UNIVERSITY *of* York

This is a repository copy of *Universal joint-measurement uncertainty relation for error bars*.

White Rose Research Online URL for this paper:
https://eprints.whiterose.ac.uk/3609/

White Rose
university consortium
Universities of Leeds, Sheffield & York

eprints@whiterose.ac.uk
https://eprints.whiterose.ac.uk/

Universal joint-measurement uncertainty relation for error bars

P. Busch[*]

Perimeter Institute for Theoretical Physics, Waterloo, Canada,
and Department of Mathematics, University of York, York, UK

D.B. Pearson[†]

Department of Mathematics, University of Hull, Hull, UK
(Dated: 20 April 2007 (small corrections: 22 June 2007))

We formulate and prove a new, universally valid uncertainty relation for the necessary errors bar widths in any approximate joint measurement of position and momentum.

I. INTRODUCTION

In his seminal paper of 1927 [1], Heisenberg envisaged not one but in fact three conceptually distinct variants of uncertainty relations for position and momentum of the general form

$$\delta q \cdot \delta p \gtrsim h \tag{1}$$

which together comprise the full content of the uncertainty: this relation can be read as describing a trade-off (a) between the widths of the probability distributions of position and momentum in a quantum state; (b) between the inaccuracies of an approximate joint measurement; and (c) between the accuracy of a measurement of (say) position and the ensuing unavoidable disturbance of the momentum (distribution).

The latter two versions have until recently lacked a rigorous formal basis and their universal validity has accordingly been questioned. Here we formulate and prove a form of the joint-measurement uncertainty relation (b) in terms of a new concept of *error bar width*. In [2] it is shown how the inaccuracy-disturbance relation (c) arises as a consequence. Our proof is an adaptation of a strategy recently developed by R. Werner [3] who proved "uncertainty" relations in the spirit of (b) and (c) for a distance measure between observables. In contrast to Werner's geometric measure of distance, our measure of error bar width is modeled in close analogy to the experimental physicists' way of estimating errors. We will also show that the notion of approximation in the sense of finite error bars is more general than that in terms of finite distance.

II. APPROXIMATE MEASUREMENTS AND ERROR BAR WIDTH

A. Preliminaries

Throughout the paper we consider a quantum particle in one spatial dimension, with Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ and canonical position and momentum operators Q, P, defined in the usual way via $(Q\psi)(x) = x\psi(x)$, $(P\psi)(x) = -i\hbar(d\psi/dx)(x)$. Generalizations to more degrees of freedom are straightforward. By Q and P we denote the spectral measures of Q and P, respectively, and $W(q,p) = e^{\frac{i}{2\hbar}qp} e^{-\frac{i}{\hbar}qP} e^{\frac{i}{\hbar}pQ}$ are the Weyl operators which comprise an irreducible unitary projective representation of the translations on phase space $\mathbb{R}^2$. States are represented as positive operators ρ of trace 1, the convex set of all states being denoted S.

Observables are represented as normalized ($E(\Omega) = I$) positive operator measures (POMs) on a measurable space (Ω, Σ), which in the present context will be one of the Borel spaces $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. An observable E is called *sharp* if it is projection valued; otherwise E is an *unsharp* observable. We write ρ^E for the probability measure induced by a state ρ and an observable E via the formula $\rho^E(X) = \operatorname{tr}[\rho E(X)]$, $X \in \Sigma$.

The *overall width* (at confidence level $1 - \varepsilon$) of a probability measure p on $\mathbb{R}$ is defined for $\varepsilon \in [0,1)$ as

$$W_\varepsilon(\mathsf{p}) := \inf\{w > 0 \,|\, \exists x \in \mathbb{R} : \mathsf{p}([x - \tfrac{w}{2}, x + \tfrac{w}{2}]) \geq 1 - \varepsilon\}. \tag{2}$$

---

[*]Electronic address: pb516@york.ac.uk
[†]Electronic address: d.b.pearson@hull.ac.uk

Note that the overall width is finite for any $\varepsilon > 0$.

In analogy to the uncertainty relation for standard deviations, the overall widths of the position and momentum distributions in a state ρ also satisfy a trade-off relation: for positive $\varepsilon_1, \varepsilon_2 > 0$, the inequality

$$W_{\varepsilon_1}(\rho^{\mathsf{Q}}) \cdot W_{\varepsilon_2}(\rho^{\mathsf{P}}) \geq 2\pi\hbar \cdot (1 - \varepsilon_1 - \varepsilon_2)^2 \tag{3}$$

holds for all $\rho \in S$ if $\varepsilon_1 + \varepsilon_2 < 1$. (For $\varepsilon_1 + \varepsilon_2 \geq 1$ there is no positive lower bound for the product on the left hand side.) Uncertainty relations of this form have been obtained by various authors, based on results of [4]. The lower bound given here was obtained in [2] using a simple argument. To our knowledge, the sharpest lower bound known so far is given by Uffink in 1990 [5]:

$$2\pi\hbar \cdot \left(\sqrt{(1 - \varepsilon_1)(1 - \varepsilon_2)} - \sqrt{\varepsilon_1 \varepsilon_2} \right)^2. \tag{4}$$

This term can be substituted for $2\pi\hbar(1 - \varepsilon_1 - \varepsilon_2)^2$ here and in all subsequent applications of (3).

B. Approximate joint measurements

A pair of observables M_1, M_2 on $\mathbb{R}$ is said to be *jointly measurable* if there is an observable M on $\mathbb{R}^2$ of which M_1, M_2 are the marginals ($M_1(X) = M(X \times \mathbb{R}), M_2(Y) = M(\mathbb{R} \times Y)$). Observable M is called a *joint observable* for M_1, M_2.

It is a fundamental fact that pairs of sharp quantum observables are jointly measurable exactly when they commute. However, there are pairs M_1, M_2 of unsharp observables that are mutually noncommuting but do have a joint observable. This opens up the general possibility of defining an *approximate joint measurement* of two noncommuting observables E_1, E_2 as a joint measurement of two observables M_1, M_2 which are approximations of E_1, E_2 in an appropriate sense. The deviation of M_i from E_i will be referred to as *error* or *inaccuracy*.

