Article:

Wei, H.L., Lang, Z.Q. and Billings, S.A. (2007) An algorithm for determining the output frequency range of Volterra models with multiple inputs. Transactions on Circuits and Systems II-Express Briefs, 54 (6). pp. 532-536. ISSN 1057-7130

https://doi.org/10.1109/TCSII.2007.894426

An Algorithm for Determining the Output Frequency Range of Volterra Models With Multiple Inputs

Hua-Liang Wei, Zi-Qiang Lang, and Stephen A. Billings

***Abstract*—A new algorithm for determining the output frequency range and the frequency components of Volterra models under multiple inputs is introduced for nonlinear system analysis. For a given Volterra model, the output frequency components corresponding to a multi-tone input can easily be calculated using the new algorithm.**

***Index Terms*—Generalized frequency response functions (GFRF), nonlinear systems, output spectrum, Volterra models.**

I. Introduction

ONE important aspect of system analysis in the frequency domain is the requirement to investigate the relationship between the system input frequencies and the output frequency behaviour. For linear systems, the output frequency function $Y(j\omega)$ is related to the input frequency spectrum $U(j\omega)$ by the system frequency response function $H(j\omega)$ via the simple linear relationship $Y(j\omega) = H(j\omega)U(j\omega)$. This simple basic result provides the foundation for all linear system analysis and design in the frequency domain. In this case, the input frequencies pass independently through the system, that is, an input at a given frequency ω produces at steady state an output at the same frequency and no energy is transferred to or from any other frequency components. The system frequency response function $H(j\omega)$ itself alone can totally characterize a given linear system. For nonlinear systems, however, this is not true. It has been observed that the output frequency components of nonlinear systems are much richer compared to the corresponding input frequencies. The input frequencies pass in a coupled way through a nonlinear system, that is, an input at given frequencies may produce quite different output frequencies. This is quite different from the case for linear systems where the output frequency range is identical in steady state to that of the inputs. This makes it difficult to give a general explicit expression connecting the input and output frequencies for most nonlinear systems.

One of the most useful representations for weakly nonlinear systems is the Volterra model. Foundations of this class of models can be found in [1]–[4], where most significant early work in this area has been summarized in detail. One property of the Volterra model is that the output frequency range can analytically be determined for some specific inputs, and several explicit algorithms are available for this purpose [2], [5]–[10]. In [2] a non-iterative method was proposed by introducing the concept of "frequency-mix vector," to determine, in an enumerative way, the output frequency components of a Volterra model driven by multitone inputs. But this method can only be used to effectively calculate the output frequencies of low-order nonlinear functions driven by multiple inputs, where the number of fundamental input components is not large. For a case where the number of the fundamental input components is large, the determination of the output frequencies becomes very complex and makes the relevant implementation very difficult, if not impossible. An iterative algorithm was given in [8], but the expression of the algorithm is not compact, and the construction of relative matrices becomes complicated when high-order submodels are involved and the number of fundamental input components is large.

This study presents a new and much simpler algorithm for the determination of the output frequency components for Volterra models under multitone inputs. As will been seen, the new proposed algorithm is very useful for the determination of the output frequency components of arbitrary-order submodels in any given Volterra model.

II. Generalized Frequency Response Functions for Nonlinear Systems

It is well known that the input–output relationship of a wide class of nonlinear systems can be approximated in the time domain by the Volterra functional series [1]–[4]

$$y(t) = \sum_{n=1}^{L} y_n(t) \tag{1}$$

where the system output $y(t)$ is expressed as a sum of the response of L parallel subsystems, each of which is related to both the system input $u(t)$ and an nth-order kernel. The output of the nth-order nonlinear subsystem, $y_n(t)$, is characterized by an extension of the familiar convolution integral of linear systems theory to higher dimensions

$$\begin{aligned} y_n(t) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) u(t-\tau_1) \cdots u(t-\tau_n) d\tau_1 \cdots d\tau_n \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} [u(t-\tau_i) d\tau_i] \end{aligned} \tag{2}$$

where the nth-*order kernel* or nth-*order impulse response* $h_n(\tau_1, \cdots, \tau_n)$ is so called because this reduces to the linear impulse response function for the simplest case $n = 1$. By introducing the concept of the nth-*order associated function*

Manuscript received September 6, 2006; revised November 28, 2006. This work was supported by the Engineering and Physical Sciences Research Council, U.K. (EPSRC-UK). This paper was recommended by Associate Editor A. Vicino.

