z}} \setminus \Gamma_X(\mathfrak{X}), \end{cases} \tag{2.29}$$

where p_X is the projection defined in (2.10). The corresponding functionals of energy $\hat{E}(\hat{\xi})$ and interaction energy $\hat{E}(\hat{\xi}, \hat{\gamma})$ ($\hat{\xi} \in \Gamma^0_{\mathcal{Z}}$, $\hat{\gamma} \in \Gamma_{\mathcal{Z}}$) are then given by (see (A.1) and (A.2))

$$\hat{E}(\hat{\xi}) := \sum_{\hat{\xi}' \subset \hat{\xi}} \hat{\Phi}(\hat{\xi}'), \tag{2.30}$$

$$\hat{E}(\hat{\xi}, \hat{\gamma}) := \begin{cases} \displaystyle\sum_{\hat{\gamma} \supset \hat{\gamma}' \in \Gamma^0_{\mathcal{Z}}} \hat{\Phi}(\hat{\xi} \cup \hat{\gamma}'), & \displaystyle\sum_{\hat{\gamma} \supset \hat{\gamma}' \in \Gamma^0_{\mathcal{Z}}} |\hat{\Phi}(\hat{\xi} \cup \hat{\gamma}')| < \infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.31}$$

The following "projection"property of the energy is obvious from the definition (2.29) of the potential $\hat{\Phi}$.

Lemma 2.4. *For any configurations $\hat{\xi} \in \Gamma^0_{\mathcal{Z}}$ and $\hat{\gamma} \in \Gamma_{\mathcal{Z}}$, we have*

$$\hat{E}(\hat{\xi}) = E(p_X(\hat{\xi})), \qquad \hat{E}(\hat{\xi}, \hat{\gamma}) = E(p_X(\hat{\xi}), p_X(\hat{\gamma})).$$

Theorem 2.5. (a) *Let $g \in \mathscr{G}(\theta, \Phi)$ be a Gibbs measure on the configuration space Γ_X, and let $\hat{g}$ be the corresponding probability measure on the configuration space $\Gamma_{\mathcal{Z}}$ (see* (2.24) *or* (2.25)*). Then $\hat{g} \in \mathscr{G}(\sigma, \hat{\Phi})$, i.e., $\hat{g}$ is a Gibbs measure on $\Gamma_{\mathcal{Z}}$ with the reference measure σ and the interaction potential $\hat{\Phi}$ defined by* (2.28) *and* (2.29)*, respectively.*

(b) *If the measure $g \in \mathscr{G}(\theta, \Phi)$ has a finite correlation function κ_g^n of some order $n \in \mathbb{N}$ (see the definition* (A.7) *in the Appendix), then the correlation function $\kappa_{\hat{g}}^n$ of the measure $\hat{g} \in \mathscr{G}(\sigma, \hat{\Phi})$ is given by*

$$\kappa_{\hat{g}}^n(z_1, \dots, z_n) = \kappa_g^n\bigl(p_X(z_1), \dots, p_X(z_n)\bigr), \qquad z_1, \dots, z_n \in \mathcal{Z}. \tag{2.32}$$

Proof. (a) In order to show that $\hat{g} \in \mathscr{G}(\sigma, \hat{\Phi})$, it suffices to check that $\hat{g}$ satisfies Nguyen–Zessin's equation on $\Gamma_{\mathcal{Z}}$ (see equation (A.3) in the Appendix), that is, for any non-negative, $\mathcal{B}(\mathcal{Z}) \times \mathcal{B}(\Gamma_{\mathcal{Z}})$-measurable function $H(z, \hat{\gamma})$ it holds

$$\int_{\Gamma_{\mathcal{Z}}} \sum_{z \in \hat{\gamma}} H(z, \hat{\gamma})\, \hat{g}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_{\mathcal{Z}}} \biggl(\int_{\mathcal{Z}} H(z, \hat{\gamma} \cup \{z\})\, \mathrm{e}^{-\hat{E}(\{z\}, \hat{\gamma})}\, \sigma(\mathrm{d}z)\biggr)\, \hat{g}(\mathrm{d}\hat{\gamma}). \tag{2.33}$$

Using the disintegration formula (2.13), the left-hand side of (2.33) can be represented as

$$\begin{aligned} &\int_{\Gamma_X} \biggl(\int_{\mathfrak{X}^\infty} \sum_{x \in \gamma} H\Bigl(x, \bar{y}_x;\ \bigcup_{x' \in \gamma} \{(x', \bar{y}_{x'})\}\Bigr) \bigotimes_{x' \in \gamma} \eta(\mathrm{d}\bar{y}_{x'})\biggr)\, g(\mathrm{d}\gamma) \\ &\quad = \int_{\Gamma_X} \Biggl(\sum_{x \in \gamma} \int_{\Gamma_{\mathfrak{X}}} \mathbf{1}_\gamma(x)\, H\Bigl(x, \bar{y}_x;\ \bigcup_{x' \in \gamma} \{(x', \bar{y}_{x'})\}\Bigr) \bigotimes_{x' \in \gamma} \eta(\mathrm{d}\bar{y}_{x'})\Biggr)\, g(\mathrm{d}\gamma). \end{aligned} \tag{2.34}$$

Applying Nguyen–Zessin's equation to the Gibbs measure g with the function

$$H_0(x, \gamma) := \int_{\Gamma_{\mathfrak{X}}} \mathbf{1}_\gamma(x)\, H\Bigl(x, \bar{y}_x;\ \bigcup_{x' \in \gamma} \{(x', \bar{y}_{x'})\}\Bigr) \bigotimes_{x' \in \gamma} \eta(\mathrm{d}\bar{y}_{x'}),$$

we see that the right-hand side of (2.34) takes the form

$$\int_{\Gamma_X} \biggl(\sum_{x \in \gamma} H_0(x, \gamma)\biggr)\, g(\mathrm{d}\gamma) = \int_{\Gamma_X} \biggl(\int_X H_0(x, \gamma \cup \{x\})\, \mathrm{e}^{-E(\{x\}, \gamma)}\, \theta(\mathrm{d}x)\biggr)\, g(\mathrm{d}\gamma). \tag{2.35}$$

Similarly, exploiting the product structure of the measures $\sigma = \theta \otimes \eta$ and

$$\bigotimes_{x' \in \gamma \cup \{x\}} \eta(\mathrm{d}\bar{y}_{x'}) = \bigotimes_{x' \in \gamma} \eta(\mathrm{d}\bar{y}_{x'}) \otimes \eta(\mathrm{d}\bar{y}_x),$$

and using Lemma 2.4, the right-hand side of (2.33) is reduced to

$$
\begin{aligned}
&\int_{\Gamma_X}\biggl(\int_{\mathfrak{X}^\infty}\biggl(\int_{X\times\mathfrak{X}} H\Bigl(x,\bar y_x;\,\bigcup_{x'\in\gamma}\{(x',\bar y_{x'})\}\cup\{(x,\bar y_x)\}\Bigr)\\
&\qquad\qquad\qquad\qquad \times \mathrm{e}^{-E(\{x\},\gamma)}\,\eta(\mathrm{d}\bar y_x)\,\theta(\mathrm{d}x)\biggr)\bigotimes_{x'\in\gamma}\eta(\mathrm{d}\bar y_{x'})\biggr)\,\boldsymbol{g}(\mathrm{d}\gamma)\\
&=\int_{\Gamma_X}\biggl(\int_X\biggl(\int_{\mathfrak{X}^\infty}\mathbf{1}_\gamma(x)\,H\Bigl(x,\bar y_x;\,\bigcup_{x'\in\gamma}\{(x',\bar y_{x'})\}\Bigr)\\
&\qquad\qquad\qquad\qquad \times \mathrm{e}^{-E(\{x\},\gamma)}\bigotimes_{x'\in\gamma}\eta(\mathrm{d}\bar y_{x'})\biggr)\,\theta(\mathrm{d}x)\biggr)\,\boldsymbol{g}(\mathrm{d}\gamma)\\
&=\int_{\Gamma_X}\biggl(\int_X H_0(x,\gamma\cup\{x\})\,\mathrm{e}^{-E(\{x\},\gamma)}\,\theta(\mathrm{d}x)\biggr)\,\boldsymbol{g}(\mathrm{d}\gamma),
\end{aligned}
$$

thus coinciding with (2.35). This proves equation (2.33), hence $\hat{g}\in\mathscr{G}(\sigma,\hat{\Phi})$.

(b) Let $f\in C_0(\mathcal{Z}^n)$ be a symmetric function. According to the disintegration formula (2.13) applied to the function

$$
F(\hat\gamma):=\sum_{\{z_1,\dots,z_n\}\subset\hat\gamma} f(z_1,\dots,z_n),
$$

we have

$$
\int_{\Gamma_{\mathcal{Z}}} F(\hat\gamma)\,\hat{\boldsymbol{g}}(\mathrm{d}\hat\gamma)=\int_{\Gamma_X}\sum_{\{x_1,\dots,x_n\}\subset\gamma}\phi(x_1,\dots,x_n)\,\boldsymbol{g}(\mathrm{d}\gamma), \tag{2.36}
$$

where

$$
\phi(x_1,\dots,x_n):=\int_{\mathfrak{X}^n} f((x_1,\bar y_1),\dots,(x_n,\bar y_n))\bigotimes_{i=1}^n\eta(\mathrm{d}\bar y_i)\in C_0(X^n).
$$

Applying the definition of the correlation function κ_g^n (see (A.7)) and using that $\theta(\mathrm{d}x)\otimes\eta(\mathrm{d}\bar y)=\sigma(\mathrm{d}x\times\mathrm{d}\bar y)$, we obtain from (2.36)

$$
\int_{\Gamma_{\mathcal{Z}}} F(\hat\gamma)\,\hat{\boldsymbol{g}}(\mathrm{d}\hat\gamma)=\frac{1}{n!}\int_{\mathcal{Z}^n} f(z_1,\dots,z_n)\,\kappa_g^n\bigl(p_X(z_1),\dots,p_X(z_n)\bigr)\bigotimes_{i=1}^n\sigma(\mathrm{d}z_i),
$$

and equality (2.32) follows. □

In the rest of this subsection, $\mathscr{G}_{\mathrm{R}}$ denotes the subclass of Gibbs measures in $\mathscr{G}$ (with a given reference measure and interaction potential) that satisfy the so-called *Ruelle bound* (see the Appendix, formula (A.11)).

Corollary 2.6. *We have $g \in \mathscr{G}_{\mathrm{R}}(\theta, \Phi)$ if and only if $\hat{g} \in \mathscr{G}_{\mathrm{R}}(\sigma, \hat{\Phi})$,*

Proof. Follows directly from formula (2.32). □

The following statement is, in a sense, converse to Theorem 2.5(a).

Theorem 2.7. *If $\varpi \in \mathscr{G}(\sigma, \hat{\Phi})$ then $g := p_X^* \varpi \in \mathscr{G}(\theta, \Phi)$. Moreover, if $g \in \mathscr{G}_{\mathrm{R}}(\theta, \Phi)$ then $\varpi = \hat{g}$.*

Proof. Applying Nguyen–Zessin's equation (A.3) to the measure ϖ and using the cylinder structure of the interaction potential, we have

$$
\begin{aligned}
\int_{\Gamma_X} \sum_{x \in \gamma} H(x, \gamma)\, p_X^* \varpi(\mathrm{d}\gamma) &= \int_{\Gamma_{\mathcal{Z}}} \sum_{x \in p_X \hat{\gamma}} H(x, p_X \hat{\gamma})\, \varpi(\mathrm{d}\hat{\gamma}) \\
&= \int_{\Gamma_{\mathcal{Z}}} \left(\int_{\mathcal{Z}} H(p_X z, p_X(\hat{\gamma} \cup \{z\})\, \mathrm{e}^{-E(\{p_X z\}, p_X \hat{\gamma})}\, \theta \otimes \eta(\mathrm{d}z) \right) \varpi(\mathrm{d}\hat{\gamma}) \\
&= \int_{\Gamma_{\mathcal{Z}}} \left(\int_X H(x, p_X \hat{\gamma} \cup \{x\})\, \mathrm{e}^{-E(\{x\}, p_X \hat{\gamma})}\, \theta(\mathrm{d}x) \right) \varpi(\mathrm{d}\hat{\gamma}) \\
&= \int_{\Gamma_X} \left(\int_X H(x, \gamma \cup \{x\})\, \mathrm{e}^{-E(\{x\}, \gamma)}\, \theta(\mathrm{d}x) \right) p_X^* \varpi(\mathrm{d}\gamma).
\end{aligned}
$$

Thus, the measure $p_X^* \varpi$ satisfies Nguyen–Zessin's equation and so, by Theorem A.1, belongs to the Gibbs class $\mathscr{G}(\theta, \Phi)$.

Next, in order to prove that $\varpi = \hat{g}$, by Proposition A.2 it suffices to show that the measures ϖ and $\hat{g}$ have the same correlation functions. Note that the correlation function κ_ϖ^n can be written in the form [16, §2.3, Lemma 2.3.8]

$$
\begin{aligned}
\kappa_\varpi^n(z_1, \dots, z_n) &= \mathrm{e}^{-\hat{E}(\{z_1, \dots, z_n\})} \int_{\Gamma_{\mathcal{Z}}} \mathrm{e}^{-\hat{E}(\{z_1, \dots, z_n\}, \hat{\gamma})}\, \varpi(\mathrm{d}\hat{\gamma}) \\
&= \mathrm{e}^{-E(\{p_X(z_1), \dots, p_X(z_n)\})} \int_{\Gamma_X} \mathrm{e}^{-E(\{p_X(z_1), \dots, p_X(z_n)\}, \gamma)}\, p_X^* \varpi(\mathrm{d}\gamma) \\
&= \mathrm{e}^{-E(\{p_X(z_1), \dots, p_X(z_n)\})} \int_{\Gamma_X} \mathrm{e}^{-E(\{p_X(z_1), \dots, p_X(z_n)\}, \gamma)}\, g(\mathrm{d}\gamma) \\
&= \kappa_g^n\bigl(p_X(z_1), \dots, p_X(z_n)\bigr).
\end{aligned}
$$

Therefore, on account of Theorem 2.5(b) we get $\kappa_\varpi^n(z_1, \dots, z_n) = \kappa_{\hat{g}}^n(z_1, \dots, z_n)$ for all $z_1, \dots, z_n \in \mathcal{Z}$ $(z_i \neq z_j)$, as required. □

In the next corollary, $\operatorname{ext}\mathscr{G}$ denotes the set of *extreme points* of the class $\mathscr{G}$ of Gibbs measures with the corresponding reference measure and interaction potential (see the Appendix).

Corollary 2.8. *Suppose that* $g \in \mathscr{G}_{\mathrm{R}}(\theta, \varPhi)$. *Then* $g \in \operatorname{ext}\mathscr{G}(\theta, \varPhi)$ *if and only if* $\hat{g} \in \operatorname{ext}\mathscr{G}(\sigma, \hat{\varPhi})$.

Proof. Let $g \in \mathscr{G}_{\mathrm{R}}(\theta, \varPhi) \cap \operatorname{ext}\mathscr{G}(\theta, \varPhi)$. Assume that $\hat{g} = \frac{1}{2}(\mu_1 + \mu_2)$ with some $\mu_1, \mu_2 \in \mathscr{G}(\sigma, \hat{\varPhi})$. Then $g = \frac{1}{2}(g_1 + g_2)$, where $g_i = p_X^* \mu_i \in \mathscr{G}(\theta, \varPhi)$. Since $g \in \operatorname{ext}\mathscr{G}(\theta, \varPhi)$, this implies that $g_1 = g_2 = g$. In particular, $g_1, g_2 \in \mathscr{G}_{\mathrm{R}}(\theta, \varPhi)$ and by Theorem 2.7 we obtain that $\mu_1 = \hat{g}_1 = \hat{g} = \hat{g}_2 = \mu_2$, which implies $\hat{g} \in \operatorname{ext}\mathscr{G}(\sigma, \hat{\varPhi})$.

