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Relations between adjacency trees

John Stell and Michael Worboys

Abstract

Adjacency trees can model the nesting structure of spatial regions. In many applications it is necessary to model foreground and background regions which exhibit changes over time such as splitting, where one region divides into two. For example, the qualitative description of the development of wildfires would use the foreground for areas on fire and the background for areas not on fire. Such dynamic behaviour can be modelled by a particular kind of relation between the nodes of two adjacency trees representing the initial and final configurations of the regions at two times. These relations, which we call bipartite, correspond to having an arbitrary relation between the foreground regions at the two times and an arbitrary relation between the background regions at the two times. We show that all bipartite relations between trees arise from sequences of atomic relations between trees. There are just four types of these atomic relations (in addition to one representing no change): inserts, splits, merges and deletes.

Keywords: adjacency trees; relations; spatial reasoning; qualitative spatial change

1 Introduction

1.1 Qualitative spatio-temporal representation

Qualitative accounts of space [CR08] provide ways of representing and reasoning about spatial phenomena by abstracting away from details, such as metric data, which are unnecessary for some applications. This abstraction enables a focus on essential features of tasks in application domains such as geographical information [Fra92, MB07], commonsense reasoning [Dav08], and robotics [ET98, SJKB10]. The combination of both spatial and temporal features remains a challenge for qualitative reasoning, but one well-motivated by practical tasks which may not currently use qualitative techniques. Such tasks include: the analysis of changing land-use [WD02], the monitoring and management of coastal areas [vVSJ05], the identification of events from traffic monitoring [KMIS99, SK08], the observation of crowd movements [DYV95, MOS09], and modelling the propagation of wildfires [CBR94, $C^{+}11$].

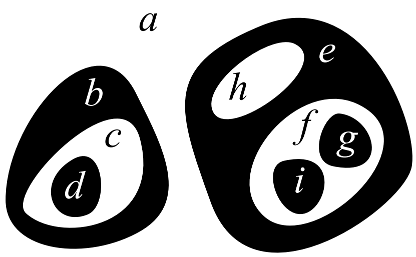
In tracking the propagation of a wildfire, we can ask which fire-regions at some earlier stage contributed to a particular fire-region at a later stage. In the context of a changing pattern of lakes or ponds, we can ask whether a particular body of water evolved into others without ever merging with any regions separate from the original one. While monitoring crowd movement, we can ask which groups contain individuals from two specific earlier groupings. In more detail, suppose we pick crowds a and b, with a observed at one time and b at a later time. We may find that all of a is present in b, but that b contains people from the earlier time not present in a. Alternatively, we may find that b only contains people from a, but that some people in a may have left the scene and are not present in any crowd observed at the same time as b.

In dealing with these kinds of relationships between entities at different times the concern here is not with changes to shape or location but with how the entities at the later stage are formed from the earlier ones. At the most abstract level a formal model would have two sets of entities and a relation between them. One entity is related to another when the first participates in the formation of the second.

We emphasize that a relationship between two entities at distinct times does not have to be a spatial relationship. In the crowd example, whether there are people common to the earlier crowd, a, and the later crowd, b, is independent of the spatial locations occupied by the crowds at the two times. The relationships we consider between entities at the same time are, however, spatial. For example, whether a is encircled by another crowd, c, does concern the spatial relationship between two entities at the same time. Our formal model thus has two aspects. A static one, where at a given time we have a system of entities with some spatial structure, and a dynamic one in which two such systems are related. The next two subsections give the background to these two aspects.

1.2 Static aspect

We will assume that at any given time the space under consideration is partitioned into foreground and background parts. These might, for example, be places that are on fire and not on fire, or they might be land and water. We also assume that any two foreground or any two background regions are disjoint. In the plane this leads to a picture such as that on the left hand side of Figure 1, where the foreground regions are shown black and the background ones white.



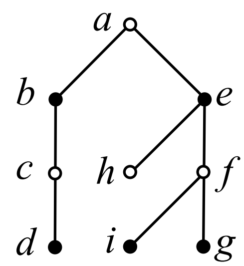


Figure 1: A system of nested regions in the plane and its adjacency tree

We do not require that regions are parts of a two-dimensional space, but we do assume that the only spatial relationship between regions is nesting. This may seem restrictive, and regions which overlap or have more elaborate relationships are certainly needed in some applications. However as the initial step for the development of the theory our choice is a natural one which is supported by its use by other authors. For example, in Milner's theory of bigraphs [Mil08] the spatial aspect is restricted to nested entities.

The nested structure leads to an associated tree, which is most easily explained in the planar case, and which is illustrated in Figure 1. In this case we can take the foreground regions to be a compact set K in the plane $\mathbb{R}^2$. If we use K' to denote the closure of the complement of K, then the nodes of the associated tree are the connected components of K together with the connected components of K'. The root of the tree is the single unbounded component of K', and two nodes are adjacent in the tree if they are distinct and have intersecting boundaries. This tree, or its analogue for the digital plane $\mathbb{Z}^2$, has been called the adjacency tree [Bun69, Ros74]. In Figure 1 the tree is shown with nodes partitioned into two classes coloured black and white corresponding to the foreground and background regions respectively. This colouring is shown only in order to make the association with the regions in the figure clearer; the trees used in the formal analysis do not come equipped with a particular choice of black or white for each node.

With a system of regions in the plane the one unbounded region naturally forms a root of the tree. If however the regions exist on a sphere, there is no intrinsic reason to prefer one node of the tree above any other. The work by Jiang and Worboys [JW09] used rooted trees, and concentrated on planar regions. Apart from Egenhofer's spherical topological relations [Ege05], there is relatively little work on qualitative distinctions which can be made on the sphere but not in the plane. Our main result, Theorem 13, is stated in terms of trees which do not have a specified root and thus applies directly to dynamic configurations of regions on the surface of the sphere. However, in deriving this result we sometimes need to single out a node for special treatment, as in the rooted-sums in Section 3.2. When we do need to consider such structures it should be noted that a rooted tree is formally just a pair consisting of a tree together with any one of its nodes.

In mathematical morphology Serra [Ser82, p89] used the term 'homotopy tree' instead of adjacency tree, although the same term is used elsewhere for a quite distinct concept [Dye79, p378]. The morphological applications include algorithms for noise removal [Kes07], and for skeletonization [RS02]. In the case of skeletonization, and several other uses of adjacency trees, the emphasis is on transformations of the image that leave the tree unchanged. That the tree remains fixed is particularly important in applications to visual markers [CR03], where the tree is used as the means of identifying a particular marker in a scene. For our purposes, however, the fact that the tree changes is essential and the ways this may happen are discussed in the next subsection.

1.3 Dynamic aspect

Particularly simple changes are those where the number of regions increases or decreases by only one. A change is detected by a difference between configurations at two times and our model does not record the process by which the change was brought about. Four kinds of primitive change are immediately evident: insertion (a new region has been created); deletion (an existing region has disappeared); merge (two regions have joined to become one); split (one region has divided into two). The simplest kind of merge in the two-dimensional setting is when two regions which are topologically discs unite to become a single region which is again topologically a disc. There are more complex kinds of merging, such as that illustrated in Figure 2 where two regions have encircled a third. In this case the region that results from the merge is not topologically a disc but has a hole containing another foreground region.

Various authors have proposed models with types of change related to the ones we examine. Spéry et al. [SCL99, p469] work with five elementary changes in a cadastral application: division, merge, extraction, passage (a kind of secession),

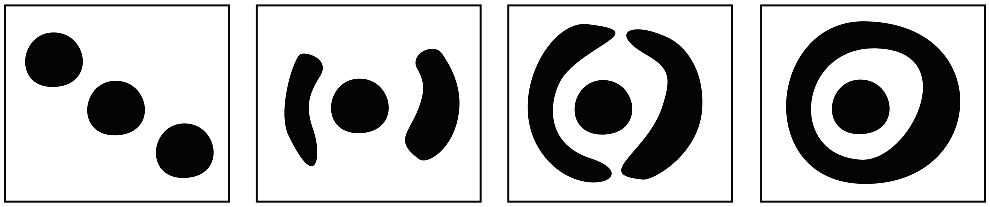


Figure 2: Sequence of changes in which two regions encircle a third

and rectification, which includes the redrawing of a shared boundary between two parcels without affecting their overall extent. There are also two complex changes ('re-allocation' and 'expropriation'). In [JYG03, p931], Jin et al. argue for the explicit modelling of 'identity changes' between objects (such as one splitting into two) in order to deal with queries such as 'Was the object O merged into another object in a given time interval?'. Medak [Med99] is also concerned with tracking identity, and works in a framework where identities may not only be created and destroyed but also suspended and resumed. A distinction is made between fusions, which are irreversible operations like forming one container of liquid out of two, and aggregations, in which constituent parts can still be identified. Robertson et al. [RNBW07] examine spatial processes, identifying five types of events: displacement, convergence, divergence, fragmentation and concentration. They provide a case study of a wildfire, and the relationships shown diagrammatically in their fig 8 [RNBW07, p223] provides a practical example to which our theoretical analysis in this paper is immediately applicable.

Our treatment is based on atomic primitives: insertion, deletion, merge and split where only one region is inserted, is deleted, splits or where just two regions merge. In addition to these four, we also need a fifth primitive for the case of no change being observed. Each of these five atomic changes gives rise to a relation between the nodes of the trees for the initial and final states. By composing sequences of such changes we obtain more general relations between the adjacency trees. The term 'relation' here means nothing more than a set of ordered pairs in the usual mathematical sense. So a relation pairs certain nodes between two trees. The relations that arise by composing the atomic changes are not arbitrary relations on the nodes, but will evidently preserve the partitioning of the nodes in a tree into foreground and background nodes.