The authors are with the Department of Automatic Control and Systems Engineering, University of Sheffield, Sheffield, S1 3JD, U.K. (e-mail: s.billings@shef.ac.uk; z.lang@sheffield.ac.uk; w.hualiang@shef.ac.uk).

Digital Object Identifier 10.1109/TCSII.2007.894426

[11] and then taking the multidimensional Fourier transform of the associated function, yields from (2)

$$\hat{Y}_n(j\omega_1,\cdots,j\omega_n)=H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i) \quad (3)$$

where $U(\cdot)$ is the input spectrum defined as the Fourier transform operator. $H_n(j\omega_1,\cdots,j\omega_n)$ is the nth-*order transfer function* or nth-*order generalized frequency response function* (GFRF) defined as

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\times e^{-j(\omega_1\tau_1+\cdots+\omega_n\tau_n)}d\tau_1\cdots d\tau_n. \quad (4)$$

Following [9] and [12], it can easily be shown that

$$y_n(t)=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\hat{Y}_n(j\omega_1,\cdots,j\omega_n)\times e^{j(\omega_1+\cdots+\omega_n)t}d\omega_1\cdots d\omega_n. \quad (5)$$

By making a change of variables

$$\begin{cases}\sigma_i=\omega_i, & 1\le i\le n-1\\ \sigma_n=\sum_{i=1}^{n}\omega_i\end{cases}. \quad (6)$$

Equation (5) becomes the equation shown at the bottom of the page, where

$$Y_n(j\omega)=\frac{1}{(2\pi)^{n-1}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\hat{Y}_n\left(j\omega_1,\cdots,j\omega_{n-1},j(\omega-\omega_1-\cdots-\omega_{n-1})\right)\times d\omega_1\cdots d\omega_{n-1}. \quad (8)$$

From (1) and (7)

$$y(t)=\sum_{n=1}^{L}\frac{1}{2\pi}\int_{-\infty}^{\infty}Y_n(j\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_{n=1}^{L}Y_n(j\omega)\right]e^{j\omega t}d\omega \quad (9)$$

Therefore, the system output frequency response or output spectrum to a given general input $u(t)$ is

$$Y(j\omega)=\sum_{n=1}^{L}Y_n(j\omega),\quad \omega\in\bigcup_{n=1}^{L}\Omega_n \quad (10)$$

where Ω_n is the effective frequency domain of the nth-order output frequency function $Y_n(j\omega)$. The family $\{\omega_1,\cdots,\omega_n;\omega\}$ in (8) was referred to as the *input–output frequency domain* in [11]. The output spectrum $Y_n(j\omega)$ can therefore be referred to as the nth-*order output frequency (response) function* or *output spectrum*. For a physical interpretation of (5) and (8), see [8], [11]. Note from the variable transform (6) that the input–output frequency domain is restricted to $\omega_1+\cdots+\omega_n=\omega$. The valid frequency range of the output spectrum can therefore be determined provided that the input frequencies are known.

III. Determining Output Frequencies Under Multiple Inputs

This section presents a useful result on calculating the output frequencies of nonlinear systems which can be described by the Volterra series.