Conversely, let $\hat{g} \in \operatorname{ext}\mathscr{G}(\sigma, \hat{\varPhi})$ and $g = \frac{1}{2}(g_1 + g_2)$ with $g_1, g_2 \in \mathscr{G}(\theta, \varPhi)$. Then $\hat{g} = \frac{1}{2}(\hat{g}_1 + \hat{g}_2)$, hence $\hat{g}_1 = \hat{g}_2 = \hat{g} \in \mathscr{G}_{\mathrm{R}}(\sigma, \hat{\varPhi})$, which implies by Theorem 2.7 that $g_1 = p_X^* \hat{g}_1 = p_X^* \hat{g}_2 = g_2$. Thus, $g \in \operatorname{ext}\mathscr{G}(\theta, \varPhi)$. □

2.5. *Criteria of local finiteness and simplicity of the Gibbs cluster process*

Let us give conditions sufficient for the Gibbs CPP to be (a) locally finite, and (b) simple. For a given Borel set $B \in \mathcal{B}(X)$, consider a set-valued function (referred to as the *droplet cluster*)

$$D_B(\bar{y}) := \bigcup_{y_i \in \bar{y}} (B - y_i), \qquad \bar{y} \in \mathfrak{X}. \tag{2.37}$$

Let us also denote by $N_B(\bar{y})$ the number of coordinates of the vector $\bar{y} = (y_i)$ falling in the set $B \in \mathcal{B}(X)$,

$$N_B(\bar{y}) := \sum_{y_i \in \bar{y}} \mathbf{1}_B(y_i), \qquad \bar{y} \in \mathfrak{X}, \tag{2.38}$$

In particular, $N_X(\bar{y})$ is the "dimension" of $\bar{y}$, that is, the total number of its coordinates (recall that $\bar{y} \in \mathfrak{X} = \bigsqcup_{n=0}^{\infty} X^n$, see (2.1)).

Theorem 2.9. *Let* g_{cl} *be a Gibbs cluster measure on the generalized configuration space* $\varGamma_X^\sharp$.

(a) *Assume that the correlation function* κ_g^1 *of the measure* $g \in \mathscr{G}(\theta, \varPhi)$ *is bounded. Then, in order that* g_{cl}*-a.a. configurations* $\gamma \in \varGamma_X^\sharp$ *be locally finite, it is sufficient that the following two conditions hold*:

(a-i) *for any compact set* $B \in \mathcal{B}(X)$*, the number of coordinates of the vector* $\bar{y} \in \mathfrak{X}$ *in* B *is a.s.-finite,*

$$N_B(\bar{y}) < \infty \quad \textit{for } \eta\textit{-a.a. } \bar{y} \in \mathfrak{X}; \tag{2.39}$$

(a-ii) *for any compact set* $B \in \mathcal{B}(X)$, *the mean* θ*-measure of the droplet cluster* $D_B(\bar{y})$ *is finite,*

$$\int_{\mathfrak{X}} \theta(D_B(\bar{y}))\,\eta(\mathrm{d}\bar{y}) < \infty. \tag{2.40}$$

(b) *In order that* g_{cl}*-a.a. configurations* $\gamma \in \Gamma_X^\sharp$ *be simple, it is sufficient that the following two conditions hold*:

(b-i) *for any* $x \in X$, *vector* $\bar{y}$ *contains a.s. no more than one coordinate* $y_i = x$,

$$\sup_{x \in X} N_{\{x\}}(\bar{y}) \le 1 \quad \textit{for } \eta\textit{-a.a. } \bar{y} \in \mathfrak{X}; \tag{2.41}$$

(b-ii) *for any* $x \in X$, *the "point" droplet cluster* $D_{\{x\}}(\bar{y})$ *has a.s. zero* θ*-measure,*

$$\theta\big(D_{\{x\}}(\bar{y})\big) = 0 \quad \textit{for } \eta\textit{-a.a. } \bar{y} \in \mathfrak{X}. \tag{2.42}$$

For the proof of part (a) of this theorem, we need a reformulation (stated as Proposition 2.10 below) of the condition (a-ii), which will also play an important role in utilizing the projection construction of the Gibbs cluster measure (see Section 3 below). For any Borel subset $B \in \mathcal{B}(X)$, denote

$$\mathcal{Z}_B := \{z \in \mathcal{Z} :\, \mathfrak{q}(z) \cap B \ne \emptyset\} \in \mathcal{B}(\mathcal{Z}), \tag{2.43}$$

where $\mathfrak{q}(z) = \bigsqcup_{y_i \in p_{\mathfrak{X}}(z)} \{y_i + p_X(z)\}$ (see (2.14)). That is to say, the set $\mathcal{Z}_B$ consists of all points $z = (x, \bar{y}) \in \mathcal{Z}$ such that, under the "projection" $\mathfrak{q}$ onto the space X, at least one coordinate $y_i + x$ ($y_i \in \bar{y}$) belongs to the set $B \subset X$.

Proposition 2.10. *For any* $B \in \mathcal{B}(X)$, *the condition* (a-ii) *of Theorem* 2.9(a) *is necessary and sufficient in order that* $\sigma(\mathcal{Z}_B) < \infty$, *where* $\sigma = \theta \otimes \eta$.

Proof of Proposition 2.10. By definition (2.43), $(x, \bar{y}) \in \mathcal{Z}_B$ if and only if $x \in \bigcup_{y_i \in \bar{y}} (B - y_i) \equiv D_B(\bar{y})$ (see (2.37)). Hence,

$$\sigma(\mathcal{Z}_B) = \int_{\mathfrak{X}} \left(\int_X \mathbf{1}_{D_B(\bar{y})}(x)\,\theta(\mathrm{d}x) \right) \eta(\mathrm{d}\bar{y}) = \int_{\mathfrak{X}} \theta\big(D_B(\bar{y})\big)\,\eta(\mathrm{d}\bar{y}),$$

and we see that the bound $\sigma(\mathcal{Z}_B) < \infty$ is nothing else but condition (2.40). □

Proof of Theorem 2.9. (a) Let $B \subset X$ be a compact set. By Proposition 2.10, condition (a-ii) is equivalent to $\sigma(\mathcal{Z}_B) < \infty$. On the other hand, by Theorem 2.5(b) we have $\kappa_{\hat{\mathrm{g}}}^1(x, \bar{y}) = \kappa_{\mathrm{g}}^1(x)$. Hence, $\kappa_{\hat{\mathrm{g}}}^1$ is bounded, and by Remark A.5 (see the Appendix) it follows that $\hat{\gamma}(\mathcal{Z}_B) < \infty$ ($\hat{g}$-a.s.). According to the projection representation $g_{\rm cl} = \mathfrak{q}^*\hat{g}$ (see (2.27)) and in view of condition (a-i), this implies that, almost surely, a projected configuration $\gamma = \mathfrak{q}(\hat{\gamma}) = \bigsqcup_{z\in\hat{\gamma}} \mathfrak{q}(z)$ contributes no more than finitely many points to the set $B \subset \mathfrak{q}(\mathcal{Z}_B)$, that is, $\gamma(B) < \infty$ ($g_{\rm cl}$-a.s.), which completes the proof of part (a).

(b) It suffices to prove that, for any compact set $\Lambda \subset X$, there are $g_{\rm cl}$-a.s. no cross-ties between the clusters whose centers belong to Λ. That is, we must show that $\mathbf{g}_{\rm cl}(A_\Lambda) = 0$, where the set $A_\Lambda \in \mathcal{B}(\Gamma_X \times \mathfrak{X}^2)$ is defined by

$$A_\Lambda := \{(\gamma, \bar{y}_1, \bar{y}_2) : \exists\, x_1, x_2 \in \gamma \cap \Lambda,\ \exists\, y_1 \in \bar{y}_1 : x_1 + y_1 - x_2 \in \bar{y}_2\}, \qquad (2.44)$$

Applying the disintegration formula (2.25), we obtain

$$\mathbf{g}_{\rm cl}(A_\Lambda) = \int_{\Gamma_X} F(\gamma)\, \mathbf{g}(\mathrm{d}\gamma), \qquad (2.45)$$

where

$$F(\gamma) := \int_{\mathfrak{X}^2} \mathbf{1}_{A_\Lambda}(\gamma, \bar{y}_1, \bar{y}_2)\, \eta(\mathrm{d}\bar{y}_1)\, \eta(\mathrm{d}\bar{y}_2), \qquad \gamma \in \Gamma_X. \qquad (2.46)$$

Note that, according to the definition (2.44), $F(\gamma) \equiv F(\gamma \cap \Lambda)$ ($\gamma \in \Gamma_X$), hence, by Proposition A.3, we can rewrite (2.45) in the form

$$\mathbf{g}_{\rm cl}(A_\Lambda) = \int_{\Gamma_\Lambda} F(\xi)\, S_\Lambda(\xi)\, \lambda_\theta(\mathrm{d}\xi), \qquad (2.47)$$

with $S_\Lambda(\xi) \in L^1(\Gamma_\Lambda, \lambda_\theta)$. Therefore, in order to show that the right-hand side of (2.47) vanishes, it suffices to check that

$$\int_{\Gamma_\Lambda} F(\xi)\, \lambda_\theta(\mathrm{d}\xi) = 0. \qquad (2.48)$$

To this end, substituting here the definition (2.46) and changing the order of integration, we can rewrite the integral in (2.48) as

$$\int_{\mathfrak{X}^2} \theta^{\otimes 2}(B_\Lambda(\bar{y}_1, \bar{y}_2))\, \eta(\mathrm{d}\bar{y}_1)\, \eta(\mathrm{d}\bar{y}_2),$$

where

$$B_\Lambda(\bar{y}_1, \bar{y}_2)) := \{(x_1, x_2) \in \Lambda^2 : x_1 + y_1 = x_2 + y_2 \text{ for some } y_1 \in \bar{y}_1,\ y_2 \in \bar{y}_2\}.$$

It remains to note that

$$
\begin{aligned}
\theta^{\otimes 2}\big(B_\Lambda(\bar y_1,\bar y_2)\big) &= \int_\Lambda \theta\left(\textstyle\bigcup_{y_1\in\bar y_1}\bigcup_{y_2\in\bar y_2}\{x_1+y_1-y_2\}\right)\theta(\mathrm{d}x_1)\\
&\le \sum_{y_1\in\bar y_1}\int_\Lambda \theta\left(\textstyle\bigcup_{y_2\in\bar y_2}\{x_1+y_1-y_2\}\right)\theta(\mathrm{d}x_1)\\
&= \sum_{y_1\in\bar y_1}\int_\Lambda \theta\big(D_{\{x_1+y_1\}}(\bar y_2)\big)\,\theta(\mathrm{d}x_1)=0 \qquad (\eta\text{-a.s.}),
\end{aligned}
$$

since, by assumption (2.42), $\theta\big(D_{\{x_1+y_1\}}(\bar y_2)\big)=0$ (η-a.s.) and $\bar y_1$ contains at most countably many coordinates. Hence, (2.48) follows and so part (b) is proved. □

Remark 2.7. In the Poisson cluster case (see [7, Theorem 2.7(a)]), conditions (a-i) and (a-ii) of Theorem 2.9(a) are not only sufficient but also necessary for the local finiteness of cluster configurations. While it is obvious that condition (a-i) is always necessary, there may be a question as to whether condition (a-ii) is such in the case of a Gibbs cluster measure g_{cl}. Inspection of the proof of Theorem 2.9(a) shows that the difficulty here lies in the questionable relationship between the conditions $\sigma(\mathcal{Z}_B)<\infty$ and $\hat\gamma(\mathcal{Z}_B)<\infty$ ($\hat g$-a.s.) (which are equivalent in the Poisson cluster case). According to Remark A.5 (see the Appendix), under the hypothesis of boundedness of the first-order correlation function κ_g^1, the former implies the latter, but the converse may not always be true. Simple counterexamples can be constructed by considering translation-invariant pair interaction potentials $\Phi(\{x_1,x_2\})=\phi_0(x_1-x_2)\equiv\phi_0(y-x)$ such that $\phi_0(x)=+\infty$ on some subset $\Lambda_\infty\subset X$ with $\theta(\Lambda_\infty)=\infty$. However, if κ_g^1 is bounded below and the mean number of configuration points in a set B is finite then the measure $\theta(B)$ must be finite (see Remark A.5).

Remark 2.8. Similarly to Remark 2.7, it is of interest to ask whether conditions (b-i) and (b-ii) of Theorem 2.9(b) are necessary for the simplicity of the cluster measure g_{cl} (as is the case for the Poisson cluster measure, see [7, Theorem 2.7(b)]). However, in the Gibbs cluster case this is not so; more precisely, (b-i) is of course necessary, but (b-ii) may not be satisfied. For a simple counterexample, let the in-cluster measure η be concentrated on a single-point configuration $\bar y=(0)$, so that the droplet cluster $D_{\{x\}}(\bar y)$ is reduced to a single-point set $\{x\}$. Here, any measure θ with atoms will not satisfy condition (b-ii). On the other hand, consider a Gibbs measure g with a hard-core translation-invariant pair interaction potential $\Phi(\{x_1,x_2\})=\phi_0(x_1-x_2)\equiv\phi_0(y-x)$, where $\phi_0(x)=+\infty$

for $|x| < r_0$ and $\phi_0(x) = 0$ for $|x| \ge r_0$; then in each admissible configuration γ any two points are at least at a distance r_0, and in particular any such γ is simple.

Remark 2.9. As suggested by Remarks 2.7 and 2.8, it is plausible that conditions (a-ii) and (b-ii) of Theorem 2.9 *are* necessary for the claims (a) and (b), respectively, if the interaction potential of the underlying Gibbs measure *g* is finite on all finite configurations, i.e., $\varPhi(\xi) < +\infty$ for all $\xi \in \varGamma_X^0$.

In conclusion of this section, let us state some criteria sufficient for conditions (a-ii) and (b-ii) of Theorem 2.9 (see details in [7, §2.4]). Assume for simplicity that the in-cluster configurations are a.s.-finite, $\eta\{N_X(\bar{y}) < \infty\} = 1$.

Proposition 2.11. *Either of the following conditions is sufficient for* (2.40):

(a-ii′) *For any compact set $B \in \mathcal{B}(X)$, the θ-measure of its translations is uniformly bounded,*

$$C_B := \sup_{x\in X} \theta(B + x) < \infty, \tag{2.49}$$

and, moreover, the mean number of in-cluster points is finite,

$$\int_{\mathfrak{X}} N_X(\bar{y})\, \eta(\mathrm{d}\bar{y}) < \infty. \tag{2.50}$$

(a-ii″) *The coordinates of vector $\bar{y}$ are a.s. uniformly bounded, that is, there is a compact set $B_0 \in \mathcal{B}(X)$ such that $N_{X\setminus B_0}(\bar{y}) = 0$ for η-a.a. $\bar{y} \in \mathfrak{X}$.*

Proposition 2.12. *Either of the following conditions is sufficient for* (2.42):

(b-ii′) *The measure θ is non-atomic, that is, $\theta\{x\} = 0$ for each $x \in X$.*

(b-ii″) *For each $x \in X$, $N_{\{x\}}(\bar{y}) = 0$ for η-a.a. $\bar{y} \in \mathfrak{X}$.*

3. Quasi-invariance and the integration-by-parts formula

From now on, we restrict ourselves to the case where $X = \mathbb{R}^d$. Henceforth, we assume that the in-cluster configurations are a.s.-finite, $\eta\{N_X(\bar{y}) < \infty\} = 1$; hence, the component X^∞ representing infinite clusters (see Section 2.1) may be dropped, so the set $\mathfrak{X}$ is now redefined as $\mathfrak{X} := \bigsqcup_{n\in\mathbb{Z}_+} X^n$ (cf. (2.1)). Note that condition (a-i) of Theorem 2.9 is then automatically satisfied.

We assume throughout that the correlation function $\kappa_g^1(x)$ is bounded, which implies by Theorem 2.9 that the same is true for the correlation function $\kappa_{\hat{g}}^1(z)$. Let us also impose conditions (2.49) and (2.50) which, by Proposition 2.11, ensure that condition (a-ii) of Theorem 2.9(a) is fulfilled and so g_{cl}-a.a. configurations $\gamma \in \varGamma_X^\sharp$ are locally finite. According to Proposition 2.10, condition (a-ii) also implies that $\sigma(\mathcal{Z}_B) < \infty$ providing that $\theta(B) < \infty$, where the set $\mathcal{Z}_B \subset \mathcal{Z}$ is defined in (2.43).