The notion of an approximate joint measurements of two noncommuting observables E_1 and E_2 draws thus on the idea of deliberately allowing inaccuracy and intrinsic unsharpness, in the hope that one can find approximations M_1 and M_2 to E_1 and E_2 which arise as marginals of some observable M. We will show that for any observable M on phase space the marginals M_1, M_2 cannot be both arbitrarily good approximations to Q, P, respectively. If they are to be approximations, they will also have to be sufficiently unsharp.

C. Error bar width

The following definition of an error measure is guided by the notion of calibrating a measuring instrument by testing it with input states that represent sharp values of the quantity to be measured. This procedure serves to estimate likely error bars.

For simplicity, we give our definitions of approximations only for sharp observables E on $\mathcal{B}(\mathbb{R})$ which are supported on $\mathbb{R}$ (meaning here that $E(J)$ differs from the null operator O for any open interval J), so that the assumption of localized input states can be described as $\rho^E(J_{x;\delta}) = 1$, for any interval $J_{x;\delta} := [x - \delta/2, x + \delta/2]$, $x \in \mathbb{R}, \delta > 0$.

Let E_1 be an observable on $\mathbb{R}$. For each $\varepsilon \in (0, 1)$, $\delta > 0$, we define the *error* of E_1 relative to E

$$\mathcal{W}_{\varepsilon,\delta}(E_1, E) := \inf\{w > 0 \mid \forall \, x \in \mathbb{R} \; \forall \rho \in S : \rho^E(J_{x;\delta}) = 1 \Rightarrow \rho^{E_1}(J_{x,w}) \geq 1 - \varepsilon\}. \tag{5}$$

The error describes the range within which the input values can be inferred from the output distributions, with confidence level $1 - \varepsilon$, given initial localizations within δ.

We say that E_1 is an *ε-approximation* to E if $\mathcal{W}_{\varepsilon,\delta}(E_1, E) < \infty$ for all $\delta > 0$.[14] We note that the error is an increasing function of δ, so that we can define the *error bar width* of E_1 relative to E:

$$\mathcal{W}_{\varepsilon}(E_1, E) := \inf_{\delta} \mathcal{W}_{\varepsilon,\delta}(E_1, E) = \lim_{\delta \to 0} \mathcal{W}_{\varepsilon,\delta}(E_1, E). \tag{6}$$

In case $\mathcal{W}_{\varepsilon,\delta}(E_1, E) = \infty$ for all $\delta > 0$, we write $\mathcal{W}_{\varepsilon}(E_1, E) = \infty$. If $E_1 = E$, then $\mathcal{W}_{\varepsilon}(E_1, E) = 0$ for all $\varepsilon \in (0, 1)$.

E_1 will be called an *approximation* to E if $\mathcal{W}_{\varepsilon}(E_1, E) < \infty$ for all $\varepsilon \in (0, 1)$.[15]

We say that an observable M on $\mathbb{R}^2$ is an *approximate joint observable* of E_1, E_2 if the marginals M_1, M_2 are approximations to E_1, E_2, respectively.

A detailed analysis of these definitions will be given elsewhere [6].

<center>D. Resolution width</center>

As an indicator of the intrinsic unsharpness of an observable E_1 on $\mathcal{B}(\mathbb{R})$, we use the *resolution width* (at confidence level $1 - \varepsilon$), defined as follows [7]:

$$\gamma_\varepsilon(E_1) := \inf\{w > 0 \,|\, \forall x \in \mathbb{R} \,\exists \rho \in S : \; \rho^{E_1}(J_{x;w}) \geq 1 - \varepsilon\}. \tag{7}$$

For a sharp observable E on $\mathcal{B}(\mathbb{R})$ with support $\mathbb{R}$ the resolution width is $\gamma_\varepsilon(E) = 0$ for all $\varepsilon \in (0,1)$.

Proposition 1 *Let E_1, E be observables on $\mathbb{R}$, and E be sharp with support $\mathbb{R}$. The error bar width of E_1 relative to E is never smaller than the resolution width of E_1:*

$$\mathcal{W}_\varepsilon(E_1, E) \geq \gamma_\varepsilon(E_1). \tag{8}$$

Proof. If $\mathcal{W}_\varepsilon(E_1, E) = \infty$, the inequality is trivially satisfied. Assume that $\mathcal{W}_\varepsilon(E_1, E)$ is finite. There is a $\delta_0 > 0$ such that $\mathcal{W}_{\varepsilon,\delta_0}(E_1, E) < \infty$. Since $\mathcal{W}_{\varepsilon,\delta}(E_1, E)$ is an increasing function of δ, we also have $\mathcal{W}_{\varepsilon,\delta}(E_1, E) < \infty$ for $\delta \leq \delta_0$. Let $w \geq \mathcal{W}_{\varepsilon,\delta}(E_1, E)$ for some δ, $0 < \delta \leq \delta_0$. Thus for all $x \in \mathbb{R}$ and all ρ with $\rho^E(J_{x;\delta}) = 1$ we have $\rho^{E_1}(J_{x;w}) \geq 1 - \varepsilon$. This entails (given that the support of E is $\mathbb{R}$) that for all $x \in \mathbb{R}$ there is some ρ such that $\rho^{E_1}(J_{x;w}) \geq 1 - \varepsilon$. Hence $w \geq \gamma_\varepsilon(E_1)$, and therefore $\mathcal{W}_{\varepsilon,\delta}(E_1, E) \geq \gamma_\varepsilon(E_1)$ for all $\delta > 0$, from which (8) follows. ∎

Corollary 1 *Let E be an observable on $\mathcal{B}(\mathbb{R})$ with support $\mathbb{R}$. Any ε-approximation E_1 of E has finite resolution width, $\gamma_\varepsilon(E_1) < \infty$.*

<center>III. UNCERTAINTY RELATIONS FOR PHASE SPACE OBSERVABLES</center>

<center>A. Approximate position and momentum</center>

An important class of candidates of approximate observables for position and momentum are obtained as smearings of Q and P, for example, by means of convolutions with probability measures μ, ν. Thus, observables $\mathsf{Q}_\mu, \mathsf{P}_\nu$ are defined via the weak integrals

$$\begin{aligned} \mathsf{Q}_\mu(X) &= \mathsf{Q} * \mu(X) = \int_{\mathbb{R}} \mu(X + q)\, \mathsf{Q}(dq), \\ \mathsf{P}_\nu(Y) &= \mathsf{P} * \nu(Y) = \int_{\mathbb{R}} \nu(Y + p)\, \mathsf{P}(dp). \end{aligned} \tag{9}$$

These shift-covariant observables will be called *approximate position and momentum*.