A. Description of Output Frequencies

As a simple example, consider a simple case, where a nonlinear system is driven by a sinusoidal signal

$$u(t)=A\cos(\omega_0 t)=\frac{A}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t}). \quad (11)$$

Substituting (11) into (2), yields [12], [13]

$$y_n(t)=\left(\frac{A}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}H_n(\underbrace{j\omega_0,\cdots,j\omega_0}_{n-k},\underbrace{-j\omega_0,\cdots,-j\omega_0}_{k})\times e^{j(n-2k)\omega_0 t}. \quad (12)$$

From (12), the input to the nth-order submodel $y_n(t)$ contains only one single principal frequency component ω_0, the output of the nth-order submodel $y_n(t)$, however, contains many frequency components distributed at $\pm n\omega_0,\pm(n-2)\omega_0,\pm(n-4)\omega_0,\cdots$. For example, for the linear submodel of the nonlinear system (1), the output frequencies include $\pm\omega_0$; for the 2nd-order nonlinear subsystem, the output frequencies will appear at 0 and $\pm 2\omega_0$.

$$\begin{aligned}y_n(t)&=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\hat{Y}_n\left(j\sigma_1,\cdots,j\sigma_{n-1},j(\sigma_n-\sigma_1-\cdots-\sigma_{n-1})\right)\times e^{j\sigma_n t}d\sigma_1\cdots d\sigma_n\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{(2\pi)^{n-1}}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n-1}\hat{Y}_n\left(j\omega_1,\cdots,j\omega_{n-1}j\left(\omega-\sum_{i=1}^{n-1}\omega_i\right)\right)d\omega_1\cdots d\omega_{n-1}\right]\times e^{j\omega t}d\omega\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}Y_n(j\omega)e^{j\omega t}d\omega(7)\end{aligned}$$

For a general case, where the input is a summation of multiple sinusoidal waves

$$u(t) = \sum_{i=1}^{K} A_i \cos(\omega_i t) = \sum_{i=-K}^{K} \frac{A_i}{2} e^{j\omega_i t} \tag{13}$$

with $\omega_0 = 0, \omega_{-i} = -\omega_i, A_0 = 0, A_{-i} = A_i$, the output of the nth-order submodel $y_n(t)$ can be calculated to be [8]

$$y_n(t) = \frac{1}{2^n} \sum_{k_1=-K}^{K} \cdots \sum_{k_n=-K}^{K} B(\omega_{k_1}) \cdots B(\omega_{k_n}) H_n(\omega_{k_1}, \cdots, \omega_{k_n}) e^{j(\omega_{k_1}+\cdots+\omega_{k_n})t} \tag{14}$$

where

$$B(\omega) = \begin{cases} A_k, & \omega \in \{\omega_k : k = \pm 1, \cdots, \pm K\} \\ 0, & \text{otherwise} \end{cases}. \tag{15}$$

Following [8], the nth-order output frequency function $Y_n(j\omega)$ can be expressed as

$$Y_n(j\omega) = \frac{1}{2^{n-1}} \underbrace{\sum_{k_1=-K}^{K} \cdots \sum_{k_n=-K}^{K}}_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} B(\omega_{k_1}) \cdots B(\omega_{k_n}) H_n(\omega_{k_1}, \cdots, \omega_{k_n}). \tag{16}$$

The output frequency components of the nth-order submodel $y_n(t)$ will be much richer compared with the input frequency components since each frequency component ω determined by the combination $\omega = \sum_{i=1}^{n} \omega_{k_i}$ with $k_i \in \{\pm 1, \pm 2, \cdots, \pm K\}$ might appear in the output frequency domain. An important point is that these possible output frequency components can be determined beforehand once the frequency components in the multiple input are given.

B. An Algorithm for Determining the Output Frequencies

It is observed that the output frequency components of nonlinear systems are much richer compared to the corresponding input frequencies. The input frequencies will pass in a coupled way through a nonlinear system, that is, an input at given frequencies may produce quite different output frequencies. Therefore, energy may be transferred to or from other frequency components. This is quite different from the case for linear systems where the output frequency range is identical in steady state to that of the input. It would be difficult to give a general explicit expression connecting the input and output frequencies for all nonlinear systems. However, for some specified inputs, explicit algorithms are available to determine the effective frequency range for arbitrary-order output frequency response functions. In [8], an algorithm to compute the frequency range of the nth-order output frequency function $Y_n(j\omega)$ defined by (7) and (8).

