Visibility of heteroclinic networks

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Abstract

The concept of stability has a long history in the field of dynamical systems: stable invariant objects are the ones that would be expected to be observed in experiments and numerical simulations. Heteroclinic networks are invariant objects in dynamical systems associated with intermittent cycling and switching behaviour, found in a range of applications. In this article, we note that the usual notions of stability, even those developed specifically for heteroclinic networks, do not provide all the information needed to determine the long-term behaviour of trajectories near heteroclinic networks. To complement the notion of stability, we introduce the concept of *visibility*, which pinpoints precisely the invariant objects that will be observed once transients have decayed. We illustrate our definitions with examples of heteroclinic networks from the literature.

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MSC codes: 34C37, 34D05, 34D20, 34D45, 37C81

1 Introduction

The concept of *stability* is an old one in dynamical systems and is invariably associated with the persistence and/or observability of an object or outcome. The work of Poincaré [33] and of Lyapunov [23] are prominent landmarks, but many other authors have made important contributions. We refer the interested reader to the historical accounts provided by Leine [22], Roque [35], Mawhin [24] and references therein. In the context of dynamical systems, we typically think of an invariant object as being *stable* when either nearby solutions do not

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move far away (Lyapunov stability [23]) or nearby solutions converge to the object (Milnor attractor [27]). When the two occur simultaneously, we talk about *asymptotic stability* (see Auslander *et al.* [5]).

In this paper we introduce the concept of *visibility*, to complement the classic notion of stability. We illustrate this with examples that have arisen in the study of heteroclinic networks. These are invariant objects that are finite unions of heteroclinic cycles, themselves the union of finitely many equilibria and trajectories connecting them in a cyclic way.

A systematic study of the asymptotic stability of heteroclinic cycles dates back to the work of Krupa and Melbourne [20]. A heteroclinic cycle that is a proper subset of a heteroclinic network can never be asymptotically stable even though it can be highly attracting. This is due to the fact that near an equilibrium that belongs to more than one cycle, there are always some trajectories that move away from any one of these cycles. In particular, initial conditions that lie exactly on a connecting orbit to an equilibrium *not* in the cycle of interest will have this property. The notions of *essential asymptotic stability* (e.a.s.), and of *fragmentary asymptotic stability* (f.a.s.), were introduced by Melbourne [26] and by Podvigina [29] respectively, to address this. A heteroclinic cycle or network is f.a.s. if it satisfies the conditions for asymptotic stability for points in a set of positive measure contained in a neighbourhood of the cycle or network, while it is e.a.s. if this set of points tends to a set of full measure as the size of the neighbourhood tends to zero. This is in contrast to the requirements of asymptotic stability, which are to be satisfied for *all* points in a neighbourhood of the invariant set of interest.

Hence, ordered by decreasing order of 'stability', heteroclinic cycles or networks can be asymptotically stable, essentially asymptotically stable, or fragmentarily asymptotically stable. All these definitions also require the conditions for Lyapunov stability, namely that trajectories do not move too far away. If this fails, the invariant set can be *quasi-asymptotically stable* [13] or a *Milnor attractor* [27]. In both cases, trajectories are attracted to the invariant set as time tends to infinity although they may move far away during a transient.

The motivation for this work appears when relating the stability of an invariant set to its observability: there are 'very stable' heteroclinic networks that can never appear in their entirety in numerical simulations or as the outcome of an experiment. On the other hand, there are heteroclinic cycles classified as 'not very stable' that invariably appear in numerical simulations. A quintessential example is the Kirk–Silber (KS) network [18], discussed in detail in Subsection 3.2 below. It is possible to choose parameters such that the entire network is e.a.s. but only one subcycle of the network ever appears in the long term in simulations. For some sets of parameters, it can even be the case that most trajectories will initially visit one cycle, but in the long term, almost all trajectories asymptote onto the other. This illustrates the need for an alternative notion that provides precise information about what we can expect to observe in simulations and experiments. We argue that what we define as *visibility*, and its various nuances, precisely pinpoints the invariant objects we can expect to observe.

This gap between a somewhat stable invariant set and its observability is not a strictly theoretical issue as we illustrate with examples from game theory and from population dy-

namics. That stability occasionally provides ambiguous information is clear in the population dynamics of the Rock–Paper–Scissors–Lizard–Spock (RPSLS) game studied by the authors in [34] and [8] as well as those of the jungle game with four species studied in [9]. In both games, species compete amongst themselves in a non-transitive manner (also known as *cyclic dominance*) and the dynamics correspond to those near a heteroclinic network. When only some of the species survive, a heteroclinic cycle involving the equilibria corresponding to these surviving species is expected as the outcome of simulations. However, under the current definitions of stability, the whole network of the RPSLS game can be asymptotically stable even when as few as three species survive. More complex outcomes not captured by the stability of the network are found in [34]. In the jungle game, although the whole network is asymptotically stable, there is always one species that becomes extinct. These two games in population dynamics illustrate the ambiguity contained in stating that the whole network is asymptotically stable in terms of survival of the species involved.

Therefore, the main purpose of this paper is to establish a concept that fills a gap left by the current definitions of stability. Visibility is more related to outcomes than stability, especially in the context of heteroclinic dynamics. The novelty of this paper is to provide explicit definitions that are unequivocally related to what would be observed in experiments or simulations. Many studies of heteroclinic networks are specifically interested in the routes that trajectories take around heteroclinic networks — see [8, 30, 34] for some examples and citations of other works. However, in this paper we are agnostic about the route a trajectory takes; we have interest only in which set of points are repeatedly visited by the trajectory.

The tools to establish visibility in cases of heteroclinic dynamics are the same we would use for stability, namely, the computation of Poincaré return maps, and the reduction to transition matrices in logarithmic coordinates. Due to this overlap in computations, previously established results on stability are already sufficient for our purposes and we are able to then relate these to our new notions of visibility.

In the next section we present and discuss some of the notions of stability from the literature, and in Section 3 we illustrate how these fail to provide complete information about what is observed in numerical experiments. These examples are known but our presentation includes new numerical simulations and presents previously unobserved/unreported behaviour in some cases. We use the examples to motivate the variations we introduce in the concept of visibility. Visibility is defined in Section 4. An invariant set that is visible always appears in numerical simulations in its entirety. For instance, the KS network briefly mentioned above is never visible. In the population dynamics examples, only the cycles corresponding to the surviving species are visible, while the network is not. We provide a discussion and illustration of visibility that clearly distinguishes it from any of the available notions of stability. Section 4.2 uses previously published computations to demonstrate the applicability of the definitions. As mentioned above, we do not repeat any stability calculations here. We end with a discussion in the last section.

2 Preliminaries

In this section we give background on commonly used definitions of stability, and also on heteroclinic cycles and networks. Let X be a compact flow-invariant set for the dynamics described by the ODE

$$\dot{x} = f(x) \tag{1}$$

for $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ smooth. Recall that an *equilibrium solution* ξ_0 satisfies $f(\xi_0) = 0$.

Definition 1 *A* heteroclinic cycle *is a finite collection of equilibria* $\{\xi_1, \ldots, \xi_m\}$ *of* (1)*, together with a set of heteroclinic connecting trajectories* $\{\gamma_1(t), \ldots, \gamma_m(t)\}$*, where* $\gamma_j(t)$ *is a solution of* (1) *such that* $\lim_{t\to-\infty} \gamma_j(t) = \xi_j$*,* $\lim_{t\to\infty} \gamma_j(t) = \xi_{j+1}$ *and* $\xi_{m+1} \equiv \xi_1$*.*

A *heteroclinic network* is a connected union of finitely many heteroclinic cycles. The literature contains numerous different definitions of heteroclinic cycles and networks: subtleties occur because there may be equilibria in the network with two-dimensional unstable manifolds, and whether or not to include the entire unstable manifold in the heteroclinic network can result in different definitions. In this paper, we use simply the definition given at the start of this paragraph, and for each example, we are specific about the object under investigation. Given Definition 1, a heteroclinic cycle is a closed flow-invariant set.

More generally, heteroclinic cycles and networks may connect invariant objects more complicated than equilibria, such as periodic orbits [1, 17] or chaotic sets [3], but we do not consider these possibilities here. In generic systems, heteroclinic cycles and networks are of high co-dimension, but when (1) contains invariant subspaces, then they may exist for open sets of parameters values, that is, they are *robust*. An early example of a robust heteroclinic cycle is the Guckenheimer–Holmes cycle in [15], addressed in Subsection 3.1 below.

Several notions of stability have been developed to deal with the properties of compact flow-invariant sets, in particular with heteroclinic cycles and networks. We present next those that are most common.

Let $\varphi(t, x)$ denote the flow of equation (1): the point in the orbit of x at time t. The *ω-limit* set of a point $x \in \mathbb{R}^n$ is

$$\omega(x) = \{y \in \mathbb{R}^n : \text{there is an increasing sequence of times } t_n \to +\infty \text{ with } \lim_{n\to+\infty} \varphi(t_n, x) = y\}.$$

Our next two definitions encompass two important properties of stability: *Lyapunov stability* requires that trajectories starting near an invariant set X stay near X, and *quasi-asymptotic stability* requires that trajectories reach X in the limit as t increases to infinity.

Definition 2 (adapted from Kuznetsov [21]) *We say that an invariant set X is* Lyapunov stable *if for any neighbourhood U of X there exists a neighbourhood V of X such that for all $x \in V$ and $t > 0$,*

$$\varphi(t, x) \in U.$$

If $d(a,b)$ denotes the distance between two points a and b, then the distance between a point a and a set B is defined as

$$d(a,B) = \inf\{d(a,b) : \ b \in B\}.$$

Definition 3 (adapted from Glendinning [13]) *We say an invariant set X is* quasi-asymptotic stable *iff there exists $\delta > 0$ such that if $d(y,X) < \delta$ then $d(\varphi(y,t),X) \to 0$ as $t \to \infty$.*

Similar to quasi-asymptotic stability, is the concept of a *Milnor attractor*.