Proposition 2 Q_μ *and* P_ν *are approximations to* Q *and* P *for any probability measures* μ *and* ν, *respectively.*

Proof. It suffices to consider the case of Q_μ.

Let $\varepsilon \in (0,1), \delta > 0$ be given. Let q_0, w_0 be such that $\mu(J_{q_0;w_0}) \geq 1 - \varepsilon$. Then, for $w \geq 2|q_0| + w_0 + \delta$, it follows that $J_{q;\delta} \subseteq x + J_{q;w}$ for all $x \in J_{q_0;w_0}$.

Now let $q \in \mathbb{R}$, and let $\rho \in S$ be such that $\rho^{\mathsf{Q}}(J_{q;\delta}) = 1$. Then $\rho^{\mathsf{Q}}(x + J_{q;w}) = 1$ for all $x \in J_{q_0;w_0}$, and therefore:

$$\begin{aligned} \rho^{\mathsf{Q}_\mu}(J_{q;w}) &= \int \mu(dx)\rho^{\mathsf{Q}}(x + J_{q;w}) \\ &\geq \int_{J_{q_0;w_0}} \mu(dx)\rho^{\mathsf{Q}}(x + J_{q;w}) = \mu(J_{q_0;w_0}) \geq 1 - \varepsilon. \end{aligned} \tag{10}$$

∎

Proposition 3 *Observables* Q_μ *and* P_ν *satisfy the following relations:*

$$\mathcal{W}_{\varepsilon_1}(\mathsf{Q}_\mu, \mathsf{Q}) \geq \gamma_{\varepsilon_1}(\mathsf{Q}_\mu) = W_{\varepsilon_1}(\mu), \quad \mathcal{W}_{\varepsilon_2}(\mathsf{P}_\nu, \mathsf{P}) \geq \gamma_{\varepsilon_2}(\mathsf{P}_\nu) = W_{\varepsilon_2}(\nu). \tag{11}$$

Proof. The inequalities are a consequence of Proposition 1. It remains to prove the equalities, which we will do for the case $\gamma_{\varepsilon_1}(\mathsf{Q}_\mu) = W_{\varepsilon_1}(\mu)$.

Assume a positive number w is given such that $w \geq \gamma_{\varepsilon_1}(\mathsf{Q}_\mu)$. Thus, for any $q \in \mathbb{R}$ there is a state ρ with

$$\rho^{\mathsf{Q}_\mu}(J_{q;w}) = \int_{\mathbb{R}} \rho^{\mathsf{Q}}(dq')\mu(J_{q;w} + q') \geq 1 - \varepsilon_1.$$

This shows that it is impossible to have $\mu(J_{q;w} + q') < 1 - \varepsilon_1$ for all q', so that there exists a q' with $\mu(J_{q;w} + q') \geq 1 - \varepsilon_1$. This means that $w \geq W_{\varepsilon_1}(\mu)$. Hence $\gamma_{\varepsilon_1}(\mathsf{Q}_\mu) \geq W_{\varepsilon_1}(\mu)$.

To show the converse inequality, $W_{\varepsilon_1}(\mu) \geq \gamma_{\varepsilon_1}(\mathsf{Q}_\mu)$, let $w > W_{\varepsilon_1}(\mu)$. Then there exists an interval K of length w such that $\mu(K) \geq 1 - \varepsilon_1$. Now let J_q be any interval of length greater than w. Since the length of J_q is greater than the length of K, it follows that the intersection of all intervals $J_q + x$, as x runs over K, is an interval of positive length. This interval, which is contained in $J_q + x$ for all $x \in K$, we denote by J_0.

Let ρ be any state concentrated in J_0, so that $\rho^{\mathsf{Q}}(J_q + x) = 1$ for $x \in K$. From formula (10), this gives $\rho^{\mathsf{Q}_\mu}(J_q) \geq \mu(K) \geq 1 - \varepsilon_1$. Hence $w \geq \gamma_{\varepsilon_1}(\mathsf{Q}_\mu)$. Since $w > W_{\varepsilon_1}(\mu)$ was arbitrary, the required result follows. ■

The question which pairs $\mathsf{Q}_\mu, \mathsf{P}_\nu$ are jointly measurable has a complete answer, proven in [8]: they have to be marginals of a covariant phase space observable.

B. Covariant phase space observables

An observable G on phase space $\mathbb{R}^2$ will be called a *phase space observable* if it satisfies the covariance condition

$$W(q,p)G(Z)W(q,p)^* = G(Z + (q,p)). \tag{12}$$

for all $Z \in \mathcal{B}(\mathbb{R}^2)$.

It is known that all covariant phase space observables are of the form $G = G^{\mathbf{m}}$,

$$\mathcal{B}(\mathbb{R}^2) \ni Z \mapsto G^{\mathbf{m}}(Z) = \frac{1}{2\pi\hbar} \int_Z W(q,p)\mathbf{m}W(q,p)^* dqdp, \tag{13}$$

where the integral is defined weakly and the operator density is generated by an arbitrary fixed positive operator $\mathbf{m}$ of trace 1. This fundamental fact has been proven and extensively studied by several authors using different techniques [9–12].

The marginal observables of $G^{\mathbf{m}}$ are of the form (9), with the probability measures $\mu_{\mathbf{m}} := \mathbf{m}_\Pi^{\mathsf{Q}}$, $\nu_{\mathbf{m}} := \mathbf{m}_\Pi^{\mathsf{P}}$, that is, $G_1^{\mathbf{m}} = \mathsf{Q} * \mu_{\mathbf{m}}$, $G_2^{\mathbf{m}} = \mathsf{P} * \nu_{\mathbf{m}}$. Here $\mathbf{m}_\Pi = \Pi\mathbf{m}\Pi^*$ is the operator obtained from $\mathbf{m}$ under the action of the parity transformation Π ($\Pi\varphi(x) = \varphi(-x)$).