It can be noted, however, that the existing algorithms are complicated in either the expression of the formulae or the iteration and calculation procedure. This may not be convenient for practical applications. Motivated by this observation, this study proposes a much improved and compact recursive algorithm for calculating the effective frequencies of arbitrary-order output frequency functions. The new algorithm is derived and formulated in an iterative manner that is significantly different from that of the existing algorithms. An important advantage of the new algorithm is that it is very simple in form, quite easy to calculate, and produces exactly the same results as those produced by existing algorithms.

From the variable transform (6) and the derivations of (7), (8) and (16), the input and output frequencies for the nth-order subsystem with a multiple input of the form (13) will be constrained by

$$\omega = \sum_{i=1}^{n} \omega_{k_i}, \qquad k_i \in \{\pm 1, \pm 2, \cdots, \pm K\}. \tag{17}$$

This will be used to determine the frequency range of the nth-order output frequency function. For convenience of description, denote

$$\begin{cases} \sigma_1 = -\omega_K \\ \quad\cdots \\ \sigma_K = -\omega_1 \\ \sigma_{K+1} = \omega_1 \\ \quad\cdots \\ \sigma_{2K} = \omega_K. \end{cases} \tag{18}$$

For the simplest case of $n = 1$, it is clear that the effective frequency range of the output spectrum is $\omega \in \Omega_1 = \{\sigma_k : k = 1, 2, \cdots, 2K\} = \{\omega_k : \pm 1, \cdots, \pm K\}$.

In order to determine the effective frequency range Ω_2 for the case of $n = 2$, consider the following combinations of two frequency components

$$\begin{cases} \sigma_1 + \sigma_1 \\ \quad\vdots \\ \sigma_1 + \sigma_{2K} \\ \quad\vdots \\ \sigma_{2K} + \sigma_1 \\ \quad\vdots \\ \sigma_{2K} + \sigma_{2K}. \end{cases} \tag{19}$$

This can be expressed in a vector form as

$$\Gamma_2 = \begin{bmatrix} \sigma_1 I_{2K} + V \\ \vdots \\ \sigma_{2K} I_{2K} + V \end{bmatrix} = V \otimes I_{2K} + I_{2K} \otimes V \tag{20}$$

where $I_{2K} = \underbrace{[1, \cdots, 1]}_{2K}^T$, $V = [\sigma_1, \cdots, \sigma_{2K}]^T$. The symbol "$\otimes$" denotes the Kronecker product, which is defined for two vectors $A = [a_1, \cdots, a_p]^T$ and $B = [b_1, \cdots, b_q]^T$ as

$$A_{p\times 1} \otimes B_{q\times 1} = \begin{bmatrix} a_1 B \\ a_2 B \\ \cdots \\ a_p B \end{bmatrix}. \tag{21}$$

For a given vector $X = [x_1, x_2, \cdots, x_p]^T$, let X^S denote a set whose elements are formed by the entities of X in the sense that $X^S = \{x_i : 1 \le i \le p\}$. It can easily be proved that all the different entities of the vector Γ_2^S are identical to all the effective frequency components of the second-order output frequency function $Y_2(j\omega)$. Note that some entities in the vector Γ_2^S may be the same. Therefore, Γ_2^S is redundant for determining the effective frequency components of $Y_2(j\omega)$.

In general, the effective frequency components of the nth-order output frequency function $Y_n(j\omega)$ can be calculated using the recursive algorithm below:

Algorithm 1: Assume that a nonlinear system is excited by a multiple input signal $u(t)$ of the form (13) with K fundamental frequency components, $\{\omega_1, \omega_2, \cdots, \omega_K\}$. The effective frequency components of the nth-order output frequency function can be determined by searching all the different entities of Ω_n, which is defined as

$$\Gamma_1 = [\sigma_1, \cdots, \sigma_{2K}]^T \tag{22}$$

$$\Gamma_n = \Gamma_{n-1} \otimes I_{2K} + I_{\langle \Gamma_{n-1}^S \rangle} \otimes \Gamma_1 \tag{23}$$