Finally, we require the probability measure η on $\mathfrak{X}$ to be absolutely continuous with respect to the Lebesgue measure $\mathrm{d}\bar{y}$,

$$\eta(\mathrm{d}\bar{y}) = h(\bar{y})\,\mathrm{d}\bar{y}, \qquad \bar{y} = (y_1, \dots, y_n) \in X^n \quad (n \in \mathbb{Z}_+). \tag{3.1}$$

By Proposition 2.12(b-ii″), this implies that Gibbs CPP configurations γ are g_{cl}-a.s. simple (i.e., have no multiple points). Altogether, the above assumptions ensure that g_{cl}-a.a. configurations γ belong to the proper configuration space $\varGamma_X$.

Our aim in this section is to prove the quasi-invariance of the measure g_{cl} with respect to compactly supported diffeomorphisms of X (Section 3.2), and to establish an integration-by-parts formula (Section 3.3). We begin in Section 3.1 with a brief description of some convenient “manifold-like” concepts and notations first introduced in [2] (see also [7, §4.1]), which furnish a suitable framework for analysis on configuration spaces.

3.1. Differentiable functions on configuration spaces

Let T_xX be the tangent space of $X = \mathbb{R}^d$ at point $x \in X$. It can be identified in the natural way with $\mathbb{R}^d$, with the corresponding (canonical) inner product denoted by a “fat” dot $\boldsymbol{\cdot}$. The gradient on X is denoted by ∇. Following [2], we define the “tangent space” of the configuration space $\varGamma_X$ at $\gamma \in \varGamma_X$ as the Hilbert space $T_\gamma\varGamma_X := L^2(X \to TX;\, \mathrm{d}\gamma)$, or equivalently $T_\gamma\varGamma_X = \bigoplus_{x\in\gamma} T_xX$. The scalar product in $T_\gamma\varGamma_X$ is denoted by $\langle\cdot,\cdot\rangle_\gamma$, with the corresponding norm $|\cdot|_\gamma$. A vector field V over $\varGamma_X$ is a map $\varGamma_X \ni \gamma \mapsto V(\gamma) = (V(\gamma)_x)_{x\in\gamma} \in T_\gamma\varGamma_X$. Thus, for vector fields V_1, V_2 over $\varGamma_X$ we have

$$\langle V_1(\gamma), V_2(\gamma)\rangle_\gamma = \sum_{x\in\gamma} V_1(\gamma)_x \boldsymbol{\cdot} V_2(\gamma)_x, \qquad \gamma \in \varGamma_X.$$

For $\gamma \in \varGamma_X$ and $x \in \gamma$, denote by $\mathcal{O}_{\gamma,x}$ an arbitrary open neighborhood of x in X such that $\mathcal{O}_{\gamma,x} \cap \gamma = \{x\}$. For any measurable function $F : \varGamma_X \to \mathbb{R}$, define the function $F_x(\gamma,\cdot) : \mathcal{O}_{\gamma,x} \to \mathbb{R}$ by $F_x(\gamma, y) := F((\gamma \setminus \{x\}) \cup \{y\})$, and set

$$\nabla_x F(\gamma) := \nabla F_x(\gamma, y)\big|_{y=x}, \qquad x \in X,$$

provided that $F_x(\gamma,\cdot)$ is differentiable at x.

Recall that for a function $\phi : X \to \mathbb{R}$ its support $\operatorname{supp}\phi$ is defined as the closure of the set $\{x \in X : \phi(x) \neq 0\}$. Denote by $\mathcal{FC}(\varGamma_X)$ the class of functions on $\varGamma_X$ of the form

$$F(\gamma) = f(\langle\phi_1, \gamma\rangle, \dots, \langle\phi_k, \gamma\rangle), \qquad \gamma \in \varGamma_X, \tag{3.2}$$

where $k \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^k)$ (:= the set of C^∞-functions on $\mathbb{R}^k$ bounded together with all their derivatives), and $\phi_1, \dots, \phi_k \in C_0^\infty(X)$ (:= the set of C^∞-functions on X with compact support). Each $F \in \mathcal{FC}(\varGamma_X)$ is local, that is, there is a compact $K \subset X$ (which may depend on F) such that $F(\gamma) = F(\gamma \cap K)$ for all $\gamma \in \varGamma_X$. Thus, for a fixed γ there are finitely many non-zero derivatives $\nabla_x F(\gamma)$.

For a function $F \in \mathcal{FC}(\varGamma_X)$ its $\varGamma$-gradient $\nabla^\varGamma F \equiv \nabla_X^\varGamma F$ is defined as

$$\nabla^\varGamma F(\gamma) := (\nabla_x F(\gamma))_{x\in\gamma} \in T_\gamma \varGamma_X, \qquad \gamma \in \varGamma_X, \tag{3.3}$$

so the directional derivative of F along a vector field V is given by

$$\nabla_V^\varGamma F(\gamma) := \langle \nabla^\varGamma F(\gamma), V(\gamma)\rangle_\gamma = \sum_{x\in\gamma} \nabla_x F(\gamma) \cdot V(\gamma)_x, \qquad \gamma \in \varGamma_X.$$

Note that the sum here contains only finitely many non-zero terms.

Further, let $\mathcal{FV}(\varGamma_X)$ be the class of cylinder vector fields V on $\varGamma_X$ of the form

$$V(\gamma)_x = \sum_{i=1}^k A_i(\gamma)\, v_i(x) \in T_x X, \qquad x \in X, \tag{3.4}$$

where $A_i \in \mathcal{FC}(\varGamma_X)$ and $v_i \in \mathrm{Vect}_0(X)$ (:= the space of compactly supported C^∞-smooth vector fields on X), $i = 1, \dots, k$ ($k \in \mathbb{N}$). Any vector filed $v \in \mathrm{Vect}_0(X)$ generates a constant vector field V on $\varGamma_X$ defined by $V(\gamma)_x := v(x)$. We shall preserve the notation v for it. Thus,

$$\nabla_v^\varGamma F(\gamma) = \sum_{x\in\gamma} \nabla_x F(\gamma) \cdot v(x), \qquad \gamma \in \varGamma_X. \tag{3.5}$$

The approach based on “lifting” the differential structure from the underlying space X to the configuration space $\varGamma_X$ as described above can also be applied to the spaces $\mathfrak{X} = \bigsqcup_{n=0}^\infty X^n$, $\mathcal{Z} = X \times \mathfrak{X}$ and $\varGamma_{\mathfrak{X}}$, $\varGamma_{\mathcal{Z}}$. For these spaces, we will use the analogous notations as above without further explanation.

3.2. Diff_0*-quasi-invariance*

In this section, we discuss the property of quasi-invariance of the measure g_{cl} with respect to diffeomorphisms of X. Let us start by describing how diffeomorphisms of X act on configuration spaces. For a measurable map $\varphi : X \to X$, its *support* $\operatorname{supp}\varphi$ is defined as the smallest closed set containing all $x \in X$ such that $\varphi(x) \neq x$. Let $\mathrm{Diff}_0(X)$ be the group of diffeomorphisms of X with *compact support*. For any $\varphi \in \mathrm{Diff}_0(X)$, consider the corresponding "diagonal" diffeomorphism $\bar{\varphi} : \mathfrak{X} \to \mathfrak{X}$ acting on each constituent space X^n ($n \in \mathbb{Z}_+$) as

$$X^n \ni \bar{y} = (y_1, \dots, y_n) \mapsto \bar{\varphi}(\bar{y}) := (\varphi(y_1), \dots, \varphi(y_n)) \in X^n. \tag{3.6}$$

For $x \in X$, we also define "shifted" diffeomorphisms

$$\bar{\varphi}_x(\bar{y}) := \bar{\varphi}(\bar{y} + x) - x, \qquad \bar{y} \in \mathfrak{X} \tag{3.7}$$

(see the shift notation (2.15)). Finally, we introduce a special class of diffeomorphisms $\hat{\varphi}$ on $\mathcal{Z}$ acting only in the $\bar{y}$-coordinate as follows,

$$\hat{\varphi}(z) := (x, \bar{\varphi}_x(\bar{y})) \equiv (x, \bar{\varphi}(\bar{y} + x) - x), \qquad z = (x, \bar{y}) \in \mathcal{Z}. \tag{3.8}$$

Remark 3.1. Note that, even though $K_\varphi := \operatorname{supp} \varphi$ is compact in X, the support of the diffeomorphism $\hat{\varphi}$ (again defined as the closure of the set $\{z \in \mathcal{Z} : \hat{\varphi}(z) \neq z\}$) is given by $\operatorname{supp} \hat{\varphi} = \mathcal{Z}_{K_\varphi}$ (see (2.43)) and hence is *not* compact in the topology of $\mathcal{Z}$ (see Section 2.1).

In the standard fashion, the maps φ and $\hat{\varphi}$ can be lifted to measurable "diagonal" transformations (denoted by the same letters) of the configuration spaces Γ_X and $\Gamma_{\mathcal{Z}}$, respectively:

$$\begin{aligned} &\Gamma_X \ni \gamma \mapsto \varphi(\gamma) := \{\varphi(x),\ x \in \gamma\} \in \Gamma_X, \\ &\Gamma_{\mathcal{Z}} \ni \hat{\gamma} \mapsto \hat{\varphi}(\hat{\gamma}) := \{\hat{\varphi}(z),\ (z) \in \hat{\gamma}\} \in \Gamma_{\mathcal{Z}}. \end{aligned} \tag{3.9}$$

The following lemma shows that the operator $\mathfrak{q}$ commutes with the action of diffeomorphisms (3.9).[5]

Lemma 3.1. *For any diffeomorphism $\varphi \in \mathrm{Diff}_0(X)$ and the corresponding diffeomorphism $\hat{\varphi}$, it holds*

$$\varphi \circ \mathfrak{q} = \mathfrak{q} \circ \hat{\varphi}. \tag{3.10}$$

Proof. The statement follows from the definition (2.16) of the map $\mathfrak{q}$ in view of the structure of diffeomorphisms φ and $\hat{\varphi}$ (see (3.8) and (3.9)). □

[5] According to relation (3.10), $\mathfrak{q}$ is an *intertwining operator* between associated diffeomorphisms φ and $\hat{\varphi}$.

Lemma 3.2. *The interaction potential* $\hat{\varPhi}$ *defined in* (2.29) *is invariant with respect to diffeomorphisms* (3.8)*, that is, for any* $\varphi \in \mathrm{Diff}_0(X)$ *we have*

$$\hat{\varPhi}(\hat{\varphi}(\hat{\gamma})) = \hat{\varPhi}(\hat{\gamma}), \qquad \hat{\gamma} \in \varGamma_{\mathcal{Z}}.$$

In particular, this implies the $\hat{\varphi}$*-invariance of the energy functionals defined in* (A.1) *and* (A.2)*, that is, for any* $\hat{\xi} \in \varGamma_{\mathcal{Z}}^0$ *and* $\hat{\gamma} \in \varGamma_{\mathcal{Z}}$*,*

$$\hat{E}(\hat{\varphi}(\hat{\xi})) = \hat{E}(\hat{\xi}), \qquad \hat{E}(\hat{\varphi}(\hat{\xi}), \hat{\varphi}(\hat{\gamma})) = \hat{E}(\hat{\xi}, \hat{\gamma}).$$

Proof. The claim readily follows by observing that a diffeomorphism (3.8) acts on the $\bar{y}$-coordinates of points $z = (x, \bar{y})$ in a configuration $\hat{\gamma} \in \varGamma_{\mathcal{Z}}$, while the interaction potential $\hat{\varPhi}$ (see (2.29)) only depends on their x-coordinates. ☐

As already mentioned (see (3.1)), we assume that the measure η is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}\bar{y}$ on $\mathfrak{X}$ and, moreover,

$$h(\bar{y}) := \frac{\eta(\mathrm{d}\bar{y})}{\mathrm{d}\bar{y}} > 0 \qquad \text{for a.a. } \bar{y} \in \mathfrak{X}. \tag{3.11}$$

This implies that the measure η is quasi-invariant with respect to the action of transformations $\bar{\varphi} : \mathfrak{X} \to \mathfrak{X}$ ($\varphi \in \mathrm{Diff}_0(X)$), that is, for any $f \in \mathrm{M}_+(\mathfrak{X})$,

$$\int_{\mathfrak{X}} f(\bar{y})\, \bar{\varphi}^*\eta(\mathrm{d}\bar{y}) = \int_{\mathfrak{X}} f(\bar{y})\, \rho_\eta^{\bar{\varphi}}(\bar{y})\, \mathrm{d}\bar{y}, \tag{3.12}$$

with the Radon–Nikodym density

$$\rho_\eta^{\bar{\varphi}}(\bar{y}) := \frac{\mathrm{d}(\bar{\varphi}^*\eta)}{\mathrm{d}\eta}(\bar{y}) = \frac{h(\bar{\varphi}^{-1}(\bar{y}))}{h(\bar{y})}\, J_{\bar{\varphi}}(\bar{y})^{-1} \tag{3.13}$$

(we set $\rho_\eta^{\bar{\varphi}}(\bar{y}) = 1$ if $h(\bar{y}) = 0$ or $h(\bar{\varphi}^{-1}(\bar{y})) = 0$). Here $J_{\bar{\varphi}}(\bar{y})$ is the Jacobian determinant of the diffeomorphism $\bar{\varphi}$; due to the diagonal structure of $\bar{\varphi}$ (see (3.6)) we have $J_{\bar{\varphi}}(\bar{y}) = \prod_{y_i \in \bar{y}} J_\varphi(y_i)$, where $J_\varphi(y)$ is the Jacobian determinant of φ.

Due to the "shift" form of diffeomorphisms (3.8), formulas (3.12), (3.13) readily imply that the product measure $\sigma(\mathrm{d}z) = \theta(\mathrm{d}x) \otimes \eta(\mathrm{d}\bar{y})$ on $\mathcal{Z} = X \times \mathfrak{X}$ is quasi-invariant with respect to $\hat{\varphi}$, that is, for each $\varphi \in \mathrm{Diff}_0(X)$ and any $f \in \mathrm{M}_+(\mathcal{Z})$,

$$\int_{\mathcal{Z}} f(z)\, \hat{\varphi}^*\sigma(\mathrm{d}z) = \int_{\mathcal{Z}} f(z)\, \rho_\varphi(z)\, \sigma(\mathrm{d}z), \tag{3.14}$$

where the Radon–Nikodym density $\rho_\varphi := \mathrm{d}(\hat{\varphi}^*\sigma)/\mathrm{d}\sigma$ is given by (see (3.13))

$$\rho_\varphi(z) = \rho_\eta^{\bar{\varphi}_x}(\bar{y}) \equiv \frac{h\big(\bar{\varphi}^{-1}(\bar{y}+x)-x\big)}{h(\bar{y})}\, J_{\bar{\varphi}}(\bar{y}+x)^{-1}, \quad z=(x,\bar{y})\in\mathcal{Z}. \tag{3.15}$$

We can now state our result on the quasi-invariance of the measure $\hat{g}$.