It should be emphasized that the relation between states in a dynamic configuration is something that exists in addition to the states themselves and is not (except in some very special cases) something that can be deduced unambiguously from the states themselves. To take a very simple example, consider the initial state of a single background region and a single foreground region. Suppose the final state consists of a single background region and two foreground regions. Between these two states are various possibilities including that the initial foreground region has split into two and that a new foreground region has been inserted. Our model assumes that the relation between these states will be given explicitly as part of the information that we work with.

Suppose now that we take two adjacency trees each with a partition of the nodes into those representing foreground regions and those representing background regions. In such a partition any two nodes of the same type must be an even number of edges apart. If we specify an arbitrary relation between the foreground nodes and another arbitrary relation between the background nodes, these two relations taken together give a single relation between the nodes of the two trees. In the formal analysis below we call relations between trees that have this form *bipartite*. The main technical result in the paper, Theorem 13, shows that all bipartite relations between trees arise from the composition of atomic primitive relations of the five types. This theorem answers a natural question from a theoretical perspective, but the question is also of practical importance. Suppose we have two systems of foreground and background regions, which might for example represent regions occupied by crowds at two times. A bipartite relation between the adjacency trees models how the crowds at the first stage participate in the formation of the crowds at the latter stage. Theorem 13 tells us that every possible relation can be explained in some way as a sequence of primitive changes, or alternatively that the primitive changes are sufficient to generate all possible relations.

1.4 Qualitative change and conceptual neighbourhoods

Qualitative spatial reasoning has produced a large number of calculi which are able to describe certain features of the spatial relationships between regions. One of the most well-known of these calculi is RCC-8 [RCC92] which is based on a primitive notion of connection. The RCC-8 recognizes eight different ways in which two regions can be related. These include being *externally connected* (informally, the regions touch at their boundaries but have no interior parts in common), and *proper overlap* (two regions both have a third region as a proper part). Among the remaining possible relations is one region being a *non-tangential* *proper part* of a second region (no region can be externally connected to the first without also overlapping the second). It should be noted that a region may consist of several disconnected pieces.

The RCC-8 allows a spatial configuration to be described by giving a set of regions and specifying which of the eight relations holds between each pair of regions. This is a similar approach to that in the static aspect of the model in this paper. In our case there are just two possibilities for each pair of regions; either they are adjacent or they are not. If regions are adjacent then the RCC-8 relation between them will be external connection, and if they are not adjacent then in RCC-8 terms they would be disconnected. A general RCC-8 configuration could be represented by a graph having a node for each region and one edge between every pair of nodes with edges labelled by the eight possible relations. This would be more general than the adjacency trees which we use, but it is in the dynamic aspect that the RCC-8 and our model have taken significantly different approaches.

Many qualitative spatial calculi, including RCC-8, allow a treatment of spatial change through the notion of a conceptual neighbourhood [Fre91]. In the RCC-8 case, two relations S_1 and S_2 of the eight are said to be conceptual neighbours if a continuous change to a spatial configuration allows the relation between two regions to change from S_1 to S_2 without passing through any other of the eight relations. The idea of 'continuous change' is often used rather informally in this setting, and Galton has shown, [Gal97], that several different definitions are possible so that an appropriate choice will depend on a particular application domain.

The kinds of change permitted by the conceptual neighbourhood approach are quite different from those discussed in this paper. This is mainly because the effects of splitting, merging, creating and deleting regions lead to changes in spatial relationships which are not conceptual neighbours. This can be seen from an example. Consider the RCC-8 description of the change observed of two regions r and s where r consists of two disconnected pieces r_1 and r_2, where r_1 is a non-tangential proper part of s and r_2 is disconnected from s. The relation of r to s is that they properly overlap. Now if r_2 is deleted the relation between r and s becomes that r is a non-tangential proper part of s. This transition from proper overlap to non-tangential proper part is not one between conceptual neighbours in the RCC-8.

Because calculi such as the RCC-8 do not model creation and deletion of regions, the kinds of change they describe through conceptual neighbourhoods will have the same set of regions in the initial and final states but with changes occurring in the spatial relations between regions in this fixed set. It would be possible to record the sequence of changes that the relation between each pair of regions undergoes. If the initial and final states were each modelled by a graph with nodes corresponding to regions and labelled edges corresponding to spatial relations, then such a sequence of changes would relate edges to edges. This would be quite different from our use of a relation between the sets of nodes.

1.5 Overview

The structure of the paper is as follows. Section 2 defines various types of relations between trees, the most important of which are the bipartite relations. Section 3 sets out constructions we need which enable trees and relations to be built up out of simpler components. The bipartite relations include the atomic relations which model primitive changes and these are defined in Section 4. This section also introduces what we call the *evolutions*, which model changes to adjacency trees that can be accomplished by a sequence of successive atomic changes.

The main technicalities in the paper appear in sections 5 and 6, where we study relations between chains which are very simple trees. We show how the issue of whether all bipartite relations between arbitrary trees are evolutions can be reduced to a question about relations between chains. Section 7 contains the main result, showing that bipartite relations are the same as evolutions. This result, Theorem 13, shows that any bipartite relation can be factorized into atomic relations, and the connections with another factorization result, obtained by Jiang and Worboys [JW09], are explained in Section 8. Finally in Section 9 we present some conclusions and suggestions for further work.

2 Relations between trees

In this section we introduce the basic definitions of trees and of relations between trees. Although the motivation for the theory is the description of qualitative spatio-temporal change, the formal development is entirely in terms of structures on trees. We use some informal examples of evolving spatial regions, but our results only concern such regions to the extent that their properties are modelled by adjacency trees, and their transitions are modelled by relations.

We include definitions of various well-known graph theoretic concepts as terminology in this area is by no means standardized [Har69, p8]. Before this we set out some terminology and notation used about relations.

2.1 Relations, graphs and trees

Given sets X, Y, Z and relations $R_1 : X \to Y$ and $R_2 : Y \to Z$ we adopt the notation $R_1 ; R_2$ for the composition of R_1 and R_2 as used, for example, in [HH02, p3]. The relation $R_1 ; R_2$ is defined by $x\ (R_1 ; R_2)\ z$ iff there is some $y \in Y$ for which $x\ R_1\ y$ and $y\ R_2\ z$.

The converse of a relation R will be denoted R^{-1}. By a subrelation of $R : X \to Y$ we will mean any relation $S : X \to Y$ for which $x\ S\ y$ implies $x\ R\ y$. The notation $S \subseteq R$ will be used in this case. Given a relation $R : X \to Y$ and a subset $A \subseteq X$ we will use $R(A)$ to denote $\{y \in Y \mid \exists a \in A\ (a\ R\ y)\}$. We say that R is functional if $x\ R\ y$ and $x\ R\ z$ imply $y = z$. A functional relation is sometimes called a partial function. The term function, used without qualification, will be assumed to be a total function.

Definition 1. *A* ***graph****, G, is a pair (N, α) where N is a set and α is a symmetric relation on N. The elements of N are called the* ***nodes*** *of G and α is called the* ***adjacency relation*** *of G. An* ***edge*** *of G is a pair of nodes $(m, n) \in N \times N$ such that $m\ \alpha\ n$. The* ***degree*** *of a node is the number of nodes to which it is adjacent.*

Trees, which we define shortly, are graphs of a particular form, but we also need to consider a larger class of graphs which includes the trees.

Definition 2. *A* ***bipartite graph*** *is a graph, $G = (N, \alpha)$, where N can be written as $N = N_1 \cup N_2$ such that $N_1 \cap N_2 = \varnothing$ and $\alpha \subseteq (N_1 \times N_2) \cup (N_2 \times N_1)$.*

In a bipartite graph the nodes can be partitioned into two disjoint sets and every pair of nodes forming an edge will have exactly one element from each of these two sets. The significance of bipartite graphs for our application to spatial configurations is that regions will correspond to nodes in a graph, and the division into foreground and background regions is then modelled by the partition of the nodes into disjoint sets where adjacent nodes must come from different sets in this partition.

Definition 3. *A* ***tree****, T, is a graph (N, α) such that given any nodes $m, n \in N$ there is a unique sequence of nodes $n_0, n_1, \ldots, n_k$ where $m = n_0$, $n = n_k$, and $n_{i-1}\ \alpha\ n_i$ for $i = 1, \ldots, k$. This sequence of nodes is called the* ***path*** *between m and n. A* ***subtree*** *of T is a subset of N which is a tree when the adjacency relation is restricted to this subset.*

We sometimes need to deal with trees where one node is singled out as having special status.

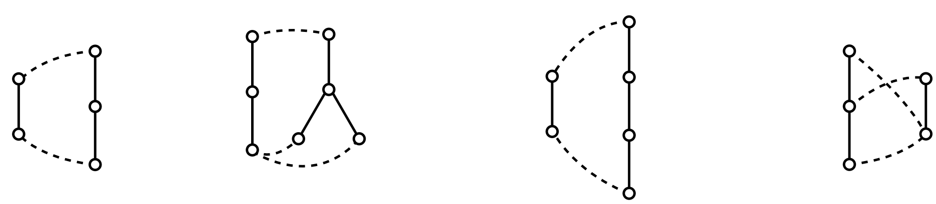
Definition 4. *A **rooted tree** is a pair* (T, r) *where* T *is a tree and* r *is a node of* T.

Since our trees are undirected graphs the root of a rooted tree has no special status with respect to the tree regarded as an abstract structure on its own. That is, we may consider the same tree but with different roots at different stages in a construction. When it is necessary to change the root of a rooted tree, we will say that the structure has been 're-rooted'.

In a tree $T = (N, \alpha)$ we can define a relation $\sim$ on N by $m \sim n$ if the path between m and n contains an odd number of nodes. The relation thus defined can also be described by saying that m and n are related if they are an even number of edges apart. It can be checked that $\sim$ is an equivalence relation and that there are two equivalence classes which partition N into disjoint subsets making T a bipartite graph. We will use the notation $[m]$ when we need to refer to the equivalence class of the node m.