As shown in [8], observables $\mathsf{Q}_\mu, \mathsf{P}_\nu$ are jointly measurable exactly when there is a *covariant* phase space observable $G^{\mathbf{m}}$ of they are the marginals. In that case the resolution widths are given by the widths of the probability measures $\mu_{\mathbf{m}}, \nu_{\mathbf{m}}$ (via Eq. (11)) which obey the uncertainty relation (3); hence,

$$\gamma_{\varepsilon_1}(\mathsf{Q}_{\mu_{\mathbf{m}}}) \cdot \gamma_{\varepsilon_2}(\mathsf{P}_{\nu_{\mathbf{m}}}) = W_{\varepsilon_1}(\mu_{\mathbf{m}}) \cdot W_{\varepsilon_2}(\nu_{\mathbf{m}}) \geq 2\pi\hbar \cdot (1 - \varepsilon_1 - \varepsilon_2)^2 \tag{14}$$

for any $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 + \varepsilon_2 < 1$.

Proposition 4 *Any covariant phase space observable $G^{\mathbf{m}}$ with generating density operator $\mathbf{m}$ is an approximate joint observable for Q, P, with the error bar widths satisfying the joint measurement uncertainty relation*

$$\mathcal{W}_{\varepsilon_1}(\mathsf{Q}_{\mu_{\mathbf{m}}}, \mathsf{Q}) \cdot \mathcal{W}_{\varepsilon_2}(\mathsf{P}_{\nu_{\mathbf{m}}}, \mathsf{P}) \geq 2\pi\hbar \cdot (1 - \varepsilon_1 - \varepsilon_2)^2 \tag{15}$$

for any $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 + \varepsilon_2 < 1$.

Proof. The first statement is a direct consequence of Proposition 2. The inequality follows from Eqs. (11) and (14). ■

IV. UNCERTAINTY RELATIONS FOR GENERAL OBSERVABLES ON PHASE SPACE

An observable M on phase space $\mathbb{R}^2$ is an $(\varepsilon_1, \varepsilon_2)$-*approximate* joint observable of position and momentum if the marginal M_1 is an ε_1-approximation to Q and the marginal M_2 is an ε_2-approximation to P. For later use we state this condition explicitly:
For any $\delta > 0$, there are positive numbers $w, w' < \infty$ such that the following conditions hold:

(α) *for all $q \in \mathbb{R}$ and all $\rho \in \mathcal{S}$, if $\rho^{\mathsf{Q}}(J_{q;\delta}) = 1$, then $\rho^{M_1}(J_{q;w}) \geq 1 - \varepsilon_1$;*

(β) *for all $p \in \mathbb{R}$ and all $\rho \in \mathcal{S}$, if $\rho^{\mathsf{P}}(J_{q;\delta}) = 1$, then $\rho^{M_2}(J_{p;w'}) \geq 1 - \varepsilon_2$.*

Our main result is the following.

Theorem 1 *Let M be an approximate joint observable for Q, P. Then, for $\varepsilon_1, \varepsilon_2 \in (0, 1)$ with $\varepsilon_1 + \varepsilon_2 < 1$, the error bar widths and resolutions widths of M_1 and M_2 satisfy the uncertainty relations*

$$
\begin{aligned}
\mathcal{W}_{\varepsilon_1}(M_1, \mathsf{Q}) \cdot \mathcal{W}_{\varepsilon_2}(M_2, \mathsf{P}) &\geq 2\pi\hbar \cdot (1 - \varepsilon_1 - \varepsilon_2)^2, \\
\gamma_{\varepsilon_1}(M_1) \cdot \gamma_{\varepsilon_2}(M_2) &\geq 2\pi\hbar \cdot (1 - \varepsilon_1 - \varepsilon_2)^2.
\end{aligned}
\tag{16}
$$

The remainder of this section develops the proof of Theorem 1. The proof strategy is adapted from recent work of R. Werner [3] who derived a Heisenberg uncertainty relation for approximate joint measurements of position and momentum in terms of a distance measure between two observables.

We set out to show that if M is an approximate joint observable of Q, P, there is a covariant phase space observable $G^{\mathbf{m}}$ whose resolutions are not worse than those of M, that is, $\mathcal{W}_{\varepsilon_1, \delta}(G_i^{\mathbf{m}}, \mathsf{Q}) \leq \mathcal{W}_{\varepsilon_1, \delta}(M_i, \mathsf{Q})$, $i = 1, 2$. The uncertainty relation (16) was already proven for $G^{\mathbf{m}}$ in Proposition 4.

Following [3], we make use of the concept of the invariant mean on the group of phase space translations to introduce a covariant phase space observable M^{av} associated with any observable M on phase space. The invariant mean is a positive linear functional η on $C(\mathbb{R}^2)$ with the invariance property $\eta(\tau_x f) = \eta(f)$. (Here τ_x, $x = (q, p) \in \mathbb{R}^2$, is the shift map on the space of bounded Borel functions f, so that $\tau_x f(y) = f(y - x)$.) This extends the operation of integrating f over an interval, dividing by the interval length, and letting that length go to infinity. While this operation only works for a very limited class of functions, the existence of η is guaranteed by the axiom of choice.

Any observable M on phase space can be viewed as a linear map from the space $C_{uc}(\mathbb{R}^2)$ of bounded uniformly continuous functions to the bounded operators on $\mathcal{H}$ via $M(f) = \int f(q, p) dM(q, p)$ [3, Lemma 2]. The marginals M_1, M_2 can then equally be defined with respect to functions $f, g \in C_{uc}(\mathbb{R})$ since such function can be extended to the functions $F, G \in C_{uc}(\mathbb{R}^2)$, where $F(q, p) := f(q)$, $G(q, p) := g(p)$; then $M_1(f) := M(F)$ and $M_2(g) := M(G)$.