Theorem 3.3. *The Gibbs measure $\hat{g}$ constructed in Section* 2.4 *is quasi-invariant with respect to the action of diagonal diffeomorphisms $\hat{\varphi}$ on $\Gamma_{\mathcal{Z}}$ ($\varphi \in \mathrm{Diff}_0(X)$) defined by formula* (3.8)*, with the Radon–Nikodym density $R_{\hat{g}}^{\hat{\varphi}} = \mathrm{d}(\hat{\varphi}^*\hat{g})/\mathrm{d}\hat{g}$ given by*

$$R_{\hat{g}}^{\hat{\varphi}}(\hat{\gamma}) = \prod_{z\in\hat{\gamma}} \rho_\varphi(z), \qquad \hat{\gamma}\in\Gamma_{\mathcal{Z}}, \tag{3.16}$$

where $\rho_\varphi(z)$ is defined in (3.15)*.*

Proof. First of all, note that $\rho_\varphi(z)=1$ for any $z=(x,\bar{y})\notin \operatorname{supp}\hat{\varphi} = \mathcal{Z}_{K_\varphi}$, where $K_\varphi = \operatorname{supp}\varphi$ (see Remark 3.1), and $\sigma(\mathcal{Z}_{K_\varphi})<\infty$ by Proposition 2.10. On the other hand, Theorem 2.5(b) implies that the correlation function $\kappa_{\hat{g}}^1$ is bounded. Therefore, by Remark A.5 (see the Appendix) we obtain that $\hat{\gamma}(\mathcal{Z}_{K_\varphi})<\infty$ for $\hat{g}$-a.a. configurations $\hat{\gamma}\in\Gamma_{\mathcal{Z}}$, hence the product in (3.16) contains finitely many terms different from 1 and so the function $R_{\hat{g}}^{\hat{\varphi}}(\hat{\gamma})$ is well defined. Moreover, it satisfies the "localization" equality

$$R_{\hat{g}}^{\hat{\varphi}}(\hat{\gamma}) = R_{\hat{g}}^{\hat{\varphi}}(\hat{\gamma}\cap\mathcal{Z}_{K_\varphi}) \qquad \text{for } \hat{g}\text{-a.a. } \hat{\gamma}\in\Gamma_{\mathcal{Z}}. \tag{3.17}$$

Following [16, §2.8, Theorem 2.8.2], the proof of the theorem will be based on the use of Ruelle's equation (see the Appendix, Theorem A.1). Namely, according to (A.4) with $\Lambda = \mathcal{Z}_{K_\varphi}$, for any function $F\in \mathrm{M}_+(\Gamma_{\mathcal{Z}})$ we have

$$\begin{aligned}
\int_{\Gamma_{\mathcal{Z}}} F(\hat{\gamma})\,\hat{\varphi}^*\hat{g}(\mathrm{d}\hat{\gamma}) &= \int_{\Gamma_{\mathcal{Z}}} F(\hat{\varphi}(\hat{\gamma}))\,\hat{g}(\mathrm{d}\hat{\gamma})\\
&= \int_{\Gamma_\Lambda}\left(\int_{\Gamma_{\mathcal{Z}\setminus\Lambda}} F(\hat{\varphi}(\hat{\xi}\cup\hat{\gamma}'))\,\mathrm{e}^{-\hat{E}(\hat{\varphi}(\hat{\xi}))-\hat{E}(\hat{\varphi}(\hat{\xi}),\hat{\varphi}(\hat{\gamma}'))}\,\hat{g}(\mathrm{d}\hat{\gamma}')\right)\lambda_\sigma(\mathrm{d}\hat{\xi})\\
&= \int_{\Gamma_\Lambda}\left(\int_{\Gamma_{\mathcal{Z}\setminus\Lambda}} F(\hat{\varphi}(\hat{\xi})\cup\hat{\gamma}')\,\mathrm{e}^{-\hat{E}(\hat{\varphi}(\hat{\xi}))-\hat{E}(\hat{\varphi}(\hat{\xi}),\hat{\gamma}')}\,\hat{g}(\mathrm{d}\hat{\gamma}')\right)\lambda_\sigma(\mathrm{d}\hat{\xi})\\
&= \int_{\Gamma_\Lambda}\left(\int_{\Gamma_{\mathcal{Z}\setminus\Lambda}} F(\hat{\xi}\cup\hat{\gamma}')\,\mathrm{e}^{-\hat{E}(\hat{\xi})-\hat{E}(\hat{\xi},\hat{\gamma}')}\,\hat{g}(\mathrm{d}\hat{\gamma}')\right)\hat{\varphi}^*\lambda_\sigma(\mathrm{d}\hat{\xi}),
\end{aligned} \tag{3.18}$$

where λ_σ is the Lebesgue–Poisson measure corresponding to the reference measure σ (see (A.5)). Since σ is quasi-invariant with respect to diffeomorphisms $\hat{\varphi}$ (see (3.14)), it readily follows from the definition (A.5) that the restriction of the Lebesgue–Poisson measure λ_σ onto the set Γ_Λ is quasi-invariant with respect to $\hat{\varphi}$, with the density given precisely by the function (3.16). Hence, using the property (3.17), the right-hand side of (3.18) is reduced to

$$\begin{aligned}\int_{\Gamma_\Lambda}&\left(\int_{\Gamma_{\mathfrak{Z}\setminus\Lambda}} F(\hat{\xi}\cup\hat{\gamma}')\,\mathrm{e}^{-\hat{E}(\hat{\xi})-\hat{E}(\hat{\xi},\hat{\gamma}')}\,\hat{g}(\mathrm{d}\gamma')\right) R_{\hat{g}}^{\hat{\varphi}}(\hat{\xi})\,\lambda_\sigma(\mathrm{d}\hat{\xi})\\ &=\int_{\Gamma_\Lambda}\left(\int_{\Gamma_{\mathfrak{Z}\setminus\Lambda}} F(\hat{\xi}\cup\hat{\gamma}')\,R_{\hat{g}}^{\hat{\varphi}}(\hat{\xi}\cup\hat{\gamma}')\,\mathrm{e}^{-E(\hat{\xi})-E(\hat{\xi},\hat{\gamma}')}\,\hat{g}(\mathrm{d}\hat{\gamma}')\right)\lambda_\sigma(\mathrm{d}\hat{\xi})\\ &\qquad=\int_{\Gamma_{\mathfrak{Z}}} F(\hat{\gamma})\,R_{\hat{g}}^{\hat{\varphi}}(\hat{\gamma})\,\hat{g}(\mathrm{d}\hat{\gamma}),\end{aligned}\tag{3.19}$$

where we have again used Ruelle's equation (A.4).

As a result, combining (3.18) and (3.19) we obtain

$$\int_{\Gamma_{\mathfrak{Z}}} F(\hat{\gamma})\,\hat{\varphi}^*\hat{g}(\mathrm{d}\hat{\gamma})=\int_{\Gamma_{\mathfrak{Z}}} F(\hat{\gamma})\,R_{\hat{g}}^{\hat{\varphi}}(\hat{\gamma})\,\hat{g}(\mathrm{d}\hat{\gamma}),\tag{3.20}$$

which proves quasi-invariance of $\hat{g}$. In particular, setting $F\equiv 1$ in (3.20) yields $\int_{\Gamma_{\mathfrak{Z}}} R_{\hat{g}}^{\hat{\varphi}}(\hat{\gamma})\,\hat{g}(\mathrm{d}\hat{\gamma})=1$, and hence $R_{\hat{g}}^{\hat{\varphi}}\in L^1(\Gamma_{\mathfrak{Z}},g_{\mathrm{cl}})$. □

Remark 3.2. Note that the Radon–Nikodym density $R_{\hat{g}}^{\hat{\varphi}}$ defined by (3.16) does not depend on the background interaction potential Φ. As should be evident from the proof above, this is due to the special "shift" form of the diffeomorphisms $\hat{\varphi}$ (see (3.8)) and the cylinder structure of the interaction potential $\hat{\Phi}$ (see (2.29)). In particular, the expression (3.16) applies to the "interaction-free" case with $\Phi\equiv 0$ (and hence $\hat{\Phi}\equiv 0$), where the Gibbs measure $g\in\mathscr{G}(\theta,\Phi=0)$ is reduced to the Poisson measure π_θ on Γ_X with intensity measure θ (see the Appendix), while the Gibbs measure $\hat{g}\in\mathscr{G}(\sigma,\hat{\Phi}=0)$ amounts to the Poisson measure π_σ on $\Gamma_{\mathfrak{Z}}$ with intensity measure σ.

Remark 3.3. As is essentially well known (see, e.g., [2, 28]), quasi-invariance of a Poisson measure on the configuration space follows directly from the quasi-invariance of its intensity measure. For a proof adapted to our slightly more general setting (where diffeomorphisms are only assumed to have the support of finite measure), we refer the reader to [7, Proposition A.1]. Incidentally, the expression for the Radon–Nikodym derivative given in [7] (see also [2, Proposition 2.2]) contained a superfluous normalizing constant, which in our context would read

$$C_\varphi := \exp\left(\int_{\mathcal{Z}} \big(1-\rho_\varphi(z)\big)\,\sigma(\mathrm{d}z)\right)$$

(cf. (3.16)). In fact, it is easy to see that $C_\varphi = 1$; indeed, $\rho_\varphi = 1$ outside the set $\operatorname{supp}\hat{\varphi} = \mathcal{Z}_{K_\varphi}$ with $\sigma(\mathcal{Z}_{K_\varphi}) < \infty$ (see Proposition 2.10), hence

$$\ln C_\varphi = \int_{\mathcal{Z}_{K_\varphi}} \big(1-\rho_\varphi(z)\big)\,\sigma(\mathrm{d}z) = \sigma(\mathcal{Z}_{K_\varphi}) - \sigma(\hat{\varphi}^{-1}(\mathcal{Z}_{K_\varphi})) = 0.$$

Let $\mathcal{I}_{\mathfrak{q}} : L^2(\Gamma_X, \mathsf{g}_{\mathrm{cl}}) \to L^2(\Gamma_{\mathcal{Z}}, \hat{\mathsf{g}})$ be the isometry defined by the map $\mathfrak{q}$ (see (2.16)),

$$(\mathcal{I}_{\mathfrak{q}} F)(\hat{\gamma}) := F \circ \mathfrak{q}(\hat{\gamma}), \qquad \hat{\gamma} \in \Gamma_{\mathcal{Z}}, \tag{3.21}$$

and consider the corresponding adjoint operator

$$\mathcal{I}_{\mathfrak{q}}^* : L^2(\Gamma_{\mathcal{Z}}, \hat{\mathsf{g}}) \to L^2(\Gamma_X, \mathsf{g}_{\mathrm{cl}}). \tag{3.22}$$

Lemma 3.4. *The operator $\mathcal{I}_{\mathfrak{q}}^*$ defined by* (3.22) *can be extended to the operator*

$$\mathcal{I}_{\mathfrak{q}}^* : L^1(\Gamma_{\mathcal{Z}}, \hat{\mathsf{g}}) \to L^1(\Gamma_X, \mathsf{g}_{\mathrm{cl}}).$$

Proof. Note that $\mathcal{I}_{\mathfrak{q}}$ can be viewed as a bounded operator acting from $L^\infty(\Gamma_X, \mathsf{g}_{\mathrm{cl}})$ to $L^\infty(\Gamma_{\mathcal{Z}}, \hat{\mathsf{g}})$. This implies that the adjoint operator $\mathcal{I}_{\mathfrak{q}}^*$ is a bounded operator on the corresponding dual spaces, $\mathcal{I}_{\mathfrak{q}}^* : L^\infty(\Gamma_{\mathcal{Z}}, \hat{\mathsf{g}})' \to L^\infty(\Gamma_X, \mathsf{g}_{\mathrm{cl}})'$.

It is known (see [20]) that, for any sigma-finite measure space (M, μ), the corresponding space $L^1(M, \mu)$ can be identified with the subspace V of the dual space $L^\infty(M, \mu)'$ consisting of all linear functionals on $L^\infty(M, \mu)$ continuous with respect to bounded convergence in $L^\infty(M, \mu)$. That is, $\ell \in V$ if and only if $\ell(\psi_n) \to 0$ for any $\psi_n \in L^\infty(M, \mu)$ such that $|\psi_n| \le 1$ and $\psi_n(x) \to 0$ as $n \to \infty$ for μ-a.a. $x \in M$. Hence, to prove the lemma it suffices to show that, for any $F \in L^1(\Gamma_{\mathcal{Z}}, \hat{\mathsf{g}})$, the functional $\mathcal{I}_{\mathfrak{q}}^* F \in L^\infty(\Gamma_{\mathcal{Z}}, \hat{\mathsf{g}})'$ is continuous with respect to bounded convergence in $L^\infty(\Gamma_{\mathcal{Z}}, \hat{\mathsf{g}})$. To this end, for any sequence (ψ_n) in $L^\infty(\Gamma_X, \mathsf{g}_{\mathrm{cl}})$ such that $|\psi_n| \le 1$ and $\psi_n(\gamma) \to 0$ for g_{cl}-a.a. $\gamma \in \Gamma_X$, we have to prove that $\mathcal{I}_{\mathfrak{q}}^* F(\psi_n) \to 0$.

Let us first show that $\mathcal{I}_{\mathfrak{q}} \psi_n(\hat{\gamma}) \equiv \psi_n(\mathfrak{q}(\hat{\gamma})) \to 0$ for $\hat{\mathsf{g}}$-a.a. $\hat{\gamma} \in \Gamma_{\mathcal{Z}}$. Set

$$A_\psi := \{\gamma \in \Gamma_X : \psi_n(\gamma) \to 0\} \in \mathcal{B}(\Gamma_X),$$

$$\hat{A}_\psi := \{\hat{\gamma} \in \Gamma_{\mathcal{Z}} : \psi_n(\mathfrak{q}(\hat{\gamma})) \to 0\} \in \mathcal{B}(\Gamma_{\mathcal{Z}}),$$

and note that $\hat{A}_\psi = \mathfrak{q}^{-1}(A_\psi)$; then, recalling the relation (2.27), we get

$$\hat{g}(\hat{A}_\psi) = \hat{g}\big(\mathfrak{q}^{-1}(A_\psi)\big) = g_{\rm cl}(A_\psi) = 1,$$

as claimed. Now, by the dominated convergence theorem this implies

$$\mathcal{I}_\mathfrak{q}^* F(\psi_n) = \int_{\Gamma_\mathcal{Z}} F(\hat\gamma)\, \mathcal{I}_\mathfrak{q}\psi_n(\hat\gamma)\, \hat{g}(\mathrm{d}\hat\gamma) \to 0,$$

and the proof is complete. □

Taking advantage of Theorem 3.3 and applying the projection construction, we obtain our main result in this section.

Theorem 3.5. *The Gibbs cluster measure $g_{\rm cl}$ is quasi-invariant with respect to the action of* $\mathrm{Diff}_0(X)$ *on* Γ_X. *The corresponding Radon–Nikodym density is given by* $R^{\varphi}_{g_{\rm cl}} = \mathcal{I}^*_\mathfrak{q} R^{\hat\varphi}_{\hat g}$.