$$\Omega_n = \Gamma_n^S, \qquad n \ge 2 \tag{24}$$

where $\langle \Gamma_{n-1}^S \rangle = \langle \Omega_{n-1} \rangle$ indicates the number of entities in the vector Γ_{n-1}, and

$$I_m = \underbrace{[1, 1, \cdots, 1]}_{m}^T. \tag{25}$$

Proof of Algorithm 1: Assume that all the different entities of the vector Γ_n are identical to all the effective frequency components of the nth-order output frequency function $Y_n(j\omega)$. Let $\Gamma_n = [\sigma_1^{(n)}, \cdots, \sigma_{\alpha(n)}^{(n)}]^T$, where $\alpha(n) = \langle \Gamma_n^S \rangle$. All the possible frequency components for the $(n+1)$th-order output frequency function $Y_{n+1}(j\omega)$ can then be determined by inspecting the following combinations:

$$\begin{cases} \sigma_1^{(n)} + \sigma_1 \\ \vdots \\ \sigma_1^{(n)} + \sigma_{2K} \\ \vdots \\ \sigma_{\alpha(n)}^{(n)} + \sigma_1 \\ \vdots \\ \sigma_{\alpha(n)}^{(n)} + \sigma_{2K}. \end{cases} \tag{26}$$

Similar to (20), the above equation can be expressed in a vector form as

$$\Gamma_{n+1} = \begin{bmatrix} \sigma_1^{(n)} I_{2K} + \Gamma_1 \\ \vdots \\ \sigma_{\alpha(n)}^{(n)} I_{2K} + \Gamma_1 \end{bmatrix} = \Gamma_n \otimes I_{2K} + I_{\alpha(n)} \otimes \Gamma_1. \tag{27}$$

This is just (23). Therefore, Algorithm 1 can be used to determine the effective frequency range for the arbitrary-order output frequency function $Y_n(j\omega)$. Note that some entities in Γ_n are the same and Γ_n is often redundant for determining the effective frequency components of the nth-order output frequency function $Y_n(j\omega)$.

It is known that the positive and negative frequencies are symmetrical about the origin, therefore only the non-negative frequencies need to be calculated. It can easily be shown that the non-negative frequency components of the nth-order output frequency function $Y_n(j\omega)$ can be calculated using the recursive algorithm below:

Algorithm 2: Assume that a nonlinear system is excited by a multiple input $u(t)$ of the form (13) with K fundamental frequency components, $\Omega_1^+ = \{\omega_1, \omega_2, \cdots, \omega_K\}$. The non-negative frequency components of the nth-order output frequency function can be determined by searching all the different entities of Ω_n^+, which is defined as

$$\Gamma_1 = [\omega_1, \omega_2, \cdots, \omega_K]^T \tag{28}$$

$$\Gamma_n = \Gamma_{n-1} \otimes I_{2K} + I_{K(2K)^{n-2}} \otimes V \tag{29}$$

$$\Omega_n^+ = |\Gamma_n|^S, \qquad n \ge 2 \tag{30}$$

where V is defined as in (20), I_m is defined by (25), and $|\Gamma_n|^S$ is a set whose elements are composed by all the different entities of the vector Γ_n by taking absolute values.

Algorithm 2 can be proved in the same way as Algorithm 1. The recursive algorithm is very simple and quite easy to implement using vector-oriented software tools. For the case of $n = 2$, (29) becomes

$$\Gamma_2 = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_K \end{bmatrix} \otimes I_{2K} + I_K \otimes V = \begin{bmatrix} \omega_1 + \sigma_1 \\ \vdots \\ \omega_1 + \sigma_{2K} \\ \vdots \\ \omega_K + \sigma_1 \\ \vdots \\ \omega_K + \sigma_{2K} \end{bmatrix}. \tag{31}$$

Clearly, the absolute values of all different entities of the vector Γ_2 are identical to all the non-negative frequency components of the second-order output frequency function $Y_2(j\omega)$.

IV. Examples

Two examples are given below to illustrate the performance and efficiency of the proposed algorithm.

