White Rose Research Online

eprints@whiterose.ac.uk
https://eprints.whiterose.ac.uk
Universities of Leeds, Sheffield and York

Deposited via The University of Sheffield.

White Rose Research Online URL for this paper:
https://eprints.whiterose.ac.uk/id/eprint/247369/

Version: Published Version

Article:
MADARÁSZ, J., STANNETT, M. and SZÉKELY, G. (2026) DEFINABLE COORDINATE GEOMETRIES OVER FIELDS. The Review of Symbolic Logic, 19 (3). pp. 459-490. ISSN: 1755-0203

https://doi.org/10.1017/s175502032610118x

Reuse
Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown
If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

UNIVERSITY OF LEEDS
University of Sheffield
UNIVERSITY of York

The Review of Symbolic Logic
Volume 19, Number 3, September 2026

DEFINABLE COORDINATE GEOMETRIES OVER FIELDS

JUDIT MADARÁSZ
HUN-REN Alfréd Rényi Institute of Mathematics
MIKE STANNETT
University of Manchester
and
GERGELY SZÉKELY
HUN-REN Alfréd Rényi Institute of Mathematics & University of Public Service

Abstract. We define general notions of coordinate geometries over fields and ordered fields, and consider coordinate geometries that are given by finitely many relations that are definable over those fields. We show that the automorphism group of such a geometry determines the geometry up to definitional equivalence; moreover, if we are given two such geometries $\mathcal{G}$ and $\mathcal{G}'$, then the concepts (explicitly definable relations) of $\mathcal{G}$ are concepts of $\mathcal{G}'$ exactly if the automorphisms of $\mathcal{G}'$ are automorphisms of $\mathcal{G}$. We show this by first proving that a relation is a concept of $\mathcal{G}$ exactly if it is closed under the automorphisms of $\mathcal{G}$ and is definable over the field; moreover, it is enough to consider automorphisms that are affine transformations.

We show how this result can be applied to quickly determine relationships and differences between various geometries and spacetimes, including ordered affine, Euclidean, Galilean, Newtonian, Late Classical, Relativistic and Minkowski spacetimes (we first define these spacetimes and geometries using a Tarskian first-order language centred on the ternary relation Bw of betweenness). We conclude with a selection of open problems related to the existence of certain intermediate geometries.

§1. Introduction. The most famous ancient work that treats geometry systematically and axiomatically is Euclid's Elements (*c*. 300 B.C.). Another important milestone was when Descartes opened a new analytic perspective to geometry by introducing coordinate systems. Later Hilbert provided a formal axiom system for Euclidean geometry, which Tarski modified to get an axiomatization within first-order logic. A typical representation theorem for an axiom system such as Hilbert's or Tarski's states that all its models are isomorphic to Cartesian spaces over certain fields, and hence coordinatizable.

In this paper, we define general notions of coordinate geometries over fields and ordered fields. This definition naturally extends to the notion of coordinatizable geometries, i.e., models that are isomorphic to coordinate geometries. For example,

Received: July 16, 2025 UTC.
2020 *Mathematics Subject Classification*: Primary 03A10, 03B10, 03B80.
Key words and phrases: definability, coordinate geometries, concepts, Erlangen program, automorphisms, Minkowski spacetime, Newtonian spacetime, Galilean spacetime.

© The Author(s), 2026. Published by Cambridge University Press on behalf of The Association for Symbolic Logic. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited. The written permission of Cambridge University Press must be obtained for commercial re-use or in order to create a derivative work.

doi:10.1017/S175502032610118X

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

by Tarski's theorem [45, theorem 1], any model of his axiom system is isomorphic to a two-dimensional coordinate geometry $\mathcal{E}ucl$ (see page 6) over some real closed field. These kinds of representation theorem show the strong connection between the synthetic (purely axiomatic) and analytic (coordinate-system-based) approaches to geometry. In [9], an axiomatization for Minkowski spacetime is given in similar style.

The coordinate geometries we consider are those given by finitely many relations definable over fields or ordered fields (constants and n-ary functions can be treated as unary and $(n+1)$-ary relations in the usual way). We show that the automorphism group of such a geometry determines the geometry up to definitional equivalence; moreover, if we are given two such geometries $\mathcal{G}$ and $\mathcal{G}'$, then the concepts (explicitly definable relations) of $\mathcal{G}$ are concepts of $\mathcal{G}'$ exactly if the automorphisms of $\mathcal{G}'$ are automorphisms of $\mathcal{G}$ (see Corollary 4.6 and Theorem 4.5). We show this by first proving that a relation is a concept of $\mathcal{G}$ exactly if it is closed under the automorphisms of $\mathcal{G}$ and is definable over the field; moreover, it is enough to consider automorphisms that are affine transformations (see Theorem 4.2).

Similar theorems were proven for affine and Euclidean geometries that are coordinatizable over real closed fields in [13, sec. 2.4] and [35]. Similar investigations on the lattices of concept-sets of various structures can be found in [30, 37, 39, 40, 41].

This work is part of an ongoing project aiming to understand the algebras of concepts of concrete mathematical structures and the conceptual 'distances' between them [19–21]. By Theorem 4.2, in the case of many historically relevant coordinate geometries, one can reduce the understanding of concepts to an understanding of (affine) automorphisms, which can be useful for solving problems associated with that project. For example, Theorem 4.2 is a key result used to prove a conjecture of Andréka and Németi stating that adding any classical concept to the concepts of special relativity results in the theory of late classical kinematics [27].

More generally, this research is part of the Andréka–Németi school's project to give a logic-based foundation and understanding of relativity theories in the spirit of the Vienna Circle and Tarski et al.'s initiative *Logic, Methodology and Philosophy of Science* [2, 10, 11].

1.1. Background. The relationship between structure, symmetry and coordinatization has long been a focus of sustained study. Since coordinates are essentially arbitrary labels attached to locations, it is argued by many—especially those working in theoretical physics (see, e.g., [28])—that the appropriate framework for expressing the structural properties of geometries and the bodies inhabiting them should be fundamentally coordinate-free.

Wallace [48] and others argue that coordinatization remains relevant. Even if phenomena exist independently of coordinates, nonetheless coordinatizations provide a framework in which to characterise and study these phenomena in detail. Moreover, different approaches to coordinatization admit different structure-preserving automorphisms of the underlying space, and this in turn reflects properties of the kind of physical world we are attempting to model. For example, North [32, pp. 74, 96, 110] notes that Newton's laws tacitly assume a world in which Cartesian coordinates are preferred, and hence there must (in his physics) be physical structure corresponding to the mathematical structures required to underpin this kind of coordinate system. Moreover, since we cannot reasonably require automorphisms to preserve in-principle-undetectable properties, the implied absence of any preferred frame suggests that

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

Newton's laws presuppose a Galilean spacetime structure. (North offers arguments both for and against this standpoint.)

Motivated by questions in the philosophy of physics, there is much ongoing research focussed on finding and analyzing criteria which can be used to determine when one mathematical object is structurally richer than another (see, e.g., [3, 5, 6, 31, 42, 49]). One such criterion is (SYM*), which is formulated in [42] as: "If $Aut(X)$ properly contains $Aut(Y)$, then X has less structure than Y." By Corollary 4.6 on p. 8, criterion (SYM*) works perfectly for a large class of coordinate geometries when 'structure' is understood in terms of explicitly definable relations.

1.2. Notation and conventions. In general, we use standard model theoretic and set theoretic notations, and we assume that the reader is familiar with basic algebraic constructs like groups, fields and vector spaces. The symbol □ indicates the end (or absence) of a proof. By a *model*, we mean a first-order structure in the sense of classical model theory.

The language of *fields* contains binary operations $+$ and $\cdot$ for addition and multiplication and constants 0 and 1; in addition to these, the language of *ordered fields* contains a binary relation $\leq$ for ordering. Other common operations, e.g., subtraction, are derived in the usual way.

We write $\mathbb{N}^+$ for the set of positive natural numbers, and fix an enumeration $(v_i)_{i\in\mathbb{N}^+} = (v_1, v_2, v_3, \ldots)$ of distinct variables.

To abbreviate formulas, we use equality and operations on sequences of variables componentwise, e.g., if $\vec{x} = \langle v_1, \ldots, v_d\rangle$ and $\vec{y} = \langle v_{d+1}, \ldots, v_{2d}\rangle$, then $\vec{x} = \vec{y}$ abbreviates formula $v_1 = v_{d+1} \wedge \cdots \wedge v_d = v_{2d}$ and $\vec{y} + v\vec{x}$ denotes the sequence $\langle v_{d+1} + v_{3d+1} \cdot v_1, \ldots, v_{2d} + v_{3d+1} \cdot v_d\rangle$ of terms (we take v to stand for the next fresh variable, in this case v_{3d+1}).

Let $\mathsf{R} \subseteq H^n$ be an arbitrary relation over a nonempty set H and let $f : H \to H$ be a map. We say that R is *closed under* f iff $(a_1, \ldots, a_n) \in \mathsf{R} \implies (f(a_1), \ldots, f(a_n)) \in \mathsf{R}$, and that f *respects* R iff $(a_1, \ldots, a_n) \in \mathsf{R} \iff (f(a_1), \ldots, f(a_n)) \in \mathsf{R}$. If G is a set of functions from H to H that form a group under composition, then R is closed under all elements of G iff all elements of G respect R. We often write "$\mathsf{R}(a_1, \ldots, a_n)$" in place of "$(a_1, \ldots, a_n) \in \mathsf{R}$."

Given models $\mathfrak{M}$ and $\mathfrak{N}$ for some first-order languages, we write M and N to denote their universes, respectively. If $(a_1, \ldots, a_n) \in M^n$ and φ is a formula in the language of $\mathfrak{M}$, we write $\mathfrak{M} \models \varphi[a_1, \ldots, a_n]$ to mean that φ is satisfied in model $\mathfrak{M}$ by any evaluation $e : \{v_1, v_2, \ldots\} \to M$ of variables for which $e(v_1) = a_1$, ..., $e(v_n) = a_n$. If φ is a formula, the expression $\varphi(v_1, \ldots, v_n)$ indicates that the free variables of φ come from the set $\{v_1, \ldots, v_n\}$.

The set of automorphisms of model $\mathfrak{M}$ is denoted by $\mathsf{Aut}\,\mathfrak{M}$. We note that, if the language of model $\mathfrak{M}$ contains only relation symbols, then function $f : M \to M$ is an automorphism of $\mathfrak{M}$ exactly if it is a bijection that respects all the relations of $\mathfrak{M}$.

The inequality $\mathfrak{A} \leq \mathfrak{B}$ denotes that algebraic structure $\mathfrak{A}$ is a subalgebra of algebraic structure $\mathfrak{B}$. If $\mathfrak{M}$ is a model and R is a relation on the universe of $\mathfrak{M}$, then model $\langle\mathfrak{M}, \mathsf{R}\rangle$ is the expansion of $\mathfrak{M}$ with relation R (to some language that contains exactly one extra relation symbol whose interpretation in $\mathfrak{M}$ is R).

If A and B are sets, then both $A \subsetneq B$ and $B \supsetneq A$ mean that A is a proper subset of B.

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

§2. Concepts. We say that an n-ary relation R on M is *definable* in $\mathfrak{M}$ iff there is a formula $\varphi(v_1, \dots, v_n)$ in the language of $\mathfrak{M}$ that defines it; i.e., for every $(a_1, \dots, a_n) \in M^n$, we have

$$(a_1, \dots, a_n) \in \mathsf{R} \iff \mathfrak{M} \models \varphi[a_1, \dots, a_n]. \tag{1}$$

Definable n-ary relations will also be called n-ary *concepts* of $\mathfrak{M}$, and the set of n-ary concepts of $\mathfrak{M}$ will be denoted by $\mathsf{Conc}_n \mathfrak{M}$. The *concept-set* of $\mathfrak{M}$, i.e., the set of all relations definable in $\mathfrak{M}$, is then

$$\mathsf{Conc}\,\mathfrak{M} \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}^+} \mathsf{Conc}_n \mathfrak{M}.$$

We call models $\mathfrak{M}$ and $\mathfrak{N}$ *definitionally equivalent* iff they have the same concepts, i.e., $\mathsf{Conc}\,\mathfrak{M} = \mathsf{Conc}\,\mathfrak{N}$. This notion of definitional equivalence is just a reformulation of the usual one that best fits the context of this paper (see, e.g., [15, p. 51] or [29, p. 453] for the standard definition).[1]

In particular, because the universe M of $\mathfrak{M}$ is definable in $\mathfrak{M}$ as a unary relation, we have

$$\mathsf{Conc}\,\mathfrak{M} \subseteq \mathsf{Conc}\,\mathfrak{N} \implies M \subseteq N,$$

and hence

$$\mathsf{Conc}\,\mathfrak{M} = \mathsf{Conc}\,\mathfrak{N} \implies M = N.$$

The cylindric-relativized set algebra obtained from model $\mathfrak{M}$ is denoted by $\mathfrak{Cs}\,\mathfrak{M}$ (see [29]). Roughly speaking, $\mathfrak{Cs}\,\mathfrak{M}$ is an algebraic structure whose universe is the set of concepts of $\mathfrak{M}$, whose operations correspond to the logical connectives, and among whose constants are the ones that correspond to logical *true* and *false*. To understand the present paper, it is not necessary to be familiar with the notion of cylindric algebras. We use them only occasionally, to reformulate some of our results in a more elegant algebraic form. We note that

$$\mathfrak{Cs}\,\mathfrak{M} \leq \mathfrak{Cs}\,\mathfrak{N} \iff \big(\mathsf{Conc}\,\mathfrak{M} \subseteq \mathsf{Conc}\,\mathfrak{N} \text{ and } M = N\big). \tag{2}$$

§3. Coordinate geometries over fields and ordered fields.

3.1. Definitions. Throughout this paper all geometries will be d-dimensional,[2] for some fixed integer $d \geq 2$, and defined over some field or ordered field $\mathfrak{F}$ with universe F. The points of coordinate space F^d will be the points of our geometries.

When $\mathfrak{F} = \langle F, +, \cdot, 0, 1\rangle$ is a field, the ternary relation Col of *collinearity* on F^d is defined for all $\vec{p}, \vec{q}, \vec{r} \in F^d$ by

$$\mathsf{Col}(\vec{p}, \vec{q}, \vec{r}\,) \stackrel{\text{def}}{\iff} \vec{q} = \vec{p} + \lambda(\vec{r} - \vec{p}\,) \text{ for some } \lambda \in F, \text{ or } \vec{r} = \vec{p}.$$

[1] A strongly related notion to definitional equivalence is interpretation. For interpretations between theories relevant to the present paper, see [1].