Definition 4 (Milnor [27]) *A closed subset $X \subset \mathbb{R}^n$ will be called an* attractor *if it satisfies two conditions:*

1. *the* realm of attraction*, $\rho(X)$, consisting of all points $x \in \mathbb{R}^n$ for which $\omega(x) \subseteq X$, must have strictly positive measure; and*
2. *there is no strictly smaller closed set $X' \subset X$ so that $\rho(X')$ coincides with $\rho(X)$ up to a set of measure zero.*

Asymptotic stability is usually defined to be the combination of Lyapunov and quasi-asymptotic stability [13, 21]. There are different definitions but they coincide when the flow-invariant set X is closed. See [32] for a discussion.

Let $V_\delta(X)$ represent the set of points within the state space at a distance at most δ from X. Podvigina [29] defines *δ-basin of attraction* of a compact flow-invariant set X as

$$\mathcal{B}_\delta(X) = \{x \in \mathbb{R}^n : \lim_{t\to+\infty} d(\varphi(t,x),X) = 0 \text{ and } \varphi(t,x) \in V_\delta(X) \text{ for } t \geq 0\}.$$

The *basin of attraction* of X is just

$$\mathcal{B}(X) = \{x \in \mathbb{R}^n : \lim_{t\to+\infty} d(\varphi(t,x),X) = 0\}.$$

This is the same as the realm of attraction in Milnor's definition (Definition 4). It is worth pointing out that the extra condition in the definition of the δ-basin of attraction is a Lyapunov stability condition. Thus in definitions 6 and 7 that follow, the use of δ-basins means that there is also a Lyapunov-type stability condition for these types of stability.

We observe that the basin of attraction is not necessarily a neighbourhood of X, and in the cases of heteroclinic cycles and networks, the basins frequently are neighbourhoods with thick or thin cusps removed [18]. Removing cusps leads to the definitions of essential asymptotic stability (Definition 6, which removes only thin cusps from the neighbourhood), and fragmentary asymptotic stability (Definition 7, which removes any type of cusp). Essential asymptotic stability is the strongest notion of stability after asymptotic stability. It originates in Melbourne [26] but is usually stated in the words of Brannath as follows.

Definition 5 (Definition 1.1 in Brannath [6]) *Given any subset N of $\mathbb{R}^n$, a closed invariant subset $X \subset \bar{N}$ ($\bar{N}$ denotes the closure of N) is said to be* asymptotically stable, relative to the set N *if for every neighbourhood U of X there is a neighbourhood V of X, such that*

$$\forall\, x \in V \cap N,\ \varphi(t,x) \in U \text{ and } \omega(x) \subseteq X.$$

Here, asymptotically stable relative to the set N means that initial conditions restricted to be in N will stay close to X (i.e., there is a Lyapunov stability condition) and will asymptote to X, but N does not need to be a neighbourhood of X. The next definition, essential asymptotic stability, requires that the relative measure of N goes to 1 as X is approached.

Definition 6 (Definition 1.2 in Brannath [6]) *A closed invariant set X is called* essentially asymptotically stable *(e.a.s.) if it is asymptotically stable relative to a set $N \subset \mathbb{R}^n$ with the property that*

$$\lim_{\delta \to 0} \frac{\ell(V_\delta(X) \cap N)}{\ell(V_\delta(X))} = 1,$$

where ℓ is the Lebesgue measure.

Definition 7 (Podvigina [29]) *We call a compact invariant set X* fragmentarily asymptotically stable *(f.a.s.) if, for any $\delta > 0$, $\ell(\mathcal{B}_\delta(X)) > 0$.*

Note that the definitions of f.a.s. and e.a.s. demand that trajectories starting near X do not go far away before converging to X. Without this restriction, f.a.s. would be the same as part 1 in Definition 4 (Milnor attractor).

3 Illustrative heteroclinic cycles and networks

In this section we present a few examples that illustrate how these existing notions of stability fail to provide complete information about what to expect when running simulations or experiments. We work in $\mathbb{R}^n$ for $n \geq 3$. The simplest way of constructing heteroclinic cycles is to have reflection symmetry in all the coordinates, so that the equations are equivariant under $x_j \to -x_j$, for $j = 1, \ldots, n$, and all the coordinate hyperplanes are invariant. Heteroclinic cycles are found in systems with more elaborate symmetries as well, for example, mode interactions with spherical symmetry [2], but we consider only examples with reflection symmetry here.

The examples in this section all have equations of the form

$$\dot{x}_j = x_j \left(1 - \chi + \sum_{k=1}^{n} a_{kj} x_k^2 \right), \quad j = 1, \ldots, n, \tag{2}$$

where $\chi = \sum_{j=1}^{n} x_j^2$, and the a_{kj} are parameters with $a_{kk} = 0$. This system has equilibria on the axes with a single coordinate equal to ± 1, and all other coordinates equal to zero.

We label the equilibria on the x_j axis as $\pm\xi_j$. Each parameter a_{jk} is the eigenvalue of the Jacobian matrix at $\pm\xi_j$ in the x_k direction. The remaining eigenvalue (in the direction x_j) is equal to -2, and does not affect the dynamics, due to the Invariant Sphere Theorem [12]. The reflection symmetry of (2) ensures that each coordinate axis and coordinate (hyper-) plane is invariant under the flow, and thus the positive orthant (which we write as $\mathbb{R}^n_+$) is also invariant under the flow.

For (2) restricted to $\mathbb{R}^n_+$, one can make the coordinate transformation $y_j = x_j^2$, and a rescaling of time, to get the equations

$$\dot{y}_j = y_j \left(1 - \chi + \sum_{k=1}^{n} a_{kj} y_k \right), \quad j = 1, \ldots, n. \tag{3}$$

where $\chi = \sum_{j=1}^{n} y_j$. These equations are the *Lotka–Volterra* equations [16] and are used in population dynamics contexts, where negative values of the coordinates have non-physical meanings. These equations are not equivariant, but still contain the same flow invariant coordinate axes and (hyper-) planes.

3.1 The Guckenheimer–Holmes cycle

The prototypical example of a robust heteroclinic cycle is the *Guckenheimer–Holmes* cycle [15], also known as the *Rock–Paper–Scissors* cycle. The governing equations are three-dimensional, and are often written as:

$$\begin{aligned} \dot{x}_1 &= x_1(1 - \chi - c_{21}x_2^2 + e_{31}x_3^2), \\ \dot{x}_2 &= x_2(1 - \chi - c_{32}x_3^2 + e_{12}x_1^2), \\ \dot{x}_3 &= x_3(1 - \chi - c_{13}x_1^2 + e_{23}x_2^2). \end{aligned} \tag{4}$$

where $\chi = \sum_{j=1}^{3} x_j^2$, and $c_{jk} > 0$ and $e_{jk} > 0$ are parameters.

Guckenheimer and Holmes imposed a three-fold symmetry $(x_1, x_2, x_3) \to (x_2, x_3, x_1)$, meaning there were only two distinct parameters. We do not impose that restriction here.

The signs of the parameters ensure the existence of heteroclinic connections from $\pm\xi_j$ to $\pm\xi_{j+1}$ (indices taken modulo 3), lying within the (invariant) (x_j, x_{j+1}) coordinate plane. Often the cycle in the positive orthant connecting the positive equilibria is considered to be the 'Guckenheimer and Holmes cycle'. It is also possible to consider the set of six equilibria and twelve connecting heteroclinic orbits as a heteroclinic network, formed from the union of eight heteroclinic cycles (see Figure 1(a)). As mentioned above, the parameters c_{jk} and e_{jl} are related to the eigenvalues at the equilibria. Each equilibrium ξ_j is a saddle point. The *contracting* eigenvalue is equal to $-c_{jk}$ in the direction x_k. The *expanding* eigenvalue is equal to e_{jl} in the direction x_l. In this way, the notation in (2) relates to that in (4) by $a_{j-1,j} = e_{j-1,j}$ and $a_{j+1,j} = -c_{j+1,j}$.

It can be shown (by constructing an appropriate Poincaré map, see for example [25] for details), that the Guckenheimer and Holmes heteroclinic *network* is asymptotically stable if

$$\rho_{123} \equiv \frac{c_{13}c_{21}c_{32}}{e_{12}e_{23}e_{31}} > 1.$$

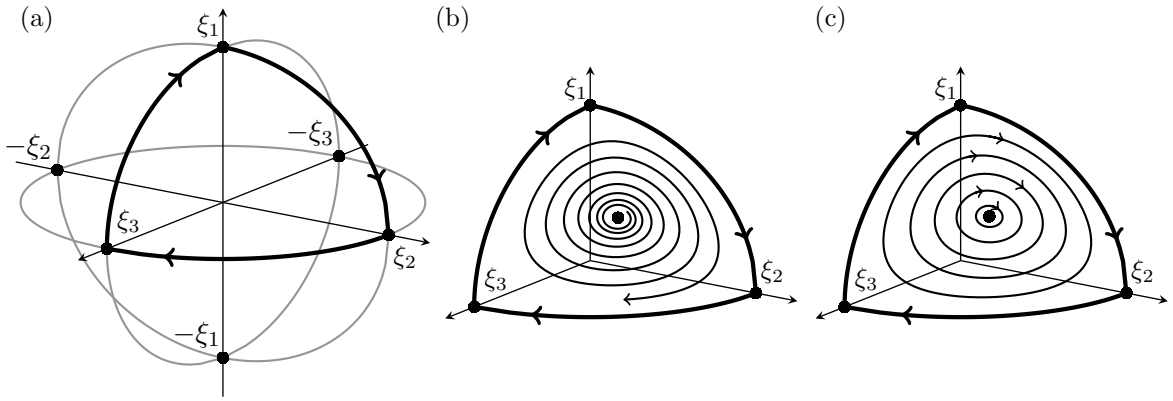


Figure 1: (a) Sketch of the Guckenheimer–Holmes heteroclinic 'network', with the cycle in the positive orthant highlighted. (b) Sketch of the dynamics in the positive orthant when $\rho_{123} > 1$. (c) Sketch of the dynamics in the positive orthant when $\rho_{123} = 1$; note that there exists an infinite family of periodic orbits.