2.2 Homomorphisms and bipartite relations

It will be assumed below that $T_1 = (N_1, \alpha_1)$ and $T_2 = (N_2, \alpha_2)$ are trees. We will speak of a relation, R, between trees whenever we have a relation between the sets of underlying nodes $R : N_1 \to N_2$. We need to consider various kinds of relation, and some representative examples are shown in Figure 3. In this and subsequent figures we indicate the relation by dashed lines and the adjacency in the trees by solid lines.



Not bipartite — Bipartite but not a function — Bipartite and a function, but not a homomorphism — A homomorphism

Figure 3: Examples of types of relations between trees

Definition 5. *A **homomorphism*** $f : T_1 \to T_2$ *is a function from* N_1 *to* N_2 *such that* $m\ \alpha_1\ n$ *implies* $(fm)\ \alpha_2\ (fn)$ *for all* $m, n \in N_1$.

The inverse of a bijective homomorphism between trees will again be a homomorphism, so we can define an **isomorphism** to be a homomorphism of this form. A homomorphism is a structure preserving function, where the preserved structure is the adjacency relation. The natural generalization to structure preserving relations is as follows.

Definition 6. *Let $R : T_1 \to T_2$ be a relation and let $m_1, n_1 \in N_1$ and $m_2, n_2 \in N_2$. We say that R* ***preserves adjacency*** *if*

$$m_1 \; R \; m_2 \textit{ and } n_1 \; R \; n_2 \textit{ and } m_1 \; \alpha_1 \; n_1 \textit{ implies } m_2 \; \alpha_2 \; n_2.$$

It is evident from Figure 4 that we need to deal with relations that are more general than the adjacency preserving ones. For the relation, R, depicted in Figure 4, we have that b and c are adjacent, but $b \; R \; d$ and $c \; R \; c$ where d and c are not adjacent.

To avoid potential confusion, we should point out that although Figure 4 uses the labels a, b, c for nodes in the source tree as well as in the target tree for the relation, this has no particular significance in the formal model. That is, any connection that exists between nodes in the two trees is only modelled by the relation between the trees and not by some of the nodes being equal. In this particular example we could replace either tree by an isomorphic one without affecting the way the relation models three regions where one splits into two.

To introduce the kind of relations we deal with, note that in any tree we have a notion of distance between nodes by counting the number of edges in the unique path joining any two nodes. Given nodes m, n we will denote this distance by $d(m, n)$. If the relation R preserves adjacency, then it is not hard to see that it need not preserve distance in general. However it will preserve distance mod 2, that is whether the distance is odd or even. This can be readily shown by induction on the distance between any two nodes.

The class of relations that preserve distance mod 2 are strictly more general than the adjacency preserving ones. We call these relations *bipartite*, for reasons that we establish after the definition.

Definition 7. *A relation $R : T_1 \to T_2$ is* ***bipartite*** *if for all $m_1, n_1 \in N_1$ and for all $m_2, n_2 \in N_2$*

$$m_1 \; R \; m_2 \textit{ and } n_1 \; R \; n_2 \textit{ implies } d(m_1, n_1) \equiv d(m_2, n_2) \pmod 2.$$

Using the observation that $[m_1] = [n_1]$ iff $d(m_1, n_1) \equiv 0 \pmod 2$ we see that the bipartite relations can be characterized as follows.

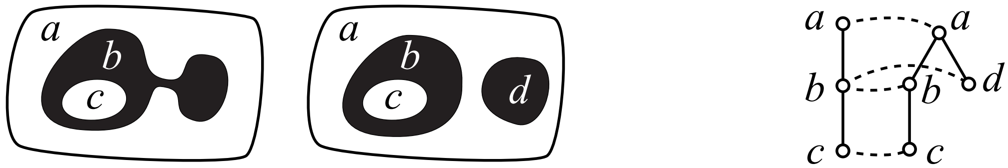


Figure 4: Showing the need for relations which do not preserve adjacency

Lemma 1. *A relation $R : T_1 \to T_2$ is bipartite if and only if for all $m_1, n_1 \in N_1$ and all $m_2, n_2 \in N_2$ where $m_1 \ R \ m_2$ and $n_1 \ R \ n_2$, we have $[m_1] = [n_1]$ iff $[m_2] = [n_2]$.* □

The lemma shows that to specify a bipartite relation it is sufficient to choose one equivalence class in each tree and to give an arbitrary relation between these classes and to give an arbitrary relation between the other two equivalence classes. In terms of our spatial interpretation, this means that if we give an arbitrary relation between the sets of foreground regions at two stages and another arbitrary relation between the sets of background regions at the same two stages then we have a bipartite relation between the adjacency trees. Thus every change in the regions can be modelled by a bipartite relation. The converse issue of whether every bipartite relation can arise from splitting, merging, inserting and deleting of regions is not so easily answered but Theorem 13 shows that this does happen.

As with the adjacency preserving relations, the bipartite relations are closed under composition, and include the identity relations. Unlike the adjacency preserving relations, the bipartite relations are also closed under the formation of converses. We record these facts as a lemma.

Lemma 2. *The identity relation on any tree is bipartite. The composite of bipartite relations is bipartite, and the converse of a bipartite relation is bipartite.* □

When dealing with rooted trees, (T_1, r_1) and (T_2, r_2), we may need to use relations which respect this additional structure as follows.

Definition 8. *A **rooted homomorphism** $f : (T_1, r_1) \to (T_2, r_2)$ is a homomorphism for which $f r_1 = r_2$. A **rooted bipartite relation** $R : (T_1, r_1) \to (T_2, r_2)$ is a bipartite relation where $r_1 \ R \ r_2$.*

In a rooted bipartite relation, the roots are required to be related to each other but note that they may also be related to other nodes as well.

Tree with one node distinguished.

Tree with two nodes and one edge distinguished

Figure 5: Components of structure diagrams

3 Constructions on trees and relations

Relations can be built up by composition, and this corresponds in our application to the succession of changes in time. In this section we introduce further means of constructing more complex trees and relations from simpler ones. The notion of rooted sums can be used to model the combination of changes to entities taking place in separate parts of some larger background.

3.1 Structure diagrams

We first introduce a diagrammatic notation for describing relations between trees. We frequently need to deal with relations $R : T_1 \to T_2$ where for some subtree S_1 of T_1 the relation R restricts to an isomorphism between S_1 and a subtree of T_2. By specifying such isomorphisms for a family of subtrees which together cover all of T_1 we may be able to describe the whole relation. The resulting diagrams will be found practically useful in describing constructions and explaining proofs later in the paper.

Figure 5 shows how these subtrees may be indicated within the structure diagrams that will be used to describe relations. On the left an arbitrary non-empty tree with a distinguished node, and on the right two nodes joined by an edge with the possibility that other edges are connected to the rightmost node. When a bipartite relation restricts to an isomorphism between subtrees this is indicated by a dotted line joining the two subtrees in the diagram. We use dotted lines in these diagrams to distinguish the lines from the dashed lines used for general relations. Figure 6 shows a structure diagram and an example of a relation having this particular structure.

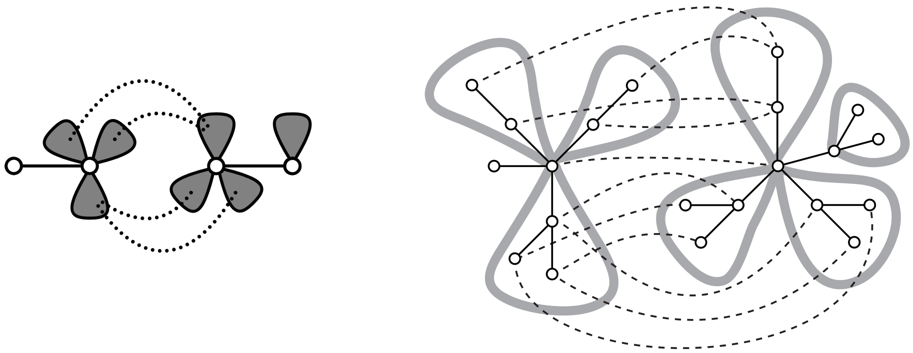


Figure 6: Example of a structure diagram and a relation having this structure

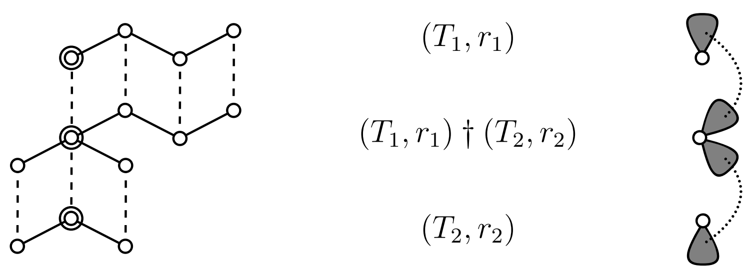


Figure 7: The rooted sum of two rooted trees and associated diagram.

3.2 Rooted sums

The idea of the rooted sum of two trees, T_1 and T_2, is that a node is specified in each tree and we form a new tree by gluing copies of T_1 and T_2 together at the specified nodes. An example is shown in Figure 7. In this and subsequent figures the specified node (i.e. the root) of a tree is indicated by an extra circle around the node.