Example 1: Consider the following nonlinear finite impulse response (NFIR) model [14]

$$y(t) = u^3(t) + u^2(t)u(t-1) + u(t)u^2(t-1). \tag{32}$$

Let the input signal $u(t)$ be given below

$$u(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) + \cos(2\pi f_3 t) \tag{33}$$

where $f_1 = 2$ Hz, $f_2 = 3$ Hz, $f_3 = 7$ Hz. This is the case of $K = 3$, $\omega_1 = 2\pi f_1$, $\omega_2 = 2\pi f_2$, $\omega_3 = 2\pi f_3$. Without any

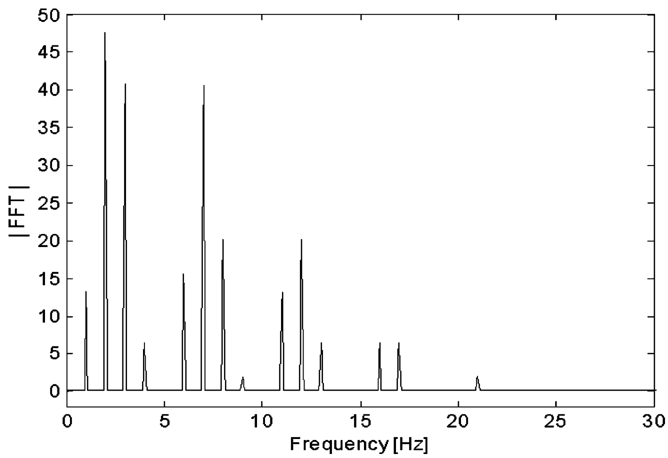

Fig. 1. Frequency response of the cubic Volterra model given by (32) and (33).

direct calculation of the Fourier transform for the output $y(t)$, it can be determined using Algorithm 2 that the non-negative output frequencies (in hertz) of the model given by (33) are: {1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 16, 17, 21}. This theoretical result can easily be verified by calculating the Fourier spectrum of the output signal $y(t)$, which is shown in Fig. 1, and where all the 14 different frequencies can clearly be observed.

Example 2: Consider the following nonlinear model:

$$y(t) = u^3(t) + u^4(t) + u^5(t). \tag{34}$$

Let the input signal $u(t)$ be given below

$$u(t) = \cos(\omega_1 t) + \cos(\omega_2 t) + \cos(\omega_3 t) + \cos(\omega_4 t) + \cos(\omega_5 t) + \cos(\omega_6 t) \tag{35}$$

where $\omega_1 = 2, \omega_2 = 3, \omega_3 = 5, \omega_4 = 8, \omega_5 = 13$, and $\omega_6 = 21$. This is the case of $K = 6$. Whilst the algorithm proposed in [8] can be used to calculate the output frequency components, the expression and construction of the relevant matrices required by the algorithm are very complex for the three nonlinear submodels in (34). The method in [2] for this problem involves constructing a total of 364, 1365, and 4368 "frequency-mix" vectors of length 12, for the three submodels, respectively. No algorithms are available to automatically produce these required frequency-mix vectors, from which all the distinct output frequencies can be determined. Using the proposed algorithm, however, all the output frequencies for the three submodels can easily be calculated in an iterative way. For example, based on Algorithm 1, the total number of all distinct output frequencies (either positive or non-positive), for the three submodels $u^3(t)$, $u^4(t)$, and $u^5(t)$, was calculated to be 87, 129, and 171, respectively. The total time for calculating all the frequencies of the three submodels was about 0.41 sec, where the algorithm was implemented using Matlab (R14) on a Sun-2500 workstation (1.28 GHz). The output frequencies for the three submodels are plotted in Fig. 2, where only non-negative frequencies are shown.