However, any trajectory will only converge to a single subcycle within the network. Each subcycle of this network cannot be asymptotically stable, since initial conditions near that subcycle but in the 'wrong' orthant will move away. In Figure 2, we show two trajectories near the Guckenheimer and Holmes cycle with different initial conditions. In panel (a), the initial condition lies in the positive orthant, and the trajectory asymptotes onto the cycle between $+\xi_1$, $+\xi_2$ and $+\xi_3$. In panel (b), the initial condition has a negative value of x_2 and the trajectory asymptotes onto a cycle between $+\xi_1$, $-\xi_2$ and $+\xi_3$. In fact, with $\rho_{123} > 1$ each subcycle is only f.a.s. The basins of attraction of each subcycle are one of the eight orthants, including the boundary planes. Note that this means that the basins of attraction of each cycle have non-empty intersection.

Generally this discussion is avoided by either identifying the equilibria ξ_j with $-\xi_j$ (invoking symmetry), or restricting the flow to the positive orthant [19]. Then there is no longer a network of connections, but only a single cycle. Most authors consider this cycle to be 'the Guckenheimer and Holmes cycle', and that it is asymptotically stable when $\rho_{123} > 1$.

However, there are some initial conditions for which the trajectories do not asymptote onto the entire cycle, namely those that lie within the invariant coordinate planes. These will instead converge to an equilibrium. Hence, even though the cycle is classified as asymptotically stable, some choices of initial conditions show only one of the equilibria in numerical simulations. These initial conditions form only a set of measure zero, which may explain why this issue has not yet been addressed.

The equivalent Lotka–Volterra equations modelling intransitive competition between three species are

$$\begin{aligned} \dot{y}_1 &= y_1(1 - \chi - c_{21}y_2 + e_{31}y_3), \\ \dot{y}_2 &= y_2(1 - \chi - c_{32}y_3 + e_{12}y_1), \\ \dot{y}_3 &= y_3(1 - \chi - c_{13}y_1 + e_{23}y_2). \end{aligned} \tag{5}$$

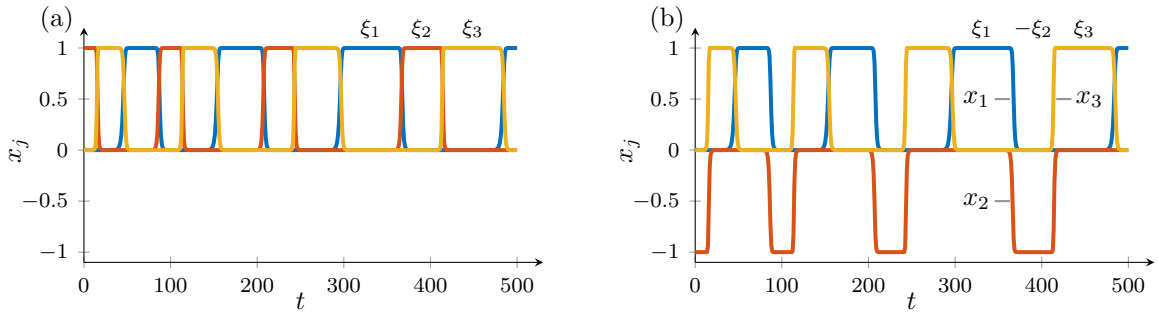


Figure 2: Trajectories near the Guckenheimer–Holmes cycle (4). In (a) the initial conditions have positive values for all coordinates, and the trajectory approaches the cycle between $+\xi_1$, $+\xi_2$ and $+\xi_3$. In (b), the initial condition has a negative value of x_2 and the trajectory asymptotes onto a cycle between $+\xi_1$, $-\xi_2$ and $+\xi_3$. Parameters are the same in both panels and are given in table 2 in Appendix A.

where $\chi = \sum_{j=1}^{3} y_j$. In these equations there is a single heteroclinic cycle, equivalent to that shown in Figure 1(b). However, trajectories that start outside of $\mathbb{R}^3_+$ will diverge to $-\infty$; there is no network of connections as in the equivariant case. Thus, when the phase space is considered to be $\mathbb{R}^n$, the cycle cannot be asymptotically stable, but can be f.a.s., again if $\rho_{123} > 1$.

For the remainder of this paper, we will consider the dynamics of equations (2) or (3) with phase space as the positive orthant ($\mathbb{R}^n_+$). Then (2) and (3) are equivalent under a coordinate transformation and we make no further distinction between the Guckenheimer and Holmes network and the Guckenheimer and Holmes cycle.

Other subtleties regarding stability can occur when $\rho_{123} = 1$. At this point we say that the Guckheimer and Holmes cycle is *at resonance*. In system (4) for these parameter values, it can be shown that an infinite family of periodic orbits exists (see Figure 1(c))—there is a Hopf bifurcation from an equilibrium with three non-zero components at the same parameter values. The Guckenheimer–Holmes cycle is Lyapunov stable: any initial condition close to the cycle will remain on a periodic orbit close to the cycle and thus remain a bounded distance from the cycle. This distance can be made smaller by choosing an initial condition closer to the cycle.

This example is one of several that motivate our definition of *asymptotic visibility* in Section 4.

3.2 The Kirk–Silber network

The Kirk–Silber network [18] is formed by joining two Guckenheimer–Holmes cycles along an edge in $\mathbb{R}^4$ (see Figure 3(a)). Although equations were not given by Kirk and Silber, the

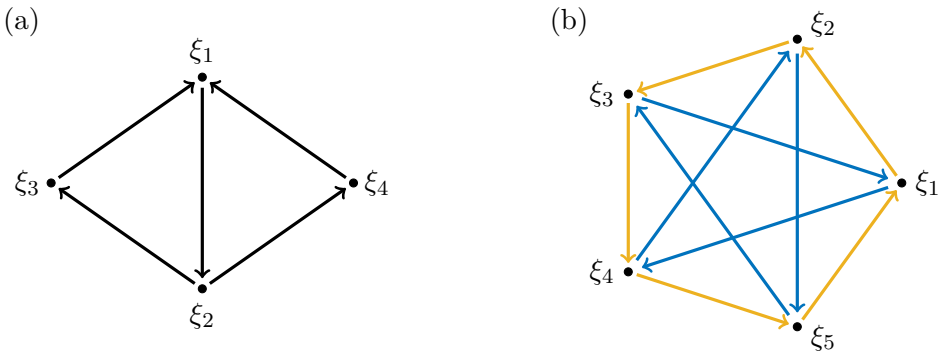


Figure 3: (a) The Kirk–Silber network, and (b) the Rock–Paper–Scissors–Lizard–Spock network, both represented as directed graphs. Dots indicate equilibria, which lie on coordinate axes, and arrows represent heteroclinic connections. In (a) all connections are one-dimensional and contained in the same plane as the equilibria they connect. The colours in (b) indicate two different types of connection: type A (amber) is a two-dimensional connection with one connecting trajectory in the plane containing the equilibria it connects and the remaining connecting trajectories in three-dimensional space; type B (blue) are one-dimensional connecting trajectories in the plane containing the equilibria. See Section 3.3 and Figure 6 for more detail on the type A (amber) connections.

network exists in the following system:

$$\begin{aligned}
\dot{x}_1 &= x_1(1-\chi-c_{21}x_2^2+e_{31}x_3^2+e_{41}x_4^2),\\
\dot{x}_2 &= x_2(1-\chi+e_{12}x_1^2-c_{32}x_3^2-c_{42}x_4^2),\\
\dot{x}_3 &= x_3(1-\chi-c_{13}x_1^2+e_{23}x_2^2-t_{43}x_4^2),\\
\dot{x}_4 &= x_4(1-\chi-c_{14}x_1^2+e_{24}x_2^2-t_{34}x_3^2).
\end{aligned} \tag{6}$$

where $\chi=\sum_{j=1}^{4}x_j^2$, and $c_{jk}>0$, $e_{jk}>0$ and $t_{jk}>0$ are parameters.

As per the discussion in the previous section, if we consider the phase space to be $\mathbb{R}^4$, no cycle or network can be asymptotically stable because of the issues of trajectories in the 'wrong' orthant. However, we will ignore those issues and restrict the phase space to $\mathbb{R}^4_+$. Even with this restriction, the Kirk–Silber system is a heteroclinic network, not a cycle. Our main focus of attention is the relative stability of the cycles ξ_1-ξ_2-ξ_3 and ξ_1-ξ_2-ξ_4, and the network formed by their union.

When considering the stability of the network or its subcycles, parameter combinations of interest are $\rho_{123}\equiv\frac{c_{13}c_{21}c_{32}}{e_{12}e_{23}e_{31}}$ and $\rho_{124}\equiv\frac{c_{14}c_{21}c_{42}}{e_{12}e_{24}e_{41}}$. These denote, as usual, the ratio between the product of the absolute value of contracting and the product of the expanding eigenvalues at the nodes of a cycle. These ratios determine whether trajectories approach the cycle, when the dynamics are restricted to the invariant subspace containing that cycle. As with the Guckenheimer–Holmes cycles, we say that if $\rho_{123}=1$ then the ξ_1-ξ_2-ξ_3 cycle is *at resonance*. If $\rho_{123}>1$, then we say the ξ_1-ξ_2-ξ_3 cycle is *past resonance*. Similar conditions apply for ρ_{124} and the ξ_1-ξ_2-ξ_4 cycle. There are another six parameter combinations ν_{ijkl} that determine

whether the cycles are stable to perturbations in the transverse direction. These are, for the ξ_1-ξ_2-ξ_3 cycle,

$$\nu_{1234} = \frac{c_{14}}{e_{12}} - \frac{c_{13}e_{24}}{e_{12}e_{23}} + \frac{c_{21}c_{13}t_{34}}{e_{12}e_{31}e_{23}},$$
$$\nu_{2314} = -\frac{e_{24}}{e_{23}} + \frac{c_{21}t_{34}}{e_{23}e_{31}} + \frac{c_{14}c_{32}c_{21}}{e_{12}e_{23}e_{31}},$$
$$\nu_{3124} = \frac{t_{34}}{e_{31}} + \frac{c_{32}c_{14}}{e_{12}e_{31}} - \frac{c_{32}c_{13}e_{24}}{e_{23}e_{12}e_{31}},$$

and the corresponding quantities for the ξ_1-ξ_2-ξ_4 cycle have the 3 and 4 swapped in all indices. If all of the $\nu_{ijk4} > 0$, then the ξ_1-ξ_2-ξ_3 cycle is stable to perturbations in the x_4 direction, and we say there is *no switching* from the ξ_1-ξ_2-ξ_3 cycle. If any of $\nu_{ijk4} < 0$, then the ξ_1-ξ_2-ξ_3 cycle is unstable to perturbations in the x_4 direction, and we say there is *switching* from the ξ_1-ξ_2-ξ_3 cycle. Similar conditions apply to define switching from the ξ_1-ξ_2-ξ_4 cycle.