[2] In general, it makes sense to discuss geometries in one-dimension. However, the definition we use in this paper may not be appropriate when $d = 1$ because in one-dimension the collinearity relation is just the universal relation (i.e., any three points are collinear) and betweenness gives less structure on a line than an ordering.

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

We call model $\langle F^d, \mathsf{Col}\rangle$ the d-dimensional *affine geometry* over field $\mathfrak{F}$. Our notion of affine geometry is a natural generalization of the affine Cartesian spaces of [43, definition 1.5.] to fields which are not necessarily ordered.

Now suppose $\mathfrak{F} = \langle F, +, \cdot, 0, 1, \leq\rangle$ is an ordered field. The ternary relation Bw of *betweenness* on F^d is defined for every $\vec{p}, \vec{q}, \vec{r} \in F^d$ by

$$\mathsf{Bw}(\vec{p}, \vec{q}, \vec{r}\,) \overset{\text{def}}{\iff} \vec{q} = \vec{p} + \lambda(\vec{r} - \vec{p}\,) \text{ for some } \lambda \in [0, 1] \subseteq F.$$

Of course, Col is also defined for ordered fields and is definable in terms of Bw by

$$\mathsf{Col}(\vec{p}, \vec{q}, \vec{r}\,) \iff \mathsf{Bw}(\vec{p}, \vec{q}, \vec{r}\,) \vee \mathsf{Bw}(\vec{q}, \vec{p}, \vec{r}\,) \vee \mathsf{Bw}(\vec{p}, \vec{r}, \vec{q}\,).$$

We call model $\langle F^d, \mathsf{Bw}\rangle$ the d-dimensional *ordered affine geometry* over ordered field $\mathfrak{F}$. Our notion of ordered affine geometry coincides with the notion of affine Cartesian space given in [43, definition 1.5.].

In much of what follows, we will need to re-express n-tuples over M^d (n-tuples of (d-tuples over M)) as (dn)-tuples over M. Given the potential confusion arising from treating the same collection of values and/or variables using tuples of different lengths, we will adopt the notational convention of denoting points in M^d (d-tuples over M) using arrows and angle-brackets (e.g., $\vec{a} = \langle a_1, \ldots, a_d\rangle$). Other tuples will be written out in the usual way using parentheses, and overlines (the context will always be clear). Given any d-tuple $\vec{p}$, we denote its i'th component by p_i, and note that $\vec{p} = \langle p_1, \ldots, p_d\rangle$.

Given any n-tuple $\bar{p}$ over M^d, we write $\mathsf{flatten}(\bar{p})$ to denote the naturally corresponding 'flattened' (dn)-tuple over M, i.e., given any n-tuple $\bar{p} = (\vec{p}_1, \ldots, \vec{p}_n)$ of d-tuples $\vec{p}_1 = \langle p_{11}, \ldots, p_{1d}\rangle$, ..., $\vec{p}_n = \langle p_{n1}, \ldots, p_{nd}\rangle$ in M^d, we define

$$\mathsf{flatten}(\bar{p}) \overset{\text{def}}{=} (p_{11}, \ldots, p_{1d}, \ldots, p_{n1}, \ldots, p_{nd}).$$

For convenience, we will generally write $(\hat{p}_1, \ldots, \hat{p}_n)$ for $\mathsf{flatten}(\bar{p})$. For example, in the case that $d = 2$, $\vec{p}_1 = \langle 1, 2\rangle$, $\vec{p}_2 = \langle 3, 4\rangle$ and $\vec{p}_3 = \langle 5, 6\rangle$, we have $(\hat{p}_1, \hat{p}_2) = (1, 2, 3, 4)$ and $(\hat{p}_1, \ldots, \hat{p}_3) = (1, 2, 3, 4, 5, 6)$. The "hatted" notation $\hat{p}$ always indicates that the underlying tuple is a d-tuple, namely, $\vec{p}$. We use the analogous notation for sequences of variables.

Since $\mathsf{flatten}$ is a bijection from $(M^d)^n$ onto M^{dn} it has an inverse (we call it $\mathsf{unflatten}$). If R is any n-ary relation on M^d, we write $\widehat{\mathsf{R}}$ to denote the corresponding (dn)-ary relation on M, i.e., for all $\bar{p} \in M^{dn}$,

$$\bar{p} \in \widehat{\mathsf{R}} \quad \overset{\text{def}}{\iff} \quad \mathsf{unflatten}(\bar{p}) \in \mathsf{R}.$$

Given any $\vec{p}_1, \ldots, \vec{p}_n$ in M^d ($n \in \mathbb{N}^+$), the bijective nature of $\mathsf{flatten}$ and $\mathsf{unflatten}$ means this can also be written as

$$(\hat{p}_1, \ldots, \hat{p}_n) \in \widehat{\mathsf{R}} \quad \iff \quad (\vec{p}_1, \ldots, \vec{p}_n) \in \mathsf{R}. \tag{3}$$

We say that n-ary relation $\mathsf{R} \subseteq \left(M^d\right)^n$ is *definable over* $\mathfrak{M}$ iff the corresponding relation $\widehat{\mathsf{R}} \subseteq M^{dn}$ is definable in $\mathfrak{M}$. In particular, Col is definable over $\mathfrak{F}$ if $\mathfrak{F}$ is a field, and Bw is definable over $\mathfrak{F}$ if $\mathfrak{F}$ is an ordered field.

DEFINITION 3.1 (Coordinate geometry over a field). *We call a model $\mathcal{G}$ a* (d-*dimensional*) coordinate geometry over a field $\mathfrak{F} = \langle F, +, \cdot, 0, 1\rangle$ *iff the following conditions are satisfied*:

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

- *the universe of* $\mathcal{G}$ *is* F^d *(called the set of points)*;
- *the ternary relation* Col *of collinearity on points is definable in* $\mathcal{G}$.

DEFINITION 3.2 (Coordinate geometry over an ordered field). *We call a model* $\mathcal{G}$ *a* (d-*dimensional*) coordinate geometry over an ordered field $\mathfrak{F} = \langle F, +, \cdot, 0, 1, \leq \rangle$ *iff the following conditions are satisfied*:

- *the universe of* $\mathcal{G}$ *is* F^d *(the set of points)*;
- *the ternary relation* Bw *of betweenness on points is definable in* $\mathcal{G}$.

From now on, because constants and n-ary functions can be treated as unary and $(n+1)$-ary relations in the usual way, we assume without loss of generality that the languages of coordinate geometries contain only relation symbols.

DEFINITION 3.3 (Field-definable and FFD coordinate geometries). *Let* $\mathcal{G}$ *be a coordinate geometry over some field or ordered field* $\mathfrak{F}$. *We call* $\mathcal{G}$ field-definable *iff all the relations of* $\mathcal{G}$ *are definable over* $\mathfrak{F}$; *and* finitely field-definable (FFD) *iff it is field-definable and it contains only finitely many relations.*

3.2. Examples. Since relations Col and Bw are definable over the corresponding (ordered) field $\mathfrak{F}$, affine and ordered affine geometries are FFD coordinate geometries for all $d \geq 2$.

Euclidean geometry can be introduced as the following FFD coordinate geometry over any ordered field $\mathfrak{F}$:

$$\mathcal{E}ucl \stackrel{\text{def}}{=} \langle F^d, \cong, \mathsf{Bw} \rangle,$$

where the 4-nary relation $\cong$ of congruence is defined by

$$(\vec{p}, \vec{q}\,) \cong (\vec{r}, \vec{s}\,) \stackrel{\text{def}}{\iff} (p_1 - q_1)^2 + \cdots + (p_d - q_d)^2 = (r_1 - s_1)^2 + \cdots + (r_d - s_d)^2.$$

Notable classical geometries such as Relativistic spacetime ($\mathcal{R}el$), Minkowski spacetime ($\mathcal{M}ink$), Galilean spacetime ($\mathcal{G}al$) and Newtonian spacetime ($\mathcal{N}ewt$), as examples for FFD coordinate geometries, will be introduced in a similar manner in §7.

The language of $\mathcal{E}ucl$ is exactly the language used by Tarski [45, 47] to axiomatize elementary Euclidean geometry in first-order logic. In Tarski's work and in the literature following his axiomatization, $\mathcal{E}ucl$ is called a d-dimensional *Cartesian space* over an ordered field $\mathfrak{F}$, and various representation theorems show that the models of suitable axiom systems are exactly those that are isomorphic to Cartesian spaces over, real-closed, Euclidean, ordered or pythagorean ordered fields (see, e.g., [14, 33, 34, 36, 45]).

Our FFD coordinate geometries can be viewed as a natural generalization of Tarski-style Cartesian spaces. Instead of restricting attention to the relations of Euclidean geometry, we allow finitely many relations that are definable over a field or an ordered field. FFD coordinate geometries also admit an axiomatic treatment.

For cardinality reasons, it is clear that there must also be coordinate geometries which are not field-definable. A concrete example of such a geometry over the ordered field of reals is the geometry:

$$\langle \mathbb{R}^d, \mathbb{Z}^d, \mathsf{Bw} \rangle.$$

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

This is because the unary relation $\mathbb{Z}^d$ on points is not definable over the field of reals; this fact follows easily from Tarski's theorem on quantifier elimination of real closed fields [46].

§4. Main contribution: Connecting automorphisms and concepts. Echoing Klein's Erlangen program,[3] many of the results in this paper rely on a correspondence between the concepts of a model and those of its natural symmetries that respect specific structural relationships.

4.1. Key theorems. In this section, we state our key theorems: they confirm that the concepts of a geometry are intimately associated with its affine automorphism group.

DEFINITION 4.1 (Affine automorphisms). *Suppose F is the universe of some field or ordered field $\mathfrak{F}$. A map $A\colon F^d \to F^d$ is called an* affine transformation *iff it is the composition of a linear bijection and a translation, i.e., $A = \tau \circ L$ for some translation τ and linear bijection L. We note that all affine transformations are bijections.*

The set of affine transformations is denoted by $\mathsf{AffineTrf}$.

Affine transformations that are automorphisms of a coordinate geometry $\mathcal{G}$ over $\mathfrak{F}$ are called affine automorphisms *of $\mathcal{G}$, and the set of these is denoted by* $\mathsf{AffAut}\,\mathcal{G}$. *Thus,*

$$\mathsf{AffAut}\,\mathcal{G} \stackrel{\text{def}}{=} \mathsf{Aut}\,\mathcal{G} \cap \mathsf{AffineTrf}. \tag{4}$$

If $\mathcal{G}$ is an affine or ordered affine geometry, we have $\mathsf{AffAut}\,\mathcal{G} = \mathsf{AffineTrf}$.

THEOREM 4.2. *Let $\mathfrak{F}$ be an ordered field or a field that has more than two elements, let $\mathcal{G}$ be an FFD coordinate geometry over $\mathfrak{F}$, and let* R *be a relation on points of F^d. Then the following statements are equivalent*:

(i) R *is a concept of $\mathcal{G}$ (i.e.,* R *is definable in $\mathcal{G}$).*
(ii) R *is definable over $\mathfrak{F}$ and is closed under automorphisms of $\mathcal{G}$.*
(iii) R *is definable over $\mathfrak{F}$ and is closed under affine automorphisms of $\mathcal{G}$.*

REMARK 4.3. *Condition (ii) of Theorem 4.2 may involve a strictly larger group than condition (iii). Let $\mathfrak{F}$ be the quadratic extension of the field of rational numbers with universe $F = \mathbb{Q}(\sqrt{2}) = \left\{a + b\sqrt{2} : a, b \in \mathbb{Q}\right\}$. Consider the affine geometry $\mathcal{G} = \left\langle F^2, \mathsf{Col}\right\rangle$ over $\mathfrak{F}$. Let α be the nontrivial automorphism of $\mathfrak{F}$, given by $\alpha\left(a + b\sqrt{2}\right) = a - b\sqrt{2}$, and let $\widetilde{\alpha} : F^2 \to F^2$ be the componentwise map induced by α, i.e., $\widetilde{\alpha}(x, y) = (\alpha(x), \alpha(y))$. Since* Col *is definable over the field $\mathfrak{F}$, the map $\widetilde{\alpha}$ is an automorphism of $\mathcal{G}$. However, $\widetilde{\alpha}$ is not affine. Indeed, it fixes the origin, so if it were affine, it would be linear. But $\widetilde{\alpha}\left(\sqrt{2}, 0\right) = \left(-\sqrt{2}, 0\right)$, whereas $\sqrt{2}\,\widetilde{\alpha}(1, 0) = \left(\sqrt{2}, 0\right)$. Thus $\widetilde{\alpha} \in \mathsf{Aut}\,\mathcal{G}$ but $\widetilde{\alpha} \notin \mathsf{AffAut}\,\mathcal{G}$. Nevertheless, Theorem 4.2 shows that, under the stated assumptions, conditions (ii) and (iii) characterize the same relations.*

Proving Theorem 4.2 requires that we first develop a number of supporting lemmas, and completing the proof is consequently relegated to §6.5 (p. 18, below). But first, we state and prove various corollaries of Theorem 4.2, showing that the subset-inclusion poset of the concept-sets associated with FFD coordinate geometries is

[3] For a logical view of Klein's Erlangen program, see [8].

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

dually isomorphic to the subgroup-inclusion poset of their respective automorphism groups.

We will apply these results in §7 to compare the sets of concepts of several historically significant spacetimes by determining their affine automorphism groups.

REMARK 4.4. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two models. Clearly,*

$$\mathsf{Conc}\,\mathfrak{M} = \mathsf{Conc}\,\mathfrak{N} \implies \mathsf{Aut}\,\mathfrak{M} = \mathsf{Aut}\,\mathfrak{N},$$

because definable relations must be closed under automorphisms. Furthermore, if the universes of $\mathfrak{M}$ and $\mathfrak{N}$ are the same, then

$$\mathsf{Conc}\,\mathfrak{M} \subseteq \mathsf{Conc}\,\mathfrak{N} \implies \mathsf{Aut}\,\mathfrak{M} \supseteq \mathsf{Aut}\,\mathfrak{N}.$$

The reverse implication ($\Longleftarrow$) does not hold in general. For example, if $\mathfrak{R}$ is the field of reals and $\mathfrak{R}'$ is the expansion of $\mathfrak{R}$ with the unary relation N of being a natural number, then $\mathsf{Aut}\,\mathfrak{R} = \mathsf{Aut}\,\mathfrak{R}'$ *because both $\mathfrak{R}$ and $\mathfrak{R}'$ have no nontrivial automorphisms, but* $\mathsf{Conc}\,\mathfrak{R}' \not\subseteq \mathsf{Conc}\,\mathfrak{R}$ *because the set of natural numbers is not definable in the field of real numbers. Nonetheless, this reverse implication* does *hold for FFD coordinate geometries—as we now show.*

THEOREM 4.5. *Assume that $\mathfrak{F}$ is an ordered field or a field with more than two elements, and that $\mathcal{G}$ and $\mathcal{G}'$ are FFD coordinate geometries over $\mathfrak{F}$. Then*:

(i) $\mathsf{Conc}\,\mathcal{G} \subseteq \mathsf{Conc}\,\mathcal{G}' \iff \mathsf{Aut}\,\mathcal{G} \supseteq \mathsf{Aut}\,\mathcal{G}'$.
(ii) $\mathsf{Conc}\,\mathcal{G} \subseteq \mathsf{Conc}\,\mathcal{G}' \iff \mathsf{AffAut}\,\mathcal{G} \supseteq \mathsf{AffAut}\,\mathcal{G}'$.