Definition 9. *Let* (T_1, r_1) *and* (T_2, r_2) *be rooted trees. Their rooted sum,* $(T_1, r_1) \dagger (T_2, r_2)$*, has the set of nodes* $((N_1 - \{r_1\}) \times \{1\}) \cup ((N_2 - \{r_2\}) \times \{2\}) \cup \{0\}$*. The adjacency relation* $\alpha_1 \dagger \alpha_2$ *is defined to be the smallest symmetric relation which*

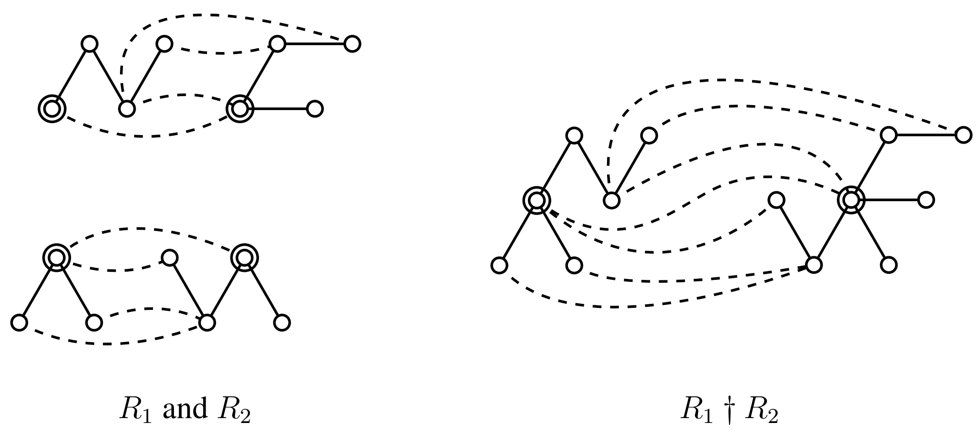


Figure 8: The rooted sum of two rooted bipartite relations.

satisfies the following conditions.

$$\begin{array}{rcl} 0\ (\alpha_1 \dagger \alpha_2)\ (m,1) & \textit{iff} & r_1\ \alpha_1\ m, \\ 0\ (\alpha_1 \dagger \alpha_2)\ (m,2) & \textit{iff} & r_2\ \alpha_2\ m, \\ (m,1)\ (\alpha_1 \dagger \alpha_2)\ (n,1) & \textit{iff} & m\ \alpha_1\ n, \\ (m,2)\ (\alpha_1 \dagger \alpha_2)\ (n,2) & \textit{iff} & m\ \alpha_2\ n. \end{array}$$

This construction may be extended from rooted trees to rooted bipartite relations.

Definition 10. *Given rooted bipartite relations* $R_i : (T_i, r_i) \to (T'_i, r'_i)$ *for* $i = 1, 2$*, their rooted sum* $R_1 \dagger R_2 : ((T_1, r_1) \dagger (T_2, r_2)) \to ((T'_1, r'_1) \dagger (T'_2, r'_2))$ *is defined to be the relation where* $x\ (R_1 \dagger R_2)\ y$ *iff*

$$\begin{cases} x = 0 \textit{ and } y = 0, \textit{ or} \\ x = (m,1), y = (n,1), \textit{ and } m\ R_1\ n \textit{ for some } m \in N_1, n \in N'_1, \textit{ or} \\ x = (m,2), y = (n,2), \textit{ and } m\ R_2\ n \textit{ for some } m \in N_2, n \in N'_2. \end{cases}$$

An example of the rooted sum of relations is shown in Figure 8. It is straightforward to check that the rooted sum of rooted bipartite relations is again bipartite.

4 Atomic relations

We next consider five particularly simple kinds of relation, which we will show in our main result, Theorem 13, are sufficient to generate all possible bipartite relations by composition. These five types may thus be considered atomic components out of which all other bipartite relations may be constructed. The motivation for the choice of these particular atomic relations comes from our intended application to qualitative changes to regions on the plane or the sphere.

4.1 Examples of atomic relations

The five types of atomic relation we consider are: merge, split, insert, delete, and no change. These are illustrated in Figure 9 which shows a sequence of changes to planar regions above the corresponding sequence of relations between each pair of successive trees. In this figure the nodes of the trees are coloured black or white to indicate whether they correspond to foreground or background regions respectively. As with Figure 1, this colouring is for illustrative purposes only and is not part of the formal structure being considered.

There are two instances of a split in Figure 9. In the first the single black region encloses a portion of the white background region, splitting the white region into two. In the second, the black foreground region grows a subsidiary part which then breaks off. Jiang and Worboys [JW09] refer to a 'self merge' when the background splits and use 'split' for the foreground case. These two cases can only be distinguished in the abstract model if the trees we deal with are equipped with some additional structure which specifies which nodes correspond to foreground regions and which to background ones. In the present paper we do not include such additional structure, and thus we do not distinguish different kinds of splits or different kinds of merges.

4.2 Atomic relations: Diagrams and definitions

Definition 11. *An **atomic split** from T_1 to T_2 and an **atomic insert** from T_1 to T_2 are bipartite relations having the forms shown in Figure 10. An **atomic merge** from T_1 to T_2 is a relation the converse of which is an atomic split from T_2 to T_1. An **atomic delete** from T_1 to T_2 is a relation the converse of which is an atomic insert from T_2 to T_1. An **atomic relation** is any relation of these four forms or an isomorphism.*

We will use the term **rooted atomic relation** to mean any atomic relation between rooted trees which is a rooted bipartite relation. If $R : (T_1, r_1) \to (T_2, r_2)$ is a rooted relation then $r_1 \ R \ r_2$, so in an atomic rooted relation the root node

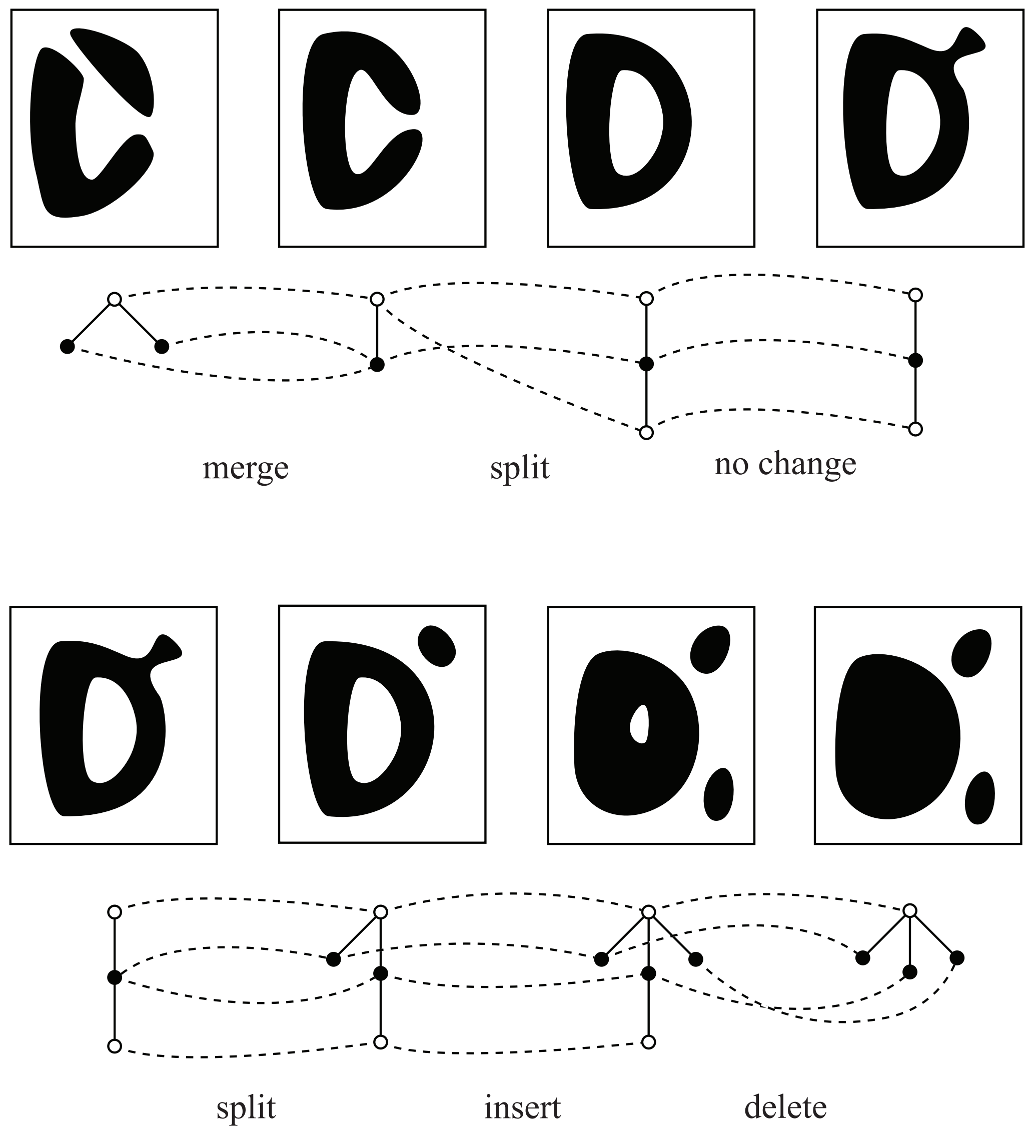


Figure 9: Sequence illustrating the five types of atomic relation

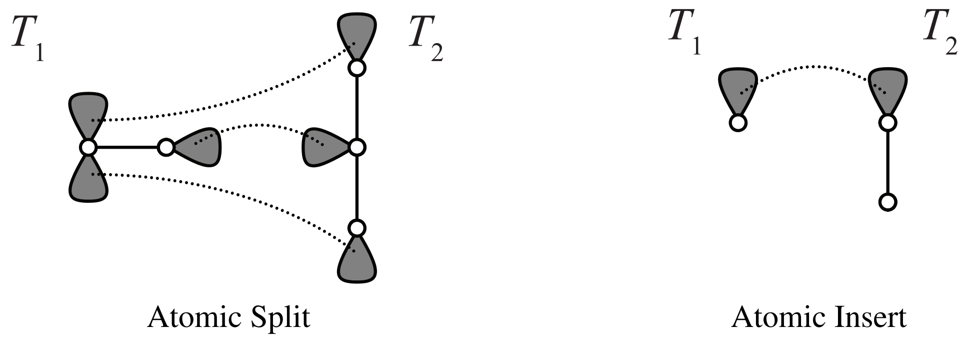


Figure 10: Atomic relations (see Definition 11).

cannot be inserted or deleted. Note that if two trees are related by an atomic relation then the number of nodes differs by at most one.