Kirk and Silber [18] show that if $\rho_{123} > 1$ and all $\nu_{ijk4} > 0$ then the ξ_1-ξ_2-ξ_3 cycle is f.a.s. (although they did not use that term). Similarly, if $\rho_{124} > 1$ and all the $\nu_{ijk3} > 0$, then the ξ_1-ξ_2-ξ_4 cycle is f.a.s. These sets of conditions can be satisfied simultaneously, and then the Kirk–Silber network is f.a.s. Kirk and Silber also show that the *network* can be f.a.s. even if one of the cycles is not f.a.s. For instance, if one of the $\nu_{ijk3} < 0$ but all $\nu_{ijk4} > 0$, then trajectories that start close to the ξ_1-ξ_2-ξ_3 cycle will eventually 'switch' to the ξ_1-ξ_2-ξ_4 cycle. However, they also show that it is not possible to choose parameters where trajectories switch in both directions. Therefore, there are no initial conditions whose trajectories repeatedly visit the entire network, for any set of parameter values.

We give four examples of parameter choices that illustrate how varied the outcomes can be, thus illustrating the deficiency in the information provided by stating that network is f.a.s. Full details of parameter regions for various types of stability can be found in [10, 18]. Each of these four examples is illustrated by a numerical trajectory shown in one panel of Figure 4. These examples are part of our motivation to introduce finer notions of *visibility* in Section 4.

3.2.1 Past resonance, no switching

For this example, we choose parameters so that $\rho_{123}, \rho_{124} > 1$, which implies that the ξ_1-ξ_2-ξ_3 and ξ_1-ξ_2-ξ_4 subcycles are asymptotically stable for the flow restricted to their own invariant subspaces. We also choose all $\nu_{ijk3}, \nu_{ijk4} > 0$, which implies that there is no switching between cycles, so each cycle is f.a.s. in the full system and is a Milnor attractor. On the other hand, the entire network is also f.a.s. and a Milnor attractor. To see the latter, let N be the network, and let Σ_3 and Σ_4 be the two subcycles. In the notation of definition 4, $\rho(N) = \rho(\Sigma_3) \cup \rho(\Sigma_4)$. However, $\rho(N)$ differs from both $\rho(\Sigma_3)$ and $\rho(\Sigma_4)$ by more than a set of measure zero, so neither Σ_3 nor Σ_4 can be used in the role of X' in part 2 of the definition.

Even though the Kirk–Silber network is f.a.s., and a Milnor attractor, only one of the two subcycles will appear as the outcome of numerical simulations: which cycle appears depends

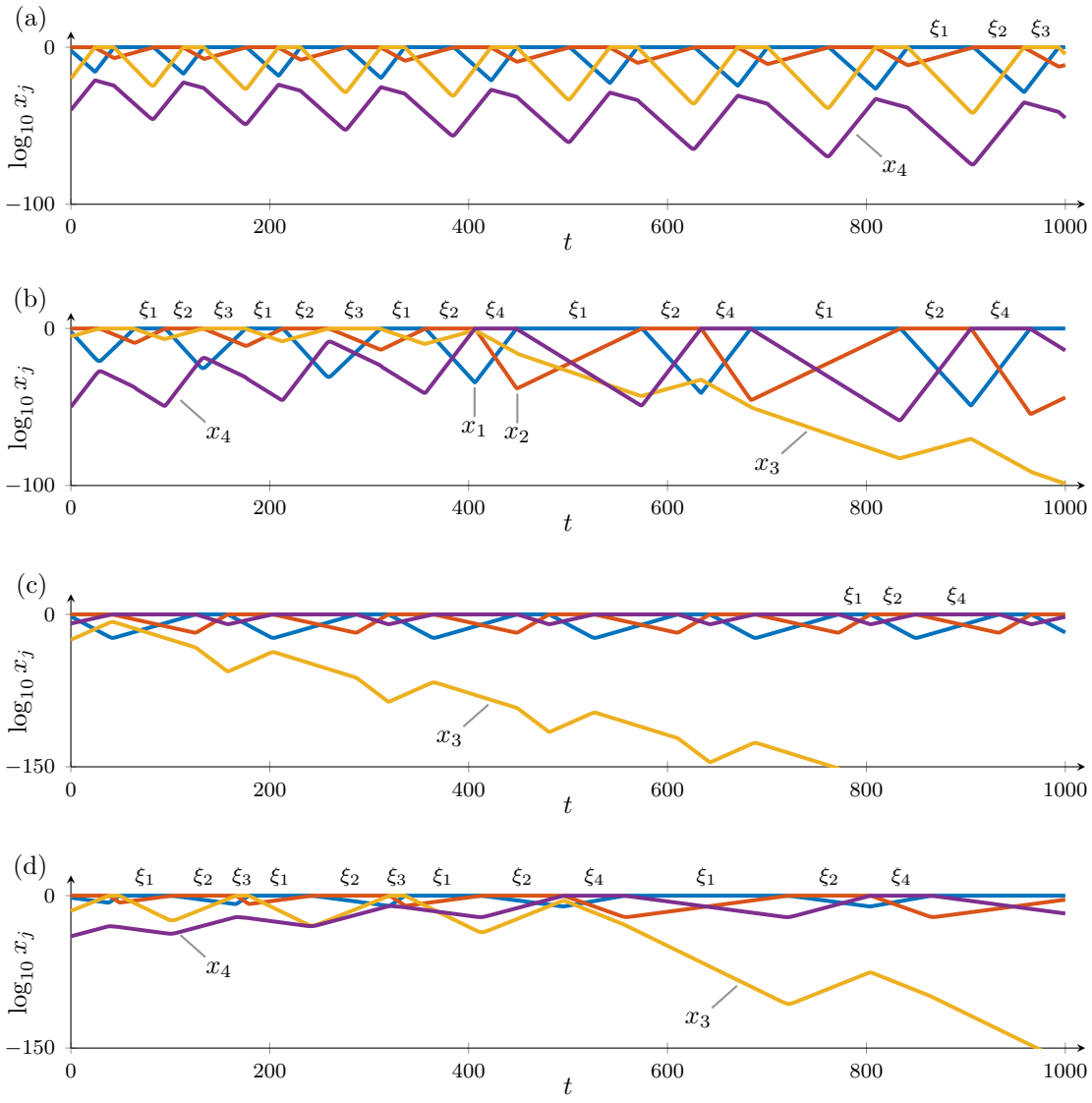


Figure 4: Trajectories near the Kirk–Silber network. In each case, the coordinates are plotted on a logarithmic axis, each colour corresponding to one coordinate as indicated. Parameters are given in table 2 in Appendix A. In (a) (section 3.2.1), the trajectory is approaching the ξ_1-ξ_2-ξ_3 cycle: the time spent near, e.g., ξ_1 increases on each loop around the cycle, the minimum values of the x_1, x_2 and x_3 coordinates decrease on each loop, and additionally the x_4 coordinate is decaying. In (b) (section 3.2.2), the trajectory initially appears to approach ξ_1-ξ_2-ξ_3: again the time spent near ξ_1 increases on each loop around the cycle. However, now the x_4 coordinate grows and the trajectory switches to the ξ_1-ξ_2-ξ_4 cycle. In (c) and (d) the ξ_1-ξ_2-ξ_4 cycle is at resonance. In (c) (section 3.2.3), the trajectory starts near the ξ_1-ξ_2-ξ_4 cycle and the x_3 coordinate decays. However, trajectories do not get closer to the ξ_1-ξ_2-ξ_4 cycle within that subspace as they remain near a periodic orbit. In (d) (section 3.2.4), the trajectory starts near the ξ_1-ξ_2-ξ_3 cycle and then switches to the ξ_1-ξ_2-ξ_4 cycle; the ξ_1-ξ_2-ξ_4 cycle is at resonance so the trajectory remains a finite distance away.

on the initial conditions. In Figure 4(a) we show a trajectory asymptoting onto the ξ_1-ξ_2-ξ_3 cycle.

3.2.2 Past resonance, with switching

For this example, we choose parameters so that $\rho_{123}, \rho_{124} > 1$, all $\nu_{ijk3} > 0$, but at least one $\nu_{ijk4} < 0$. Then each of the ξ_1-ξ_2-ξ_3 and ξ_1-ξ_2-ξ_4 cycles are again asymptotically stable for the flow restricted to their own invariant subspaces. However, now there is switching from the ξ_3 cycle to the ξ_4 cycle.

The ξ_1-ξ_2-ξ_4 cycle is f.a.s., and the ξ_1-ξ_2-ξ_3 cycle is not. However, almost all initial conditions near the ξ_1-ξ_2-ξ_4 cycle eventually asymptote onto that cycle, although some may visit the ξ_1-ξ_2-ξ_3 cycle first. The ξ_1-ξ_2-ξ_4 cycle will appear in most numerical simulations; even though it is 'only' f.a.s., if nearby trajectories move away and make some loops around the ξ_1-ξ_2-ξ_3 cycle, after a transient these trajectories will eventually asymptote onto the ξ_1-ξ_2-ξ_4 cycle.