Proof. Note that $\mathcal{G}$ and $\mathcal{G}'$ have the same universe. So, as we have already seen, the $\Longrightarrow$ implications of both items hold. Thus we have to prove only the $\Longleftarrow$ directions.

(i): Assume that $\mathsf{Aut}\,\mathcal{G} \supseteq \mathsf{Aut}\,\mathcal{G}'$ and that $\boldsymbol{\rho} \in \mathsf{Conc}\,\mathcal{G}$. Then, by Theorem 4.2, $\boldsymbol{\rho}$ is definable over $\mathfrak{F}$ and is closed under automorphisms of $\mathcal{G}$. Then $\boldsymbol{\rho}$ is closed under automorphisms of $\mathcal{G}'$ since $\mathsf{Aut}\,\mathcal{G} \supseteq \mathsf{Aut}\,\mathcal{G}'$. And so, by Theorem 4.2 again, $\boldsymbol{\rho}$ is definable in $\mathcal{G}'$, i.e., $\boldsymbol{\rho} \in \mathsf{Conc}\,\mathcal{G}'$.

(ii): The proof mirrors that of (i); the only difference is that here one has to refer to affine automorphisms instead of automorphisms. □

COROLLARY 4.6. *Assume that $\mathfrak{F}$ is an ordered field or a field that has more than two elements, and that $\mathcal{G}$ and $\mathcal{G}'$ are FFD coordinate geometries over $\mathfrak{F}$. Then*:

(i) $\mathsf{Conc}\,\mathcal{G} = \mathsf{Conc}\,\mathcal{G}' \iff \mathsf{Aut}\,\mathcal{G} = \mathsf{Aut}\,\mathcal{G}'$.
(ii) $\mathsf{Conc}\,\mathcal{G} = \mathsf{Conc}\,\mathcal{G}' \iff \mathsf{AffAut}\,\mathcal{G} = \mathsf{AffAut}\,\mathcal{G}'$.
(iii) $\mathsf{Conc}\,\mathcal{G} \subsetneq \mathsf{Conc}\,\mathcal{G}' \iff \mathsf{Aut}\,\mathcal{G} \supsetneq \mathsf{Aut}\,\mathcal{G}'$.
(iv) $\mathsf{Conc}\,\mathcal{G} \subsetneq \mathsf{Conc}\,\mathcal{G}' \iff \mathsf{AffAut}\,\mathcal{G} \supsetneq \mathsf{AffAut}\,\mathcal{G}'$.

Observe that $\langle \mathsf{Aut}\,\mathfrak{M}, \circ\rangle$ is a group for every model $\mathfrak{M}$, and similarly $\langle \mathsf{AffAut}\,\mathcal{G}, \circ\rangle$ is a group for every coordinate geometry $\mathcal{G}$, where $\circ$ is the standard composition of functions. By (2), the following is a reformulation of Theorem 4.5 in terms of cylindric algebras.

COROLLARY 4.7. *Assume that $\mathfrak{F}$ is an ordered field or a field that has more than two elements, and that $\mathcal{G}$ and $\mathcal{G}'$ are FFD coordinate geometries over $\mathfrak{F}$. Then*:

(i) $\mathfrak{Cs}\,\mathcal{G} \leq \mathfrak{Cs}\,\mathcal{G}' \iff \langle \mathsf{Aut}\,\mathcal{G}, \circ\rangle \geq \langle \mathsf{Aut}\,\mathcal{G}', \circ\rangle$.
(ii) $\mathfrak{Cs}\,\mathcal{G} \leq \mathfrak{Cs}\,\mathcal{G}' \iff \langle \mathsf{AffAut}\,\mathcal{G}, \circ\rangle \geq \langle \mathsf{AffAut}\,\mathcal{G}', \circ\rangle$.

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

REMARK 4.8. *The condition that* $\mathfrak{F}$ *has more than two elements cannot be omitted from Theorem* 4.2 *when* $\mathfrak{F}$ *is an* (*unordered*) *field. To see this, let us first note that every permutation of* F^d *is an automorphism of the* d*-dimensional affine geometry over the two element field. This is so because, in this geometry,* $\mathsf{Col}(\vec{p},\vec{q},\vec{r}\,)$ *holds iff* $\vec{q}=\vec{p}$ *or* $\vec{q}=\vec{r}$ *or* $\vec{p}=\vec{r}$ *since the value* λ *in the definition of* Col *can only take the values* 0 *and* 1. *So lines here are exactly the two element subsets of* F^d, *and every permutation of* F^d *respects the relation* Col. *In other words,* Col *here is definable from equality, and hence every structure over the two element field is a coordinate geometry.*

Bearing this in mind, let us consider the following three-dimensional geometry over the two element field: $\mathcal{G}=\langle\{0,1\}^3, O, U_x, U_y, U_z\rangle$, *where unary relations* $O=\{\langle 0,0,0\rangle\}$, $U_x=\{\langle 1,0,0\rangle\}$, $U_y=\{\langle 0,1,0\rangle\}$, $U_z=\{\langle 0,0,1\rangle\}$ *color the origin and the three standard basis vectors in* F^3, *respectively. Now* $\alpha\in\mathsf{Aut}\,\mathcal{G}$ *iff* α *is a bijection that fixes points* $\langle 0,0,0\rangle$, $\langle 1,0,0\rangle$, $\langle 0,1,0\rangle$, $\langle 0,0,1\rangle$, *while* $\mathsf{AffAut}\,\mathcal{G}=\{\mathsf{Id}\}$ *because no nontrivial affine transformation fixes the origin and all the three standard basis vectors. Let* $\mathsf{R}=\{\langle 1,1,1\rangle\}$. *Then* R *is not definable in* $\mathcal{G}$ *because it is not respected by the automorphisms of* $\mathcal{G}$, *yet it is field-definable and respected by all the affine automorphisms of* $\mathcal{G}$. *So* (*iii*) *holds for* R *but* (*i*) *does not.*

Nevertheless, the equivalence between (*i*) *and* (*ii*) *holds even over the two element field because, if* $\mathcal{G}$ *is finite, then the definable relations of* $\mathcal{G}$ *are exactly those which are closed under the automorphisms of* $\mathcal{G}$ (*see Remark* 6.13).

As observed above, proving (and fully explaining) Theorem 4.2 requires that we first consider a number of preliminary results; the full proof concludes in §6.5.

§5. Outline proofs of Theorem 4.2. In this section, we provide two different outline proofs of Theorem 4.2. The first proof, the more elementary of the two, would require too much space to present in detail, but the outline is nonetheless useful because it provides insight as to *why* the theorem is true. It also provides an explicit definition (thereby substantiating parts (ii) and (iii) of the theorem) for the relation R. The second proof relies on deep model-theoretic results and is more succinct when written out in full; its complete proof is presented in §6. Here, we give only its outline, which likewise provides insight into *why* the theorem holds, from a more abstract perspective.

To begin, we recall the well-known constructions and results of [12, 16, 36, 45] that if an affine geometry satisfies certain axioms then it is coordinatizable over some field,[4] and moreover this coordinatization can be defined from the affine collinearity relation. This is because the underlying field operations can be reconstructed (up to isomorphism) using simple parallel-line constructions in subplanes of the geometry [12, pp. 23–27]—this in turn allows us to assign coordinates to arbitrary locations (see Figure 1). In very rough terms, it is this reconstructibility which allows our proof to go through, since it provides us with a way to re-express statements defined over the field in purely geometric terms.

Let us look at this in more detail. We assume that $\mathfrak{F}$ is a field with more than two elements, and for illustrative purposes we take $d=2$ so that the affine geometry in question is the plane $\langle F^2, \mathsf{Col}\rangle$.

[4] An affine plane is field-coordinatizable if and only if it is Pappian [12, p. 23].

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

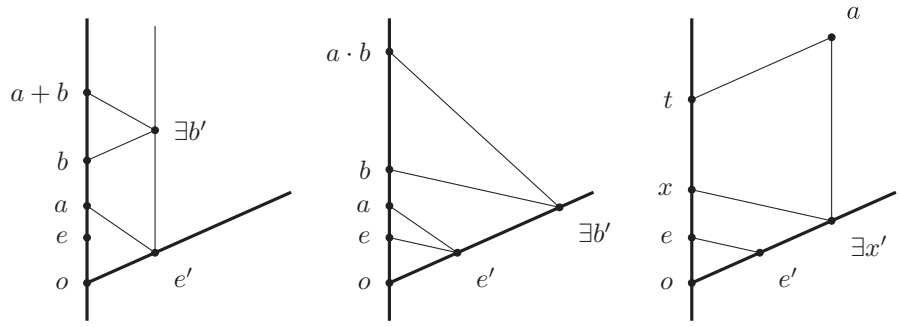


Figure 1. *Addition*: Draw the line parallel to oe through e'. Intersect this with the line through b parallel to oe' to generate point b'. Finally, draw the line through b' parallel to ae' to obtain $(a + b)$. *Multiplication*: Draw bb' parallel to ee', then the line through b' parallel to ae' to find $(a \cdot b)$. *Coordinates*: Thinking of the geometry as a spacetime tx-plane, we obtain the t coordinate of point a using the line through a parallel to oe'; we obtain the x-coordinate via the point x' obtained by intersecting the vertical through a with the oe' axis. (Figures adapted from [12, pp. 23–25].)

Following Goldblatt, let $o, e, e' \in F^2$ be three points that are not collinear [12, pp. 23–25].[5] Goldblatt shows how addition and multiplication can be defined on the points of the line oe, thereby turning it into a field isomorphic to $\mathfrak{F}$ with additive identity o and multiplicative identity e (see Figure 1). He then shows how to determine the coordinates of any point. This construction assigns the coordinates $\langle o, o\rangle$, $\langle e, o\rangle$ and $\langle o, e\rangle$ to the points o, e and e'.

These definitions can all be expressed by formulas containing only Col, together with the parameters o, e and e'.[6] For example, the first coordinate $\mathsf{crd}_1(a)$ of point a in coordinate system o, e, e' is given by

$$\mathsf{crd}_1(a) = t \iff^{\text{def}} \mathsf{Col}(o, e, t) \wedge$$
$$\big((\mathsf{Col}(o, e, a) \wedge t = a) \vee (\mathsf{Col}(o, e', a) \wedge t = o) \vee \neg\exists c(\mathsf{Col}(t, a, c) \wedge \mathsf{Col}(o, e', c))\big),$$

see the right-hand side of Figure 1. By these definitions of addition and multiplication, every formula in the language of $\mathfrak{F}$ can be translated to a formula containing only Col and parameters o, e and e'.

Now suppose that R is an arbitrary relation on F^2 that is definable over $\mathfrak{F}$. By definition, this means there is a formula in the language of $\mathfrak{F}$ that defines R. Translating this formula yields a relation $\mathsf{R}^*_{oee'}$ on F^2 defined in terms of Col and parameters o, e, e' of the same arity as R, which is intuitively a re-expression of R from the basic coordinate system $\langle 0, 0\rangle, \langle 1, 0\rangle, \langle 0, 1\rangle$ into the coordinate system o, e, e'.

[5] Goldblatt includes lines as a second sort and uses parallelism between them, but this introduces no complications because parallelism can easily be defined from Col.

[6] Formulas defining addition and multiplication in terms of Col and parameters o, e and e' can be found in [36, p. 144].

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

As Col is definable in any coordinate geometry, for any relation R on F^2 definable over $\mathfrak{F}$, we have that

$$\mathsf{R}^*_{oee'} \text{ is definable in any coordinate geometry using parameters } o, e, e'. \tag{5}$$

For any three noncollinear points o, e and e', let $A_{oee'}$ denote the affine transformation that takes $\langle 0,0\rangle$, $\langle 1,0\rangle$ and $\langle 0,1\rangle$ to o, e and e', respectively. One can prove that

$$\mathsf{R}^*_{oee'}(a^1,\dots,a^n) \iff \mathsf{R}\left(A^{-1}_{oee'}(a^1),\dots,A^{-1}_{oee'}(a^n)\right) \tag{6}$$

if R is an n-ary relation definable over $\mathfrak{F}$.

Let R be a relation definable over $\mathfrak{F}$. We say that three points o, e and e' form a coordinate system for R iff they are not collinear and $\mathsf{R}^*_{oee'}$ coincides with R. Points o, e and e' form a coordinate system for an FFD coordinate geometry $\mathcal{G}$, in symbols $\mathsf{CoSys}_{\mathcal{G}}(o,e,e')$, iff they form a coordinate system for every relation of $\mathcal{G}$. For example, consider the binary relations S of absolute simultaneity, and Rest of being at rest, on F^2 defined by

$$\langle t,s\rangle\ \mathsf{S}\ \langle t',s'\rangle \overset{\text{def}}{\iff} t = t',$$
$$\langle t,s\rangle\ \mathsf{Rest}\ \langle t',s'\rangle \overset{\text{def}}{\iff} s = s'.$$

We illustrate $\mathsf{S}^*_{oee'}$, $\mathsf{Rest}^*_{oee'}$ and coordinate systems for S and Rest in Figure 2.

Because any FFD geometry cites only finitely many relations, we find by (5) that

$$\text{ternary relation } \mathsf{CoSys}_{\mathcal{G}} \text{ is definable in } \mathcal{G}. \tag{7}$$

Furthermore, by (6), one can prove that

$$\mathsf{CoSys}_{\mathcal{G}}(o,e,e') \iff A_{oee'} \in \mathsf{AffAut}\,\mathcal{G}. \tag{8}$$

We are now ready to present the two outline proofs. The proofs of (i) ⇒(ii) and (ii) ⇒(iii) are the same in both outlines.

Outline proof of Theorem 4.2: (i) ⇒(ii): Assume that R is definable in $\mathcal{G}$. Relation R is definable over $\mathfrak{F}$, because it is definable from the relations of $\mathcal{G}$ which are all definable over $\mathfrak{F}$. It is closed under automorphisms, because definable relations are such in any model. (ii) ⇒ (iii) holds because affine automorphisms are automorphisms.