Proof. Note that, due to (2.27) and (3.10),

$$g_{\rm cl} \circ \varphi^{-1} = \hat g \circ \mathfrak{q}^{-1} \circ \varphi^{-1} = \hat g \circ \hat\varphi^{-1} \circ \mathfrak{q}^{-1}.$$

That is, $\varphi^* g_{\rm cl} = g_{\rm cl} \circ \varphi^{-1}$ is a push-forward of the measure $\hat\varphi^*\hat g = \hat g \circ \hat\varphi^{-1}$ under the map $\mathfrak{q}$, that is, $\varphi^* g_{\rm cl} = \mathfrak{q}^*\hat\varphi^*\hat g$. In particular, if $\hat\varphi^*\hat g$ is absolutely continuous with respect to $\hat g$ then so is $\varphi^* g_{\rm cl}$ with respect to $g_{\rm cl}$. Moreover, by the change of measure (2.27) and by Theorem 3.3, for any $F \in L^\infty(\Gamma_X, g_{\rm cl})$ we have

$$\int_{\Gamma_X} F(\gamma)\, \varphi^* g_{\rm cl}(\mathrm{d}\gamma) = \int_{\Gamma_\mathcal{Z}} \mathcal{I}_\mathfrak{q} F(\hat\gamma)\, \hat\varphi^*\hat g(\mathrm{d}\hat\gamma) = \int_{\Gamma_\mathcal{Z}} \mathcal{I}_\mathfrak{q} F(\hat\gamma)\, R^{\hat\varphi}_{\hat g}(\hat\gamma)\, \hat g(\mathrm{d}\hat\gamma). \tag{3.23}$$

By Lemma 3.4, the operator $\mathcal{I}^*_\mathfrak{q}$ acts from $L^1(\Gamma_\mathcal{Z}, \hat g)$ to $L^1(\Gamma_X, g_{\rm cl})$. Therefore, by the change of measure (2.25) the right-hand side of (3.23) can be rewritten as

$$\int_{\Gamma_X} F(\gamma)\, (\mathcal{I}^*_\mathfrak{q} R^{\hat\varphi}_{\hat g})(\gamma)\, g_{\rm cl}(\mathrm{d}\gamma),$$

which completes the proof. □

Remark 3.4. The Gibbs cluster measure $g_{\rm cl}$ on the configuration space Γ_X can be used to construct a unitary representation U of the diffeomorphism group $\mathrm{Diff}_0(X)$ by operators in $L^2(\Gamma_X, g_{\rm cl})$, given by the formula

$$U_\varphi F(\gamma) = \sqrt{R^{\varphi}_{g_{\rm cl}}(\gamma)}\, F(\varphi^{-1}(\gamma)), \qquad F \in L^2(\Gamma_X, g_{\rm cl}). \tag{3.24}$$

Such representations, which can be defined for arbitrary quasi-invariant measures on Γ_X, play a significant role in the representation theory of the group $\mathrm{Diff}_0(X)$ [14, 29] and quantum field theory [11, 12]. An important question is whether the representation (3.24) is irreducible. According to [29], this is equivalent to the $\mathrm{Diff}_0(X)$-ergodicity of the measure g_{cl}, which in our case is equivalent to the ergodicity of the measure $\hat{g}$ with respect to the group of transformations $\hat{\varphi}$ ($\varphi \in \mathrm{Diff}_0(X)$). Adapting the technique developed in [18], it can be shown that the aforementioned ergodicity of $\hat{g}$ is valid if and only if $\hat{g} \in \operatorname{ext} \mathscr{G}(\sigma, \hat{\Phi})$. In turn, the latter is equivalent to $g \in \operatorname{ext} \mathscr{G}(\theta, \Phi)$, provided that $g \in \mathscr{G}_{\mathrm{R}}(\theta, \Phi)$ (see Corollary 2.8).

3.3. Integration-by-parts formula

Let us first prove simple sufficient conditions for our measures on configuration spaces to belong to the corresponding moment classes $\mathcal{M}^n$ (see the Appendix, formula (A.10)).

Lemma 3.6. (a) *Let $g \in \mathscr{G}(\theta, \Phi)$, and suppose that the correlation functions κ_g^m are bounded for all $m = 1, \dots, n$. Then $\hat{g} \in \mathcal{M}^n(\Gamma_{\mathcal{Z}})$, that is,*

$$\int_{\Gamma_{\mathcal{Z}}} |\langle f, \hat{\gamma} \rangle|^n \, \hat{g}(\mathrm{d}\hat{\gamma}) < \infty, \qquad f \in C_0(\mathcal{Z}). \tag{3.25}$$

Moreover, the bound (3.25) *is valid for any function $f \in \bigcap_{m=1}^n L^m(\mathcal{Z}, \sigma)$.*

(b) *If, in addition, the total number of components of a random vector $\bar{y} \in \mathfrak{X}$ has a finite n-th moment*[6] *with respect to the measure η,*

$$\int_{\mathfrak{X}} N_X(\bar{y})^n \, \eta(\mathrm{d}\bar{y}) < \infty, \tag{3.26}$$

then $g_{\mathrm{cl}} \in \mathcal{M}^n(\Gamma_X)$.

Proof. (a) Using the multinomial expansion, for any $f \in C_0(\mathcal{Z})$ we have

$$\begin{aligned} \int_{\Gamma_{\mathcal{Z}}} |\langle f, \hat{\gamma} \rangle|^n \, \hat{g}(\mathrm{d}\hat{\gamma}) &\le \int_{\Gamma_{\mathcal{Z}}} \Bigl(\sum\nolimits_{z \in \hat{\gamma}} |f(z)| \Bigr)^n \, \hat{g}(\mathrm{d}\hat{\gamma}) \\ &= \sum_{m=1}^n \int_{\Gamma_{\mathcal{Z}}} \sum_{\{z_1, \dots, z_m\} \subset \hat{\gamma}} \phi_n(z_1, \dots, z_m) \, \hat{g}(\mathrm{d}\hat{\gamma}), \end{aligned} \tag{3.27}$$

[6] Cf. our standard assumption (2.50), where $n = 1$.

where $\phi_n(z_1, \dots, z_m)$ is a symmetric function given by

$$\phi_n(z_1, \dots, z_m) := \sum_{\substack{i_1, \dots, i_m \ge 1 \\ i_1 + \cdots + i_m = n}} \frac{n!}{i_1! \cdots i_m!} \, |f(z_1)|^{i_1} \cdots |f(z_m)|^{i_m}. \tag{3.28}$$

By the definition (A.7), the integral on the right-hand side of (3.27) is reduced to

$$\frac{1}{m!} \int_{\mathcal{Z}^m} \phi_n(z_1, \dots, z_m) \, \kappa_{\hat{g}}^m(z_1, \dots, z_m) \, \sigma(\mathrm{d}z_1) \cdots \sigma(\mathrm{d}z_m). \tag{3.29}$$

By Theorem 2.5(b), the hypotheses of the lemma imply that $0 \le \kappa_{\hat{g}}^m \le a_{\hat{g}}$ ($m = 1, \dots, n$) with some constant $a_{\hat{g}} < \infty$. Hence, substituting (3.28) we obtain that the integral in (3.29) is bounded by

$$a_{\hat{g}} \sum_{\substack{i_1, \dots, i_m \ge 1 \\ i_1 + \cdots + i_m = n}} \frac{n!}{i_1! \cdots i_m!} \prod_{j=1}^{m} \int_{\mathcal{Z}} |f(z_j)|^{i_j} \, \sigma(\mathrm{d}z_j) < \infty, \tag{3.30}$$

since each integral in (3.30) is finite owing to the assumption $f \in C_0(\mathcal{Z})$. Moreover, the bound (3.30) is valid for any function $f \in \bigcap_{m=1}^{n} L^m(\mathcal{Z}, \sigma)$. Returning to (3.27), this yields (3.25).

(b) Using the change of measure (2.27), for any $\phi \in C_0(X)$ we obtain

$$\int_{\Gamma_X} |\langle \phi, \gamma \rangle|^n \, \mathsf{g}_{\mathrm{cl}}(\mathrm{d}\gamma) = \int_{\Gamma_{\mathcal{Z}}} |\langle \phi, \mathfrak{q}(\hat{\gamma}) \rangle|^n \, \hat{g}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_{\mathcal{Z}}} |\langle \mathfrak{q}^* \phi, \hat{\gamma} \rangle|^n \, \hat{g}(\mathrm{d}\hat{\gamma}), \tag{3.31}$$

where

$$\mathfrak{q}^* \phi(x, \bar{y}) := \sum_{y_i \in \bar{y}} \phi(y_i + x), \qquad (x, \bar{y}) \in \mathcal{Z}. \tag{3.32}$$

Due to part (a) of the lemma, it suffices to show that $\mathfrak{q}^* \phi \in L^m(\mathcal{Z}, \sigma)$ for any $m = 1, \dots, n$. By the elementary inequality $(a_1 + \cdots + a_k)^m \le k^{m-1}(a_1^m + \cdots + a_k^m)$, from (3.32) we have

$$\int_{\mathcal{Z}} |\mathfrak{q}^* \phi(z)|^m \, \sigma(\mathrm{d}z) \le \int_{\mathcal{Z}} N_X(\bar{y})^{m-1} \sum_{y_i \in \bar{y}} |\phi(y_i + x)|^m \, \sigma(\mathrm{d}x \times \mathrm{d}\bar{y}). \tag{3.33}$$

Recalling that $\sigma = \theta \otimes \eta$ and denoting $b_\phi := \sup_{x \in X} |\phi(x)| < \infty$ and $K_\phi :=$

$\operatorname{supp}\phi \subset X$, the right-hand side of (3.33) is dominated by

$$\int_{\mathfrak{X}} N_X(\bar{y})^{m-1} \left((b_\phi)^m \sum_{y_i \in \bar{y}} \int_X \mathbf{1}_{K_\phi - y_i}(x)\, \theta(\mathrm{d}x) \right) \eta(\mathrm{d}\bar{y})$$
$$= (b_\phi)^m \int_{\mathfrak{X}} N_X(\bar{y})^{m-1} \sum_{y_i \in \bar{y}} \theta(K_\phi - y_i)\; \eta(\mathrm{d}\bar{y})$$
$$\le (b_\phi)^m \sup_{y \in X} \theta(K_\phi - y) \int_{\mathfrak{X}} N_X(\bar{y})^m\, \eta(\mathrm{d}\bar{y}) < \infty,$$

according to the assumptions (2.49) and (3.26). □

In the rest of this section, we shall assume that the conditions of Lemma 3.6 are satisfied with $n = 1$. Thus, the measures g, $\hat{g}$ and g_{cl} belong to the corresponding $\mathcal{M}^1$-classes.

Let $v \in \operatorname{Vect}_0(X)$ (:= the space of compactly supported smooth vector fields on X), and define a vector field $\hat{v}_x$ on $\mathfrak{X}$ by the formula

$$\hat{v}_x(\bar{y}) := \left(v(y_1 + x), \dots, v(y_n + x)\right), \qquad \bar{y} = (y_1, \dots, y_n) \in \mathfrak{X}. \tag{3.34}$$

Observe that the measure η satisfies the following integration-by-parts formula,

$$\int_{\mathfrak{X}} \nabla^{\hat{v}_x} f(\bar{y})\, \eta(\mathrm{d}\bar{y}) = -\int_{\mathfrak{X}} f(\bar{y})\, \beta_\eta^{\hat{v}}(x, \bar{y})\, \eta(\mathrm{d}\bar{y}), \qquad f \in C_0^\infty(\mathfrak{X}), \tag{3.35}$$

where $\nabla^{\hat{v}_x}$ is the derivative along the vector field $\hat{v}_x$ and

$$\beta_\eta^{\hat{v}}(x, \bar{y}) := (\beta_\eta(\bar{y}), \hat{v}_x(\bar{y}))_{T_{\bar{y}}\mathfrak{X}} + \operatorname{div} \hat{v}_x(\bar{y}) \tag{3.36}$$

is the logarithmic derivative of $\eta(\mathrm{d}\bar{y}) = h(\bar{y})\,\mathrm{d}\bar{y}$ along $\hat{v}_x$, expressed in terms of the vector logarithmic derivative

$$\beta_\eta(\bar{y}) := \frac{\nabla h(\bar{y})}{h(\bar{y})}, \qquad \bar{y} \in \mathfrak{X}. \tag{3.37}$$

Let us define the space $H^{1,n}(\mathfrak{X})$ ($n \ge 1$) as the set of functions $f \in L^n(\mathfrak{X}, \mathrm{d}\bar{y})$ satisfying the condition

$$\int_{\mathfrak{X}} \left(\sum_{y_i \in \bar{y}} \left| \nabla_{y_i} f(\bar{y}) \right| \right)^n \mathrm{d}\bar{y} < \infty. \tag{3.38}$$

Note that $H^{1,n}(\mathfrak{X})$ is a linear space, due to the elementary inequality $(|a|+|b|)^n \le 2^{n-1}\bigl(|a|^n + |b|^n\bigr)$.

Lemma 3.7. *Assume that* $h^{1/n} \in H^{1,n}(\mathfrak{X})$ *for some integer* $n \geq 1$*, and let the condition* (3.26) *hold. Then* $\beta_\eta^{\hat v} \in L^m(\mathcal{Z}, \sigma)$ *for any* $m = 1, \dots, n$*.*

Proof. Firth of all, observe that the condition $h^{1/n} \in H^{1,n}(\mathfrak{X})$ implies that $h^{1/m} \in H^{1,m}(\mathfrak{X})$ for any $m = 1, \dots, n$. Indeed, $h^{1/m} \in L^m(\mathfrak{X}, \mathrm{d}\bar y)$ if and only if $h \in L^1(\mathfrak{X}, \mathrm{d}\bar y)$; furthermore, using the definition (3.37) of $\beta_\eta(\bar y)$ we see that

$$\begin{aligned}\int_{\mathfrak{X}} \Bigg(\sum_{y_i \in \bar y} \big|\nabla_{y_i}\big(h(\bar y)^{1/m}\big)\big|\Bigg)^m \mathrm{d}\bar y &= m^{-m} \int_{\mathfrak{X}} \Bigg(\sum_{y_i \in \bar y} \frac{|\nabla_{y_i} h(\bar y)|}{h(\bar y)^{1-1/m}}\Bigg)^m \mathrm{d}\bar y \\ &= m^{-m} \int_{\mathfrak{X}} \Bigg(\sum_{y_i \in \bar y} |\beta_\eta(\bar y)_i|\Bigg)^m \eta(\mathrm{d}\bar y) < \infty,\end{aligned} \tag{3.39}$$

since η is a probability measure and hence $L^m(\mathfrak{X}, \eta) \subset L^n(\mathfrak{X}, \eta)$ ($m = 1, \dots, n$).

To show that $\beta_\eta^{\hat v} \in L^m(\mathcal{Z}, \sigma)$, it suffices to check that each of the two terms on the right-hand side of (3.36) belongs to $L^m(\mathcal{Z}, \sigma)$. Denote $b_v := \sup_{x \in X} |v(x)| < \infty$, $K_v := \operatorname{supp} v \subset X$, and recall that $C_{K_v} := \sup_{y \in X} \theta(K_v - y) < \infty$ by condition (2.49). Using (3.34), we have

$$\begin{aligned}\int_{\mathcal{Z}} |(\beta_\eta(\bar y), \hat v_x(\bar y))|^m \,\sigma(\mathrm{d}x \times \mathrm{d}\bar y) &\leq \int_{\mathcal{Z}} \Bigg(\sum_{y_i \in \bar y} |\beta_\eta(\bar y)_i| \cdot |v(y_i + x)|\Bigg)^m \theta(\mathrm{d}x)\,\eta(\mathrm{d}\bar y) \\ \leq (b_v)^{m-1} \int_{\mathfrak{X}} \Bigg(\sum_{y_i \in \bar y} |\beta_\eta(\bar y)_i|\Bigg)^{m-1} & \sum_{y_i \in \bar y} |\beta_\eta(\bar y)_i| \left(\int_X |v(y_i + x)|\,\theta(\mathrm{d}x)\right) \eta(\mathrm{d}\bar y) \\ \leq (b_v)^m \int_{\mathfrak{X}} \Bigg(\sum_{y_i \in \bar y} |\beta_\eta(\bar y)_i|\Bigg)^{m-1} & \sum_{y_i \in \bar y} |\beta_\eta(\bar y)_i|\,\theta(K_v - y_i)\,\eta(\mathrm{d}\bar y) \\ \leq (b_v)^m\, C_{K_v} \int_{\mathfrak{X}} \Bigg(\sum_{y_i \in \bar y} |\beta_\eta(\bar y)_i|\Bigg)^m & \eta(\mathrm{d}\bar y) < \infty,\end{aligned} \tag{3.40}$$

according to (3.39). Similarly, denoting $d_v := \sup_{x \in X} |\operatorname{div} v(x)| < \infty$ and again using (3.34), we obtain

$$
\begin{aligned}
\int_{\mathcal{Z}} \bigl|\operatorname{div} \hat{v}_x(\bar{y})\bigr|^m \, \sigma(\mathrm{d}x \times \mathrm{d}\bar{y}) &= \int_{\mathcal{Z}} \Biggl(\sum_{y_i \in \bar{y}} \bigl|(\operatorname{div} v)(y_i + x)\bigr|\Biggr)^m \theta(\mathrm{d}x)\, \eta(\mathrm{d}\bar{y}) \\
&\le (d_v)^{m-1} \int_{\mathfrak{X}} N_X(\bar{y})^{m-1} \Biggl(\sum_{y_i \in \bar{y}} \int_X \bigl|(\operatorname{div} v)(y_i + x)\bigr| \, \theta(\mathrm{d}x)\Biggr) \eta(\mathrm{d}\bar{y}) \\
&\le (d_v)^m \int_{\mathfrak{X}} N_X(\bar{y})^{m-1} \sum_{y_i \in \bar{y}} \theta(K_v - y_i)\, \eta(\mathrm{d}\bar{y}) \\
&\le (d_v)^m \, C_{K_v} \int_{\mathfrak{X}} N_X(\bar{y})^m \, \eta(\mathrm{d}\bar{y}) < \infty, \qquad (3.41)
\end{aligned}
$$

according to the assumption (3.26). As a result, combining the bounds (3.40) and (3.41), we see that $\beta_\eta^{\hat{v}} \in L^m(\mathcal{Z}, \sigma)$, as claimed. $\square$

The next two theorems are our main results in this section.