In Figure 4(b) we show a trajectory that starts near ξ_2, so the initial condition is close to both the ξ_1-ξ_2-ξ_3 cycle, and the ξ_1-ξ_2-ξ_4 cycle. The trajectory makes three excursions around the ξ_1-ξ_2-ξ_3 cycle, and then switches onto the ξ_1-ξ_2-ξ_4 cycle.

3.2.3 At resonance, no switching

For this example, we choose parameters so that $\rho_{124} = 1$, $\rho_{123} > 1$, and all $\nu_{ijk3}, \nu_{ijk4} > 0$. Then the ξ_1-ξ_2-ξ_4 cycle is at resonance, so as with the Guckenheimer–Holmes resonance example (Figure 1(c)), numerical simulations near the ξ_1-ξ_2-ξ_4 cycle show trajectories approaching a periodic orbit close to that cycle in the subspace $x_3 = 0$. The ξ_1-ξ_2-ξ_4 cycle is not f.a.s. In Figure 4(c) we show a trajectory of this type, and in Figure 5(c) we show the same trajectory in linear coordinates. The ξ_1-ξ_2-ξ_4 cycle is easily identifiable in the numerics, but unlike in panels (a) and (b), the period of the oscillations does not lengthen. The dynamics near the ξ_1-ξ_2-ξ_3 cycle are as in Section 3.2.1, that is, the ξ_1-ξ_2-ξ_3 cycle is f.a.s. Trajectories that start near the ξ_1-ξ_2-ξ_3 cycle approach that cycle.

3.2.4 At resonance, switching

For our last example, we choose parameters so that $\rho_{124} = 1$, $\rho_{123} > 1$, all $\nu_{ijk3} > 0$, and at least one $\nu_{ijk4} < 0$. So again, the ξ_1-ξ_2-ξ_4 cycle is at resonance and now there is switching from the ξ_3 cycle to the ξ_4 cycle.

Trajectories that start near the ξ_1-ξ_2-ξ_4 cycle eventually approach a periodic orbit close to that cycle in the subspace $x_3 = 0$. Some trajectories (specifically, an open set starting near ξ_2) will first make one, or several, excursions around the ξ_1-ξ_2-ξ_3 cycle before switching back to the ξ_1-ξ_2-ξ_4 cycle. Thus the ξ_1-ξ_2-ξ_4 cycle does not satisfy the definitions of either Lyapunov stability or quasi-asymptotic stability. However, trajectories near the ξ_1-ξ_2-ξ_4 cycle will be observable in numerical simulations. Figure 4(d) shows such a trajectory. The trajectory is shown again in linear coordinates in Figure 5(d). It can be seen here that

the period of oscillations as the trajectory goes around ξ_1-ξ_2-ξ_3 is increasing, but after the switch to ξ_1-ξ_2-ξ_4, the period remains constant. The ξ_1-ξ_2-ξ_3 cycle also does not satisfy any definition of stability as all trajectories that start near it will eventually switch to be near the ξ_1-ξ_2-ξ_4 cycle.

3.3 Rock–Paper–Scissors–Lizard–Spock

The Rock–Paper–Scissors–Lizard–Spock network is a union of five Guckenheimer–Holmes cycles and is shown in Figure 3(b). The dynamics near this network are highly complex and some results on the stability of subcycles and omnicycles can be found in [34] and [8] (see [30] for the definition of omnicycle). Here we give four example numerical simulations where different parts of the network, but not necessarily the whole network, appear in the long term. In each case, we are in a parameter region where the entire network has been shown to be asymptotically stable, that is, all nearby trajectories approach the network, and remain close to it. A sufficient condition for asymptotic stability is that the parameters in the following equations satisfy $0 < e_B < e_A < \min(c_A, c_B)$ and $e_A \leq 1$ [8].

The network exists in the following equations:

$$\begin{aligned}
\dot{x}_1 &= x_1\left(1-\chi-c_Ax_2^2+e_Bx_3^2-c_Bx_4^2+e_Ax_5^2\right)\\
\dot{x}_2 &= x_2\left(1-\chi-c_Ax_3^2+e_Bx_4^2-c_Bx_5^2+e_Ax_1^2\right)\\
\dot{x}_3 &= x_3\left(1-\chi-c_Ax_4^2+e_Bx_5^2-c_Bx_1^2+e_Ax_2^2\right)\\
\dot{x}_4 &= x_4\left(1-\chi-c_Ax_5^2+e_Bx_1^2-c_Bx_2^2+e_Ax_3^2\right)\\
\dot{x}_5 &= x_5\left(1-\chi-c_Ax_1^2+e_Bx_2^2-c_Bx_3^2+e_Ax_4^2\right)
\end{aligned} \tag{7}$$

where again $\chi = \sum_{j=1}^{5} x_j^2$ and $e_A, e_B, c_A, c_B > 0$ are parameters. As with our other examples, there are equilibria lying on each coordinate axis, each with one non-zero coordinate. We refer to these equilibria as ξ_j $(j = 1, \ldots, 5)$.

In [34], the edges of the Rock–Paper–Scissors–Lizard–Spock network are divided into two types: A, which are heteroclinic connections between equilibria ξ_j and ξ_{j+1}, and B, which are heteroclinic connections between equilibria ξ_j and ξ_{j-2} (with indices taken mod 5). The type B connections are one-dimensional, but the type A connections are actually a two-dimensional manifold of connecting orbits. The boundary of this two-dimensional manifold consists of a one-dimensional connecting trajectory from ξ_j to ξ_{j+1}, and two type B connections from ξ_j to ξ_{j-2} and ξ_{j-2} to $\xi_{j-4} \equiv \xi_{j+1}$. Given our definition of heteroclinic cycle, when confronted with higher-dimensional connections, we select the one-dimensional connecting trajectory on the boundary to construct the heteroclinic cycle. This trajectory is contained in a coordinate plane. We still use type A when referring to it and explicitly provide the dimension when necessary.

In [8, 34], parameter regions are calculated for f.a.s. of various cycles and omnicycles. For instance, the A cycle, the heteroclinic cycle consisting only of one-dimensional type A connections, is f.a.s. if $c_Ae_A > c_Be_B$ (in addition to the conditions for asymptotic stability of the network). The B cycle, similarly, is f.a.s. if $c_Be_B > c_Ae_A$ and $c_A^3e_B > c_Be_A^3$. In [34],

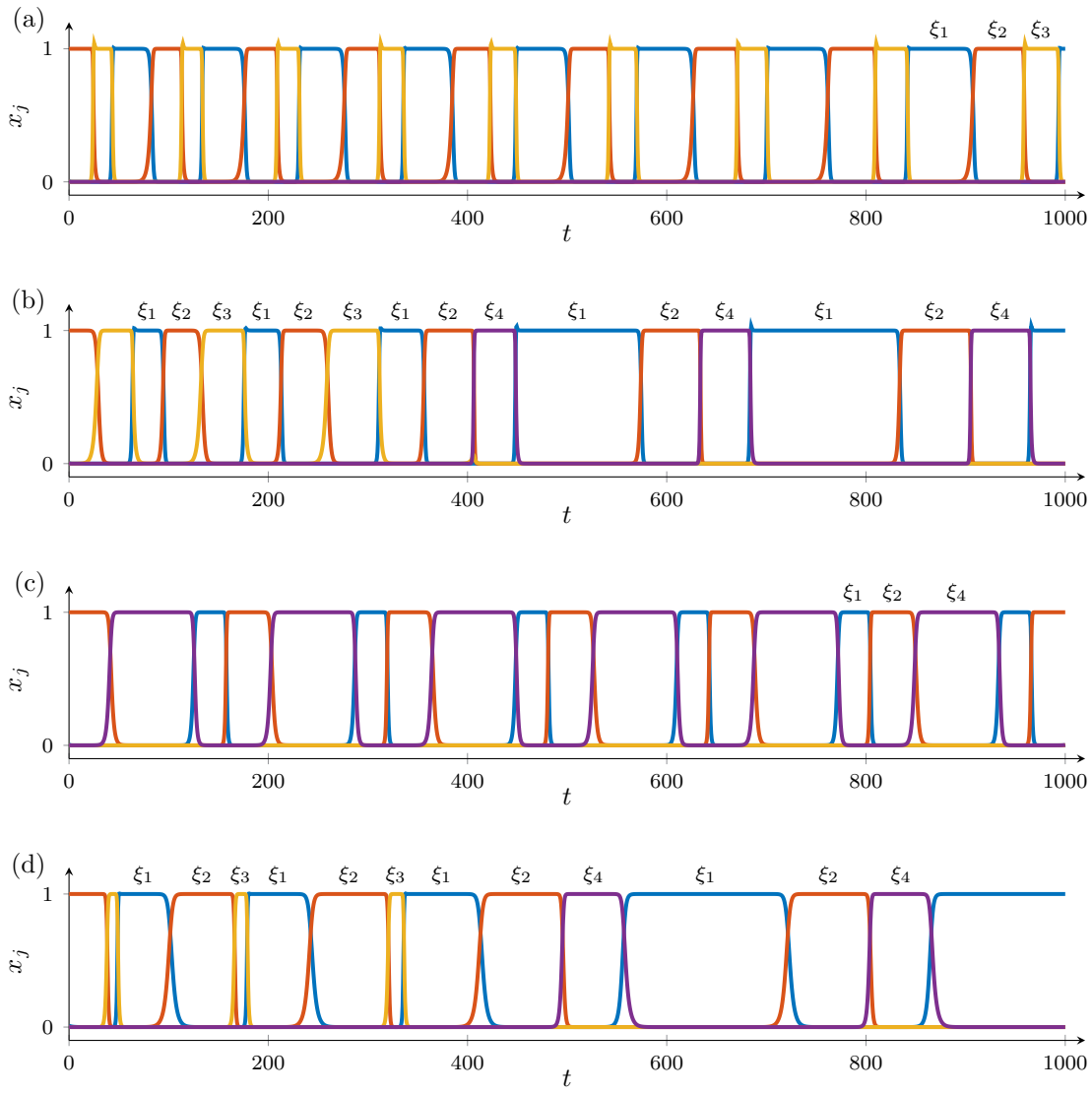


Figure 5: Trajectories near the Kirk–Silber network: the same data as in Figure 4 but on a linear scale.