Although atomic relations are the focus of our approach, it may be argued that these fail to model all possible changes to regions because there might be simultaneous merging or splitting. For example, in Figure 2 the two encircling regions might join together in two places at the same time. This is clearly a physical possibility that could be important in some applications. In this particular example the capture of the central region by the two outermost ones can readily be expressed as the composite of two atomic relations. Should it be necessary to capture a notion of concurrency then an appropriate equivalence relation on sequences of atomic relations could be introduced.

4.3 Evolutions

By composing atomic relations we can generate more complex relations.

Definition 12. *An* ***evolution*** *between trees is any relation that arises from composing a sequence of atomic relations. Similarly, a* ***rooted evolution*** *between rooted trees is a rooted relation obtained by composing atomic rooted relations.*

As the atomic relations are bipartite, the evolutions are bipartite by Lemma 2. The evolutions are clearly closed under composition and under taking converses and they include all isomorphisms. We will also need the fact that they are closed under rooted sums. If $R : (T_1, r_1) \to (T_1', r_1')$ is a rooted atomic relation, and I is the identity relation on (T_2, r_2) then it is easily checked that $R \dagger I$ is an atomic rooted relation. Hence by composition we get the following.

Lemma 3. *Let* $R : (T_1, r_1) \to (T_1', r_1')$ *be a rooted evolution, let* I *be the identity relation on* (T_2, r_2)*. Then* $R \dagger I : (T_1, r_1) \dagger (T_2, r_2) \to (T_1', r_1') \dagger (T_2, r_2)$ *is an evolution.* □

Corollary 4. *Let* $R_i : (T_i, r_i) \to (T_i', r_i')$ *be rooted evolutions for* $i = 1, 2$*. Then* $R_1 \dagger R_2$ *is a rooted evolution.*

Proof. We can write $R_1 \dagger R_2 = (R_1 \dagger I_2) ; (I_1' \dagger R_2) = (I_1 \dagger R_2) ; (R_1 \dagger I_2')$, where the I_i and I_i' denote the identity relations on the trees T_i and T_i' respectively. □

5 Chains and ladders

In this section we consider trees of a particularly simple kind in which the nodes with their adjacency relation constitute a linearly ordered set, also called a chain.

5.1 Definitions and examples

Definition 13. *The tree with nodes* $\{1, \ldots, n\}$ *where* $n \geqslant 1$ *and adjacency* α *where* $i \; \alpha \; i+1$ *will be denoted* $\underline{n}$*. Any tree isomorphic to some* $\underline{n}$ *will be called a* ***chain****. A chain in which one of the nodes of degree 1 is distinguished as the root will be called a* ***directed chain****.*

Definition 14. *A* ***ladder*** *is any relation which is isomorphic to a subrelation of the identity relation on a chain.*

Relations which are ladders can be drawn (see Figure 11) so that the only nodes which may be related are those which align horizontally. Not every pair of horizontally aligned nodes need be related so the effect is of a ladder in which some of the rungs may be missing. If no rungs are missing then the ladder is an isomorphism between chains.

Definition 15. *A* ***directed ladder*** *is any relation which is isomorphic to a subrelation of the identity relation on a directed chain.*

Note that there is no requirement that a directed ladder should be a rooted relation. That is, the two root nodes on the two chains forming the two sides of the relation need not be related.

If λ is a subrelation of the identity relation on $\underline{n}$, then we can represent λ by a sequence of 0s and 1s of length n. In this sequence a 1 in the i-th place indicates

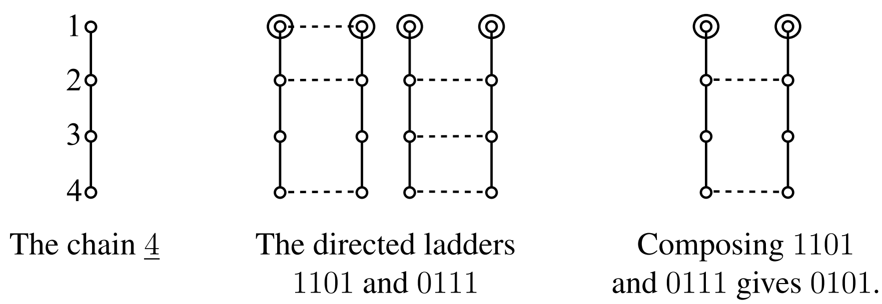


Figure 11: Examples of ladders

that $i\ \lambda\ i$, and a 0 indicates that this is not the case. Figure 11 provides examples. If $\lambda = \lambda_1\lambda_2\cdots\lambda_n$ and $\mu = \mu_1\mu_2\cdots\mu_n$ are both ladders of length n then the composite is given by $\lambda\,;\mu = (\lambda_1 \wedge \mu_1)(\lambda_2 \wedge \mu_2)\cdots(\lambda_n \wedge \mu_n)$, where $\lambda_i \wedge \mu_i$ is 0 unless both λ_i and μ_i are 1. This operation on sequences is known [Knu98, p111] as the *bitwise and* of the sequences.

5.2 All ladders are evolutions

Figure 12 shows the directed ladder 101, which is the simplest case where it is not immediately obvious how to express the directed ladder as a composite of atomic relations.

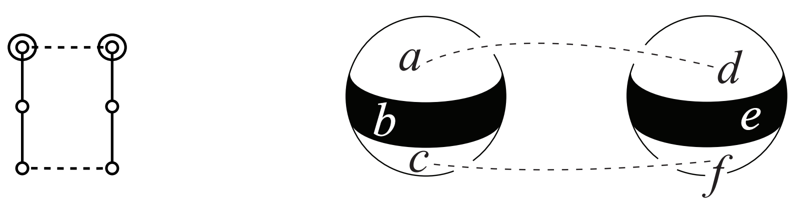


Figure 12: The directed ladder 101 and a possible interpretation.

Before showing that 101 can be factorized into atomic relations, it is instructive to consider what this relation might mean in terms of qualitative spatial change. We can use the chain $\underline{3}$ to represent three regions on the sphere as shown in Figure 12. The central node in the chain represents the region forming a band around the equator of the sphere and shown black in the figure.

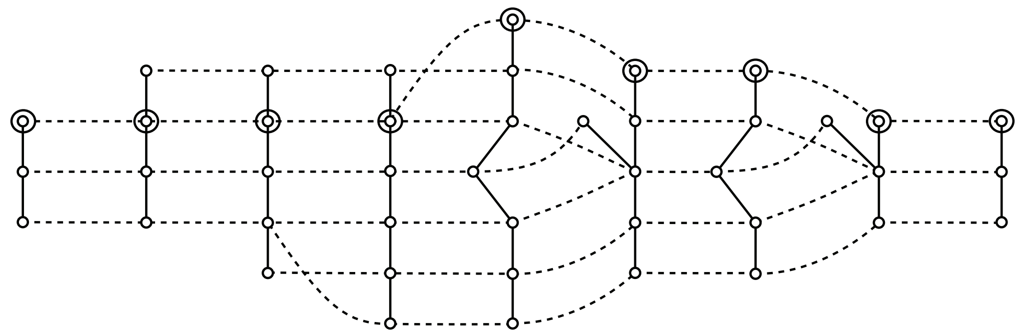


Figure 13: The directed ladder 101 as a composite of eight atomic relations.

To visualize a possible interpretation, imagine the surface of the sphere covered by lakes of two types of substance coloured black and white. The entity labelled a in the figure should be understood as a quantity of white material rather than the part of the surface of the sphere that this material occupies. The atomic relations available to us mean that lakes of opposing colours cannot merge with each other, but two lakes of the same colour may combine to form a single lake. The properties of composition of relations imply that once two lakes of the same colour have merged they cannot be separated again. This is because if lakes x and y merge into z we would have $x\ R\ z$ and $y\ R\ z$ for some relation R. Then for any relation S we will have $x\ (R\,;S)\ w$ iff $y\ (R\,;S)\ w$.

As the atomic relations include inserts and deletes, new disc-like lakes may appear within lakes of the opposite colour, and lakes may disappear provided they have just a one-piece boundary. The black equatorial band has a two-piece boundary and thus cannot disappear without the two white lakes a and c first merging with each other. This means, in particular, that to obtain the ladder 101 as a composite of atomic relations we require something more complex than the deletion of b followed by the insertion of e.

When the factorization in terms of atomic relations shown in Figure 13 is examined, we see that a portion of a has merged with a portion of c and the resultant entity has been deleted by the last atomic relation in the factorization. This highlights a potential danger in interpreting the fact that a and d are related to each other and neither is related to any other entity. It might seem that the exclusive link between a and d in Figure 12 should mean that lake a evolves unchanged to

d. Consideration of the properties of the operation of composing relations shows that the exclusive link does not exclude the possibility that parts of a may have split off, then merged with parts of c, and then been deleted. Additionally, we cannot infer from this exclusive link that the lake d contains material only present in a. This is because a new lake might have been created between the initial and final stages and this new region might have merged with the region that became d.

Lemma 5. *The directed ladder* 101 *is an evolution.*

Proof. A factorization into atomic relations is provided in Figure 13. The eight atomic relations are: insert; insert; split; split; merge; delete; merge; delete. □

We can use the idea of rooted sums of relations to show that ladders can always be expressed as sequences of atomic relations.

Theorem 6. *Every ladder is an evolution.*

Proof. Given an arbitrary ladder λ, we can choose a direction and assume we have a directed ladder. We have noted that directed ladders correspond to binary sequences and that the composition of relations corresponds to taking the bitwise *and* of such sequences.

Now every binary sequence is expressible as the bitwise *and* of a number of sequences each of which contains at most one zero. Thus the result follows if we can show that every ladder with exactly one rung missing is an evolution.