Theorem 3.8. *For any function $F \in \mathcal{FC}(\Gamma_X)$, the Gibbs cluster measure g_{cl} satisfies the integration-by-parts formula*

$$
\int_{\Gamma_X} \sum_{x \in \gamma} \nabla_x F(\gamma) \cdot v(x) \, \mathsf{g}_{\mathrm{cl}}(\mathrm{d}\gamma) = -\int_{\Gamma_X} F(\gamma)\, B^v_{\mathsf{g}_{\mathrm{cl}}}(\gamma)\, \mathsf{g}_{\mathrm{cl}}(\mathrm{d}\gamma), \qquad (3.42)
$$

where $B^v_{\mathsf{g}_{\mathrm{cl}}}(\gamma) := \mathcal{I}^*_{\mathfrak{q}} \langle \beta_\eta^{\hat{v}}, \hat{\gamma}\rangle \in L^1(\Gamma_X, \mathsf{g}_{\mathrm{cl}})$ and $\beta_\eta^{\hat{v}}$ is the logarithmic derivative defined in (3.36).

Proof. For any function $F \in \mathcal{FC}(\Gamma_X)$ and vector field $v \in \mathrm{Vect}_0(X)$, let us denote for brevity

$$
H(x,\gamma) := \nabla_x F(\gamma) \cdot v(x), \qquad x \in X, \ \ \gamma \in \Gamma_X. \qquad (3.43)
$$

Furthermore, setting $\hat{F} = \mathcal{I}_{\mathfrak{q}} F : \Gamma_{\mathcal{Z}} \to \mathbb{R}$ we introduce the notation

$$
\hat{H}(z,\hat{\gamma}) := \nabla_{\bar{y}} \hat{F}(\hat{\gamma}) \cdot \hat{v}_x(\bar{y}), \qquad z = (x,\bar{y}) \in \mathcal{Z}, \ \ \hat{\gamma} \in \Gamma_{\mathcal{Z}}. \qquad (3.44)
$$

From these definitions, it is clear that

$$
\mathcal{I}_{\mathfrak{q}} \Biggl(\sum_{x \in \gamma} H(x,\gamma)\Biggr)(\hat{\gamma}) = \sum_{z \in \hat{\gamma}} \hat{H}(z,\hat{\gamma}), \qquad \hat{\gamma} \in \Gamma_{\mathcal{Z}}. \qquad (3.45)
$$

Let us show that the Gibbs measure $\hat{g}$ on $\Gamma_{\mathcal{Z}}$ satisfies the following integration-by-parts formula:

$$\int_{\Gamma_{\mathcal{Z}}} \sum_{z\in\hat{\gamma}} \hat{H}(z,\hat{\gamma})\,\hat{\mathbf{g}}(\mathrm{d}\hat{\gamma}) = -\int_{\Gamma_{\mathcal{Z}}} \hat{F}(\hat{\gamma}) B_{\hat{\mathbf{g}}}^{\hat{v}}(\hat{\gamma})\,\hat{\mathbf{g}}(\mathrm{d}\hat{\gamma}), \tag{3.46}$$

where the logarithmic derivative $B_{\hat{g}}^{\hat{v}}(\hat{\gamma}) := \langle \beta_{\eta}^{\hat{v}}, \hat{\gamma}\rangle$ belongs to $L^1(\Gamma_{\mathcal{Z}},\hat{g})$ (by Lemmas 3.6(b) and 3.7 with $n=1$). By the change of measure (2.25) and due to relation (3.45), we have

$$\begin{aligned}\int_{\Gamma_{\mathcal{Z}}} \sum_{z\in\hat{\gamma}} |\hat{H}(z,\hat{\gamma})|\,\hat{\mathbf{g}}(\mathrm{d}\hat{\gamma}) &= \int_{\Gamma_X} \sum_{x\in\gamma} |H(x,\gamma)|\,\mathbf{g}_{\mathrm{cl}}(\mathrm{d}\gamma)\\ &\le \sup_{(x,\gamma)} |H(x,\gamma)| \int_{\Gamma_X} \sum_{x\in\gamma} \mathbf{1}_{K_v}(x)\,\mathbf{g}_{\mathrm{cl}}(\mathrm{d}\gamma),\end{aligned} \tag{3.47}$$

where $K_v := \operatorname{supp} v$ is a compact set in X. Note that the right-hand side of (3.47) is finite, since the function H is bounded (see (3.43)) and, by Lemma 3.6(b), $g_{\mathrm{cl}} \in \mathcal{M}^1(\Gamma_X)$. Therefore, by Remark A.1 we can apply Nguyen–Zessin's equation (A.3) with the function $\hat{H}$ to obtain

$$\int_{\Gamma_{\mathcal{Z}}} \sum_{z\in\hat{\gamma}} \hat{H}(z,\hat{\gamma})\,\hat{\mathbf{g}}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_{\mathcal{Z}}} \left(\int_{\mathcal{Z}} \hat{H}(z,\hat{\gamma}\cup\{z\})\,\mathrm{e}^{-\hat{E}(\{z\},\hat{\gamma})}\,\sigma(\mathrm{d}z)\right) \hat{\mathbf{g}}(\mathrm{d}\hat{\gamma}). \tag{3.48}$$

Inserting the definition (3.43), using Lemma 3.2 and recalling that $\sigma = \theta\otimes\eta$ (see (2.28)), let us apply the integration-by-parts formula (3.35) for the measure η to rewrite the internal integral in (3.48) as

$$\begin{aligned}&\int_X \mathrm{e}^{-E(\{x\},p_X(\hat{\gamma}))} \left(\int_{\mathfrak{X}} \nabla_{\bar{y}} \hat{F}(\hat{\gamma}\cup\{(x,\bar{y})\}) \cdot \hat{v}_x(\bar{y})\,\eta(\mathrm{d}\bar{y})\right) \theta(\mathrm{d}x)\\ &\quad = -\int_X \mathrm{e}^{-E(\{x\},p_X(\hat{\gamma}))} \left(\int_{\mathfrak{X}} \hat{F}(\hat{\gamma}\cup\{(x,\bar{y})\})\,\beta_{\eta}^{\hat{v}}(x,\bar{y})\,\eta(\mathrm{d}\bar{y})\right) \theta(\mathrm{d}x)\\ &\quad = -\int_{\mathcal{Z}} \mathrm{e}^{-E(\{p_X(z)\},p_X(\hat{\gamma}))} \hat{F}(\hat{\gamma}\cup\{z\})\,\beta_{\eta}^{\hat{v}}(z)\,\sigma(\mathrm{d}z).\end{aligned}$$

Returning to (3.48) and again using Nguyen–Zessin's equation (A.3), we see that the right-hand side of (3.48) is reduced to

$$-\int_{\Gamma_{\mathcal{Z}}} \sum_{z\in\hat{\gamma}} \hat{F}(\hat{\gamma})\,\beta_{\eta}^{\hat{v}}(z)\,\hat{\mathbf{g}}(\mathrm{d}\hat{\gamma}) = -\int_{\Gamma_{\mathcal{Z}}} \hat{F}(\hat{\gamma})\,B_{\hat{\mathbf{g}}}^{\hat{v}}\,\hat{\gamma}\rangle\,\hat{\mathbf{g}}(\mathrm{d}\hat{\gamma}),$$

which proves formula (3.46).

Now, using equality (3.45), we obtain

$$\int_{\Gamma_X}\sum_{x\in\gamma}H(x,\gamma)\,\mathsf{g}_{\mathrm{cl}}(\mathrm{d}\gamma)=\int_{\Gamma_{\mathcal Z}}\Bigg(\sum_{(x,\bar y)\in\hat\gamma}\nabla_{\bar y}\mathcal I_{\mathfrak q}F(\hat\gamma)\cdot\hat v_x(\bar y)\Bigg)\,\hat{\mathsf g}(\mathrm d\hat\gamma)$$
$$=-\int_{\Gamma_{\mathcal Z}}\mathcal I_{\mathfrak q}F(\hat\gamma)\,B_{\hat{\mathsf g}}^{\hat v}(\hat\gamma)\,\hat{\mathsf g}(\mathrm d\hat\gamma)$$
$$=-\int_{\Gamma_X}F(\gamma)\,\mathcal I_{\mathfrak q}^{*}B_{\hat{\mathsf g}}^{\hat v}(\gamma)\,\mathsf g_{\mathrm{cl}}(\mathrm d\gamma),$$

where $\mathcal I_{\mathfrak q}^{*}B_{\hat{\mathsf g}}^{\hat v}\in L^1(\Gamma_X,\mathsf g_{\mathrm{cl}})$ by Lemma 3.4. Thus, formula (3.42) is proved. □

Remark 3.5. Observe that the logarithmic derivative $B_{\hat{\mathsf g}}^{\hat v}=\langle\beta_\eta^{\hat v},\hat\gamma\rangle$ (see (3.46)) does not depend on the interaction potential Φ, and in particular coincides with that in the case $\Phi\equiv 0$, where the Gibbs measure $\mathsf g$ is reduced to the Poisson measure π_θ. Nevertheless, the logarithmic derivative $B_{\mathsf g_{\mathrm{cl}}}^{v}$ does depend on Φ via the map $\mathcal I_{\mathfrak q}^{*}$.

Remark 3.6. Note that in Theorem 3.8 the reference measure θ does not have to be differentiable with respect to v.

According to Theorem 3.8, $B_{\mathsf g_{\mathrm{cl}}}^{v}\in L^1(\Gamma_{\mathcal Z},\mathsf g_{\mathrm{cl}})$. However, under the conditions of Lemma 3.7 with $n\ge 2$, this statement can be enhanced.

Lemma 3.9. *Assume that $h^{1/n}\in H^{1,n}(\mathfrak X)$ for some integer $n\ge 2$, and let the condition* (3.26) *hold. Then $B_{\mathsf g_{\mathrm{cl}}}^{v}\in L^n(\Gamma_{\mathcal Z},\mathsf g_{\mathrm{cl}})$.*

Proof. By Lemmas 3.6(a) and 3.7, it follows that $\langle\beta_\eta^{\hat v},\hat\gamma\rangle\in L^n(\Gamma_{\mathcal Z},\hat{\mathsf g})$. Let $s:=n/(n-1)$, so that $n^{-1}+s^{-1}=1$. Note that $\mathcal I_{\mathfrak q}$ can be treated as a bounded operator acting from $L^s(\Gamma_X,\mathsf g_{\mathrm{cl}})$ to $L^s(\Gamma_{\mathcal Z},\hat{\mathsf g})$. Hence, $\mathcal I_{\mathfrak q}^{*}$ is a bounded operator from $L^s(\Gamma_{\mathcal Z},\hat{\mathsf g})'=L^n(\Gamma_{\mathcal Z},\hat{\mathsf g})$ to $L^s(\Gamma_X,\mathsf g_{\mathrm{cl}})'=L^n(\Gamma_X,\mathsf g_{\mathrm{cl}})$, which implies that $B_{\mathsf g_{\mathrm{cl}}}^{v}=\mathcal I_{\mathfrak q}^{*}\langle\beta_\eta^{\hat v},\hat\gamma\rangle\in L^n(\Gamma_{\mathcal Z},\mathsf g_{\mathrm{cl}})$. □

Formula (3.42) can be extended to more general vector fields on Γ_X. Let $\mathcal{FV}(\Gamma_X)$ be the class of vector fields V of the form $V(\gamma)=(V(\gamma)_x)_{x\in\gamma}$,

$$V(\gamma)_x=\sum_{j=1}^{N}G_j(\gamma)\,v_j(x)\in T_xX,$$

where $G_j \in \mathcal{FC}(\Gamma_X)$ and $v_j \in \mathrm{Vect}_0(X),\ j = 1, \dots, N$. For any such V we set

$$B^V_{g_{\mathrm{cl}}}(\gamma) := (\mathcal{I}^*_{\mathfrak{q}} B^{\mathcal{I}_{\mathfrak{q}} V}_{\hat{g}})(\gamma),$$

where $B^{\mathcal{I}_{\mathfrak{q}} V}_{\hat{g}}(\hat{\gamma})$ is the logarithmic derivative of $\hat{g}$ along $\mathcal{I}_{\mathfrak{q}} V(\hat{\gamma}) := V(\mathfrak{q}(\hat{\gamma}))$ (see [2]). Note that $\mathcal{I}_{\mathfrak{q}} V$ is a vector field on $\Gamma_{\mathcal{Z}}$ owing to the obvious equality

$$T_{\hat{\gamma}} \Gamma_{\mathcal{Z}} = T_{\mathfrak{q}(\hat{\gamma})} \Gamma_X.$$

Clearly,

$$B^V_{g_{\mathrm{cl}}}(\gamma) = \sum_{j=1}^{N} \Big(G_j(\gamma) B^{v_j}_{g_{\mathrm{cl}}}(\gamma) + \sum_{x \in \gamma} \nabla_x G_j(\gamma) \cdot v_j(x) \Big).$$

Theorem 3.10. *For any $F_1, F_2 \in \mathcal{FC}(\Gamma_X)$ and $V \in \mathcal{FV}(\Gamma_X)$, we have*

$$\begin{aligned} \int_{\Gamma_X} \sum_{x \in \gamma} \nabla_x F_1(\gamma) \cdot V(\gamma)_x \, F_2(\gamma) \, g_{\mathrm{cl}}(\mathrm{d}\gamma) &= - \int_{\Gamma_X} F_1(\gamma) \sum_{x \in \gamma} \nabla_x F_2(\gamma) \cdot V(\gamma)_x \, g_{\mathrm{cl}}(\mathrm{d}\gamma) \\ &\quad - \int_{\Gamma_X} F_1(\gamma) F_2(\gamma) B^V_{g_{\mathrm{cl}}}(\gamma) \, g_{\mathrm{cl}}(\mathrm{d}\gamma). \end{aligned}$$

Proof. The proof can be obtained by a straightforward generalization of the arguments used in the proof of Theorem 3.8. □

We define the *vector logarithmic derivative* of g_{cl} as a linear operator

$$B_{g_{\mathrm{cl}}} \colon \mathcal{FV}(\Gamma_X) \to L^1(\Gamma_X, g_{\mathrm{cl}})$$

via the formula

$$B_{g_{\mathrm{cl}}} V(\gamma) := B^V_{g_{\mathrm{cl}}}(\gamma).$$

This notation will be used in the next section.

4. The Dirichlet form and equilibrium stochastic dynamics

Throughout this section, we assume that the conditions of Lemma 3.6 are satisfied with $n = 2$. Thus, the measures g, $\hat{g}$ and g_{cl} belong to the corresponding $\mathcal{M}^2$-classes. Our considerations will involve the Γ-gradients (see Section 3.1) on different configuration spaces, such as Γ_X, $\Gamma_{\mathfrak{X}}$ and $\Gamma_{\mathcal{Z}}$; to avoid confusion, we shall denote them by ∇^Γ_X, $\nabla^\Gamma_{\mathfrak{X}}$ and $\nabla^\Gamma_{\mathcal{Z}}$, respectively.