For a POM M on $\mathcal{B}(\mathbb{R}^2)$, an associated linear map M^{av} is defined via the following equations, required to hold for any $f \in C_{uc}(\mathbb{R}^2)$ and all $\rho \in S$:

$$
\begin{aligned}
\mathrm{tr}\left[\rho M^{\mathrm{av}}(f)\right] &= \eta(u(\rho, f)), \\
u(\rho, f)(q, p) &= \mathrm{tr}\left[W(q, p)\rho W(q, p)^* M(\tau_{(q,p)} f)\right] =: \mathrm{tr}\left[\rho M^{(q,p)}(f)\right].
\end{aligned}
\tag{17}
$$

The covariance of M^{av},

$$
W(q, p) M^{\mathrm{av}}(f) W(q, p)^* = M^{\mathrm{av}}(\tau_{(q,p)} f),
\tag{18}
$$

is an immediate consequence of the invariance of η. The marginals $M_1^{\mathrm{av}}, M_2^{\mathrm{av}}$ are defined according to the prescription given in the preceding paragraph.

In order to apply and check the conditions of an approximate joint measurement to M^{av}, we need to restate the definition in terms of $M(f)$, $f \in C_{uc}(\mathbb{R}^2)$. In fact, we only need to refer to $M_1(f), M_2(g)$ with $f, g \in C_{uc}(\mathbb{R})$. Let χ_J denote the characteristic function of the set J.

Lemma 1 *Let $\varepsilon_1, \varepsilon_2 \in (0, \frac{1}{2})$ be given. An observable M on phase space $\mathbb{R}^2$ is an $(\varepsilon_1, \varepsilon_2)$-approximate joint observable for Q, P if and only if the following conditions hold: for any $\delta > 0$, there are positive finite numbers w, w' such that:*

(α') *for all $q \in \mathbb{R}$, all $f \in C_{uc}(\mathbb{R})$ with $\chi_{J_{q;w}} \leq f \leq 1$ and all ρ with $\rho^{\mathsf{Q}}(J_{q;\delta}) = 1$, one has $\rho^{M_1}(f) \geq 1 - \varepsilon_1$;*

(β') *for all $p \in \mathbb{R}$, all $g \in C_{uc}(\mathbb{R})$ with $\chi_{J_{p;w'}} \leq g \leq 1$ and all ρ with $\rho^{\mathsf{P}}(J_{p;\delta}) = 1$, one has $\rho^{M_2}(g) \geq 1 - \varepsilon_2$.*

Proof. Assume that M is an $(\varepsilon_1, \varepsilon_2)$-approximate joint observable for Q, P. For given δ, there exist $w, w' < \infty$ such that the conditions (α), (β) (formulated just before Theorem 1) hold. Then (α'), (β') follow immediately since due to the monotonicity of M_1 we have $\rho^{M_1}(J_{q;w}) \leq \rho^{M_1}(f) \leq 1$ for any measurable function f with $\chi_{J_{q;w}} \leq f \leq 1$; and similarly for M_2.

Conversely, assume that M is such that for given $\varepsilon_1, \varepsilon_2, \delta$, there exist $w, w' < \infty$ such that (α'), (β') hold. We show that (α), (β) hold. It suffices to consider the case of (α') implying (α).

For each $q \in \mathbb{R}$, the functions $f \in C_{uc}(\mathbb{R})$ with $\chi_{J_{q;w}} \leq f \leq 1$ form a decreasingly directed set which converges to $\chi_{J_{q;w}}$. In fact, one can easily construct a decreasing sequence of uniformly continuous functions f_n with $\chi_{J_{q;w}} \leq f_n \leq 1$

and support in $[q - \delta/2 - 1/n, q + \delta + 1/n]$ that converges to $\chi_{J_{q;w}}$. It follows that for every ρ, the sequence of numbers $\rho^{M_1}(f_n) \to \rho^{M_1}(J_{q;w})$ as $n \to \infty$. (See [13, Theorem 11.(iii)].) Since for all ρ with $\rho^{\mathsf{Q}}(J_{q;\delta}) = 1$ we have $\rho^{M_1}(f_n) \geq 1 - \varepsilon_1$, then also $\text{tr}\,[\rho M_1(J_{q;w})] \geq 1 - \varepsilon_1$ for such ρ. ∎

Lemma 2 *Let M be an $(\varepsilon_1, \varepsilon_2)$-approximate joint observable for Q, P. Then the covariant linear map M^{av} obtained from M satisfies the conditions described in the preceding Lemma for the given $\varepsilon_1, \varepsilon_2$.*

Proof. It suffices to consider the statement for M_1^{av}, that is: we show that for any $\varepsilon_1 \in (0,1)$, $\delta > 0$, there is a positive $w < \infty$ such that (α') holds for M_1^{av}.

Thus, given $\varepsilon_1 \in (0,1)$, $\delta > 0$, there is $w < \infty$ such that (α') holds for M_1. Now note that for $f \in C_{uc}(\mathbb{R})$ with $\chi_{J_{q;w}} \leq f \leq 1$ the function F on $\mathbb{R}^2$, defined by $F(q,p) = f(q)$, is also uniformly continuous and satisfies $\chi_{J_{q;w} \times I} \leq F \leq 1$ and $M(F) = M_1(f)$. Then the property (α') can be expressed equivalently as follows: for all $q \in \mathbb{R}$, all $F \in C_{uc}(\mathbb{R}^2)$ with $\chi_{J_{q;w} \times \mathbb{R}} \leq F \leq 1$ and all ρ with $\rho^{\mathsf{Q}}(J_{q;\delta}) = 1$, we have $\rho^M(F) \geq 1 - \varepsilon_1$.

Consider the terms

$$\text{tr}\left[\rho M^{(q',p')}(F)\right] = \text{tr}\left[\rho W(q',p')^* M(\tau_{(q'p')}F)W(q',p')\right]$$
$$= \text{tr}\left[W(q',p')\rho W(q',p')^* M(\tau_{(q'p')}F)\right]$$

for any state ρ, any $(q', p') \in \mathbb{R}^2$, and any $F \in C_{uc}(\mathbb{R}^2)$. If F runs through all such functions satisfying $\chi_{J_{q;w} \times \mathbb{R}} \leq F \leq 1$, and ρ is any state with $\rho^{\mathsf{Q}}(J_{q;\delta}) = 1$, then $\tau_{(q',p')}F$ runs through all uniformly continuous functions with the property $\chi_{\tau_{(q',p')}J_{q;w} \times \mathbb{R}} \leq \tau_{(q',p')}F \leq 1$, and $W(q',p')\rho W(q',p')^*$ runs through all states localized in $J_{q+q';\delta}$.