If the missing rung is the top or bottom one, a delete followed by an insert gives us what we require. If the missing rung is in the i-th place in a ladder of length n and $1 < i < n$ then we use Lemmas 5 and 3 as follows. The directed ladder, Λ, of length $i + 1$ which lacks only the second rung can be obtained by re-rooting $I \dagger 101$ where I is the identity relation on the chain $\underline{i-1}$. The ladder we require is then obtained from $\Lambda \dagger J$, where J is the identity relation on the chain $\underline{n-i}$. □

6 Relations between chains

A chain is a particularly simple form of tree – one with exactly two nodes of degree one, or none in the case of a single node chain. In terms of regions, a chain represents a sequence of regions nested within each other. The importance of chains is that we are able to reduce the problem of showing that arbitrary bipartite relations on trees are evolutions to a problem about bipartite relations only

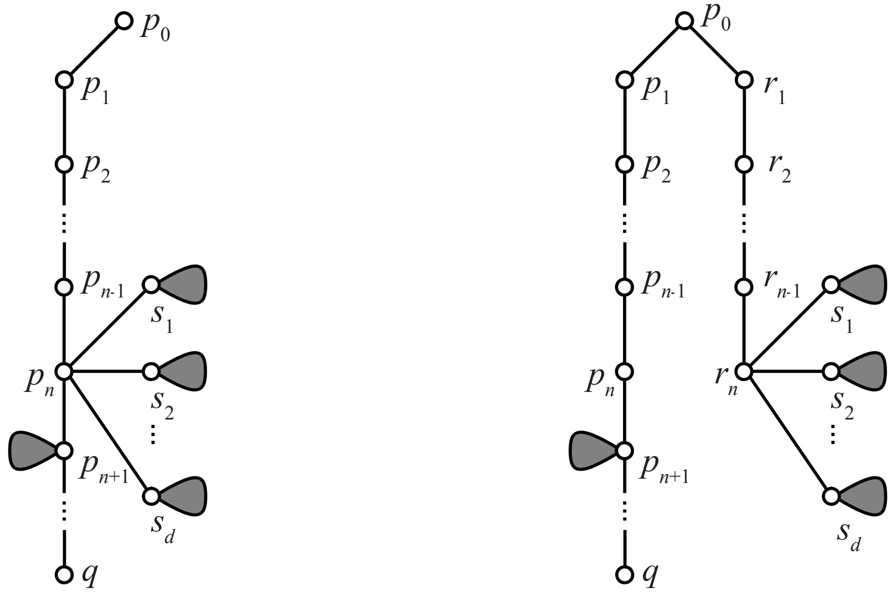


Figure 14: Construction in proof of Lemma 7

between chains. We have already met some simple relations between chains in the ladder relations, and these were all shown to be evolutions in Theorem 6. This section generalizes this result to show in Theorem 12 that all bipartite relations between chains are evolutions.

6.1 Reduction to chains

First, we recall an observation about ordinary relations. If $\alpha : A \to B$ is any relation between sets A and B, and α is injective and total then $\alpha\,;\alpha^{-1} = I$, where α^{-1} is the converse of α, and I is the identity relation on A. This is because α is total iff $I \subseteq \alpha\,;\alpha^{-1}$, and α is injective iff $\alpha\,;\alpha^{-1} \subseteq I$.

Lemma 7. *For any tree, T, there is a chain C and an evolution $\alpha : T \to C$ which is injective and total.*

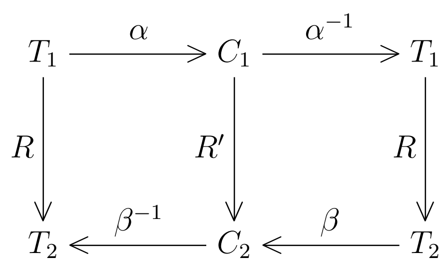
Proof. The relation α is constructed as the composite of a sequence of evolutions $\alpha_i : T_{i-1} \to T_i$ for $i = 1, \ldots, m$, and where $T = T_0$ and $T_m = C$.

Assume inductively that we have constructed $\alpha_i : T_{i-1} \to T_i$ for $i = 1, \ldots, k$, and that each α_i is total and injective. If T_k has only two nodes of degree one then it is a chain, and we are done. If there are more than two nodes of degree one, then let p_0 and q be any two distinct such nodes in T_k. Consider the path from p_0 to q. This will have an initial segment of the form $p_0, p_1, \ldots, p_n, p_{n+1}$ where p_j has degree two for $j = 1, \ldots, n-1$, and p_n has degree strictly greater than 2. Let the nodes adjacent to p_n be $\{p_{n-1}, p_{n+1}, s_1, s_2, \ldots, s_d\}$. The tree T_{k+1} has the same nodes as T_k together with n new nodes $r_1, r_2, \ldots, r_n$ attached as shown in Figure 14.

The relation $\alpha_{k+1} : T_k \to T_{k+1}$ is the union of the identity relation on the nodes of T_k with the relation $p_j \ \alpha_k \ r_j$ for $j = 1, \ldots, n$. Clearly this is injective and total. It is easily checked that α_{k+1} is an evolution by using a sequence of splits at each of $p_n, p_{n-1}, \ldots, p_1$ to obtain T_{k+1} from T_k. Since T_{k+1} has exactly one fewer node of degree one than T_k, (i.e. p_0 in the above construction) we must eventually obtain a chain. □

From the lemma we obtain the following result which shows that if every bipartite relation between chains is an evolution, then every bipartite relation between arbitrary trees is an evolution.

Corollary 8. *Any bipartite relation $R : T_1 \to T_2$ can be expressed as $\alpha \, ; R' \, ; \beta^{-1}$ where R' is a bipartite relation between chains, and where α and β are evolutions as in the diagram with $\alpha \, ; \alpha^{-1}$ and $\beta \, ; \beta^{-1}$ being the identities on T_1 and T_2 respectively.*



Proof. Construct $\alpha : T_1 \to C_1$ and $\beta : T_2 \to C_2$ as the injective and total evolutions from the two trees to chains by the method in Lemma 7. The relation R' is defined to be the composite $\alpha^{-1}; R; \beta$, and the above diagram commutes. □

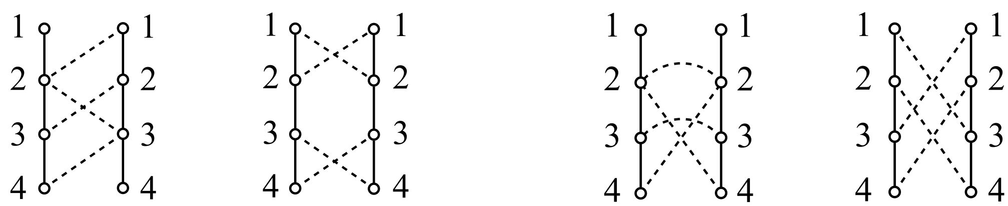
We proceed by looking first at certain simple relations between chains: the bijective ones in section 6.2 and then more generally in section 6.3 at the injective and functional relations. These cases are then used to show that all relations between chains are evolutions.

6.2 Tangled pairings

When dealing with relations between trees where the source and target are the same we can distinguish two kinds of bipartite relations.

Definition 16. *Let $R : T \to T$ be a bipartite relation. We say R is* ***direct*** *if for all nodes x, y, we have $x \ R \ y$ implies $[x] = [y]$. If R is not direct, it is said to be* ***reverse****.*

If R is a direct bipartite relation and $x \ R \ y$ then the distance $d(x, y)$, as defined in Section 2.2, will be even. If R is reverse then this distance will be odd. Some examples of direct and reverse bipartite relations appear in Figure 15.



Reverse bipartite relations.
The right hand one is
a permutation of $\{1, 2, 3, 4\}$.

Direct bipartite relations.
The right hand one is
a tangled pairing.

Figure 15: Examples of direct and reverse bipartite relations on the chain $\underline{4}$

Definition 17. *A* ***tangled pairing*** *on a chain C is any direct bipartite relation $R : C \to C$ which is a bijective total function on the set of nodes in C.*

A ***transposition*** *on a chain C is any tangled pairing $R : C \to C$ for which there are exactly two nodes n such that $n \ R \ n$ does not hold.*

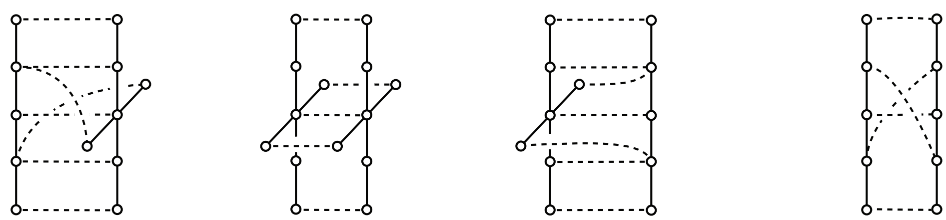
A tangled pairing is thus a permutation on the set of nodes. We will use (n_1, n_2) to denote the transposition that swaps nodes n_1 and n_2 while leaving all others fixed. Since transpositions are direct bipartite relations the two nodes that are transposed must be an even number of edges apart in the chain.

Lemma 9. *Let $R : C \to C$ be any direct bipartite relation on a chain which is functional and injective. Then there is a relation $R' : C \to C$ such that $R \subseteq R'$ and R' is a tangled pairing on C.*

Proof. Let the two equivalence classes in C be K_1 and K_2. The sets $K_1 \cap (C - R^{-1}(C))$ and $K_1 \cap (C - R(C))$ have equal numbers of members, and the same is true of $K_2 \cap (C - R^{-1}(C))$ and $K_2 \cap (C - R(C))$. So we form R' by bijectively pairing, in any way, elements of $K_1 \cap (C - R^{-1}(C))$ with elements of $K_1 \cap (C - R(C))$, and elements of $K_2 \cap (C - R^{-1}(C))$ with elements of $K_2 \cap (C - R(C))$. □

Lemma 10. *Every tangled pairing is an evolution.*

Proof. First we show that all transpositions are evolutions. We have seen in Theorem 6 that all ladders are evolutions. Combining this result with Lemma 3, we know that the relation R_2 shown in Figure 16 is an evolution. The relations R_1 and R_3 in this figure are also evolutions, being respectively two splits and two merges. Hence the composite $R_1 \,;\, R_2 \,;\, R_3$, that is the transposition $(2, 4)$ on the chain $\underline{5}$, is an evolution.