4.1. The Dirichlet form associated with $\mathsf{g}_{\rm cl}$

Let us introduce a pre-Dirichlet form $\mathcal{E}_{\mathsf{g}_{\rm cl}}$ associated with the Gibbs cluster measure $\mathsf{g}_{\rm cl}$, defined on functions $F_1, F_2 \in \mathcal{FC}(\Gamma_X) \subset L^2(\Gamma_X, \mathsf{g}_{\rm cl})$ by

$$\mathcal{E}_{\mathsf{g}_{\rm cl}}(F_1, F_2) := \int_{\Gamma_X} \langle \nabla_X^{\Gamma} F_1(\gamma), \nabla_X^{\Gamma} F_2(\gamma) \rangle_\gamma \, \mathsf{g}_{\rm cl}(\mathrm{d}\gamma). \tag{4.1}$$

Let us also consider the operator $H_{\mathsf{g}_{\rm cl}}$ defined by

$$H_{\mathsf{g}_{\rm cl}} F := -\Delta^{\Gamma} F + B_{\mathsf{g}_{\rm cl}} \nabla_X^{\Gamma} F, \qquad F \in \mathcal{FC}(\Gamma_X), \tag{4.2}$$

where $\Delta^{\Gamma} F(\gamma) := \sum_{x \in \gamma} \Delta_x F(\gamma)$.

The next theorem readily follows from the general theory of (pre-)Dirichlet forms associated with measures from the class $\mathcal{M}^2(\Gamma_X)$ (see [3, 22]).

Theorem 4.1. (a) *The pre-Dirichlet form* (4.1) *is well defined, i.e.,* $\mathcal{E}_{\mathsf{g}_{\rm cl}}(F_1, F_2) < \infty$ *for all* $F_1, F_2 \in \mathcal{FC}(\Gamma_X)$;

(b) *The expression* (4.2) *defines a symmetric operator* $H_{\mathsf{g}_{\rm cl}}$ *in* $L^2(\Gamma_X, \mathsf{g}_{\rm cl})$ *whose domain includes* $\mathcal{FC}(\Gamma_X)$;

(c) *The operator* $H_{\mathsf{g}_{\rm cl}}$ *is the generator of the pre-Dirichlet form* $\mathcal{E}_{\mathsf{g}_{\rm cl}}$*, i.e.,*

$$\mathcal{E}_{\mathsf{g}_{\rm cl}}(F_1, F_2) = \int_{\Gamma_X} F_1(\gamma) \, H_{\mathsf{g}_{\rm cl}} F_2(\gamma) \, \mathsf{g}_{\rm cl}(\mathrm{d}\gamma), \qquad F_1, F_2 \in \mathcal{FC}(\Gamma_X). \tag{4.3}$$

Formula (4.3) implies that the form $\mathcal{E}_{\mathsf{g}_{\rm cl}}$ is closable. It follows from the properties of the *carré du champ* $\sum_{x \in \gamma} \nabla_x F_1(\gamma) \cdot \nabla_x F_2(\gamma)$ that the closure of $\mathcal{E}_{\mathsf{g}_{\rm cl}}$ (for which we shall keep the same notation) is a quasi-regular local Dirichlet form on a bigger state space $\ddot{\Gamma}_X$ consisting of all integer-valued Radon measures on X (see [22]). By the general theory of Dirichlet forms (see [21]), this implies the following result (cf. [2, 3, 7]).

Theorem 4.2. *There exists a conservative diffusion process* $\mathbf{X} = (\mathbf{X}_t, \, t \geq 0)$ *on* $\ddot{\Gamma}_X$*, properly associated with the Dirichlet form* $\mathcal{E}_{\mathsf{g}_{\rm cl}}$*, that is, for any function* $F \in L^2(\ddot{\Gamma}_X, \mathsf{g}_{\rm cl})$ *and all* $t \geq 0$*, the map*

$$\ddot{\Gamma}_X \ni \gamma \mapsto p_t F(\gamma) := \int_{\Omega} F(\mathbf{X}_t) \, \mathrm{d}P_\gamma$$

is an $\mathcal{E}_{\mathsf{g}_{\rm cl}}$*-quasi-continuous version of* $\exp(-tH_{\mathsf{g}_{\rm cl}})F$*. Here* Ω *is the canonical sample space* (*of* $\ddot{\Gamma}_X$*-valued continuous functions on* $\mathbb{R}_+$) *and* $(P_\gamma, \, \gamma \in \ddot{\Gamma}_X)$ *is* *the family of probability distributions of the process* $\mathbf{X}$ *conditioned on the initial value* $\gamma = \mathbf{X}_0$. *The process* $\mathbf{X}$ *is unique up to* g_{cl}*-equivalence. In particular,* $\mathbf{X}$ *is* g_{cl}*-symmetric (i.e.,* $\int F_1\, p_t F_2 \,\mathrm{d}\mathrm{g}_{\mathrm{cl}} = \int F_2\, p_t F_1 \,\mathrm{d}\mathrm{g}_{\mathrm{cl}}$ *for all measurable functions* $F_1, F_2 : \ddot{\Gamma}_X \to \mathbb{R}_+$*) and* g_{cl} *is its invariant measure.*

4.2. Irreducibility of the Dirichlet form

Similarly to (4.1), let $\mathcal{E}_{\hat{\mathrm{g}}}$ be the pre-Dirichlet form associated with the Gibbs measure $\hat{g}$, defined on functions $F_1, F_2 \in \mathcal{FC}(\Gamma_{\mathcal{Z}}) \subset L^2(\Gamma_{\mathcal{Z}}, \hat{\mathrm{g}})$ by

$$\mathcal{E}_{\hat{\mathrm{g}}}(F_1, F_2) := \int_{\Gamma_{\mathcal{Z}}} \langle \nabla^{\Gamma}_{\mathcal{Z}} F_1(\hat{\gamma}), \nabla^{\Gamma}_{\mathcal{Z}} F_2(\hat{\gamma})\rangle_{\hat{\gamma}}\, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}). \tag{4.4}$$

The integral on the right-hand side of (4.4) is well defined because $\hat{g} \in \mathcal{M}^2 \subset \mathcal{M}^1$. The latter fact also implies that the gradient operator $\nabla^{\Gamma}_{\mathcal{Z}}$ can be considered as an (unbounded) operator $L^2(\Gamma_{\mathcal{Z}}, \hat{\mathrm{g}}) \to L^2V(\Gamma_{\mathcal{Z}}, \hat{\mathrm{g}})$ with domain $\mathcal{FC}(\Gamma_{\mathcal{Z}})$, where $L^2V(\Gamma_{\mathcal{Z}}, \hat{\mathrm{g}})$ is the space of square-integrable vector fields on $\Gamma_{\mathcal{Z}}$. Since the form $\mathcal{E}_{\hat{g}}$ belongs to the class $\mathcal{M}^2$, it is closable [3] (we keep the same notation for the closure and denote by $\mathcal{D}(\mathcal{E}_{\hat{\mathrm{g}}})$ its domain).

Our aim is to study a relationship between the forms $\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}$ and $\mathcal{E}_{\hat{\mathrm{g}}}$ and to characterize in this way the kernel of $\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}$. We need some preparations. Let us recall that the projection map $\mathfrak{q} : \mathcal{Z} \to \Gamma^{\sharp}_X$ was defined in (2.14) as $\mathfrak{q} := \mathfrak{p} \circ s$, where

$$s : \mathcal{Z} \ni (x, \bar{y}) \mapsto \bar{y} + x \in \mathfrak{X}.$$

As usual, we preserve the same notations for the induced maps of the corresponding configuration spaces. It follows directly from the definition (2.3) of the map $\mathfrak{p}$ that

$$(\nabla^{\Gamma}_X F) \circ \mathfrak{p} = \nabla^{\Gamma}_{\mathfrak{X}} (F \circ \mathfrak{p}), \qquad F \in \mathcal{FC}(\Gamma_X), \tag{4.5}$$

where we use the identification of the tangent spaces

$$T_{\bar{\gamma}}\Gamma_{\mathfrak{X}} = \bigoplus_{\bar{y}\in\bar{\gamma}} T_{\bar{y}}\mathfrak{X} = \bigoplus_{\bar{y}\in\bar{\gamma}} \bigoplus_{y_i\in\bar{y}} T_{y_i}X = \bigoplus_{y_i\in\mathfrak{p}(\bar{\gamma})} T_{y_i}X = T_{\mathfrak{p}(\bar{\gamma})}X. \tag{4.6}$$

Theorem 4.3. *For the Dirichlet forms* $\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}$ *and* $\mathcal{E}_{\hat{\mathrm{g}}}$ *defined in* (4.3) *and* (4.4)*, respectively, their domains satisfy the relation* $\mathcal{I}_{\mathfrak{q}}(\mathcal{D}(\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}})) \subset \mathcal{D}(\mathcal{E}_{\hat{\mathrm{g}}})$. *Furthermore,* $F \in \operatorname{Ker} \mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}$ *if and only if* $\mathcal{I}_{\mathfrak{q}} F \in \operatorname{Ker} \mathcal{E}_{\hat{\mathrm{g}}}$.

Proof. Let us introduce a map $\mathrm{d}s^* : \mathfrak{X} \to \mathcal{Z}$ by the formula

$$\mathrm{d}s^*(\bar{y}) := \Big(\textstyle\sum_{y_i\in\bar{y}} y_i,\ \bar{y}\Big), \qquad \bar{y} \in \mathfrak{X}.$$

As suggested by the notation, this map coincides with the adjoint of the derivative

$$\mathrm{d}s(z)\colon\, T_z\mathcal{Z} \to T_{s(z)}\mathfrak{X}$$

under the identification $T_{\bar y}\mathfrak{X} = \mathfrak{X}$ and $T_z\mathcal{Z} = \mathcal{Z}$. A direct calculation shows that for any differentiable function f on $\mathfrak{X}$ the following commutation relation holds:

$$(\mathrm{d}s^*\nabla f)\circ s = \nabla(f\circ s). \tag{4.7}$$

Here the symbol ∇ denotes the gradient on the corresponding space (i.e., $\mathfrak{X}$ on the left and $\mathcal{Z}$ on the right).

Let

$$\mathrm{d}s^*(\hat\gamma)\,:\, T_{s(\hat\gamma)}\Gamma_{\mathfrak{X}} = \bigoplus_{\bar y\in s(\hat\gamma)} T_{\bar y}\mathfrak{X} \to \bigoplus_{z\in\hat\gamma} T_z\mathcal{Z} = T_{\hat\gamma}\Gamma_{\mathcal{Z}}$$

be the natural lifting of the operator $\mathrm{d}s^*$. Further, using (4.6), it can be interpreted as the operator

$$\mathrm{d}s^*(\hat\gamma)\,:\, T_{\mathfrak{q}(\hat\gamma)}\Gamma_X \to T_{\hat\gamma}\Gamma_{\mathcal{Z}},$$

which induces the (bounded) operator

$$\mathcal{I}_{\mathfrak{q}}\,\mathrm{d}s^*\,:\, L^2V(\Gamma_X, g_{\mathrm{cl}}) \to L^2V(\Gamma_{\mathcal{Z}}, \hat g) \tag{4.8}$$

acting according to the formula

$$(\mathcal{I}_{\mathfrak{q}}\,\mathrm{d}s^*V)(\hat\gamma) = \mathrm{d}s^*(\hat\gamma)V(\mathfrak{q}(\hat\gamma)), \qquad V\in L^2V(\Gamma_X, g_{\mathrm{cl}}).$$

Formula (4.7) together with (4.5) implies that

$$\big(\mathrm{d}s^*\nabla_X^\Gamma F\big)\circ\mathfrak{q} = \nabla_{\mathcal{Z}}^\Gamma(F\circ\mathfrak{q}), \qquad F\in\mathcal{FC}(\Gamma_X),$$

or, in terms of operators acting in the corresponding L^2-spaces,

$$\mathcal{I}_{\mathfrak{q}}\,\mathrm{d}s^*\nabla_X^\Gamma F = \nabla_{\mathcal{Z}}^\Gamma\mathcal{I}_{\mathfrak{q}}F, \qquad F\in\mathcal{FC}(\Gamma_X).$$

Therefore, for any $F\in\mathcal{FC}(\Gamma_X)$

$$\begin{aligned}
\mathcal{E}_{\hat g}(\mathcal{I}_{\mathfrak{q}}F,\mathcal{I}_{\mathfrak{q}}F) &= \int_{\Gamma_{\mathcal{Z}}} |(\mathcal{I}_{\mathfrak{q}}\,\mathrm{d}s^*\nabla_X^\Gamma F)(\hat\gamma)|_{\hat\gamma}^2\,\hat g(\mathrm{d}\hat\gamma) && (4.9)\\
&= \int_{\Gamma_{\mathcal{Z}}} |\mathrm{d}s^*\nabla_X^\Gamma F(\mathfrak{q}(\hat\gamma))|_{\hat\gamma}^2\,\hat g(\mathrm{d}\hat\gamma)\\
&= \int_{\Gamma_X} |\mathrm{d}s^*\nabla_X^\Gamma F(\gamma)|_{\gamma}^2\,g_{\mathrm{cl}}(\mathrm{d}\gamma)\\
&\ge \int_{\Gamma_X} |\nabla_X^\Gamma F(\gamma)|_{\gamma}^2\,g_{\mathrm{cl}}(\mathrm{d}\gamma) = \mathcal{E}_{g_{\mathrm{cl}}}(F,F), && (4.10)
\end{aligned}$$

where in (4.10) we used the obvious inequality $|\mathrm{d}s^*(\bar y)| \ge |\bar y|$ $(\bar y \in \mathfrak{X})$. Hence,

$$\begin{aligned}\|F\|^2_{\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}} &:= \mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}(F,F) + \int_{\Gamma_X} F^2 \,\mathrm{d}\mathbf{g}_{\mathrm{cl}}\\ &\le \mathcal{E}_{\hat{\mathrm{g}}}(\mathcal{I}_{\mathfrak{q}}F, \mathcal{I}_{\mathfrak{q}}F) + \int_{\Gamma_{\mathfrak{Z}}} (\mathcal{I}_{\mathfrak{q}}F)^2 \,\mathrm{d}\hat{\mathbf{g}} = \|\mathcal{I}_{\mathfrak{q}}F\|^2_{\mathcal{E}_{\hat{\mathrm{g}}}},\end{aligned}$$

which implies that $\mathcal{I}_{\mathfrak{q}}(\mathcal{D}(\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}})) \subset \mathcal{D}(\mathcal{E}_{\hat{\mathrm{g}}})$, thus proving the first part of the theorem.

Further, using approximation arguments and continuity of the operator (4.8), one can show that the equality (4.9) extends to the domain $\mathcal{D}(\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}})$,

$$\mathcal{E}_{\hat{\mathrm{g}}}(\mathcal{I}_{\mathfrak{q}}F, \mathcal{I}_{\mathfrak{q}}F) = \int_{\Gamma_{\mathfrak{Z}}} |\mathcal{I}_{\mathfrak{q}}\,\mathrm{d}s^*\nabla^{\Gamma}_X F|^2_{\hat\gamma}\,\mathrm{d}\hat{\mathbf{g}}, \qquad F \in \mathcal{D}(\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}). \tag{4.11}$$

Since $\mathrm{Ker}(\mathcal{I}_{\mathfrak{q}}\,\mathrm{d}s^*) = \{0\}$, formula (4.11) readily implies that $\mathcal{I}_{\mathfrak{q}}F \in \mathrm{Ker}\,\mathcal{E}_{\hat{\mathrm{g}}}$ if and only if $\nabla^{\Gamma}_X F = 0$. In turn, due to equality (4.10), the latter is equivalent to $F \in \mathrm{Ker}\,\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}$. □

Let us recall that a Dirichlet form $\mathcal{E}$ is called *irreducible* if the condition $\mathcal{E}(F,F) = 0$ implies that $F = \mathrm{const}$.

Corollary 4.4. *The Dirichlet form $\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}$ is irreducible if $\mathcal{E}_{\hat{\mathrm{g}}}$ is so.*

Proof. Follows immediately from Theorem 4.3 and the obvious fact that if $\mathcal{I}_{\mathfrak{q}}F = \mathrm{const}$ ($\hat{g}$-a.s.) then $F = \mathrm{const}$ (g_{cl}-a.s.). □

Remark 4.1. It follows from the general theory of Gibbs measures (see, e.g., [3]) that the form $\mathcal{E}_{\hat{\mathrm{g}}}$ is irreducible if and only if $\hat{g} \in \mathrm{ext}\,\mathscr{G}(\sigma, \hat{\Phi})$, which is in turn equivalent to $g \in \mathrm{ext}\,\mathscr{G}(\theta, \Phi)$ (provided that $g \in \mathscr{G}_{\mathrm{R}}(\theta,\Phi)$, see Corollary 2.8).