First proof of (iii) ⇒ (i): Assume that $\mathcal{G}$ is an FFD coordinate geometry over $\mathfrak{F}$ and R is an n-ary relation that is closed under affine automorphisms and definable over $\mathfrak{F}$. We will prove that

$$\mathsf{R}(a^1,\dots,a^n) \iff \exists o,e,e'\left(\mathsf{CoSys}_{\mathcal{G}}(o,e,e') \wedge \mathsf{R}^*_{oee'}(a^1,\dots,a^n)\right). \tag{9}$$

To prove (9), first assume that $\mathsf{R}(a^1,\dots,a^n)$. By (6) and (8),

$$\mathsf{CoSys}_{\mathcal{G}}\left(\langle 0,0\rangle,\langle 1,0\rangle,\langle 0,1\rangle\right) \wedge \mathsf{R}^*_{\langle 0,0\rangle\langle 1,0\rangle\langle 0,1\rangle}(a^1,\dots,a^n)$$

because $A_{\langle 0,0\rangle\langle 1,0\rangle\langle 0,1\rangle}$ is the identity function and is an affine automorphism of any coordinate geometry. To prove the reverse direction of (9), assume that $\mathsf{CoSys}_{\mathcal{G}}(o,e,e') \wedge \mathsf{R}^*_{oee'}(a^1,\dots,a^n)$. We have to prove that $\mathsf{R}(a^1,\dots,a^n)$. By (8), we have that $A_{oee'} \in \mathsf{AffAut}\,\mathcal{G}$. By (6), $\mathsf{R}\left(A^{-1}_{oee'}(a^1),\dots,A^{-1}_{oee'}(a^n)\right)$. Relation R is closed under affine automorphisms by assumption. Therefore, $\mathsf{R}(a^1,\dots,a^n)$ holds as required, and this completes the proof of (9).

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

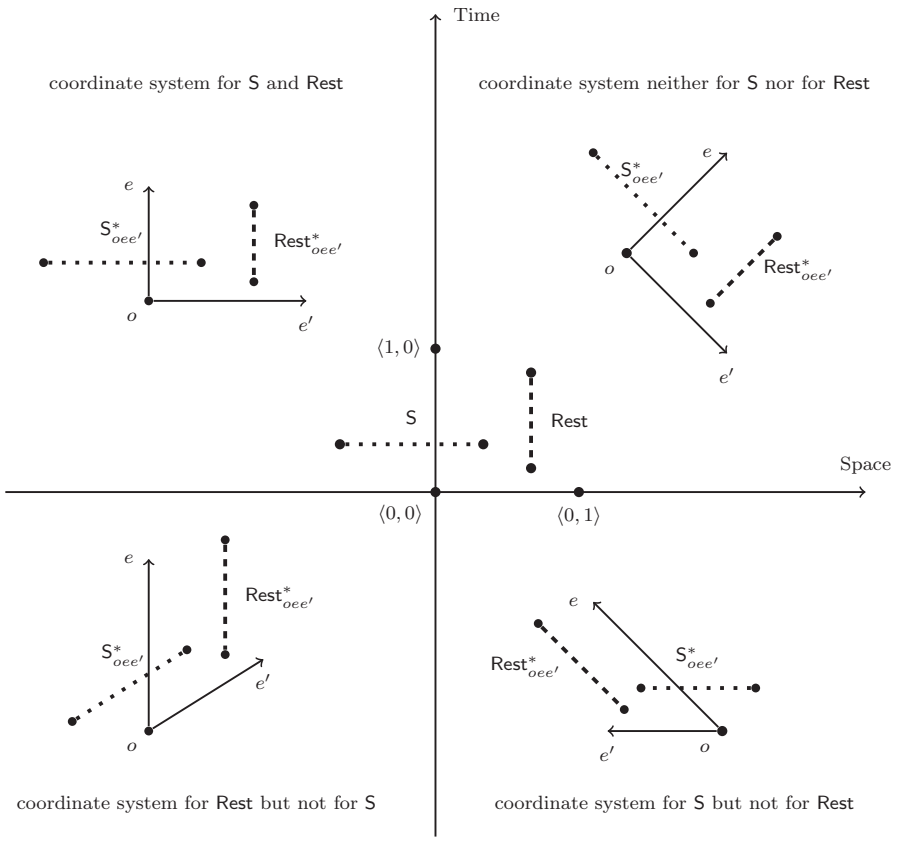


Figure 2. *What it means to be a coordinate system for a relation.* Dotted lines show points that are related by the relations which label them. The central axes show the standard $\{\langle 0,0\rangle, \langle 1,0\rangle, \langle 0,1\rangle\}$ system, together with the relations S and Rest. Each quadrant shows a transformed version of the axes and relations. For example, in the lower left quadrant the transformed version of Rest remains vertical (it is the same as the original) even though the axes have changed, but S is no longer horizontal; so this is a coordinate system for Rest but not for S.

By (5), (7) and (9), we get that R is definable in $\mathcal{G}$, and this completes the outline of the first proof.

If the explicit definition of Col in $\mathcal{G}$ and the formulas in the language of $\mathfrak{F}$ that define R and the relations of $\mathcal{G}$ are given, then using (9) one can obtain an explicit definition for R in $\mathcal{G}$.

Second proof of (iii) ⇒(i): We provide only an outline here. The full proof is included in §6.

The proof relies on the following results:

(A) Theorem 6.12, which says that a relation R on the universe of a model $\mathfrak{M}$ is definable in $\mathfrak{M}$ iff, for every ultrapower[7] $\langle \mathfrak{M}^{\mathcal{U}}, \mathsf{R}^{\mathcal{U}}\rangle$ of $\langle \mathfrak{M}, \mathsf{R}\rangle$, every automorphism of $\mathfrak{M}^{\mathcal{U}}$ is an automorphism of $\langle \mathfrak{M}^{\mathcal{U}}, \mathsf{R}^{\mathcal{U}}\rangle$.

[7] This notation for the ultrapower $\mathfrak{M}^{\mathcal{U}}$ of model $\mathfrak{M}$ according to ultrafilter $\mathcal{U}$ is introduced formally above equation (12) (p. 17).

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

(B) The Fundamental Theorem of Affine Geometry, which implies that every automorphism of a field-definable geometry is the composition of an affine automorphism and a map induced by an (ordered) field automorphism, if the field has more than two elements (see Proposition 6.1).

(C) Lemma 6.11, which implies that for any FFD coordinate geometry $\mathcal{G}$ and any relation R on the points definable over $\mathfrak{F}$, we can write up a formula φ in the language of $\mathfrak{F}$ expressing that every affine automorphism of $\mathcal{G}$ respects R, i.e., $\mathfrak{F} \models \varphi$ iff every affine automorphism of $\mathcal{G}$ is an affine automorphism of $\langle \mathcal{G}, \mathsf{R} \rangle$. The formula φ is constructed from the formulas defining R and the relations of $\mathcal{G}$ over $\mathfrak{F}$.[8]

Let $\mathcal{G}$ be an FFD coordinate geometry and let R be a relation on the points definable over $\mathfrak{F}$ and closed under the affine automorphisms of $\mathcal{G}$. Since the affine automorphisms of $\mathcal{G}$ form a group, relation R is also respected by the affine automorphisms. Let φ be the formula provided by (C). Then $\mathfrak{F} \models \varphi$.

To show that R is definable in $\mathcal{G}$, we apply Theorem 6.12. Consider the ultrapower $\langle \mathcal{G}^{\mathcal{U}}, \mathsf{R}^{\mathcal{U}} \rangle$ of $\langle \mathcal{G}, \mathsf{R} \rangle$ according to an ultrafilter $\mathcal{U}$. Thus, it is enough to prove that every automorphism of $\mathcal{G}^{\mathcal{U}}$ is an automorphism $\langle \mathcal{G}^{\mathcal{U}}, \mathsf{R}^{\mathcal{U}} \rangle$.

Let us consider the ultrapower $\mathfrak{F}^{\mathcal{U}}$ of $\mathfrak{F}$ according to the same ultrafilter $\mathcal{U}$. There exist unique FFD coordinate geometries $\mathcal{G}^*$ and $\langle \mathcal{G}^*, \mathsf{R}^* \rangle$ over $\mathfrak{F}^{\mathcal{U}}$, whose languages coincide with those of $\mathcal{G}$ and $\langle \mathcal{G}, \mathsf{R} \rangle$, respectively, such that each pair of corresponding relations is definable by the same formula in the language of (ordered) fields.

It can be shown that $\langle \mathcal{G}^*, \mathsf{R}^* \rangle$ is isomorphic to $\langle \mathcal{G}^{\mathcal{U}}, \mathsf{R}^{\mathcal{U}} \rangle$ (cf. Lemma 6.15). Therefore, it is enough to show that every automorphism of $\mathcal{G}^*$ is an automorphism of $\langle \mathcal{G}^*, \mathsf{R}^* \rangle$. By Łoś's lemma and $\mathfrak{F} \models \varphi$, we get that $\mathfrak{F}^{\mathcal{U}} \models \varphi$. Therefore, by (C), every affine automorphism of $\mathcal{G}^*$ is an affine automorphism of $\langle \mathcal{G}^*, \mathsf{R}^* \rangle$. By (B) every automorphism of $\mathcal{G}^*$ is a composition of an affine automorphism and the map induced by an (ordered) field automorphism. Maps induced by (ordered) field automorphisms respect any relation definable over the given (ordered) field. Therefore, as required, every automorphism of $\mathcal{G}^*$ is an automorphism of $\langle \mathcal{G}^*, \mathsf{R}^* \rangle$.

§6. Formal proof of Theorem 4.2. In this section, we introduce the necessary definitions and present the results needed for the proof of Theorem 4.2, and conclude with the proof of the theorem. Proofs of most propositions and lemmas are omitted here and can be found in [25].

6.1. Automorphisms and affine automorphisms. Suppose F is the universe of some field or ordered field $\mathfrak{F}$. Given a function $f : F \to F$, we write $\widetilde{f} : F^d \to F^d$ for the map induced on F^d by f componentwise, i.e., $\widetilde{f}\big(\langle p_1, \ldots, p_d \rangle\big) \stackrel{\text{def}}{=} \langle f(p_1), \ldots, f(p_d) \rangle$. We note that if f is a bijection, then so is $\widetilde{f}$, and we define the induced automorphisms on F^d to be the members of the set

$$\widetilde{\mathsf{Aut}}\ \mathfrak{F} \stackrel{\text{def}}{=} \{\widetilde{\alpha} : \alpha \in \mathsf{Aut}\ \mathfrak{F}\}.$$

[8] We note that, by Proposition 6.1, formula φ also expresses that every automorphism of $\mathcal{G}$ respects R.

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

PROPOSITION 6.1. *Assume that $\mathfrak{F}$ is an ordered field or a field that has more than two elements, and that $\mathcal{G}$ is a field-definable coordinate geometry over $\mathfrak{F}$. Then*

$$\mathsf{Aut}\,\mathcal{G} = \mathsf{AffAut}\,\mathcal{G} \circ \widetilde{\mathsf{Aut}}\,\mathfrak{F}.$$

Moreover, the decomposition of an automorphism of $\mathcal{G}$ into an affine automorphism following a map induced by an automorphism of $\mathfrak{F}$ is unique.

Proof. Note that $\widetilde{\mathsf{Aut}}\,\mathfrak{F} \subseteq \mathsf{Aut}\,\mathcal{G}$, since every relation of $\mathcal{G}$ is definable over $\mathfrak{F}$. The proposition then follows from the Fundamental Theorem of Affine Geometry; for the latter, see, e.g., [7, theorem 2.6.3, p. 52], [25, lemma 5.2.1], and [44, theorem 2.46, p. 81]. For a complete proof, see [25]. □

6.2. Definability of relations on d-tuples in arbitrary models. With geometries as our main application in mind, we now make general observations that are useful when investigating the definability of relations on d-tuples in arbitrary models.

Let $\mathcal{L}$ be an arbitrary first-order language and let $Fm_{\mathcal{L}}$ be the set of first-order formulas in the language $\mathcal{L}$. Then we define the following set of relation symbols for $\mathcal{L}$-definable relations of d-tuples:

$$\mathcal{RS}ymb_{\mathcal{L}} \stackrel{\text{def}}{=} \left\{ \langle \varrho, n\rangle \in Fm_{\mathcal{L}} \times \mathbb{N}^{+} \;:\; \text{the free variables of } \varrho \text{ are in } \{v_1, \ldots, v_{dn}\} \right\}.$$

Let $\mathfrak{M}$ be an arbitrary model for language $\mathcal{L}$ with universe M. For each relation symbol $R = \langle \varrho, n\rangle \in \mathcal{RS}ymb_{\mathcal{L}}$, we define a corresponding n-ary relation $R_{\mathfrak{M}d}$ on M^d by

$$(\vec{p}_1, \ldots, \vec{p}_n) \in R_{\mathfrak{M}d} \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mathfrak{M} \models \varrho[\hat{p}_1, \ldots, \hat{p}_n]. \tag{10}$$

REMARK 6.2. *Clearly, $R_{\mathfrak{M}d}$ is definable over $\mathfrak{M}$ for every $R \in \mathcal{RS}ymb_{\mathcal{L}}$. Conversely, given any relation R on M^d that is definable over $\mathfrak{M}$, there is some $R \in \mathcal{RS}ymb_{\mathcal{L}}$ for which $\mathsf{R} = R_{\mathfrak{M}d}$. For suppose ϱ is a formula defining R over $\mathfrak{M}$, and let n be R's arity. Taking $R = \langle \varrho, n\rangle$ then yields $\mathsf{R} = R_{\mathfrak{M}d}$. Thus, given any relation R on M^d, it is definable over $\mathfrak{M}$ iff $\mathsf{R} = R_{\mathfrak{M}d}$ for some $R \in \mathcal{RS}ymb_{\mathcal{L}}$.*

For every $\Psi \subseteq \mathcal{RS}ymb_{\mathcal{L}}$, let $\mathcal{L}_{\Psi}$ denote the first-order language with relation symbols in Ψ such that the arity of each $R = \langle \varrho, n\rangle \in \Psi$ is n.