R_1 R_2 R_3 $(2, 4)$

Figure 16: Composing R_1, R_2 and R_3 gives the transposition $(2, 4)$.

Knowing that the transposition $(2, 4)$ is an evolution, we see (making use of Lemma 3) that all transpositions in which the two transposed nodes are exactly two edges apart are also evolutions. That is, if we number the nodes in the chain $n_0, n_1, \ldots, n_k$ then we can obtain every transposition of the form (n_i, n_{i+2}). From this we get that all transpositions are evolutions, since letting $\pi = (n_i, n_{i+2}) \,;\, (n_{i+2}, n_{i+4}) \,;\, \cdots \,;\, (n_{i+2j-4}, n_{i+2j-2})$ we can express an arbitrary transposition as $(n_i, n_{i+2j}) = \pi \,;\, (n_{i+2j-2}, n_{i+2j}) \,;\, \pi^{-1}$.

By the well-known result that all permutations arise by composing transpositions (see for example, [Jac85, p49]), we thus obtain all tangled pairings in which one of the two equivalence classes of nodes has every element fixed. Arbitrary tangled pairings arise by composing two relations of this special form. □

6.3 Arbitrary relations between chains

We can now demonstrate that all bipartite relations between chains are evolutions. This result is Theorem 12 below, before which we need a lemma. Note that the lemma includes the case of permutations of the nodes of a chain which are reverse bipartite relations and thus not covered by Lemma 10 above.

Lemma 11. *If $R : C \to D$ is an injective and functional bipartite relation between chains then R is an evolution.*

Proof. Suppose that C and D have m and n nodes respectively. We can assume that C and D are the chains $\underline{m}$ and $\underline{n}$. The proof proceeds by writing R as a composite $R_1 ; R_2 ; R_3$ where R_2 is a tangled pairing and R_1 and R_3 have simple forms which are evidently evolutions. The relation R_2 will be constructed by extending another relation S.

Let i be the least element of $\underline{m}$ for which there is a j such that $i \; R \; j$. As R is functional, this j is unique. We now consider two cases according as $i \geqslant j$ or not.

When $i \geqslant j$, take ℓ to be the maximum of m and $n+i-j$ and define $S : \underline{\ell} \to \underline{\ell}$ by $x \; S \; y$ iff $x \; R \; (y - i + j)$. Define $R_1 : \underline{m} \to \underline{\ell}$ by $R_1 = \{(x, x) \mid \exists y \; (x \; R \; y)\}$, and define $R_3 : \underline{\ell} \to \underline{n}$ by $R_3 = \{(x, x - i + j) \mid x = i - j + 1, \cdots, i - j + n\}$.

When $i < j$, define ℓ to be the maximum of n and $m + j - i$ and define $S : \underline{\ell} \to \underline{\ell}$ by $x \; S \; y$ iff $(x + i - j) \; R \; y$. We define $R_1 : \underline{m} \to \underline{\ell}$ by

$$R_1 = \{(x, x + j - i) \mid \exists y \; (x \; R \; y)\},$$

and we define $R_3 : \underline{\ell} \to \underline{n}$ by $R_3 = \{(x, x) \mid x = 1, \cdots, n\}$.

In each case S is bipartite and is injective and functional since R is, but $i \; S \; i$ so S is a direct bipartite relation. Thus by Lemma 9 we can extend S to a tangled pairing R_2 on ℓ. We have thus expressed R as a composite of three evolutions. □

Theorem 12. *Any bipartite relation $R : C \to D$ between chains is an evolution.*

Proof. The technique is to write $R = R_{\text{inj}} ; R_{\text{fnl}}$, where R_{inj} is injective and R_{fnl} is functional.

Let $z \in D$ be any node for which $x \; R \; z$ and $y \; R \; z$ for distinct x and y. Make a new chain from D by replacing the node z by a chain the nodes of which are pairs

$$(x_1, z), (w_1, z), (x_2, z), (w_2, z), \ldots, (w_{n-1}, z), (x_n, z)$$

where $x_1, w_1, \ldots, x_n$ is the interval in C with endpoints the extreme elements x for which $x \; R \; z$ (i.e. any x such that $x \; R \; z$ lies in the interval from x_1 to x_n). Denote this new chain by K.

Now let $R' : C \to K$ act as R except that whenever $x \; R \; z$ we now have $x \; R' \; (x, z)$ and, in general, $x \; R' \; (x', z)$ iff $x = x'$ and $x \; R \; z$. We can also define $R'' : K \to D$ by $a \; R'' \; b$ iff $a = b$ or $a = (x_i, b)$ for some x_i. We then have $R = R' \; ; R''$ and by repeating this process on R' we will eventually arrive at a stage where R' is injective and the composition of all the R''s provides the required functional relation R_{fnl}. We can see that each R'' is an evolution as it arises by merging all the (x_i, z) with each other, and deleting the node that results from merging together all the (w_i, z) with each other. We will need the fact that in this construction if R is functional then R_{inj} will be functional as well as injective.

We have factorized R into an injective part, R_{inj}, and a functional part,R_{fnl}. The functional part has been shown to be an evolution so we are left with an arbitrary injective bipartite relation to deal with.

Consider the converse of this relation ${R_{\text{inj}}}^{-1}$. By applying the above process to this relation, we arrive at ${R_{\text{inj}}}^{-1} = S_1 \; ; S_2$ where S_1 is injective and S_2 is an evolution. Since ${R_{\text{inj}}}^{-1}$ is functional we have that S_1 is both injective and functional. The result then follows from Lemma 11. □

7 The characterization of evolutions

The main result now follows from Theorem 12 and Corollary 8.

Theorem 13. *For any two trees T_1, T_2, the evolutions from T_1 to T_2 are exactly the bipartite relations from T_1 to T_2.* □

Relations between abstract trees (without any additional structure, such as a choice of root) correspond most naturally to bounded regions evolving on a sphere, such as the surface of the Earth. For some applications, however, it can be more natural to consider bounded regions evolving against an unbounded plane background. This is the case examined in [JW09] and corresponds to using rooted trees because the background has a special status. The background may not be deleted, although it may be involved in splitting and merging. We can use our result Theorem 13 to show that the analogous statement holds in the rooted case.

Corollary 14. *For any two rooted trees (T_1, r_1), (T_2, r_2), the rooted evolutions from (T_1, r_1) to (T_2, r_2) are exactly the rooted bipartite relations from (T_1, r_1) to (T_2, r_2).*

Proof. Suppose we are given a rooted bipartite relation $R : (T_1, r_1) \to (T_2, r_2)$. By Theorem 13 we can express this as a sequence of atomic relations.

$$T_1 = S_1 \xrightarrow{R_1} S_2 \xrightarrow{R_2} S_3 \quad \cdots \quad S_{n-1} \xrightarrow{R_{k-1}} S_k = T_2.$$

Since the relation R preserves the root (i.e. $r_1 \; R \; r_2$) we can identify a node n_i in each S_i for $i = 2, \ldots, k-1$ such that $r_1 \; R_1 \; n_2 \; R_2 \; n_3 \cdots n_{k-1} \; R_{k-1} \; r_2$. By designating these nodes as the roots we make each R_i into a rooted atomic relation and so have that R is a rooted evolution. □

8 Fourfold factorizations

We have shown that arbitrary bipartite relations can always be factorized into atomic relations. Previous work by Jiang and Worboys [JW09] deals with a different kind of factorization. In this section we show how the two approaches are related.

8.1 Homomorphic relations

So far we have introduced atomic relations and the more general bipartite relations. In order to understand how our approach relates to the results in [JW09] we need to introduce further kinds of relations.

Definition 18. *A* ***homomorphic insert*** *is a relation between trees $f : T_1 \to T_2$ which is an injective homomorphism. A* ***homomorphic delete*** *is a relation the converse of which is a homomorphic insert.*

A ***homomorphic merge*** *is a relation between trees $f : T_1 \to T_2$ which is a surjective homomorphism. A* ***homomorphic split*** *is a relation the converse of which is a homomorphic merge.*

Any relation of one of the above four forms which is not an isomorphism will be called ***non-degenerate****.*

In general homomorphic deletes and splits will not be functions, let alone homomorphisms. Jiang and Worboys use the terms ‘insert’ and ‘merge’ for what we have just defined as a homomorphic insert and a homomorphic merge. We have introduced these new terms to avoid confusion with the atomic inserts and merges we introduced earlier. Jiang and Worboys use the terms ‘split’ and ‘delete’ for the converses of what we call a homomorphic split and a homomorphic delete.