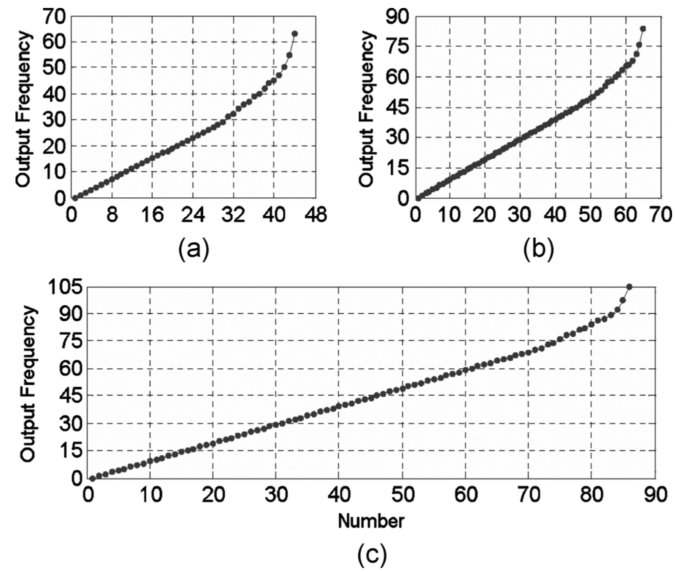

Fig. 2. Output frequency components for the model given by (34) and (35): (a) for the third-order function $u^3(t)$, (b) for the forth-order function $u^4(t)$, and (c) for the fifth-order function $u^5(t)$.

V. Conclusion

A new algorithm has been introduced to determine the output frequency components for the Volterra class of nonlinear systems with multitone inputs. The main advantage of the new algorithm, compared with existing algorithms, is that it is very compact and simple in form, and thus is quite easy to implement using vector and matrix-oriented software tools. This will greatly facilitate the determination of effective output frequencies of any Volterra models.

References

[1] M. Schetzen, *The Volterra and Wiener Theories of Nonlinear Systems*. New York: Wiley, 1980.
[2] D. D. Weiner and J. F. Spina, *Sinusoidal Analysis and Modeling of Weakly Nonlinear Circuits: With Application to NonlinearIinterference Effects*. New York: Van Nostrand Reinhold, 1980.
[3] W. J. Rugh, *Nonlinear System Theory—The Voterra/Wiener Approach*. Baltimore, MD: The John Hopkins University Press, 1981.
[4] S. Boyd, L. O. Chua, and C. A. Desoer, "Analytical foundations of Volterra series," *IMA J. Math. Contr. Inf.*, vol. 1, pp. 243–282, 1984.
[5] A. Ushida and L. O. Chua, "Frequency-domain analysis of nonlinear circuits driven by multi-tone signals," *IEEE Trans. Circuits Syst.*, vol. CAS-31, no. 9, pp. 766–779, 1984.
[6] W. A. Frank, "Sampling requirements for Volterra system identification," *IEEE Signal Process. Lett.*, vol. 3, no. 9, pp. 266–268, Sep. 1996.
[7] A. Buonomo and C. DiBello, "Asymptotic formulas in nearly sinusoidal nonlinear oscillators," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 43, no. 12, pp. 953–963, Dec. 1996.
[8] Z. Q. Lang and S. A. Billings, "Output frequency characteristics of nonlinear systems," *Int. J. Contr.*, vol. 64, no. 6, pp. 1049–1067, 1996.
[9] ——, "Output frequencies of nonlinear systems," *Int. J. Contr.*, vol. 67, no. 5, pp. 713–730, Jul. 1997.
[10] G. M. Raz and B. D. Van Veen, "Baseband Volterra filters for implementing carrier based nonlinearities," *IEEE Trans. Signal Process.*, vol. 46, no. 1, pp. 103–114, Jan. 1998.
[11] J. C. P. Jones and S. A. Billings, "Interpretation of nonlinear frequency response functions," *Int. J. Contr.*, vol. 52, no. 2, pp. 319–346, 1990.
[12] G. M. Lee, "Estimation of non-linear system parameters using high-order frequency response functions," *Mech. Syst. Signal Process.*, vol. 11, pp. 219–228, Mar. 1997.
[13] J. A. Cherry and W. M. Snelgrove, "On the characterization and reduction of distortion in bandpass filters," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 45, no. 5, pp. 523–537, May 1998.
[14] V. J. Mathews and G. L. Sicuranza, *Polynomial Signal Procession*. New York: Wiley, 2000.