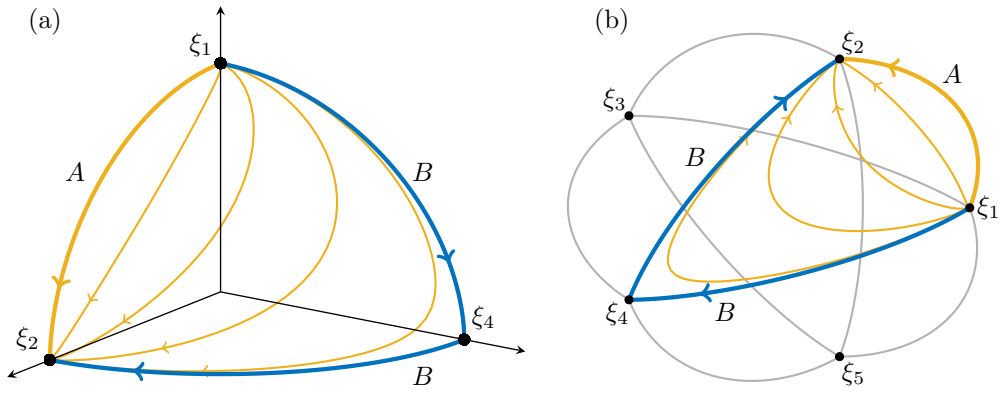


Figure 6: The figures show the two-dimensional unstable manifold of ξ_1 in the RPSLS network. In (a), a set of trajectories within this manifold is shown in the three-dimensional x_1-x_2-x_4-subspace. In (b), the same trajectories are shown using the pentacle projection described in the text (the origin is in the center of the figure, and ξ_1 is at $(1, 0)$). Connections of types A and B are labelled in each figure. The amber trajectories are part of a two-dimensional connection of type A. The connecting trajectory in the plane containing the equilibria ξ_1 and ξ_2 is the one-dimensional representative trajectory of type A. The choice of colours is the same as in Figure 3(b). The light gray lines show the remaining one-dimensional A and B connections (compare with Figure 3(b)).

it is shown that the omnicycle $AABBB$ is f.a.s. for an open region of parameter space. Expressions for the boundaries of this region can in theory be written down but are not enlightening.

For the clarity of displaying our numerical results, we use what we term the *pentacle projection* [4] to project the five-dimensional phase space onto two dimensions to provide a convenient visualisation for the dynamics. Consider a trajectory $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)) \in \mathbb{R}^5$. The pentacle projection is the map:

$$P : \mathbb{R}^5 \to \mathbb{R}^2,$$

where we write

$$y_1(t) = \sum_{j=1}^{5} x_j(t) \cos\left(\frac{2\pi j}{5}\right), \qquad y_2(t) = \sum_{j=1}^{5} x_j(t) \sin\left(\frac{2\pi j}{5}\right),$$

and then $P(\mathbf{x}) = \mathbf{y} \equiv (y_1(t), y_2(t)) \in \mathbb{R}^2$. Notice that the points $P(\xi_1), \ldots, P(\xi_5)$ lie on the vertices of a pentagon. In Figure 6 we show the two-dimensional unstable manifold of ξ_1 both in a three-dimensional subspace and also in the pentacle projection.

3.3.1 A cycle

Here we choose parameters so that the one-dimensional A cycle, consisting of one-dimensional connecting trajectories on the coordinate planes, is f.a.s. In Figure 7(a) and (b) we show

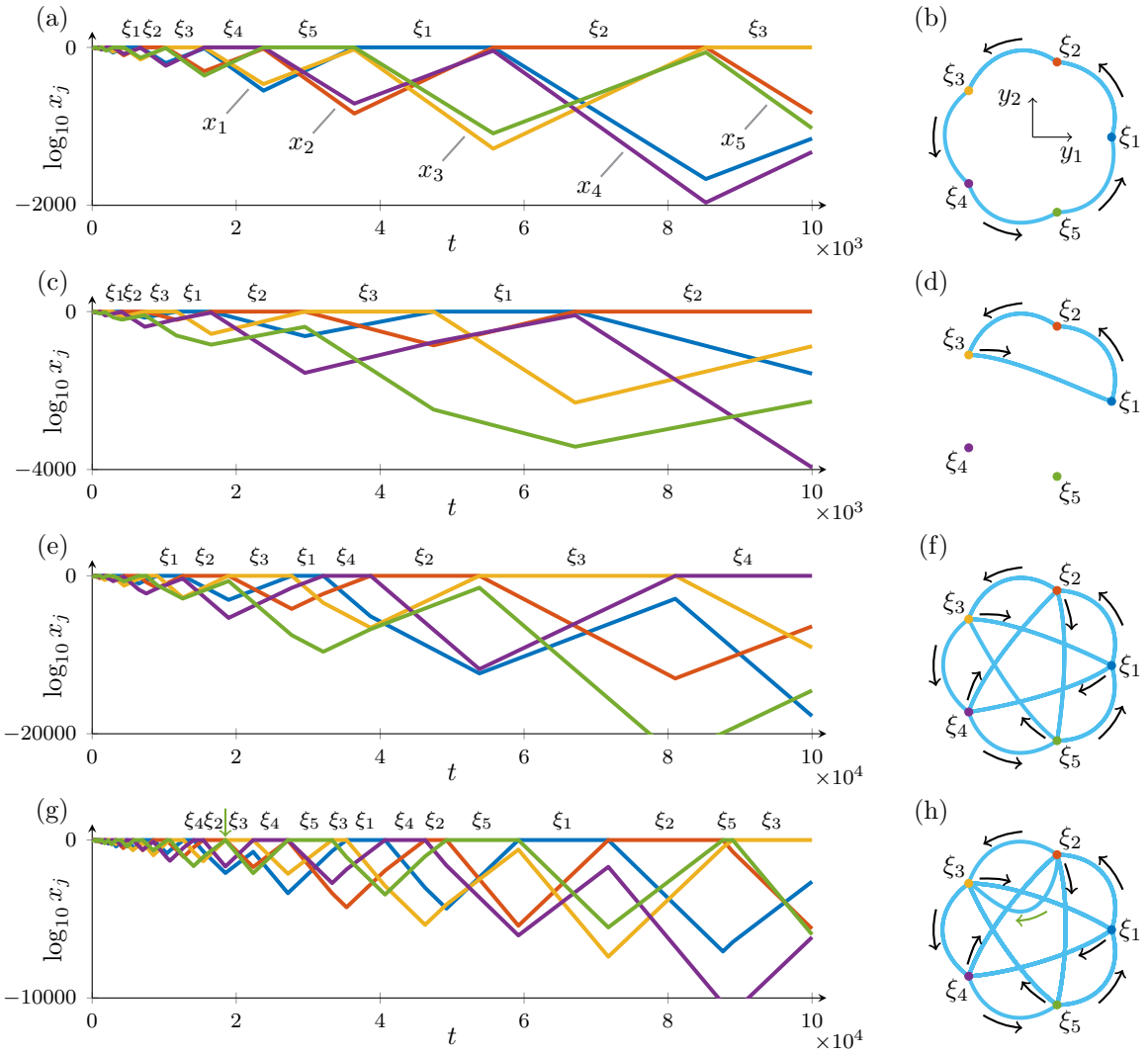


Figure 7: In each pair of panels, one trajectory is shown: on the left as a time series in logarithmic coordinates, and on the right in the pentacle projection. In (a) and (b) (section 3.3.1) the trajectory approaches the A-subcycle. In (c) and (d) (section 3.3.2) the trajectory approaches one of the AAB-cycles, in this case, the ξ_1-ξ_2-ξ_3 cycle. In (e) and (f) (section 3.3.3), the $AABBB$ omnicycle is f.a.s., and the trajectory visits the entire one-dimensional RPSLS network. In (g) and (h) (section 3.3.4), the trajectory shown visits the equilibria in an aperiodic fashion. The entire one-dimensional RPSLS network is visited, but in addition the trajectory sometimes makes excursions through the interior of the two-dimensional unstable manifolds, indicated by the green arrow between ξ_2 and ξ_3 in (g) and (h). In all cases, the panels in the right column have the transient up to $t = 500$ removed. Parameters are given in table 2 in Appendix A.

a trajectory that asymptotes onto this subcycle (ξ_1-ξ_2-ξ_3-ξ_4-ξ_5). Although it has not been proven that there are no other attracting dynamics for these parameter values, extensive numerical investigations have been unable to find any. Numerical experiments strongly suggest that almost all initial conditions asymptote onto the A-cycle, and as such, only the A-type edges of the network are seen in the long term.

3.3.2 AAB cycle

Here we choose parameters so that each AAB cycle (of which there are five) is f.a.s. Note that the AAB cycle, when considering one-dimensional connecting trajectories in coordinate planes, is equivalent to a Guckenheimer and Holmes cycle (with a slight restriction on the choice of parameter values). The cycle that appears in numerical simulations will depend on the initial conditions. In Figure 7(c) and (d) we show a trajectory that asymptotes onto the ξ_1-ξ_2-ξ_3 cycle. Again, it has not been proven that there are no other attracting dynamics for these parameter values but numerical experiments strongly suggest this is the case. Trajectories will not explore the entire network in the long term, but will be attracted to a subcycle of length three.

3.3.3 $AABBB$ cycling

Here we choose parameter such that the $AABBB$ omnicycle is f.a.s. That is, there is an open set of initial conditions such that trajectories repeated follow the edges in the order $AABBB$ and also get closer to the network. A trajectory of this type is shown in Figure 7(e) and (f). In this case, the entire one-dimensional RPSLS network is visited by the trajectory.