In particular, we define the following two formulas γ and β, the first in the languages of fields and the second in that of ordered fields, and each having $3d$ free variables, $v_1, \ldots, v_{3d}$. Given $\vec{x} = \langle v_1, \ldots, v_d\rangle$, $\vec{y} = \langle v_{d+1}, \ldots, v_{2d}\rangle$, $\vec{z} = \langle v_{2d+1}, \ldots, v_{3d}\rangle$ and $v = v_{3d+1}$, we define

$$\gamma(\hat{x}, \hat{y}, \hat{z}) := \exists v\ (\vec{y} + v\vec{x} = \vec{x} + v\vec{z} \ \vee\ \vec{z} = \vec{x}\,) \text{ and}$$
$$\beta(\hat{x}, \hat{y}, \hat{z}) := \exists v\ (\vec{y} + v\vec{x} = \vec{x} + v\vec{z} \ \wedge\ 0 \leq v \leq 1)\,.$$

REMARK 6.3. *Suppose $\mathcal{L}$ is the language of $\mathfrak{F}$. Then*

- *$\langle \gamma, 3\rangle \in \mathcal{RS}ymb_{\mathcal{L}}$ and if $\mathfrak{F}$ is a field, then $\langle \gamma, 3\rangle_{\mathfrak{F}d}$ coincides with Col of the affine geometry over $\mathfrak{F}$.*
- *$\langle \beta, 3\rangle \in \mathcal{RS}ymb_{\mathcal{L}}$ and, if $\mathfrak{F}$ is an ordered field, then $\langle \beta, 3\rangle_{\mathfrak{F}d}$ coincides with Bw of the ordered affine geometry over $\mathfrak{F}$.*

Throughout the remainder of this paper, we consider two cases in parallel: the case where $\mathfrak{F}$ is a field, $\mathcal{L}$ is the language of fields, γ is the relevant formula and Col the relevant relation; and the case where $\mathfrak{F}$ is an ordered field, $\mathcal{L}$ is the language of ordered fields, β is the relevant formula and Bw the relevant relation. To simplify the

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

presentation of results below, we will write κ ('key formula') in place of the formulas γ and β, i.e.,

$$\kappa = \begin{cases} \gamma & \text{if the language of } \mathfrak{F} \text{ is the language of fields; and} \\ \beta & \text{if the language of } \mathfrak{F} \text{ is the language of ordered fields.} \end{cases}$$

We likewise define the 'key relation' to be $\mathsf{K} = \langle \kappa, 3 \rangle_{\mathfrak{F}d}$, and note that K coincides either with Col or Bw depending on the language of $\mathfrak{F}$ (cf. Remark 6.3). In both cases, we continue to write F for the universe of $\mathfrak{F}$ and note that the language of $\mathfrak{F}$ is usually clear from context.

For every $\Psi \subseteq \mathcal{RS}ymb_{\mathcal{L}}$, we define a model $\mathcal{G}_\Psi(\mathfrak{M})$ for language $\mathcal{L}_\Psi$ to be the model with universe M^d in which the interpretation of every $R \in \Psi$ is $R_{\mathfrak{M}d}$, i.e.,

$$\mathcal{G}_\Psi(\mathfrak{M}) = \langle M^d, R_{\mathfrak{M}d} \rangle_{R \in \Psi}. \tag{11}$$

To express that two models $\mathfrak{M}$ and $\mathfrak{N}$ for possibly different languages are the same, we introduce the following notation: we write $\mathfrak{M} \doteq \mathfrak{N}$ iff their universes are the same and there is a one-to-one correspondence between their languages such that the corresponding relation symbols and function symbols have the same interpretations in the two models.

REMARK 6.4. *In some models of the form $\mathcal{G}_\Psi(\mathfrak{M})$, the same concept appears more than once, but named using different relation symbols. This is because, if φ and ψ are different but logically equivalent formulas, then the symbols $\langle \varphi, n \rangle$ and $\langle \psi, n \rangle$ of $\mathcal{RS}ymb_{\mathcal{L}}$ differ but the corresponding relations are the same. One extreme case is the conceptually richest model $\mathcal{G}_{\mathcal{RS}ymb_{\mathcal{L}}}(\mathfrak{M})$, where every field-definable concept appears as a primitive concept infinitely many times.*

REMARK 6.5. *Suppose that $\mathcal{L}$ is the language of $\mathfrak{F}$. By Definitions 3.1 and 3.2, model $\mathcal{G}$ is a field-definable coordinate geometry over $\mathfrak{F}$ iff there is some $\Psi \subseteq \mathcal{RS}ymb_{\mathcal{L}}$ such that $\mathcal{G} \doteq \mathcal{G}_\Psi(\mathfrak{F})$ and the key relation K is definable in $\mathcal{G}$. Moreover, $\mathcal{G}$ is FFD iff Ψ can be chosen finite.*

REMARK 6.6. *For arbitrary models $\mathfrak{M}_1$ and $\mathfrak{M}_2$, and (a possibly infinite) subset Ψ of $\mathcal{RS}ymb_{\mathcal{L}}$, we have that $\mathcal{G}_\Psi(\mathfrak{M}_1)$ and $\mathcal{G}_\Psi(\mathfrak{M}_2)$ are elementarily equivalent whenever $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are elementarily equivalent. This is because, using the formula parts of the symbols of Ψ, one can define a translation* Tr *from the common language of $\mathcal{G}_\Psi(\mathfrak{M}_1)$ and $\mathcal{G}_\Psi(\mathfrak{M}_2)$ to the common language of $\mathfrak{M}_1$ and $\mathfrak{M}_2$, and it can be shown by formula induction that*

$$\mathcal{G}_\Psi(\mathfrak{M}) \models \sigma \iff \mathfrak{M} \models \mathsf{Tr}(\sigma)$$

holds for every model $\mathfrak{M}$ and sentence σ of common language of $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

6.3. Extending automorphisms from one geometry to another. In this section, we establish the conditions under which affine automorphisms of certain coordinate geometries are also automorphisms of their finite extensions. In particular, if $\mathcal{G}_\Psi(\mathfrak{F})$ is a coordinate geometry for some finite $\Psi \subseteq \mathcal{RS}ymb_{\mathcal{L}}$, and $R \in \mathcal{RS}ymb_{\mathcal{L}}$, Lemma 6.11 establishes conditions under which every affine automorphism of $\mathcal{G}_\Psi(\mathfrak{F})$ must also be an affine automorphism of $\mathcal{G}_{\Psi \cup \{R\}}(\mathfrak{F})$.

DEFINITION 6.7. *We write $\vec{\mathsf{o}}$ to denote the* origin $\langle 0, \ldots, 0 \rangle \in F^d$, *and $\vec{\mathsf{E}}_1, \ldots, \vec{\mathsf{E}}_d$ to denote the* unit vectors $\langle 1, 0, \ldots, 0 \rangle, \ldots, \langle 0, \ldots, 0, 1 \rangle \in F^d$, *respectively.*

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

The following simple proposition is well-known from linear algebra. We restate it here to fix notation, and then re-express the concepts embedded within it in a form suitable for our purposes.

PROPOSITION 6.8. *Suppose* $\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_d \in F^d$. *Then there is an affine transformation that takes the origin* $\vec{o}$ *to* $\vec{e}_0$ *and each unit vector* $\vec{E}_j$ *to the corresponding* $\vec{e}_j$ *iff the vectors* $(\vec{e}_1 - \vec{e}_0)$, ..., $(\vec{e}_d - \vec{e}_0)$ *are linearly independent. Moreover, this affine transformation is unique.*

DEFINITION 6.9. *Suppose* $\vec{e}_0 \in F^d$ *and* $\vec{\boldsymbol{e}} = (\vec{e}_1, \ldots, \vec{e}_d)$ *where each* $\vec{e}_j \in F^d$ *for* $1 \leq j \leq d$. *Provided the vectors* $(\vec{e}_1 - \vec{e}_0)$, ..., $(\vec{e}_d - \vec{e}_0)$ *are linearly independent, we write* $A_{\vec{e}_0:\vec{\boldsymbol{e}}}$ *for the unique affine transformation of Proposition 6.8 that maps* $\vec{o}$ *to* $\vec{e}_0$ *and each* $\vec{E}_j$ *to the corresponding* $\vec{e}_j$.

We now introduce various formulas, in the language $\mathcal{L}$ of $\mathfrak{F}$, needed to capture the concepts expressed in Proposition 6.8.[9] These are:

ι: Formula $\iota \in Fm_{\mathcal{L}}$ has $d^2 + d$ free variables and expresses the idea that vectors $(\vec{x}_1 - \vec{x}_0), \ldots, (\vec{x}_d - \vec{x}_0)$ are linearly independent:

$$\iota(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_d) \stackrel{\text{def}}{=} \left[\forall \lambda_1 \ldots \forall \lambda_d \left(\textstyle\sum_{i=1}^{d} \lambda_i(\vec{x}_i - \vec{x}_0) = 0 \;\rightarrow\; \bigwedge_{i=1}^{d} \lambda_i = 0\right)\right].$$

θ: Taken together with ι, formula $\theta \in Fm_{\mathcal{L}}$ (with $d^2 + 3d$ free variables) expresses the idea that affine map $A_{\vec{e}_0:\vec{\boldsymbol{e}}}$ takes vector $\vec{x} = \langle x_1, \ldots, x_d \rangle$ to vector $\vec{y} = \langle y_1, \ldots, y_d \rangle$:

$$\theta(\hat{e}_0, \hat{e}_1, \ldots, \hat{e}_d, \hat{x}, \hat{y}) \stackrel{\text{def}}{=} \left[\vec{y} = \vec{e}_0 + \textstyle\sum_{i=1}^{d} x_i \cdot (\vec{e}_i - \vec{e}_0)\right].$$

θ_R: For each $R = \langle \varrho, n \rangle \in \mathcal{RS}ymb_{\mathcal{L}}$, we define a formula θ_R which expresses[10] in the language of $\mathfrak{F}$ that $A_{\vec{e}_0:\vec{\boldsymbol{e}}}$ exists and respects R:

$$\begin{aligned}\theta_R(\hat{e}_0, \hat{e}_1, \ldots, \hat{e}_d) \stackrel{\text{def}}{=}\; & \iota(\hat{e}_0, \hat{e}_1, \ldots, \hat{e}_d) \\ \wedge\; & [\,\forall \hat{x}_1 \ldots \forall \hat{x}_n \forall \hat{y}_1 \ldots \forall \hat{y}_n \\ & \left(\left(\textstyle\bigwedge_{i=1}^{n} \theta(\hat{e}_0, \hat{e}_1, \ldots, \hat{e}_d, \hat{x}_i, \hat{y}_i)\right) \rightarrow (\varrho(\hat{x}_1, \ldots, \hat{x}_n) \leftrightarrow \varrho(\hat{y}_1, \ldots, \hat{y}_n))\right)\,].\end{aligned}$$

θ_Ψ: Finally, if Ψ is a finite subset of $\mathcal{RS}ymb_{\mathcal{L}}$, we define

$$\theta_\Psi(\hat{e}_0, \hat{e}_1, \ldots, \hat{e}_d) \stackrel{\text{def}}{=} \bigwedge_{R \in \Psi} \theta_R(\hat{e}_0, \hat{e}_1, \ldots, \hat{e}_d).$$

LEMMA 6.10. *Suppose* $R \in \mathcal{RS}ymb_{\mathcal{L}}$ *and let* Ψ *be a finite subset of* $\mathcal{RS}ymb_{\mathcal{L}}$, *where* $\mathcal{L}$ *is the language of* $\mathfrak{F}$. *Suppose* $\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_d \in F^d$. *Then*

(i) $\mathfrak{F} \models \theta_R[\hat{e}_0, \hat{e}_1, \ldots, \hat{e}_d]$ *iff* $(\vec{e}_1 - \vec{e}_0), \ldots, (\vec{e}_d - \vec{e}_0)$ *are linearly independent and* $A_{\vec{e}_0:\vec{\boldsymbol{e}}}$ *respects* $R_{\mathfrak{F}d}$.

[9] Technically, we should write these using variable names $v_1, v_2, \ldots$. However, to simplify our formulas and make them more intelligible, we will use metavariables such as $\vec{x}_1$ and $\hat{x}_0$ for the various v_n's and sequences thereof.

[10] We use Tarskian substitution of variables as follows: for formula $\varphi(v_1, \ldots, v_n)$ and $x_1, \ldots, x_n$ metavariables, we define $\varphi(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \exists v_1(v_1 = x_1 \wedge \ldots (\exists v_n(v_n = x_n \wedge \varphi(v_1, \ldots, v_n)) \ldots)$.

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

(ii) $\mathfrak{F} \models \theta_\Psi[\hat{e}_0, \hat{e}_1, \ldots, \hat{e}_d]$ *iff* $(\vec{e}_1 - \vec{e}_0), \ldots, (\vec{e}_d - \vec{e}_0)$ *are linearly independent and* $A_{\vec{e}_0;\vec{e}}$ *is an affine automorphism of FFD coordinate geometry* $\mathcal{G}_\Psi(\mathfrak{F})$.

LEMMA 6.11. *Let* $\mathcal{G}_\Psi(\mathfrak{F})$ *be a coordinate geometry for some finite* $\Psi \subseteq \mathcal{RS}ymb_{\mathcal{L}}$ *and suppose* $R \in \mathcal{RS}ymb_{\mathcal{L}}$. *Then* $\mathfrak{F} \models \theta_\Psi \to \theta_R$ *iff every affine automorphism of* $\mathcal{G}_\Psi(\mathfrak{F})$ *is an affine automorphism of* $\mathcal{G}_{\Psi\cup\{R\}}(\mathfrak{F})$.

Proof. Follows from Lemma 6.10. □

6.4. Results obtained using ultrapowers. The next theorem and lemma concern the ultrapowers of a model $\mathfrak{M}$, and will be used below to complete our proof of Theorem 4.2. For the benefit of readers who are unfamiliar with this concept, and also to fix our notation, we briefly recall the following definitions. For a more detailed account, see e.g., [17].

Suppose that M is the universe of a model $\mathfrak{M}$ of some FOL language $\mathcal{L}$, and that $\mathcal{U}$ is an ultrafilter on some set I (called an *index set*); that is, $\mathcal{U}$ is a collection of subsets of I satisfying (i) $\varnothing \notin \mathcal{U}$; (ii) whenever S_1 and S_2 are members of $\mathcal{U}$, so is $S_1 \cap S_2$; (iii) whenever $S \in \mathcal{U}$ and $S \subseteq T \subseteq I$, then $T \in \mathcal{U}$; and (iv) given any $S \subseteq I$, either S or $I \setminus S$ is in $\mathcal{U}$. One can use $\mathcal{U}$ to define an equivalence relation $\sim_{\mathcal{U}}$ on the collection ${}^IM$ of all functions from I to M by defining, for all $f_1, f_2 \colon I \to M$,

$$f_1 \sim_{\mathcal{U}} f_2 \quad \overset{\text{def}}{\iff} \quad \{i : f_1(i) = f_2(i)\} \in \mathcal{U}.$$

Given any $f : I \to M$, we write $f/\mathcal{U}$ for the $\sim_{\mathcal{U}}$-equivalence class containing f, and ${}^IM/\mathcal{U}$ for the set of all such equivalence classes.