We can thus conclude that the functions $u(\rho, F)$ used in (17) to define M^{av} satisfy $u(\rho, F)(q', p') \geq 1 - \varepsilon_1$, and therefore $\text{tr}\left[\rho M^{\text{av}}(F)\right] \geq 1 - \varepsilon_1$ for all uniformly continuous F with $\chi_{J_{q;w} \times \mathbb{R}} \leq F \leq 1$ and all ρ localized in $J_{q;\delta}$. ∎

We will show that under the assumptions of Theorem 1 for M, which are now seen to apply to M^{av} in the form described in Lemma 2, the functional M^{av} extends to a normalized POM which is thus a covariant phase space observable, and which inherits the property of being an approximate joint measurement. According to [3, Lemma 3], these results will follow if M^{av} can be shown to have zero weight at infinity.

The set of operators

$$\{M^{\text{av}}(f) \,:\, f \in C_{uc}(\mathbb{R}^2), \; f \text{ has compact support}, \; 0 \leq f \leq 1\}$$

forms an increasingly directed net with upper bound $M^{\text{av}}(1)$, so that there is a supremum which we denote $I - M^{\text{av}}(\infty)$. We have to show that $M^{\text{av}}(\infty) = O$, that is, the supremum of the above set is the unit operator $I = M^{\text{av}}(1)$. (This is the statement that the functional M^{av} has zero weight at infinity.) According to part 2 of Lemma 2 in [3], this follows if one can show that $M_1(\infty) = M_2(\infty) = O$ (where these operators are similarly defined).

Lemma 3 *Let M be an approximate joint observable for Q, P, with associated covariant M^{av}. Then the associated linear maps $M_1^{\text{av}}, M_2^{\text{av}}$ have zero weight at infinity, in the following sense: for all $\rho \in S$,*

$$\sup\{\text{tr}\,[\rho M_i^{\text{av}}(f)] \,:\, f \in C(\mathbb{R}), \; f \text{ has compact support}, \; 0 \leq f \leq 1\} = 1. \tag{19}$$

Thus $M_1^{\text{av}}(\infty) = M_2^{\text{av}}(\infty) = O$ and therefore $M^{\text{av}}(\infty) = O$.

Proof. It is sufficient to carry out the proof for M_1^{av}, using the fact that M^{av} is also an approximate joint observable. Let ρ be any state. Let $\varepsilon_1 \in (0,1)$ be given. We have to show that there is a nonnegative function $f \in C_{uc}(\mathbb{R})$, $0 \leq f \leq 1$, with compact support such that $\text{tr}\,[\rho M_1^{\text{av}}(f)] \geq 1 - \varepsilon_1$.

We show this first for ρ with $\rho^{\mathsf{Q}}(J_{q;\delta}) = 1$ for some q, δ. In that case, given $\varepsilon_1 \in (0,1)$, there is a positive finite w and a function $f \in C_{uc}(\mathbb{R})$ having compact support with $\chi_{J_{q;w}} \leq f \leq 1$ such that $\text{tr}\,[\rho M_1^{\text{av}}(f)] \geq 1 - \varepsilon_1$. Thus Eq. (19) holds.

Now consider any state ρ. Let $J_N = [-N, N]$, put $\mathsf{Q}_N = \mathsf{Q}(J_N)$. Then, since Q_N converges to I ultraweakly, we have eventually $\text{tr}\,[\rho \mathsf{Q}_N] \neq 0$, and we can define $\rho_N = \mathsf{Q}_N \rho \mathsf{Q}_N / \text{tr}\,[\rho \mathsf{Q}_N]$. Then $\rho - \rho_N \to O$ in trace norm. (Write $\mathsf{Q}'_N = I - \mathsf{Q}_N$ and $\rho = \mathsf{Q}_N \rho \mathsf{Q}_N + \mathsf{Q}'_N \rho \mathsf{Q}'_N + \mathsf{Q}_N \rho \mathsf{Q}'_N + \mathsf{Q}'_N \rho \mathsf{Q}_N$. For any effect F, we can estimate:

$$|\text{tr}\,[(\rho - \rho_N)F]| \leq \left|\frac{1}{\text{tr}\,[\rho \mathsf{Q}_N]} - 1\right| \text{tr}\,[\mathsf{Q}_N \rho \mathsf{Q}_N F] + |\text{tr}\,[\mathsf{Q}'_N \rho \mathsf{Q}'_N F]|$$
$$+ |\text{tr}\,[\mathsf{Q}_N \rho \mathsf{Q}'_N F]| + |\text{tr}\,[\mathsf{Q}'_N \rho \mathsf{Q}_N F]|$$
$$\leq \left|\frac{1}{\text{tr}\,[\rho \mathsf{Q}_N]} - 1\right| \text{tr}\,[\mathsf{Q}_N \rho \mathsf{Q}_N] + |\text{tr}\,[\mathsf{Q}'_N \rho \mathsf{Q}'_N]|$$
$$+ 2\left(\text{tr}\,[F^2 \mathsf{Q}_N \rho \mathsf{Q}_N]\right)^{1/2} \left(\text{tr}\,[\rho \mathsf{Q}'_N]\right)^{1/2}$$
$$\leq 2\text{tr}\,[\rho \mathsf{Q}'_N] + 2\left(\text{tr}\,[\rho \mathsf{Q}'_N]\right)^{1/2}.$$

In the second line we have used the Cauchy-Schwarz inequality for Hilbert-Schmidt operators and $O \leq F \leq I$, and in the last line we used $O \leq F^2 \leq I$. All terms in the last line tend to 0 as $N \to \infty$ (since $\text{tr}\,[\rho Q'_N] \to 0$), and their sum is an upper bound for the l.h.s. for all effects F. Since $\rho - \rho_N$ has zero trace, the trace norm is given by $\|\rho - \rho_N\|_{\text{tr}} = 2 \sup_{O \leq F \leq I} |\text{tr}\,[(\rho - \rho_N)F]|$, and this tends to zero as $N \to \infty$.)