Lemma 15. *The non-degenerate homomorphic inserts are exactly the composites of atomic inserts.*

Proof. Composing atomic inserts clearly gives a homomorphic insert. Conversely, let $f : T_1 \to T_2$ be a non-degenerate homomorphic insert. There must be a node n in T_2 of degree 1 which is not in the image of f, because if two distinct nodes are in the image f then every node on the path between them is too. Let T_2' be the tree T_2 with node n removed, and $\alpha : T_2' \to T_2$ the atomic insert which inserts this node. We can write f as $f' ; \alpha$ where f' is a homomorphic insert with one fewer node inserted than f, so the result follows by induction. □

Corollary 16. *The non-degenerate homomorphic deletes are exactly the composites of atomic deletes.* □

Lemma 17. *A relation $f : T_1 \to T_2$ is a non-degenerate homomorphic merge if and only if it is a composite of one or more atomic merges.*

Proof. Composites of atomic merges are clearly homomorphic merges. For the converse, we use induction on the number of sets of nodes $\{a, b\}$ in T_1 which are distance 2 apart and where $fa = fb$. If there are no such sets of nodes then f must be an isomorphism. Let T_1' be the tree with nodes $(N_1 - \{a, b\}) \cup \{t\}$, where t is a new node not in N_1. The node t is adjacent in T_1' to those nodes of T_1 adjacent to at least one of a and b. Other nodes of T_1' are adjacent to each other as they are in T_1. Now let $\alpha : T_1 \to T_1'$ be the atomic relation which merges a and b with t. We can then write $f = \alpha ; f'$ with $f'x = fx$ whenever $x \neq t$, and $f't = fa$. □

Corollary 18. *The non-degenerate homomorphic splits are exactly the composites of atomic splits.* □

The following theorem is a slight reformulation of a result established in [JW09].

Theorem 19 (Jiang and Worboys)**.** *Suppose the rooted bipartite relation $R : T_1 \to T_2$ between rooted trees is a composite of an arbitrary sequence of relations each one of which is a homomorphic insert, a homomorphic delete, a homomorphic merge or a homomorphic split. Then R can be expressed as a composite of just four homomorphic relations*

$$T_1 \xrightarrow{R_I} U \xrightarrow{R_S} V \xrightarrow{R_M} W \xrightarrow{R_D} T_2$$

in which R_I, R_S, R_M, and R_D are respectively a homomorphic insert a homomorphic split, a homomorphic merge, and a homomorphic delete. □

From Lemmas 15 and 17 and Corollaries 16 and 18, it follows by using Corollary 14 that the relations R of the form described in Theorem 19 comprise all possible rooted bipartite relations. So we deduce that all rooted bipartite relations admit such fourfold factorizations.

In the remainder of this section we investigate related factorizations in our framework. To start with, we need to recall some basic properties of relations between sets rather than trees.

8.2 Factorizing relations between sets

For a relation R from set X to set Y, there are four especially simple kinds of relation which we can identify.

Definition 19. *For a relation $R : X \to Y$ with converse $R^{-1} : Y \to X$,*

$$R \textit{ is } \left\{ \begin{array}{c} \textit{inserting} \\ \textit{deleting} \\ \textit{splitting} \\ \textit{merging} \end{array} \right\} \textit{ iff } \left\{ \begin{array}{c} R \textit{ and } R^{-1} \textit{ are injective, and } R^{-1} \textit{ is surjective} \\ R \textit{ and } R^{-1} \textit{ are injective, and } R \textit{ is surjective} \\ R \textit{ and } R^{-1} \textit{ are surjective, and } R \textit{ is injective} \\ R \textit{ and } R^{-1} \textit{ are surjective, and } R^{-1} \textit{ is injective} \end{array} \right\}$$

*A relation of any one of these forms is called a **basic relation**.*

Note that R is inserting iff R^{-1} is deleting, and R is splitting iff R^{-1} is merging. The following result can probably be described as well-known folklore, but we have included a detailed proof because we need to understand how it extends to trees.

Lemma 20. *Any relation $R : X \to Y$ between sets admits a factorization into basic relations*

$$X \xrightarrow{R_I} A \xrightarrow{R_S} B \xrightarrow{R_M} C \xrightarrow{R_D} Y$$

in which R_I, R_S, R_M, and R_D are respectively inserting, splitting, merging, and deleting.

Proof. Assume that X and Y are disjoint, since if not we can find a relation isomorphic to R in which they are. Define $E_X = \{x \in X \mid \nexists y \in Y \cdot x \; R \; y\}$ and $E_Y = \{y \in Y \mid \nexists x \in X \cdot x \; R \; y\}$. The required factorization comes from the following diagram of sets and functions

$$X \xrightarrow{\iota} X \cup E_Y \xleftarrow{\sigma} E_X \cup R \cup E_Y \xrightarrow{\mu} E_X \cup Y \xleftarrow{\delta} Y$$

where R is the given relation as a set of ordered pairs. The functions ι and δ are the evident inclusions; these give $R_I = \iota$, which is inserting, and $R_D = \delta^{-1}$, which is deleting. The function σ acts as the identity on $E_X \cup E_Y$ and sends $(x, y) \in R$ to x. The function μ also acts as the identity on $E_X \cup E_Y$, but sends $(x, y) \in R$ to y. These provide $R_S = \sigma^{-1}$ which is splitting, and $R_M = \mu$ which is merging.

It is sufficient to check that ι and μ are inserting and merging in order to justify that the four relations R_I, R_S, R_M and R_D have the required properties. This is because the inserting component of R is the deleting component of R^{-1}, that is $R_I = (R^{-1})_D$, and also $R_M = (R^{-1})_S$. It is also easily checked that the composite $R_I; R_S; R_M; R_D$ yields the original relation R. □

8.3 Factorizing relations between trees

We return now to bipartite relations between trees. The terminology of Definition 19 can be applied directly to these relations.

Suppose $R : T_1 \to T_2$ is a bipartite relation. It is possible to colour each tree so that every node is either black or white and so that any pair of nodes related by R both have the same colour. We can express this colouring by writing $N_1 = B_1 \cup W_1$ and $N_2 = B_2 \cup W_2$, where B_i is the set of black nodes for $i = 1, 2$, and W_i is the set of white nodes. Then the bipartite relation R from T_1 to T_2 is equivalent to two ordinary relations between sets $R_B : B_1 \to B_2$, and $R_W : W_1 \to W_2$. Note that R is an inserting if and only if both R_B and R_W are both insertings, and similarly for the other types.

Now, R_B and R_W each admits a factorization as in Lemma 20 and taking the unions of the corresponding parts yields a factorization of R. That is, the inserting component of R is the union of the inserting components of R_B and R_W etc. If a set is partitioned into two then the two subsets can form the two differently coloured sets of nodes of a tree except when one set is empty and the other has at least two elements. Because our T_1 and T_2 are trees to start with, and from the properties of basic relations, it follows that the factorization obtained for R between the sets of nodes allows all the intermediate sets to be made into trees in a way respecting the colours. Thus we have established the following.

Theorem 21. *Any bipartite relation $R : T_1 \to T_2$ admits a factorization into basic relations*

$$T_1 \xrightarrow{R_I} U \xrightarrow{R_S} V \xrightarrow{R_M} W \xrightarrow{R_D} T_2$$

in which R_I, R_S, R_M, and R_D are respectively inserting, splitting, merging, and deleting. □

The rooted case is easily obtained from this. If T_1 and T_2 have specified root nodes n_1, n_2 and $n_1 \; R \; n_2$ then it will be possible to identify a root node in each of U, V, and W so that the four basic components are rooted bipartite relations.

Corollary 22. *Any rooted bipartite relation $R : (T_1, n_1) \to (T_2, n_2)$ admits a factorization into rooted basic relations*

$$(T_1, n_1) \xrightarrow{R_I} (U, u) \xrightarrow{R_S} (V, v) \xrightarrow{R_M} (W, w) \xrightarrow{R_D} (T_2, n_2)$$

in which R_I, R_S, R_M, and R_D are respectively inserting, splitting, merging, and deleting. □

The interest of these results lies in the way they depend only on the corresponding result for relations between sets. They are however weaker than Theorem 19 because the basic relations need not be homomorphic. Also it should be noted that this theorem does not subsume our main result, Theorem 13, as it would be necessary to show that the four components it includes are themselves sequences of atomic relations.

9 Conclusions and further work

We have shown that evolutions, or composites of atomic relations, are the same as bipartite relations between trees. The motivation for studying such relations is that if we interpret the trees as adjacency trees of spatial entities, then the bipartite relations can be interpreted as descriptions of how entities present at an initial stage have contributed to the formation of the entities present at a final stage. Being able to equate bipartite relations with compositions of atomic relations shows that any pattern of formation for regions expressible as a relation has an explanation in terms of the intuitively simple ideas of inserts, splits, merges and deletes.

We have not addressed the issue of whether some factorizations of bipartite relations into atomic relations are preferable over others, but this would be a natural direction for further work. For example, it could be asked whether there is a simplest factorization in some sense. The factorization of the ladder 101 given in Figure 13 requires eight atomic relations and this appears likely to be a minimum. However, the minimum number of atomic relations might not be the most appropriate measure of simplicity for some applications.

A further direction would be an analysis of how more complex patterns of behaviour, such as the encircling illustrated in Figure 2, could be expressed using sequences of atomic relations. In terms of practical applications the identification of these higher-order patterns might be used to model changes in which entities composed of individual people or animals could move with the intent of achieving certain ends.

The use of adjacency trees means that we cannot account for changes of shape to regions which do not affect their topological properties. However in practical applications a less abstract representation would often be required. For example, in the monitoring of spatial change by wireless sensor networks [WD06]. In such a setting regions could be modelled by vertices, edges and faces, and changes might be detected at the level of addition and removal of such components. Not all these changes would induce changes in the adjacency tree, but primitive operations for the changes would be closely related to the Euler operations used in geometric modelling (see for example [ADF85]). Euler operations provide a limited number of actions which are used to construct complex solids in terms of the two dimensional surface bounding a three dimensional solid. The use of the operations ensures that a description in terms of vertices, edges and faces is topologically a valid surface. An implementation of a system for monitoring qualitative spatial change could use similar operations, working at the level of concrete representations of regions.

Our treatment has been purely in terms of trees, but it is natural to ask whether the theory might be extended to more general kinds of graphs. One possibility would be to consider bipartite planar graphs. Moving away from trees seems to require new kinds of atomic change in which an edge may be added or deleted between two nodes in distinct equivalence classes without there being any change to the nodes themselves.

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