The *ultrapower* of $\mathfrak{M}$ according to $\mathcal{U}$ (written $\mathfrak{M}^{\mathcal{U}}$) is the model of $\mathcal{L}$ whose universe is the set of $\sim_{\mathcal{U}}$-equivalence classes, in which the interpretation $\mathsf{R}^{\mathcal{U}}$ of any (n-ary) relation symbol R is defined, for $f_1, \ldots, f_n \in {}^IM$, by

$$(f_1/\mathcal{U}, \ldots, f_n/\mathcal{U}) \in \mathsf{R}^{\mathcal{U}} \quad \overset{\text{def}}{\iff} \quad \{i : (f_1(i), \ldots, f_n(i)) \in \mathsf{R}\} \in \mathcal{U}, \tag{12}$$

where R is the interpretation of R in $\mathfrak{M}$. The interpretations of function symbols and constants are defined analogously. Given any (n-ary) formula ϕ, Łos's Theorem [17, theorem 8.5.1] tells us that

$$\mathfrak{M}^{\mathcal{U}} \models \phi[f_1/\mathcal{U}, \ldots, f_n/\mathcal{U}] \quad \iff \quad \{i : \mathfrak{M} \models \phi[f_1(i), \ldots, f_n(i)]\} \in \mathcal{U}. \tag{13}$$

An easy but important consequence of (12) and (13) is that whenever $\mathfrak{F}$ is a field or ordered field, so is $\mathfrak{F}^{\mathcal{U}}$.

The following theorem is due to András Simon.[11]

THEOREM 6.12. *Let* $\mathfrak{M}$ *be an FOL model and* R *a relation on its universe. Then* R *is definable in* $\mathfrak{M}$ *iff, for every ultrapower* $\langle \mathfrak{M}^{\mathcal{U}}, \mathsf{R}^{\mathcal{U}} \rangle$ *of* $\langle \mathfrak{M}, \mathsf{R} \rangle$, *every automorphism of* $\mathfrak{M}^{\mathcal{U}}$ *is an automorphism of* $\langle \mathfrak{M}^{\mathcal{U}}, \mathsf{R}^{\mathcal{U}} \rangle$.

Proof. By [38, corollary 1 on p. 969], if R is not definable in $\mathfrak{M}$, then there is a model $\langle \mathfrak{N}, S \rangle$ elementarily equivalent to $\langle \mathfrak{M}, \mathsf{R} \rangle$ such that relation S is not respected by some automorphism f of $\mathfrak{N}$. By the Keisler–Shelah isomorphism theorem, there is an ultrafilter $\mathcal{U}$ such that $\langle \mathfrak{M}^{\mathcal{U}}, \mathsf{R}^{\mathcal{U}} \rangle$ is isomorphic to $\langle \mathfrak{N}^{\mathcal{U}}, S^{\mathcal{U}} \rangle$. The automorphism f of $\mathfrak{N}$ extends componentwise to an automorphism of $\mathfrak{N}^{\mathcal{U}}$ which does not respect $S^{\mathcal{U}}$,

[11] Here we give an independent proof. Simon's proof (personal communication) uses Keisler–Shelah's theorem applied twice.

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

and hence it is not an automorphism of $\langle \mathfrak{N}^{\mathcal{U}}, S^{\mathcal{U}} \rangle$. Since $\langle \mathfrak{N}^{\mathcal{U}}, S^{\mathcal{U}} \rangle$ is isomorphic to $\langle \mathfrak{M}^{\mathcal{U}}, \mathsf{R}^{\mathcal{U}} \rangle$, there must also be an automorphism of $\mathfrak{M}^{\mathcal{U}}$ that is not an automorphism of $\langle \mathfrak{M}^{\mathcal{U}}, \mathsf{R}^{\mathcal{U}} \rangle$. The other direction follows immediately because automorphisms have to respect definable relations in every model. □

REMARK 6.13. *The fact that in finite structures a relation is definable iff it is respected by all automorphisms of the structure is a simple corollary of Theorem 6.12 since any ultrapower of a finite structure is isomorphic to the original structure.*

DEFINITION 6.14. *If M is some set and $j \leq d$ ($j \in \mathbb{N}^+$), we write π_j for the* j*-th projection function from M^d to M, i.e., $\pi_j : \langle p_1, \ldots, p_j, \ldots, p_d \rangle \mapsto p_j$. Given a set M and an ultrafilter $\mathcal{U}$ over some index set I, we define the function $\vec{\boldsymbol{\pi}}_{M\mathcal{U}} : (M^d)^{\mathcal{U}} \to (M^{\mathcal{U}})^d$ by*

$$\vec{\boldsymbol{\pi}}_{M\mathcal{U}}(f/\mathcal{U}) \stackrel{\text{def}}{=} \langle (\pi_1 \circ f)/\mathcal{U}, \ldots, (\pi_d \circ f)/\mathcal{U} \rangle$$

for all $f : I \to M^d$. Since the context is always clear where we use this function below, we will generally abbreviate $\vec{\boldsymbol{\pi}}_{M\mathcal{U}}$ to $\vec{\boldsymbol{\pi}}$ in what follows.

LEMMA 6.15. *Let $\mathfrak{M}$ be a model for some first-order language $\mathcal{L}$, let M be its universe, let $\mathcal{U}$ be an ultrafilter over some index set I, and $\Psi \subseteq \mathcal{RS}ymb_{\mathcal{L}}$. Then,*

(i) *$\vec{\boldsymbol{\pi}}$ is well-defined;*
(ii) *$\vec{\boldsymbol{\pi}}$ is an isomorphism from $\mathcal{G}_\Psi(\mathfrak{M})^{\mathcal{U}}$ to $\mathcal{G}_\Psi\left(\mathfrak{M}^{\mathcal{U}}\right)$, and hence these models are isomorphic.*

Proof. See [25]. □

6.5. Proof of Theorem 4.2. We are now in a position to complete the proof of Theorem 4.2.

THEOREM 4.2. *Let $\mathfrak{F}$ be an ordered field or a field that has more than two elements, let $\mathcal{G}$ be an FFD coordinate geometry over $\mathfrak{F}$, and let* R *be a relation on points of F^d. Then the following statements are equivalent*:

(i) R *is a concept of $\mathcal{G}$ (i.e.,* R *is definable in $\mathcal{G}$).*
(ii) R *is definable over $\mathfrak{F}$ and is closed under automorphisms of $\mathcal{G}$.*
(iii) R *is definable over $\mathfrak{F}$ and is closed under affine automorphisms of $\mathcal{G}$.*

Proof. (i) ⇒ (ii): Since R is definable in $\mathcal{G}$ it must be closed under automorphisms of $\mathcal{G}$. So to prove (ii), it is enough to show that R is definable over $\mathfrak{F}$. To do this, we note that it's possible to define a translation Tr from the language of $\mathcal{G}$ to the language of $\mathfrak{F}$ and prove (see [25] for details) that, for every formula $\varphi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ in the language of $\mathcal{G}$ and points $\vec{p}_1, \ldots, \vec{p}_n$, we have

$$\mathcal{G} \models \varphi[\vec{p}_1, \ldots, \vec{p}_n] \iff \mathfrak{F} \models \mathsf{Tr}(\varphi)[\hat{p}_1, \ldots, \hat{p}_n]. \tag{14}$$

Now let ϱ be the formula defining R in $\mathcal{G}$. Then, by (14), $\mathcal{G} \models \varrho[\vec{p}_1, \ldots, \vec{p}_n]$ iff $\mathfrak{F} \models \mathsf{Tr}(\varrho)[\hat{p}_1, \ldots, \hat{p}_n]$, i.e., $\mathsf{Tr}(\varrho)$ defines $\widehat{\mathsf{R}}$ in $\mathfrak{F}$, and hence R is definable over $\mathfrak{F}$ as stated.

(ii) ⇒ (iii): Trivial as affine automorphisms are also automorphisms.

(iii) ⇒ (i): Assume that R is definable over $\mathfrak{F}$ and closed under affine automorphisms of $\mathcal{G}$. We have to prove that R is definable in $\mathcal{G}$. As affine automorphisms of $\mathcal{G}$ form a group under composition, they respect R.

Let $\mathcal{L}$ be the language of $\mathfrak{F}$, let $R \in \mathcal{RS}ymb_{\mathcal{L}}$ be the relation symbol for which $\mathsf{R} = R_{\mathfrak{F}d}$, and let Ψ be a finite subset of $\mathcal{RS}ymb_{\mathcal{L}}$ containing $\langle \kappa, 3 \rangle$, such that $\langle \mathcal{G}, \mathsf{K} \rangle \doteq$

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

$\mathcal{G}_\Psi(\mathfrak{F})$. Such R and Ψ exist by Remarks 6.2, 6.3 and 6.5. Since Ψ contains $\langle\kappa, 3\rangle$, and $\langle\kappa, 3\rangle_{\mathfrak{F}d}$ coincides with K, $\mathcal{G}_\Psi(\mathfrak{F})$ is a coordinate geometry as well, and it follows easily that the definable relations, automorphisms and affine automorphisms of $\mathcal{G}$ and $\mathcal{G}_\Psi(\mathfrak{F})$ coincide. Hence affine automorphisms of $\mathcal{G}_\Psi(\mathfrak{F})$ respect $\mathsf{R} = R_{\mathfrak{F}d}$, and to prove that $\mathsf{R} = R_{\mathfrak{F}d}$ is definable in $\mathcal{G}$, it is enough to prove that $R_{\mathfrak{F}d}$ is definable in $\mathcal{G}_\Psi(\mathfrak{F})$. Clearly, the expansion of $\mathcal{G}_\Psi(\mathfrak{F})$ with $R_{\mathfrak{F}d}$ to the language $\mathcal{L}_{\Psi\cup\{R\}}$ is $\mathcal{G}_{\Psi\cup\{R\}}(\mathfrak{F})$ and every affine automorphism of $\mathcal{G}_\Psi(\mathfrak{F})$ is an affine automorphism of $\mathcal{G}_{\Psi\cup\{R\}}(\mathfrak{F})$, so by Lemma 6.11, $\mathfrak{F} \models (\theta_\Psi \to \theta_R)$. We will apply Theorem 6.12 to obtain a proof that $R_{\mathfrak{F}d}$ is definable in $\mathcal{G}_\Psi(\mathfrak{F})$, namely, we will prove that for any ultrafilter $\mathcal{U}$, every automorphism of $(\mathcal{G}_\Psi(\mathfrak{F}))^{\mathcal{U}}$ is an automorphism of $(\mathcal{G}_{\Psi\cup\{R\}}(\mathfrak{F}))^{\mathcal{U}}$. To prove this, let $\mathcal{U}$ be any ultrafilter (on any index set). Applying Lemma 6.15 (twice), we observe that the function $\vec{\boldsymbol{\pi}}$ is both an isomorphism from $(\mathcal{G}_\Psi(\mathfrak{F}))^{\mathcal{U}}$ to $\mathcal{G}_\Psi(\mathfrak{F}^{\mathcal{U}})$, and also an isomorphism from $(\mathcal{G}_{\Psi\cup\{R\}}(\mathfrak{F}))^{\mathcal{U}}$ to $\mathcal{G}_{\Psi\cup\{R\}}(\mathfrak{F}^{\mathcal{U}})$.

It is therefore enough to prove that every automorphism of $\mathcal{G}_\Psi(\mathfrak{F}^{\mathcal{U}})$ is an automorphism of $\mathcal{G}_{\Psi\cup\{R\}}(\mathfrak{F}^{\mathcal{U}})$. To this end, we observe that the model $\mathcal{G}_\Psi(\mathfrak{F}^{\mathcal{U}})$ is a coordinate geometry over $\mathfrak{F}^{\mathcal{U}}$ because Ψ contains $\langle\kappa, 3\rangle$ and relation $\langle\kappa, 3\rangle_{\mathfrak{F}^{\mathcal{U}}d}$ of $\mathcal{G}_\Psi(\mathfrak{F}^{\mathcal{U}})$ coincides with the relation K of the affine geometry over $\mathfrak{F}^{\mathcal{U}}$ (cf. Remark 6.3).

Moreover, we know that $\mathfrak{F}^{\mathcal{U}} \models (\theta_\Psi \to \theta_R)$ because $\mathfrak{F} \models (\theta_\Psi \to \theta_R)$. By Lemma 6.11, this implies that every affine automorphism of $\mathcal{G}_\Psi(\mathfrak{F}^{\mathcal{U}})$ is an affine automorphism of $\mathcal{G}_{\Psi\cup\{R\}}(\mathfrak{F}^{\mathcal{U}})$ as well.

Assume therefore that α is an arbitrary automorphism of $\mathcal{G}_\Psi(\mathfrak{F}^{\mathcal{U}})$. By Proposition 6.1, we can write $\alpha = A \circ \widetilde{\varphi}$ for some affine automorphism A of $\mathcal{G}_\Psi(\mathfrak{F}^{\mathcal{U}})$ and some automorphism φ of $\mathfrak{F}^{\mathcal{U}}$. Since A is also an affine automorphism of $\mathcal{G}_{\Psi\cup\{R\}}(\mathfrak{F}^{\mathcal{U}})$, applying Proposition 6.1 once again shows that $\alpha = A \circ \widetilde{\varphi}$ is an automorphism of $\mathcal{G}_{\Psi\cup\{R\}}(\mathfrak{F}^{\mathcal{U}})$ as required. □

§7. Applications. These results yield a simple but powerful strategy for comparing the conceptual content of different geometries: *If we wish to compare the sets of concepts of FFD coordinate geometries over the same field and of the same dimension, it is enough to compare their affine automorphism groups.*

We demonstrate the power of this approach by showing the connection between the concept-sets of various classical geometries, including Galilean, Newtonian and Relativistic spacetimes and Euclidean geometry, as illustrated in Figures 3 and 4. We define these spacetimes and geometries using a Tarskian first-order language centred around the ternary relation Bw of betweenness. The language we use to model Galilean spacetime is the same as the one used by [18], who gives a second-order axiomatization for Galilean spacetime using the basic notions of betweenness (Bw), simultaneity (S), and congruence on simultaneity ($\cong_{\mathsf{S}}$).

The power of this approach shows itself, for example, Theorem 7.8(ii) below states that affine automorphisms respect lightlike relatedness if and only if they respect Minkowski equidistance, which implies that these two concepts are interdefinable (using betweenness). Finding the explicit formula that defines Minkowski equidistance from lightlike relatedness and betweenness is not trivial, whereas comparing the relevant affine automorphism groups is relatively straightforward.