Given $\varepsilon_1 \in (0,1)$ and $\rho \in S$, choose N such that $\|\rho - \rho_N\|_{\text{tr}} \leq \varepsilon_1/2$. We know that for ρ_N there is a uniformly continuous f_N with $0 \leq f_N \leq 1$ such that $\text{tr}\,[\rho_N M_1^{\text{av}}(f_N)] \geq 1 - \varepsilon_1/2)$. Then $\text{tr}\,[\rho M_1^{\text{av}}(f_N)] \geq \text{tr}\,[\rho_N M_1^{\text{av}}(f_N)] - \varepsilon_1/2 \geq 1 - \varepsilon_1$. ∎

We summarize the above considerations:

Lemma 4 *Let M be an approximate joint observable for Q, P. The associated M^{av} extends to a covariant phase space observable of the form (13), denoted again M^{av}, and this is in turn an approximate joint observable for Q, P with*

$$\mathcal{W}_{\varepsilon_1,\delta}(M_1, Q) \geq \mathcal{W}_{\varepsilon_1,\delta}(M_1^{\text{av}}, Q) \geq \mathcal{W}_{\varepsilon_1}(M_1^{\text{av}}, Q),$$
$$\mathcal{W}_{\varepsilon_2,\delta}(M_2, P) \geq \mathcal{W}_{\varepsilon_2,\delta}(M_2^{\text{av}}, P) \geq \mathcal{W}_{\varepsilon_2}(M_2^{\text{av}}, P). \tag{20}$$

Proof. It remains to verify the inequalities, and here it suffices to show that $\mathcal{W}_{\varepsilon_1,\delta}(M_1, Q) \geq \mathcal{W}_{\varepsilon_1,\delta}(M_1^{\text{av}}, Q)$.

Let $\varepsilon_1 \in (0,1), \delta > 0$ be given. Let w be a positive finite number such that for any $q \in \mathbb{R}$ and all ρ with $\rho^Q(J_{q;\delta}) = 1$, we have $\rho^{M_1}(J_{q;w}) \geq 1 - \varepsilon_1$. We conclude that for any $F \in C_{uc}(\mathbb{R}^2)$ with $\chi_{J_{q;w} \times \mathbb{R}} \leq F \leq 1$, we obtain $\rho^M(F) \geq 1 - \varepsilon_1$, and therefore, following the reasoning of the proof of Lemma 2, also $\rho^{M^{\text{av}}}(F) \geq 1 - \varepsilon_1$. Since these functions F form a decreasingly directed set converging to $\chi_{J_{q;w} \times \mathbb{R}}$, it follows also that $\rho^{M_1^{\text{av}}}(J_{q;w}) \geq 1 - \varepsilon_1$.

So we have shown that $w \geq \mathcal{W}_{\varepsilon_1,\delta}(M_1, Q)$ implies $w \geq \mathcal{W}_{\varepsilon_1,\delta}(M_1^{\text{av}}, Q)$. ∎

Since M^{av} is a covariant phase space observable, Proposition 4 applies and we have the measurement uncertainty relation (15) for M^{av}. The inequalities (20) finally yield the general uncertainty relation for error bars (16).

The inequality (16) for resolution widths follows similarly as a consequence of the inequalities

$$\gamma_{\varepsilon_1}(M_1) \geq \gamma_{\varepsilon_1}(M_1^{\text{av}}), \quad \gamma_{\varepsilon_2}(M_2) \geq \gamma_{\varepsilon_2}(M_2^{\text{av}}), \tag{21}$$

the proof of which is analogous to the argument in the proof of Lemma 4 and will thus be omitted. Theorem 1 is thus proven.

An investigation of the scope and applications of this result will be given elsewhere [6]. Here we conclude with a comparison of the present approach with that of R. Werner [3] from which we have adopted the proof strategy for our Theorem 1. Werner defines a distance $d(E_1, E_2)$ on the set of observables on $\mathbb{R}$ as follows.

First recall that for any bounded measurable function $h : \mathbb{R} \to \mathbb{R}$, the integral $\int_{\mathbb{R}} h \, dE$ defines (in the weak sense) a bounded selfadjoint operator, which we denote by $E[h]$. Thus, for any vector state φ the number $\langle \varphi | E[h]) \varphi \rangle = \int_{\mathbb{R}} h \, d\mathsf{p}_\varphi^E$ is well-defined.

Denoting by Λ the set of bounded measurable functions $h : \mathbb{R} \to \mathbb{R}$ for which $|h(x) - h(y)| \leq |x - y|$, the distance between the observables E_1 and E_2 is defined as

$$d(E_1, E_2) := \sup_{\rho \in S} \sup_{h \in \Lambda} |\text{tr}\,[\rho E_1[h]] - \text{tr}\,[\rho E_2[h]]|. \tag{22}$$

Werner proved the following joint-measurement uncertainty relation, valid for any observable M on phase space with marginals M_1, M_2:

$$d(M_1, Q) \cdot d(M_2, P) \geq C\hbar. \tag{23}$$

The tightest lower bound for the product of distances can be determined within the class of covariant phase space observables and has a value of approximately 0.3047.

We show that the condition of finite distance is stricter than that of finite error bar width.