3.3.4 Aperiodic behaviour

Our final example in this section uses parameters for which the A-cycle is f.a.s. In addition, numerical experiments have found trajectories that perform apparently irregular cycling around the network: the observed sequence of edges A and B that is followed by the trajectory is aperiodic. An example of such as trajectory is shown in Figure 7(g) and (h). The entire network of one-dimensional connections is visited, but in addition, the trajectory occasionally moves away from this set of one-dimensional connections and passes through the interior of the two-dimensional connection from ξ_j to ξ_{j+1}. This can be seen in Figure 7(h) where there is a curve from ξ_2 to ξ_3 that is not close to the one-dimensional RPSLS network.

We suspect, but have not proved, that for a long enough trajectory, the entirety of the two-dimensional RPSLS network would be visited by the trajectory, in the sense that the trajectory would eventually come arbitrarily close to any point on the two-dimensional manifolds.

4 Visibility

The examples in Section 3 clearly indicate that establishing the stability properties of a heteroclinic network and its cycles does not always provide information about what to expect from numerical simulations, especially in the long-run. In this section, without attempting to cover all possible instances, we suggest the use of the notion of *visibility* to express the likely outcome of the dynamics near a heteroclinic network. We are particularly concerned with the fact that a network can be called 'asymptotically stable' (or a weaker version of this) while trajectories could visit only a part of the network, with the remaining part never seen in numerical simulations, or only seen during a transient. We introduce the concept of 'visibility' to mean a set that is attracting and nearby trajectories repeatedly visit the *entire* set.

For example, in the context of population dynamics when the concern is to establish whether various species survive, the stability of the entire network does not give this information. Take the case of five species competing as in the RPSLS network. In all four of the examples we give above, the entire RPSLS network is asymptotically stable, but this does not establish the survival of all the species. For example in the second example (Section 3.3.2), only three of the species survive, and which ones depends on the initial conditions. In this case, the concept of visibility will classify each of the five three-species subcycles as visible, but not the whole network.

4.1 Definitions of visibility

We align our definitions of visibility (and all its variants) along the lines of the previous definitions of stability. In particular:

(i) we maintain the use of prefixes 'fragmentary-' or 'essential-' when describing visible invariant sets to clarify *from where* the set is visible, that is, from which initial conditions near the set. If neither are used the invariant set is visible from a neighbourhood of the set. In addition, we introduce the prefix 'almost-' to mean a set is visible from a neighbourhood, excluding a set of measure zero.

(ii) we maintain the use of the prefix 'quasi-' when trajectories may contain an initial transient that moves away from the visible invariant set. That is, we allow visible sets to not be visible for an initial transient. This clarifies *from when* the set is visible.

(iii) we maintain the use of the prefix 'Lyapunov' to indicate that trajectories should 'stay close' to the visible set.

In parallel with the corresponding definitions for stability, we define *Lyapunov-visibility* to mean sets for which a neighbourhood of initial conditions stay close to the set – and visit the entire set (but may not asymptote onto that set), and *quasi-visibility* to mean sets for which a neighbourhood of initial conditions eventually asymptote onto the entire set (but may move a large distance away from the set initially). Similarly, if a set is both Lyapunov-

and quasi-visible, then we say it is *asymptotically visible*. The precise definitions we propose are given next, along with a summary in Table 1. The table includes shortened versions of each definition and pointers to examples illustrating each definition. As discussed above, initial conditions that start in an invariant subspace near a heteroclinic cycle will not visit the whole cycle, so some of our examples are not heteroclinic cycles. These are expressed in polar coordinates, and include infinite families of periodic orbits, saddle-node on an invariant circle (SNIC) bifurcations, etc.

Let $N_\delta = \{x \notin X : d(x, X) < \delta\}$. This set is a ball around X excluding X itself. We do this because if X is a heteroclinic cycle or network, points in X (that is, equilibria and connecting trajectories), never visit the whole of X.

Definition 8 *A closed invariant set X is called* Lyapunov-visible*, or* L-visible*, if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\bar{x} \in X$ and for all $x \in N_\delta$, $d(\varphi(t, x), X) < \varepsilon$ for all $t > 0$, and there exists an increasing sequence (t_n) with $\lim_{n\to\infty} t_n = \infty$ such that $d(\varphi(t_n, x), \bar{x}) < \varepsilon$.*

Demanding that trajectories repeatedly come within ε of every point in X ensures that trajectories starting at initial conditions near X visit the whole of X (and not just a subset of X).

Definition 9 *A closed invariant set X is called* quasi-visible*, or* Q-visible*, if there exists $\delta > 0$ such that if an initial condition $x \in N_\delta$ then $\omega(x) = X$.*

Notice that this definition differs from Definition 3 by requiring that $\omega(x) = X$, so trajectories must visit the whole of X.

These two definitions (definitions 8 and 9) of types of visibility imply the corresponding versions of stability (definitions 2 and 3), but not vice-versa.

Definition 10 *A closed invariant set X is called* asymptotically visible*, or* A-visible *if it is both Q-visible and L-visible.*

In some examples, trajectories which start close to one subcycle of a network loop around a different cycle a potentially unbounded number of times before eventually being attracted back to the first cycle. The first cycle is what will be seen in experiments after a long enough transient. This leads to the following definition of *visible*.

Definition 11 *A closed invariant set X is called* visible *if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\bar{x} \in X$, and for all $x \in N_\delta$, there exists a $T > 0$, such that $d(\varphi(t, x), X) < \varepsilon$ for all $t > T$, and there exists an increasing sequence (t_n) with $\lim_{n\to\infty} t_n = \infty$ such that $d(\varphi(t_n, x), \bar{x}) < \varepsilon$.*

The difference between Definitions 8 and 11 is that in 11, we remove a transient, whose length may depend on the initial condition, before insisting that the trajectory remains close to the entire set X. It is important to note that a visible set could be neither Q- nor L-visible. With this definition it is possible to have some nearby points move away first, and come back, but no points ever asymptote onto the set.

Table 1: Summary of the definitions. Recall that the prefixes L-, Q- and A- mean Lyapunov-, quasi- and asymptotically-. Recall X is a closed invariant set, and "start near X" means the initial condition is close to X but not in X. In the almost, essentially and fragmentarily definitions, the meaning of "start near X" is refined, and V-visible refers to L-visible, Q-visible, A-visible or visible (V is a placeholder for L, Q, A or 'nothing').

Definition	Qualitative statement	Examples
Definition 8 L-visible	Trajectories that start near X stay close to X, and repeatedly come close to every point in X.	$\dot{\theta} = 1$, $\dot{r} = 0$, $X = \{\text{the circle } r = 1\}$.
Definition 9 Q-visible	Trajectories that start near X have ω-limit set that is X.	$\dot{\theta} = 1 - \cos\theta$, $\dot{r} = r(1 - r^2)$, $X = \{(\theta = 0, r = 1)\}$.
Definition 10 A-visible (A = L and Q)	Trajectories that start near X stay close to X, and have ω-limit set that is X.	$\dot{\theta} = 1$, $\dot{r} = r(1 - r^2)$, $X = \{\text{the circle } r = 1\}$.
Definition 11 visible	Trajectories that start near X stay close to X after a transient, and repeatedly come close to every point in X.	$\dot{\theta} = 1 - \cos\theta$, $\dot{r} = 0$, $X = \{(\theta = 0, r = 1)\}$.
Definition 12 almost V-visible	As for V-visible, except the set of allowed initial conditions excludes a set of measure zero. $(V \in \{L, Q, A, \emptyset\})$	The GH cycle in §3.1 is almost A-visible or almost L-visible. One KS subcycle in §3.2.2 is almost Q-visible.
Definition 13 essentially V-visible	As for V-visible, except the set of allowed initial conditions excludes a set that has measure zero as X is approached.	One of the KS subcycles in §3.2.1 or the A cycle in §3.3.1 can be essentially A-visible if parameters are chosen appropriately.
Definition 14 fragmentarily V-visible	As for V-visible, except the set of allowed initial conditions has positive measure as X is approached.	Both KS subcycles in §3.2.1 are fragmentarily A-visible. One KS subcycle in §3.2.3 is fragmentarily A-visible, the other is fragmentarily L-visible. One KS subcycle in §3.2.4 is fragmentarily visible.

In Definitions 8–11 we use neighbourhoods of X excluding points within X itself. Thus a set X is Q-visible if it eventually attracts a neighbourhood of points, not in X, which are allowed to diverge from X for a finite time. Analogously, L-visible sets are such that initial conditions in a neighbourhood of X, but not in X, do not move far away. When studying heteroclinic cycles in a network the use of neighbourhoods is excessively strong even when X itself is excluded, for two reasons. Firstly, systems containing robust heteroclinic connections typically contain invariant subspaces. Trajectories that start in these subspaces will necessarily not visit the entire network (due to the invariance of that subspace). These subspaces will have measure zero in the phase space. Secondly, a neighbourhood of an equilibrium with more than one outgoing direction necessarily has points following both heteroclinic connections, and thus requiring all trajectories in a neighbourhood to follow a particular path is overly restrictive. In order to include heteroclinic cycles in networks in our study, we use the prefixes almost-, essentially- and fragmentarily- to allow for the exclusion of sets (of various sizes) from a neighbourhood of X. It is these examples of dynamics near heteroclinic networks which motivates our study, as here visibility is different from stability. In the first four rows of table 1, we give examples of systems which satisfy the visibility definitions without any of these qualifying prefixes. However, these examples do not use heteroclinic dynamics, and here visibility is equivalent to stability.

Definition 12 *A closed invariant set X is called* almost *Lyapunov-/quasi-/asymptotically-visible, if the set N_δ in the definitions for Lyapunov-/quasi-/asymptotic- visibility is replaced by a set M_δ that differs from N_δ by only a set of measure zero.*

Definition 13 *A closed invariant set X is called* essentially *Lyapunov-/quasi-/asymptotically- visible, if the set N_δ in the definitions for Lyapunov-/quasi-/asymptotic- visibility is replaced by a set $M_\delta \subset N_\delta$ that satisfies*

$$\lim_{\varepsilon\to 0} \frac{\ell(V_\varepsilon(X)\cap M_\delta)}{\ell(V_\varepsilon(X))} = 1,$$

where V_ε is an ε-neighbourhood of X.