In the remainder of this section, we show how this strategy can be used to demonstrate the conceptual relationships between various historically significant geometries (see Figure 3), thereby establishing which of these geometries are and are not definitionally equivalent to one another. The main geometries considered here

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

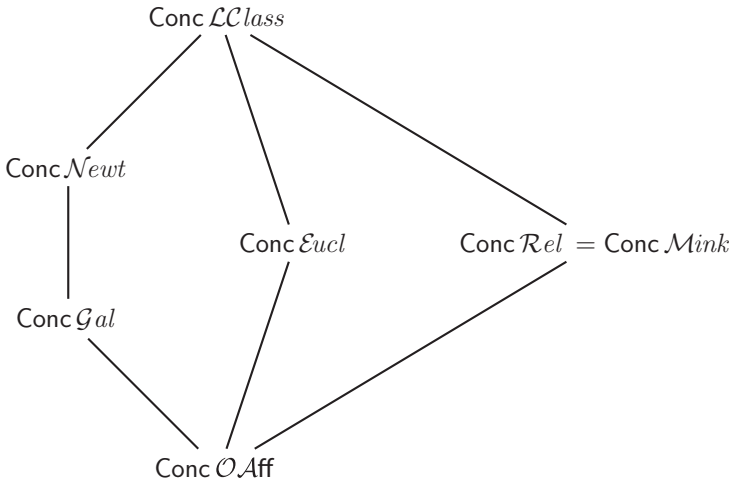


Figure 3. Hasse diagram showing how the concept-sets associated with various historically significant geometries are related to one another by subset inclusion.

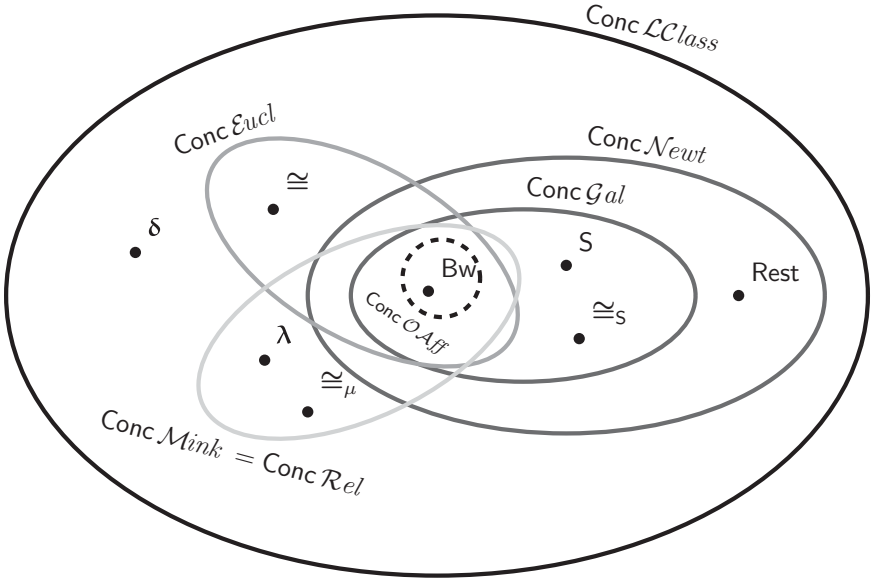


Figure 4. Venn diagram showing how the concept-sets associated with various geometries are related to one another. The diagram shows which geometries do and do not contain various key concepts (see Table 1).

are Ordered Affine ($\mathcal{OA}ff$) and Euclidean ($\mathcal{E}ucl$) geometries, and Relativistic ($\mathcal{R}el$), Minkowski ($\mathcal{M}ink$), Galilean ($\mathcal{G}al$), Newtonian ($\mathcal{N}ewt$) and Late Classical ($\mathcal{LC}lass$) spacetimes.

7.1. Assumptions. Throughout, we restrict attention to d-dimensional coordinate geometries over an ordered field $\mathfrak{F} = \langle F, +, \cdot, 0, 1, \leq \rangle$ where, as always, we assume that $d \geq 2$.

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

DEFINITION 7.1 (Products, lengths and distances). *A function* $\otimes\colon F^d \times F^d \to F$ *will be called a* product *on* F^d *iff it is both symmetric and bilinear. If* $\otimes$ *is a product on* F^d, *its corresponding* squared length *function is the function* $\|\vec{p}\,\|^2_{\otimes} \stackrel{\text{def}}{=} \vec{p} \otimes \vec{p}$ *and its corresponding* squared distance *function is the function* $\mathsf{d}^2_{\otimes}(\vec{p},\vec{q}\,) \stackrel{\text{def}}{=} \|\vec{p}-\vec{q}\,\|^2_{\otimes}$, *for all* $\vec{p},\vec{q} \in F^d$.

In particular, the Euclidean *scalar product* $\cdot$ of vectors $\vec{p},\vec{q} \in F^d$ is defined, as usual, by

$$\vec{p}\cdot\vec{q} \stackrel{\text{def}}{=} p_1q_1 + \cdots + p_dq_d,$$

and their *Minkowski product* ($\diamond$) by

$$\vec{p}\diamond\vec{q} \stackrel{\text{def}}{=} p_1q_1 - p_2q_2 - \cdots - p_dq_d.$$

The squared distance functions corresponding to the Euclidean and Minkowski products will be denoted by d^2 and d^2_μ, respectively.

7.2. Unit-free spacetimes. Thinking of F^d as the set of spacetime points, we regard the first component, p_1, of any vector $\vec{p} = \langle p_1, p_2, \ldots, p_d\rangle$ as its time component, and the rest, $(p_2, \ldots, p_d)$ as its spatial component. This is reflected in the following terminology.

On the set of points F^d, the binary relations S of *absolute simultaneity*, Rest of *being at rest* and $\boldsymbol{\lambda}$ of *lightlike relatedness* are defined for $\vec{p},\vec{q} \in F^d$ by

$$\begin{aligned}
\vec{p}\,\mathsf{S}\,\vec{q} &\iff^{\text{def}} p_1 = q_1,\\
\vec{p}\,\mathsf{Rest}\,\vec{q} &\iff^{\text{def}} (p_2,\ldots,p_d) = (q_2,\ldots,q_d),\\
\vec{p}\,\boldsymbol{\lambda}\,\vec{q} &\iff^{\text{def}} (p_1-q_1)^2 = (p_2-q_2)^2 + \cdots + (p_d-q_d)^2.
\end{aligned}$$

We note that

$$\vec{p}\,\boldsymbol{\lambda}\,\vec{q} \iff \mathsf{d}^2_\mu(\vec{p},\vec{q}\,) = 0. \tag{15}$$

On the set of points F^d, the 4-ary relations $\cong$ of *(Euclidean) congruence*, $\cong_\mu$ of *Minkowski congruence*, and $\cong_{\mathsf{S}}$ of *congruence on simultaneity* are defined for $\vec{p},\vec{q},\vec{r},\vec{s} \in F^d$ by

$$\begin{aligned}
(\vec{p},\vec{q}\,) \cong (\vec{r},\vec{s}\,) &\iff^{\text{def}} \mathsf{d}^2(\vec{p},\vec{q}\,) = \mathsf{d}^2(\vec{r},\vec{s}\,),\\
(\vec{p},\vec{q}\,) \cong_\mu (\vec{r},\vec{s}\,) &\iff^{\text{def}} \mathsf{d}^2_\mu(\vec{p},\vec{q}\,) = \mathsf{d}^2_\mu(\vec{r},\vec{s}\,),\\
(\vec{p},\vec{q}\,) \cong_{\mathsf{S}} (\vec{r},\vec{s}\,) &\iff^{\text{def}} (\vec{p},\vec{q}\,) \cong (\vec{r},\vec{s}\,) \wedge \vec{p}\,\mathsf{S}\,\vec{q} \wedge \vec{r}\,\mathsf{S}\,\vec{s}.
\end{aligned}$$

When $(\vec{p},\vec{q}\,) \cong (\vec{r},\vec{s}\,)$ holds, we say that $(\vec{p},\vec{q}\,)$ and $(\vec{r},\vec{s}\,)$ are *congruent segments*, and when $(\vec{p},\vec{q}\,) \cong_{\mathsf{S}} (\vec{r},\vec{s}\,)$ holds, we say that they are *spatially congruent*.

Note that the relations $\boldsymbol{\lambda}$, S, Rest, $\cong$, $\cong_\mu$ and $\cong_{\mathsf{S}}$ are all definable over $\mathfrak{F}$.

We also define a ternary relation $\boldsymbol{\delta}$ on F^d by

$$\boldsymbol{\delta}(\vec{p},\vec{q},\vec{r}\,) \iff^{\text{def}} \vec{p}\,\mathsf{S}\,\vec{q} \wedge \vec{p}\,\boldsymbol{\lambda}\,\vec{r}.$$

Notice that $\langle F^d, \mathsf{S}, \boldsymbol{\lambda}\rangle$ and $\langle F^d, \boldsymbol{\delta}\rangle$ are definitionally equivalent, i.e.,

$$\mathsf{Conc}\,\langle F^d, \mathsf{S}, \boldsymbol{\lambda}\rangle = \mathsf{Conc}\,\langle F^d, \boldsymbol{\delta}\rangle \tag{16}$$

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

because $\boldsymbol{\delta}$ is defined in terms of S and $\boldsymbol{\lambda}$, and it is not difficult to prove that both S and $\boldsymbol{\lambda}$ can be defined in terms of $\boldsymbol{\delta}$ by $\vec{p}\,\mathsf{S}\,\vec{q} \Leftrightarrow \exists \vec{r}\,\boldsymbol{\delta}(\vec{p},\vec{q},\vec{r}\,)$ and $\vec{p}\,\boldsymbol{\lambda}\,\vec{r} \Leftrightarrow \exists \vec{q}\,\boldsymbol{\delta}(\vec{p},\vec{q},\vec{r}\,)$.

All spacetimes and geometries considered in this section are unit-free.[12] We now recall the definition of (d-dimensional) ordered affine geometry, and introduce Euclidean geometry and, respectively, Relativistic, Minkowski, Galilean, Newtonian and Late Classical[13] spacetimes over $\mathfrak{F}$:

$$\begin{aligned}
\mathcal{OAff} &\stackrel{\text{def}}{=} \langle F^d, \mathsf{Bw}\rangle,\\
\mathcal{Eucl} &\stackrel{\text{def}}{=} \langle F^d, \cong, \mathsf{Bw}\rangle,\\
\mathcal{Rel} &\stackrel{\text{def}}{=} \langle F^d, \boldsymbol{\lambda}, \mathsf{Bw}\rangle,\\
\mathcal{Mink} &\stackrel{\text{def}}{=} \langle F^d, \cong_\mu, \mathsf{Bw}\rangle,\\
\mathcal{Gal} &\stackrel{\text{def}}{=} \langle F^d, \cong_\mathsf{S}, \mathsf{Bw}\rangle,\\
\mathcal{Newt} &\stackrel{\text{def}}{=} \langle \mathcal{Gal}, \mathsf{Rest}\rangle,\\
\mathcal{LClass} &\stackrel{\text{def}}{=} \langle \mathcal{Gal}, \boldsymbol{\lambda}\rangle.
\end{aligned}$$

Note that these are all FFD coordinate geometries over $\mathfrak{F}$.

REMARK 7.2. *If $\mathfrak{F}$ is a* Euclidean *field* (*i.e., one in which positive elements have square roots*), *then* Bw *is definable in the* Bw*-free reduct $\langle F^d, \cong\rangle$ of $\mathcal{Eucl} := \langle F^d, \cong, \mathsf{Bw}\rangle$; if $d \geq 3$ and $\mathfrak{F}$ is Euclidean, then* Bw *is also definable in the* Bw*-free reducts of $\mathcal{Rel}$, $\mathcal{Mink}$ and $\mathcal{LClass}$. On the other hand,* Bw *is not definable in the* Bw*-free reducts of $\mathcal{Gal}$ and $\mathcal{Newt}$ for any $d \geq 2$.*

Given the close historical relationship between Relativistic spacetime and Minkowski spacetime, one would naturally expect to find, as shown in Figure 4, that $\mathcal{Mink}$ and $\mathcal{Rel}$ are conceptually identical. Both this relationship and the other subset relationships shown in Figure 4 are summarised in the following theorem.

THEOREM 7.3. *The subset relations between* $\mathsf{Conc}\,\mathcal{Rel}$, $\mathsf{Conc}\,\mathcal{Mink}$, $\mathsf{Conc}\,\mathcal{Eucl}$, $\mathsf{Conc}\,\mathcal{Gal}$, $\mathsf{Conc}\,\mathcal{Newt}$, $\mathsf{Conc}\,\mathcal{LClass}$ *and* $\mathsf{Conc}\,\mathcal{OAff}$ *are as shown in Figure 4. In particular, the concepts* S, Rest, $\boldsymbol{\lambda}$, $\cong$, $\cong_\mathsf{S}$, $\cong_\mu$ *and* $\boldsymbol{\delta}$ *allow us to distinguish between different geometries, as shown explicitly in Table 1. In more detail*:

(i) $\mathsf{Conc}\,\mathcal{Rel} = \mathsf{Conc}\,\mathcal{Mink}$, *i.e., $\mathcal{Rel}$ and $\mathcal{Mink}$ are definitionally equivalent.*
(ii) $\mathsf{Conc}\,\mathcal{Gal} \subseteq \mathsf{Conc}\,\mathcal{Newt}$.
(iii) $\mathsf{Conc}\,\mathcal{OAff} \subseteq (\mathsf{Conc}\,\mathcal{Eucl} \cap \mathsf{Conc}\,\mathcal{Rel} \cap \mathsf{Conc}\,\mathcal{Gal})$.
(iv) $(\mathsf{Conc}\,\mathcal{Eucl} \cup \mathsf{Conc}\,\mathcal{Rel} \cup \mathsf{Conc}\,\mathcal{Newt}) \subseteq \mathsf{Conc}\,\mathcal{LClass}$.
(v) *Table 1 correctly identifies which of the cited geometries do and do not contain the concepts* S, Rest, $\boldsymbol{\lambda}$, $\cong$, $\cong_\mathsf{S}$, $\cong_\mu$ *and* $\boldsymbol{\delta}$.

The next theorem tells us that, with just a few exceptions, if we expand any of the geometries we have been studying with any of the concepts from

[12] By "unit-free" we mean that there is no specific units fixed for measuring spatial or temporal distances.

[13] Late classical spacetime is a geometry corresponding to the understanding of classical kinematics during the time of the search for the luminiferous ether before Einstein's introduction of special relativity in 1905, cf. also [22, 23] for an alternative approach to this kinematics.

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

Table 1. *The geometries shown in Figure 4 which do and do not contain the concepts* S, Rest, $\boldsymbol{\lambda}$, $\cong$, $\cong_{\mathsf{S}}$, $\cong_{\mu}$ *and* $\boldsymbol{\delta}$

	$\mathsf{Conc}\,\mathcal{E}ucl$	$\mathsf{Conc}\,\mathcal{R}el$	$\mathsf{Conc}\,\mathcal{G}al$	$\mathsf{Conc}\,\mathcal{N}ewt$	$\mathsf{Conc}\,\mathcal{LC}lass$
$\cong$	$\in$	$\notin$	$\notin$	$\notin$	$\in$
$\cong_{\mu}$	$\notin$	$\in$	$\notin$	$\notin$	$\in$
$\boldsymbol{\lambda}$	$\notin$	$\in$	$\notin$	$\notin$	$\in$
$\cong_{\mathsf{S}}$	$\notin$	$\notin$	$\in$	$\in$	$\in$
S	$\notin$	$\notin$	$\in$	$\in$	$\in$
Rest	$\notin$	$\notin$	$\notin$	$\in$	$\in$
$\boldsymbol{\delta}$	$\notin$	$\notin$	$\notin$	$\notin$	$\in$

Note: This information confirms which geometries are definitionally distinct from one another.