Proposition 5 *Any observable E_1 on $\mathbb{R}$ that satisfies the condition $d(E_1, E) < \infty$ for a sharp observable E on $\mathbb{R}$ is an approximation to E in the sense of finite error bars. In that case the following inequality holds:*

$$\mathcal{W}_\varepsilon(E_1, E) \leq \frac{2}{\varepsilon} d(E_1, E). \tag{24}$$

Proof. We are given that

$$\left| \text{tr}\,[\rho E_1(h)] - \text{tr}\,[\rho E(h)] \right| \leq d(E_1, E) =: c \quad \text{for all } \rho \in S, \; h \in \Lambda. \qquad (+)$$

Let $\varepsilon \in (0,1)$ and $\delta > 0$ be given. Put $w = \delta + 2n$, with $n \in \mathbb{N}$, $n \geq c/\varepsilon$. Consider an interval $J_{q;\delta}$ and a state ρ with $\rho^E(J_{q;\delta}) = 1$. Define the functions h_n via

$$h_n(x) := \begin{cases} n & \text{if } |x - q| \leq \delta/2; \\ n + \delta/2 - |x - q| & \text{if } \delta/2 < |x - q| \leq \delta/2 + n; \\ 0 & \text{if } \delta/2 + n < |x - q|. \end{cases}$$

Note that $h_n \in \Lambda$. Condition (+) for h_n entails for $g_n = h_n/n$ that $\left|\rho^{E_1}(g_n) - \rho^E(g_n)\right| \leq c/n$. We then have $\chi_{J_{q;\delta}} \leq g_n \leq \chi_{J_{q;w}}$.

Now $\rho^E(J_{q;\delta}) = 1$ implies $\operatorname{tr}[\rho E(g_n)] = 1$, and so, using the assumption $n \geq c/\varepsilon$, we obtain

$$\operatorname{tr}[\rho E_1(J_{q;w})] \geq \operatorname{tr}[\rho E_1(g_n)] \geq \operatorname{tr}[\rho E(g_n)] - c/n \geq 1 - \varepsilon.$$

To prove the inequality (24), we note that on putting $w = \delta + 2c/\varepsilon$, one still obtains $\operatorname{tr}[\rho E_1(J_{q;w})] \geq 1 - \varepsilon$. This yields $\mathcal{W}_{\varepsilon,\delta}(E_1, E) \leq \delta + 2d(E_1, E)/\varepsilon$, and on letting δ approach 0, then (24) follows. ■

An immediate consequence of Eqs. (24) and (11) for an approximate position observable Q_μ is the following:

$$W_{\varepsilon_1}(\mu) = \gamma_{\varepsilon_1}(\mathsf{Q}_\mu) \leq \mathcal{W}_{\varepsilon_1}(\mathsf{Q}_\mu, \mathsf{Q}) \leq \frac{2}{\varepsilon_1}\, d(\mathsf{Q}_\mu, \mathsf{Q}). \tag{25}$$

This gives a bound on the resolution width of Q_μ and on the overall width of the unsharpness measure μ, showing the behaviour of these quantities as $\varepsilon_1 \to 0$.

There are instances of joint measurements for which Werner's distances are infinite while the error bar widths are finite. This can be seen in the case of covariant phase space observables where the relevant distance between (say) the marginal $\mathsf{Q}_{\mu_{\mathbf{m}}}$ and Q is $d(\mathsf{Q}_{\mu_{\mathbf{m}}}, \mathsf{Q}) = \int |q| \mu_{\mathbf{m}}(dq)$ (see [3]).

Finally we note that there exist non-covariant observables on phase space which are approximate joint observables for Q and P. An example is $M := G^{\mathbf{m}} \circ \gamma^{-1}$, where $\gamma = (\gamma_1, \gamma_2)$ is a bijective measurable map of $\mathbb{R}^2$ onto itself; M is an approximate joint observable if $\gamma_1(q) - q$ and $\gamma_2(p) - p$ are bounded functions, and M is non-covariant if γ_1 or γ_2 is not an affine map (see [6] for details).

V. CONCLUSION

We have introduced an operationally significant and experimentally relevant criterion, based on the new concept of error bar width, of what constitutes an approximate joint observable of position and momentum. The associated error bar widths obey a Heisenberg uncertainty relation. This shows that the approximations of position and momentum in terms of marginals of an observable on phase space cannot both be arbitrarily good.

We also considered the resolution width as an indicator of the degree of intrinsic unsharpness. It was found that the resolution widths of the marginals of any approximate joint observable for position and momentum cannot both be arbitrarily small but must obey a Heisenberg uncertainty relation.

[1] W. Heisenberg, Z. Phys. **43**, 172 (1927).
[2] P. Busch, T. Heinonen, and P. Lahti, quant-ph/0609185 (2006).
[3] R. Werner, Qu. Inf. Comp. **4**, 546 (2004).
[4] H. Landau and H. Pollak, Bell System Tech. J. **40**, 65 (1961).
[5] J.B.M. Uffink, *Measures of Uncertainty and the Uncertainty Principle*, PhD Thesis, University of Utrecht (1990).
[6] P. Busch and D. Pearson, *Inaccuracy and unsharpness in approximate joint measurements of position and momentum*, in preparation (2007).
[7] C. Carmeli, T. Heinonen, and A. Toigo, J. Phys. A **40**, 1303 (2007).
[8] C. Carmeli, T. Heinonen, and A. Toigo, J. Phys. A **38**, 5253 (2005).
[9] A. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland Publishing Co., Amsterdam, 1982).
[10] R. Werner, J. Math. Phys. **25**, 1404 (1984).
[11] G. Cassinelli, E. De Vito, and A. Toigo, J. Math. Phys. **44**, 4768 (2003).
[12] J. Kiukas, P. Lahti, and K. Ylinen, J. Math. Anal. Appl. **319**, 783 (2006).

[13] S. Berberian, *Notes on Spectral Theory* (D. Van Nostrand Company, Princeton, New Jersey, 1966).

[14] The fact that this condition is required for *all* δ reflects the idea that calibrations at confidence level $1 - \varepsilon$ should be valid on all scales.

[15] One will only consider E_1 to be an approximation of E if for a given input distribution ρ^E supported within $J_{x;\delta}$ the output distribution ρ^{E_1} is *concentrated* around the set $J_{x;\delta}$, that is, assigns probability greater than $1/2$ to some interval $J_{x;w}$. This means that the definition of an ε-approximation is only of interest for $\varepsilon \in (0, 1/2)$. However, since the quantity $\mathcal{W}_\varepsilon(E_1, E)$ is a decreasing function of ε, the defining condition would still be satisfied for all $\varepsilon \in (0, 1)$ if it holds for $\varepsilon \in (0, 1/2)$.