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Appendix A. Gibbs measures on configuration spaces

Let us briefly recall the definition and some properties of (grand canonical) Gibbs measures on the configuration space Γ_X. For a more systematic exposition and further details, see the classical books [10, 25, 27]; more recent useful references include [3, 16, 19].

Denote by $\Gamma_X^0 := \{\gamma \in \Gamma_X : \gamma(X) < \infty\}$ the subspace of finite configurations in X. Let $\Phi : \Gamma_X^0 \to \mathbb{R} \cup \{+\infty\}$ be a measurable function (called the *interaction potential*) such that $\Phi(\emptyset) = 0$. A simple, most common example is that of the *pair interaction potential*, i.e., such that $\Phi(\gamma) = 0$ unless configuration γ consists of exactly two points.

Definition A.1. The *energy* $E : \Gamma_X^0 \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$E(\xi) := \sum_{\zeta \subset \xi} \Phi(\zeta) \qquad (\xi \in \Gamma_X^0), \qquad E(\emptyset) := 0. \tag{A.1}$$

The *interaction energy* between configurations $\xi \in \Gamma_X^0$ and $\gamma \in \Gamma_X$ is given by

$$E(\xi, \gamma) := \begin{cases} \displaystyle\sum_{\gamma \supset \gamma' \in \Gamma_X^0} \Phi(\xi \cup \gamma') & \text{if } \displaystyle\sum_{\gamma \supset \gamma' \in \Gamma_X^0} |\Phi(\xi \cup \gamma')| < \infty, \\ +\infty & \text{otherwise}. \end{cases} \tag{A.2}$$

Definition A.2. Let $\mathscr{G}(\theta, \Phi)$ denote the class of all *grand canonical Gibbs measures* corresponding to the reference measure θ and the interaction potential Φ, that is, the probability measures on Γ_X that satisfy the *Dobrushin–Lanford–Ruelle* (*DLR*) *equation* (see, e.g., [3, Eq. (2.17), p. 251]).

In the present paper, we use an equivalent characterization of Gibbs measures based on the following theorem, first proved by Nguyen and Zessin [23, Theorem 2].

Theorem A.1. *A measure* g *on the configuration space* Γ_X *belongs to the Gibbs class* $\mathscr{G}(\theta, \Phi)$ *if and only if either of the following conditions holds*:

(i) (Nguyen–Zessin's equation) *For any function* $H \in \mathrm{M}_+(X \times \Gamma_X)$,

$$\int_{\Gamma_X} \sum_{x_i \in \gamma} H(x_i, \gamma)\, \mathsf{g}(\mathrm{d}\gamma) = \int_{\Gamma_X} \left(\int_X H(x, \gamma \cup \{x\})\, \mathrm{e}^{-E(\{x\}, \gamma)}\, \theta(\mathrm{d}x) \right) \mathsf{g}(\mathrm{d}\gamma). \tag{A.3}$$

(ii) (Ruelle's equation) *For any bounded function* $F \in \mathrm{M}_+(\Gamma_X)$ *and any compact set* $\Lambda \subset X$,

$$\int_{\Gamma_X} F(\gamma)\, \mathsf{g}(\mathrm{d}\gamma) = \int_{\Gamma_\Lambda} \mathrm{e}^{-E(\xi)} \left(\int_{\Gamma_{X \setminus \Lambda}} F(\xi \cup \gamma')\, \mathrm{e}^{-E(\xi, \gamma')}\, \mathsf{g}(\mathrm{d}\gamma') \right) \lambda_\theta(\mathrm{d}\xi), \tag{A.4}$$

where λ_θ *is the* Lebesgue–Poisson measure *on* Γ_X^0 *defined by the formula*

$$\lambda_\theta(\mathrm{d}\xi) = \sum_{n=0}^{\infty} \mathbf{1}\{\xi(\Lambda) = n\}\, \frac{1}{n!} \bigotimes_{x_i \in \xi} \theta(\mathrm{d}x_i), \qquad \xi \in \Gamma_\Lambda^0. \tag{A.5}$$

Remark A.1. Using a standard argument based on the decomposition $H = H^+ - H^-$, $|H| = H^+ + H^-$ with $H^+ := \max\{H, 0\}$, $H^- := \max\{-H, 0\}$, one can see that equation (A.3) is also valid for an arbitrary measurable function $H : X \times \Gamma_X \to \mathbb{R}$ provided that

$$\int_{\Gamma_X} \sum_{x_i \in \gamma} |H(x_i, \gamma)|\, g(\mathrm{d}\gamma) < \infty. \tag{A.6}$$

Remark A.2. In the original paper [23], the authors proved the result of Theorem A.1 under additional assumptions of *stability* of the interaction potential Φ and *temperedness* of the measure g. In subsequent work by Kuna [16, Theorems 2.2.4, A.1.1], these assumptions have been removed.

Remark A.3. Inspection of [23, Theorem 2] or [16, Theorem A.1.1] reveals that the proof of the implication (A.3) $\Rightarrow$ (A.4) is valid for *any* set $\Lambda \in \mathcal{B}(X)$ satisfying a priori conditions $\theta(\Lambda) < \infty$ and $\gamma(\Lambda) < \infty$ (g-a.s.). Hence, Ruelle's equation (A.4) is valid for such sets as well.

In the "interaction-free" case where $\Phi \equiv 0$, the unique grand canonical Gibbs measure coincides with the Poisson measure π_θ (with intensity measure θ). In the general situation, there are various types of conditions to ensure that the class $\mathscr{G}(\theta, \Phi)$ is non-empty (see [10, 25, 27] and also [16, 17, 19]).

Example A.1. The following are four classical examples of translation-invariant pair interaction potentials (i.e., such that $\Phi(\{x, y\}) = \phi_0(x - y) \equiv \phi_0(y - x)$), for which $\mathscr{G}(\theta, \Phi) \neq \emptyset$.

(1) (*Hard core potential*) $\phi_0(x) = +\infty$ for $|x| \leq r_0$, otherwise $\phi_0(x) = 0$ ($r_0 > 0$).
(2) (*Purely repulsive potential*) $\phi_0 \in C_0^2(\mathbb{R}^d)$, $\phi_0 \geq 0$ on $\mathbb{R}^d$, and $\phi_0(0) > 0$.
(3) (*Lennard–Jones type potential*) $\phi_0 \in C^2(\mathbb{R}^d \setminus \{0\})$, $\phi_0 \geq -a > -\infty$ on $\mathbb{R}^d$, $\phi_0(x) := c|x|^{-\alpha}$ for $|x| \leq r_1$ ($c > 0$, $\alpha > d$), and $\phi_0(x) = 0$ for $|x| > r_2$ ($0 < r_1 < r_2 < \infty$).
(4) (Lennard–Jones "6–12" potential) $d = 3$, $\phi_0(x) = c(|x|^{-12} - |x|^{-6})$ for $x \neq 0$ ($c > 0$) and $\phi_0(0) = +\infty$.

Definition A.3. For a Gibbs measure g on Γ_X, its *correlation function* $\kappa_g^n : X^n \to \mathbb{R}_+$ *of the* n*-th order* ($n \in \mathbb{N}$) is defined by the following property: for any function

$\phi \in C_0(X^n)$, symmetric with respect to permutations of its arguments, it holds

$$\int_{\Gamma_X} \sum_{\{x_1,\dots,x_n\}\subset\gamma} \phi(x_1,\dots,x_n)\, \boldsymbol{g}(\mathrm{d}\gamma) \\ = \frac{1}{n!}\int_{X^n} \phi(x_1,\dots,x_n)\, \kappa_{\boldsymbol{g}}^n(x_1,\dots,x_n)\, \theta(\mathrm{d}x_1)\cdots\theta(\mathrm{d}x_n). \quad \text{(A.7)}$$

By a standard approximation argument, equation (A.7) can be extended to any (symmetric) bounded measurable functions $\phi : X^n \to \mathbb{R}$ with support of finite $\theta^{\otimes n}$-measure. For $n = 1$ and $\phi(x) = \mathbf{1}_B(x)$, the definition (A.7) specializes to

$$\int_{\Gamma_X} \gamma(B)\, \boldsymbol{g}(\mathrm{d}\gamma) = \int_B \kappa_{\boldsymbol{g}}^1(x)\, \theta(\mathrm{d}x). \quad \text{(A.8)}$$

More generally, choosing $\phi(x_1,\dots,x_n) = \prod_{i=1}^n \mathbf{1}_{B_i}(x_i)$ with arbitrary test sets $B_i \in \mathcal{B}(X)$, it is easy to see that the definition (A.7) is equivalent to the following more explicit description (cf. [2, p. 266]),

$$\kappa_{\boldsymbol{g}}^n(x_1,\dots,x_n)\, \theta(\mathrm{d}x_1)\cdots\theta(\mathrm{d}x_n) = n! \cdot \boldsymbol{g}\{\gamma \in \Gamma_X : \gamma(\mathrm{d}x_i) \geq 1,\ i = 1,\dots,n\},$$

also showing that indeed $\kappa_{\boldsymbol{g}}^n \geq 0$.

Example A.2. In the Poisson case (i.e., $\Phi \equiv 0$), we have $\kappa_{\pi_\theta}^n(x) \equiv n!$ $(n \in \mathbb{N})$.

Remark A.4. Using Nguyen–Zessin's equation (A.3) with $H(x,\gamma) = \phi(x)$, from the definition (A.8) it follows that

$$\kappa_{\boldsymbol{g}}^1(x) = \int_{\Gamma_X} \mathrm{e}^{-E(\{x\},\gamma)}\, \boldsymbol{g}(\mathrm{d}\gamma), \qquad x \in X. \quad \text{(A.9)}$$

In particular, the representation (A.9) implies that if $\Phi \geq 0$ (non-attractive interaction potential) then $\kappa_{\boldsymbol{g}}^1(x) \leq 1$ for al $x \in X$, so that $\kappa_{\boldsymbol{g}}^1$ is bounded.

Remark A.5. If the first-order correlation function $\kappa_{\boldsymbol{g}}^1(x)$ is integrable on any set $B \in \mathcal{B}(X)$ of finite θ-measure (for instance, if $\kappa_{\boldsymbol{g}}^1$ is bounded on X, cf. Remark A.4), then, according to (A.8), the mean number of points in $\gamma \cap B$ is finite, also implying that $\gamma(B) < \infty$ for $\boldsymbol{g}$-a.a. configurations $\gamma \in \Gamma_X$ (cf. Remark A.3). Conversely, if $\kappa_{\boldsymbol{g}}^1$ is bounded below (i.e., $\kappa_{\boldsymbol{g}}^1(x) \geq c > 0$ for all $x \in X$) and the mean number of points in $\gamma\cap B$ is finite, then it follows from (A.8) that $\theta(B) < \infty$.

Definition A.4. For a probability measure μ on Γ_X, the notation $\mu \in \mathcal{M}^n(\Gamma_X)$ signifies that

$$\int_{\Gamma_X} |\langle\phi,\gamma\rangle|^n\, \mu(\mathrm{d}\gamma) < \infty, \qquad \phi \in C_0(X). \quad \text{(A.10)}$$

Definition A.5. We denote by $\mathscr{G}_{\mathrm{R}}(\theta,\Phi)$ the set of all Gibbs measures $g \in \mathscr{G}(\theta,\Phi)$ such that all its correlation functions κ_g^n are well defined and satisfy the *Ruelle bound*, that is, for some constant $R \in \mathbb{R}_+$ and all $n \in \mathbb{N}$,

$$|\kappa_g^n(x_1,\dots,x_n)| \le R^n, \qquad (x_1,\dots,x_n) \in X^n. \tag{A.11}$$

Proposition A.2. *Let* $g_1, g_2 \in \mathscr{G}_{\mathrm{R}}(\theta,\Phi)$ *and* $\kappa_{g_1}^n = \kappa_{g_2}^n$ *for all* $n \in \mathbb{N}$. *Then* $g_1 = g_2$.

Proof. For any measure $g \in \mathscr{G}_{\mathrm{R}}(\theta,\Phi)$, its Laplace transform $L_g(f)$ on functions $f \in C_0(X)$ may be represented in the form

$$\begin{aligned} L_g(f) &= \int_{\Gamma_X} \prod_{x_i\in\gamma} \bigl(1 + (\mathrm{e}^{-f(x_i)} - 1)\bigr)\, g(\mathrm{d}\gamma) \\ &= 1 + \int_{\Gamma_X} \sum_{n=1}^{\infty} \sum_{\{x_1,\dots,x_n\}\subset\gamma} \prod_{i=1}^{n} \bigl(\mathrm{e}^{-f(x_i)} - 1\bigr)\, g(\mathrm{d}\gamma) && \text{(A.12)} \\ &= 1 + \sum_{n=1}^{\infty} \int_{\Gamma_X} \sum_{\{x_1,\dots,x_n\}\subset\gamma} \prod_{i=1}^{n} \bigl(\mathrm{e}^{-f(x_i)} - 1\bigr)\, g(\mathrm{d}\gamma) && \text{(A.13)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{X^n} \prod_{i=1}^{n} \bigl(\mathrm{e}^{-f(x_i)} - 1\bigr)\, \kappa_g^n(x_1,\dots,x_n)\, \theta(\mathrm{d}x_1)\cdots\theta(\mathrm{d}x_n), && \text{(A.14)} \end{aligned}$$

where (A.14) is obtained from (A.13) using formula (A.7). Interchanging the order of integration and summation in (A.12) is justified by the dominated convergence theorem; indeed, using that $|f(x)| \le C_f$ on $K_f := \operatorname{supp} f$ with some $C_f > 0$ and recalling that the correlation functions κ_g^n satisfy the Ruelle bound (A.11), we see that the right-hand side of (A.14) is dominated by

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!}\, (\mathrm{e}^{C_f} + 1)^n\, R^n\, \theta(K_f)^n = \exp\bigl\{R\,(\mathrm{e}^{C_f} + 1)\,\theta(K_f)\bigr\} < \infty.$$

Now, formula (A.14) implies that if measures $g_1, g_2 \in \mathscr{G}_{\mathrm{R}}(\theta,\Phi)$ have the same correlation functions, then their Laplace transforms coincide with each other, $L_{g_1}(f) = L_{g_2}(f)$ for any $f \in \mathrm{M}_+(X)$, hence $g_1 = g_2$. □

Definition A.6. It is well known that $\mathscr{G}(\theta,\Phi)$ is a convex set [25]. We denote by $\operatorname{ext}\mathscr{G}(\theta,\Phi)$ the set of its extreme elements, that is, those measures $g \in \mathscr{G}(\theta,\Phi)$ that cannot be written as $g = \frac{1}{2}(g_1 + g_2)$ with $g_1, g_2 \in \mathscr{G}(\theta,\Phi)$ and $g_1 \ne g_2$.

Using Ruelle's equation (A.4), it is easy to obtain the following result (cf. [16, Corollary 2.2.6]).

Proposition A.3. *Let $g \in \mathscr{G}(\theta, \Phi)$, and let $\Lambda \in \mathcal{B}(X)$ be a compact set. Then the restriction of the Gibbs measure $g \in \mathscr{G}(\theta, \Phi)$ onto the space Γ_Λ, defined by*

$$g_\Lambda(A) := g(A \cap \Gamma_\Lambda), \qquad A \in \mathcal{B}(\Gamma_X),$$

is absolutely continuous with respect to the Lebesgue–Poisson measure λ_θ, with the Radon–Nikodym density $S_\Lambda := \mathrm{d}g_\Lambda / \mathrm{d}\lambda_\theta \in L^1(\Gamma_\Lambda, \lambda_\theta)$ given by

$$S_\Lambda(\gamma) = \mathrm{e}^{-E(\gamma)} \int_{\Gamma_{X \setminus \Lambda}} \mathrm{e}^{-E(\gamma, \gamma')}\, g(\mathrm{d}\gamma'), \qquad \gamma \in \Gamma_\Lambda. \tag{A.15}$$

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