Definition 14 *A closed invariant set X is called* fragmentarily *Lyapunov-/quasi-/asymptotically visible, if the set N_δ in the definitions for Lyapunov-/quasi-/asymptotic- visibility is replaced by a set $M_\delta \subset N_\delta$ that has positive measure.*

The size of M_δ in each of these definitions means that 'almost-' implies 'essentially-' implies 'fragmentarily-'. Note that we are using the prefixes fragmentarily- and essentially-*without* requiring the Lyapunov prefix. This is different to the original definition of f.a.s. in [29]. For instance, in Example 3.2.2, the ξ_1-ξ_2-ξ_4 cycle is fragmentarily-quasi-visible. A positive measure set of points that start in a neighbourhood of that cycle eventually asymptote onto that cycle, but some of these trajectories will first make an excursion around the ξ_1-ξ_2-ξ_3 cycle. Kirk and Silber [18] refer to this behaviour as 'asymptotically stable in spirit'.

4.2 Application of visibility definitions to examples

The Guckenheimer–Holmes cycle in Section 3.1 is almost asymptotically visible when $\rho_{123} > 1$ and almost Lyapunov visible when $\rho_{123} = 1$. In this example there is a correspondence between stability and almost visibility due to the simplicity of the system: there is only one route trajectories can take around the cycle.

For the Kirk–Silber network, we note that it is never possible for the entire network to be visible in any sense: trajectories only ever asymptote onto one subcycle of the network. However, the subcycles can individually have some type of visibility. For the example in Subsection 3.2.1, both cycles are fragmentarily asymptotically-visible. In Subsection 3.2.2, the ξ_1-ξ_2-ξ_4 cycle is almost quasi-visible; the ξ_1-ξ_2-ξ_3 cycle is not visible. In Subsection 3.2.3, the ξ_1-ξ_2-ξ_3 cycle is fragmentarily asymptotically-visible and the ξ_1-ξ_2-ξ_4 cycle is fragmentarily Lyapunov-visible. Finally, in Subsection 3.2.4, the ξ_1-ξ_2-ξ_3 cycle is not visible, but the ξ_1-ξ_2-ξ_4 cycle is fragmentarily visible. Here the ξ_1-ξ_2-ξ_4 cycle is not fragmentarily-quasi-visible, because trajectories remain a finite distance away from the cycle due to the resonance condition. It is fragmentarily-Lyapunov-visible, but the measure of the set from which it is fragmentarily-Lyapunov-visible is different from the measure of the set from which it is fragmentarily visible, because some nearby trajectories will make one or more excursions around the ξ_1-ξ_2-ξ_3 cycle first. This could be made precise with a variant of stability indices [31], but that is beyond the scope of this paper.

The one-dimensional RPSLS network, in contrast to the KS network, can be fragmentarily-asymptotically visible, depending on parameters. For the parameters chosen in subsection 3.3.1, results from [8, 34] prove that the A-cycle is fragmentarily-asymptotically visible. Our numerical experiments strongly suggest that the network does not satisfy even the weakest definition of visible, because the B connections are not visited in the long term, and in fact that the A-cycle has stronger visibility: that it is almost asymptotically visible. For the parameters chosen in subsection 3.3.2, each of the five AAB-cycles is fragmentarily-asymptotically visible. Again, numerical experiments strongly suggest that the entire network satisfies none of the definitions of any type of visibility. For the parameters chosen in subsection 3.3.3, results from [34] can be used to prove the entire network is fragmentarily visible, but again we conjecture that in fact the network is almost asymptotically visible. For the parameters chosen in subsection 3.3.4 the numerical experiments suggest the network is not visible. Although the irregular trajectories repeatedly visit all parts of the network, there are rare excursions away from the one-dimensional network. We conjecture that these excursions continue to occur indefinitely, thus the 1D network does not satisfy any definitions of visibility. Furthermore, we conjecture that the 2D RPSLS network is fragmentarily visible, as eventually all parts of the two-dimensional unstable manifold will be visited, but a proof of this is beyond the scope of this paper.

5 Discussion

In this paper we have introduced a new concept, visibility, which we believe is as important as the stability of an invariant object when one is concerned with predicting observations of a dynamical system after transients have decayed. Our examples are all heteroclinic cycles or networks; it has been clear for some years that traditional stability measures such as asymptotic stability are not necessarily the most useful in these cases, which is why weaker properties such as fragmentary asymptotic stability have been introduced. To determine the visibility of an object one must ask additional questions to determine whether the entire object will be visited by trajectories after transients have decayed.

This is of course not to say that stability is unimportant or not useful. For example, in game theory the outcome of a game is frequently a Nash equilibrium. In a class of games known as 'coordination games' there is more than one Nash equilibrium and deciding which one occurs is still an open and relevant question. Recently, Castro [7] has used techniques from heteroclinic dynamics to, under some assumptions, decide which Nash equilibrium is chosen (in this case, one that is asymptotically stable in an extension of the original game).

In this paper we have not provided any proofs, but we note that proofs of some of our statements can be deduced from previously known results. For example, results in the original paper by Kirk and Silber [18] show that the Kirk–Silber network cannot be visible, for any choices of parameters. Results from [34] showing that the $AABBB$ cycle of the RPSLS network is f.a.s. imply that the network is fragmentarily-asymptotically visible.

We conclude by offering several avenues of work that require further investigation. First, we note that proving whether or not a network is visible will typically require some graph theory as well as stability calculations. For example, in the RPSLS network, stability of any of the omnicycles A, B or AAB does *not* imply visibility of the whole network because each of these omnicycles is itself a heteroclinic cycle which is a proper subset of the whole network, but stability of $AABBB$ does, because it is not. In general, one would need to examine the structure of the digraph representing the network and ask whether repeating a certain sequence of types of edges generates a path that covers the entire graph. For larger networks/digraphs, determining this could be computationally difficult.

Second, the issue of two-dimensional unstable manifolds requires special care. Our definition of heteroclinic networks in the introduction implies that they are one-dimensional objects. However, many heteroclinic networks have at least one equilibrium point that has an unstable manifold of dimension two or higher. In some cases, such as the KS-network, it makes sense to study the visibility, or stability, of the one-dimensional set of heteroclinic connections that intersect with the invariant coordinate planes. However, our numerical computations near the RPSLS network, as well as in some other larger networks, have indicated that there exists aperiodic switching where trajectories repeatedly move away from this one-dimensional set of connections and pass through the interior of the two-dimensional unstable manifold. In this case, the original (one-dimensional) network does not satisfy our definition of visible as trajectories repeatedly move away from it. We might ask the question of whether we could include the entire two-dimensional manifold in the invariant object we study, but then a proof of visibility would require proving that trajectories repeatedly visit

the entire two-dimensional manifold. How to construct such a proof is beyond our current understanding.

Some of these issues may be better understood using techniques recently introduced to examine Poincaré maps near heteroclinic networks using techniques from piecewise dynamical systems [14, 28]. These methods allows us both to have some level of control over the description of the observed transients and also potentially the opportunity to prove statements about the existence of chaotic attractors.

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The data associated with this paper are openly available from the University of Leeds Data Repository (`http://doi.org/10.5518/1652`) [11], as are the programs that generated the data. For the purpose of open access, the authors have applied a Creative Commons Attribution (CC BY) license to any Author Accepted Manuscript version arising from this submission.

A Parameters used in figures

Table 2 gives the parameters used in the creation of Figures 2, 4, 5 and 7.

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Table 2: Parameter values for the Guckenheimer–Holmes example (Figure 2), the Kirk–Silber examples (Figures 4 and 5) and the Rock–Paper–Scissors–Lizard–Spock examples (Figure 7). The data are available from [11].

Cycle and figure	Parameter values
Guckenheimer– –Holmes Figure 2	$e_{12} = 0.9$, $c_{13} = 1.0$. $e_{23} = 1.5$, $c_{21} = 0.9$. $e_{31} = 0.6$, $c_{32} = 1.2$.
Kirk–Silber Figures 4 and 5	(a) $e_{12} = 0.4$, $c_{13} = 1.5$, $c_{14} = 1.3$, $c_{21} = 1.3$, $e_{23} = 1.9$, $e_{24} = 1.8$, $e_{31} = 1.9$, $c_{32} = 0.8$, $c_{34} = 0.4$, $e_{41} = 1.8$, $c_{42} = 0.8$, $c_{43} = 1.2$. (b) $e_{12} = 0.7$, $c_{13} = 0.5$, $c_{14} = 0.9$, $c_{21} = 1.6$, $e_{23} = 0.4$, $e_{24} = 1.9$, $e_{31} = 1.4$, $c_{32} = 0.6$, $c_{34} = 0.7$, $e_{41} = 1.9$, $c_{42} = 2.1$, $c_{43} = 0.8$. (c) $e_{12} = 1.3$, $c_{13} = 1.7$, $c_{14} = 0.7$, $c_{21} = 1.2$, $e_{23} = 1.0$, $e_{24} = 0.5$, $e_{31} = 2.0$, $c_{32} = 1.4$, $c_{34} = 0.5$, $e_{41} = 0.646$, $c_{42} = 0.5$, $c_{43} = 0.7$. (d) $e_{12} = 0.3$, $c_{13} = 1.1$, $c_{14} = 0.3$, $c_{21} = 0.3$, $e_{23} = 0.9$, $e_{24} = 0.6$, $e_{31} = 1.5$, $c_{32} = 1.5$, $c_{34} = 0.2$, $e_{41} = 0.4$, $c_{42} = 0.8$, $c_{43} = 0.9$.
Rock–Paper– –Scissors–Lizard– –Spock Figure 7	(a) $c_A = 1.30$, $e_A = 1.00$, $c_B = 1.50$, $e_B = 0.80$. (b) $c_A = 1.10$, $e_A = 1.00$, $c_B = 2.70$, $e_B = 0.80$. (c) $c_A = 1.10$, $e_A = 1.00$, $c_B = 1.80$, $e_B = 0.80$. (d) $c_A = 1.02$, $e_A = 1.00$, $c_B = 1.25$, $e_B = 0.80$.

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