$\{\cong, \cong_{\mu}, \boldsymbol{\lambda}, \cong_{\mathsf{S}}, \mathsf{S}, \mathsf{Rest}, \boldsymbol{\delta}\}$ that are not concepts of the geometry in question, the result is definitionally equivalent to $\mathcal{LC}lass$.

THEOREM 7.4. *Let* $\mathcal{G} \in \{\mathcal{R}el, \mathcal{E}ucl, \mathcal{G}al, \mathcal{N}ewt\}$ *and* $\mathsf{R} \in \{\cong, \cong_{\mu}, \boldsymbol{\lambda}, \cong_{\mathsf{S}}, \mathsf{S}, \mathsf{Rest}, \boldsymbol{\delta}\}$ *and suppose that* $\mathsf{R} \notin \mathsf{Conc}\,\mathcal{G}$ *and* $\mathsf{R} \neq \mathsf{Rest}$ *if* $\mathcal{G} = \mathcal{G}al$. *Table 1 shows the possible choices of* $\mathcal{G}$ *and* R. *Then*:

(i) *If* $d \geq 3$, *then* $\langle \mathcal{G}, \mathsf{R} \rangle$ *is definitionally equivalent to* $\mathcal{LC}lass$, *i.e.*, $\mathsf{Conc}\,\langle \mathcal{G}, \mathsf{R} \rangle = \mathsf{Conc}\,\mathcal{LC}lass$.
(ii) *If* $d = 2$ *and* $\langle \mathcal{G}, \mathsf{R} \rangle \notin \{\langle \mathcal{E}ucl, \boldsymbol{\lambda} \rangle, \langle \mathcal{E}ucl, \cong_{\mu} \rangle, \langle \mathcal{R}el, \cong \rangle\}$, *then* $\langle \mathcal{G}, \mathsf{R} \rangle$ *is definitionally equivalent to* $\mathcal{LC}lass$, *i.e.*, $\mathsf{Conc}\,\langle \mathcal{G}, \mathsf{R} \rangle = \mathsf{Conc}\,\mathcal{LC}lass$.
(iii) *If* $d = 2$, *then* $\langle \mathcal{R}el, \cong \rangle$, $\langle \mathcal{E}ucl, \cong_{\mu} \rangle$ *and* $\langle \mathcal{E}ucl, \boldsymbol{\lambda} \rangle$ *are definitionally equivalent, but they are not definitionally equivalent to* $\mathcal{LC}lass$.
(iv) $\langle \mathcal{OA}ff, \boldsymbol{\delta} \rangle$ *is definitionally equivalent to* $\mathcal{LC}lass$, *i.e.*, $\mathsf{Conc}\,\langle \mathcal{OA}ff, \boldsymbol{\delta} \rangle = \mathsf{Conc}\,\mathcal{LC}lass$.

7.3. Affine automorphisms and similarities. In this section, we show that the affine automorphisms of various geometries can be characterised as similarity transformations. We begin by defining the various types of similarities we will be using.

DEFINITION 7.5 ($\otimes$-similarities). *Suppose* $A\colon F^d \to F^d$ *is a function and* $\otimes$ *is a product on* F^d. *As a natural generalization of the notion to arbitrary ordered fields* $\mathfrak{F}$, *we call* A *a* $\otimes$-similarity *iff it is an affine transformation that respects squared* $\otimes$*-distances up to scaling, i.e., there is* $a \in F$ *such that*

$$\mathsf{d}^2_{\otimes}\big(A(\vec{p}\,), A(\vec{q}\,)\big) = a \cdot \mathsf{d}^2_{\otimes}(\vec{p}, \vec{q}\,) \quad \textit{for every } \vec{p}, \vec{q} \in F^d. \tag{17}$$

The value a *is called the* squared scale factor (*or* square-factor).

In particular, if $\otimes$ is the Euclidean scalar product, we call A a *Euclidean similarity*, and if $\otimes$ is the Minkowski product, we call A a *Poincaré similarity*. When the square-factor a is 1, these similarities are traditionally called Euclidean and Poincaré *transformations*.

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

REMARK 7.6. *We note, in the Euclidean case, that* a *is necessarily positive,*[14] *and if* $d > 2$, a *must also be positive in the Poincaré case.*[15] *However, this need not be the case when* $d = 2$; *for example, the transformation that interchanges 'time' and 'space' in* F^2 *is a Poincaré similarity with* $a = -1$.

If $\mathfrak{F}$ is a Euclidean field and A a Euclidean similarity with square-factor a, then A is the composition of a Euclidean transformation with the uniform scaling $\vec{p} \mapsto \sqrt{a}\,\vec{p}$ with scale factor $\sqrt{a}$. For non-Euclidean fields this need not hold, e.g., the map $L(t, x) = (t + x, t - x)$ is a Euclidean similarity with square-factor 2 which cannot be decomposed in this way over the field of rationals. We also note that, for any ordered field, the composition of a Euclidean transformation with a uniform scaling of factor $\lambda \neq 0$ is a Euclidean similarity with square-factor λ^2. Analogous statements hold for Poincaré transformations and similarities when $d > 2$.

DEFINITION 7.7 (Trivial similarities). *A map is called a* trivial Euclidean similarity *iff it is a Euclidean similarity that respects* Rest.

The sets of Euclidean, Poincaré and trivial Euclidean similarities are denoted by EuclSim, PoiSim and TrivEuclSim, respectively.

The following key theorem shows that these similarities are precisely the affine automorphisms of their associated spacetimes. It is this result which makes it easy for us to check for definitional equivalence between spacetimes.[16]

THEOREM 7.8.

(i) AffAut $\mathcal{E}ucl$ = EuclSim.
(ii) AffAut $\mathcal{R}el$ = AffAut $\mathcal{M}ink$ = PoiSim.
(iii) AffAut $\mathcal{LC}lass$ = TrivEuclSim.

Because Aut $\mathfrak{M}$ is always a group under composition for any model $\mathfrak{M}$, it immediately follows that all of these similarity sets are groups.

COROLLARY 7.9. EuclSim, PoiSim *and* TrivEuclSim *all form groups under composition*.

The results collected in the following lemma can all be proven using standard linear algebraic methods; full details are provided in [26].

LEMMA 7.10.

(i) EuclSim $\cap$ Aut $\langle F^d, \mathsf{S}\rangle$ = EuclSim $\cap$ Aut $\langle F^d, \mathsf{Rest}\rangle$ = TrivEuclSim.
(ii) PoiSim $\cap$ Aut $\langle F^d, \mathsf{S}\rangle$ = PoiSim $\cap$ Aut $\langle F^d, \mathsf{Rest}\rangle$ = TrivEuclSim.
(iii) EuclSim $\cap$ PoiSim = TrivEuclSim *if* $d > 2$.

7.4. Proof sketches for Theorem 7.8. Theorem 7.8 identifies which similarity groups act as automorphism groups for which geometries. We provide here sketch versions of the proofs, and note that full details are provided in [26].

[14] Because it can be calculated from $a = \mathsf{d}^2(A(\vec{0}), A(\vec{\mathrm{E}}_1))$, which is a non-empty sum of squares.
[15] To see this, it is enough to show that two non-zero 'timelike' vectors cannot be Minkowski-orthogonal (where a vector $\vec{p}$ is said to be 'timelike' if $\vec{p} \diamond \vec{p} > 0$). This latter follows easily from the Cauchy–Schwarz inequality, and we omit the details.
[16] In [26] we also introduce the sets of Galilean and trivial Galilean similarities, and show that AffAut $\mathcal{G}al$ = GalSim and AffAut $\mathcal{N}ewt$ = TrivGalSim.

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

Proof of (i) $\mathsf{AffAut}\,\mathit{Eucl} = \mathsf{EuclSim}$. It follows from the definitions that $\mathsf{EuclSim} \subseteq \mathsf{AffAut}\,\mathit{Eucl}$. To prove the reverse inclusion, let $A \in \mathsf{AffAut}\,\mathit{Eucl}$. Then A takes segments of squared-length 1 to segments of squared-length a for some positive a. If $\mathfrak{F}$ is Euclidean it is easy to show that A is a Euclidean similarity with square-factor a because any straight line contains segments of length 1 and affine transformations preserve segment length ratios. For the complete proof, including the case where $\mathfrak{F}$ is not Euclidean, see [26]. □

Proof of (ii) $(\mathsf{AffAut}\,\mathcal{R}el = \mathsf{AffAut}\,\mathit{Mink} = \mathsf{PoiSim})$. It follows easily from the definitions that $\mathsf{PoiSim} \subseteq \mathsf{AffAut}\,\mathit{Mink}$, and we know that $\mathsf{AffAut}\,\mathit{Mink} \subseteq \mathsf{AffAut}\,\mathcal{R}el$ because $\boldsymbol{\lambda}$ is definable in terms of $\cong_\mu$ (as $\vec{p}\,\boldsymbol{\lambda}\,\vec{q} \iff (\vec{p},\vec{q}\,) \cong_\mu (\vec{p},\vec{p}\,)$). This shows that

$$\mathsf{PoiSim} \subseteq \mathsf{AffAut}\,\mathit{Mink} \subseteq \mathsf{AffAut}\,\mathcal{R}el,$$

so we only need to show the reverse inclusion, $\mathsf{AffAut}\,\mathcal{R}el \subseteq \mathsf{PoiSim}$. For the case $d > 2$, this follows easily from Lester [24]. In [26] we provide a self-contained proof which works for the case when $d = 2$ as well. □

Proof of (iii) $\mathsf{AffAut}\,\mathcal{LC}lass = \mathsf{TrivEuclSim}$. By definition, $\mathcal{LC}lass \doteq \langle \mathcal{R}el, \cong_{\mathsf{S}} \rangle$, and we know that S is definable from $\cong_{\mathsf{S}}$ (as $\vec{p}\,\mathsf{S}\vec{q} \iff (\vec{p},\vec{q}\,) \cong_{\mathsf{S}} (\vec{p},\vec{q}\,)$). It follows from this, from Lemma 7.10(ii) and Theorem 7.8(ii) that

$$\mathsf{AffAut}\,\mathcal{LC}lass = \mathsf{AffAut}\,\mathcal{R}el \cap \mathsf{Aut}\,\langle F^d, \cong_{\mathsf{S}} \rangle \subseteq \mathsf{PoiSim} \cap \langle F^d, \mathsf{S} \rangle = \mathsf{TrivEuclSim}.$$

On the other hand, by Lemma 7.10(i) and item (i) we have

$$\mathsf{TrivEuclSim} = \mathsf{EuclSim} \cap \mathsf{Aut}\,\langle F^d, \mathsf{S} \rangle = \mathsf{AffAut}\,\mathit{Eucl} \cap \mathsf{Aut}\,\langle F^d, \mathsf{S} \rangle,$$

whence elements of TrivEuclSim respect both $\cong$ and S, and hence also $\cong_{\mathsf{S}}$ (which is defined in terms of them). So $\mathsf{TrivEuclSim} \subseteq \mathsf{Aut}\,\langle F^d, \cong_{\mathsf{S}} \rangle$. By Lemma 7.10(ii), $\mathsf{TrivEuclSim} \subseteq \mathsf{PoiSim}$. It therefore follows, as required, that

$$\mathsf{TrivEuclSim} \subseteq \mathsf{PoiSim} \cap \mathsf{Aut}\,\langle F^d, \cong_{\mathsf{S}} \rangle = \mathsf{AffAut}\,\mathcal{R}el \cap \mathsf{Aut}\,\langle F^d, \cong_{\mathsf{S}} \rangle = \mathsf{AffAut}\,\mathcal{LC}lass.$$

□

7.5. Proof sketches for Theorems 7.3 and 7.4. In the proofs of Theorems 7.3 and 7.4, we use Corollary 4.6(ii), according to which, for any FFD coordinate geometries $\mathcal{G}$ and $\mathcal{G}'$ over $\mathfrak{F}$,

$$\mathsf{Conc}\,\mathcal{G} = \mathsf{Conc}\,\mathcal{G}' \iff \mathsf{AffAut}\,\mathcal{G} = \mathsf{AffAut}\,\mathcal{G}',$$

together with Theorem 4.2 which implies that for any FFD coordinate geometry $\mathcal{G}$ and relation R on F^d which is definable over $\mathfrak{F}$,

$$\mathsf{R} \in \mathsf{Conc}\,\mathcal{G} \iff \mathsf{R} \text{ is closed under the elements of } \mathsf{AffAut}\,\mathcal{G}.$$

The proofs as to which concept sets are contained within one another are generally straightforward (full details are provided in [26]). For example:

Proof of 7.3(i) $\mathsf{Conc}\,\mathcal{R}el = \mathsf{Conc}\,\mathit{Mink}$. Follows from Corollary 4.6(ii) and Theorem 7.8(ii), which says that $\mathsf{AffAut}\,\mathcal{R}el = \mathsf{AffAut}\,\mathit{Mink}$. □

Proof of 7.3(iv) $((\mathsf{Conc}\,\mathit{Eucl} \cup \mathsf{Conc}\,\mathcal{R}el \cup \mathsf{Conc}\,\mathit{Newt}) \subseteq \mathsf{Conc}\,\mathcal{LC}lass)$. We know that $\{\boldsymbol{\lambda}, \cong_{\mathsf{S}}, \mathsf{Bw}\} \subseteq \mathsf{Conc}\,\mathcal{LC}lass$ by definition of $\mathcal{LC}lass$. The affine automorphism group of $\mathcal{LC}lass$ is TrivEuclSim by Theorem 7.8. Elements of TrivEuclSim respect Rest

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

and $\cong$ because $\mathsf{TrivEuclSim} = \mathsf{EuclSim} \cap \mathsf{Aut}\,\langle F^d, \mathsf{Rest}\rangle$ by definition and elements of EuclSim respect $\cong$ by Theorem 7.8. Therefore, by Theorem 4.2, $\{\mathsf{Rest}, \cong\} \subseteq \mathsf{Conc}\,\mathcal{LC}lass$. We have that $\mathsf{Conc}\,\mathcal{E}ucl \subseteq \mathsf{Conc}\,\mathcal{LC}lass$ by $\{\cong, \mathsf{Bw}\} \subseteq \mathsf{Conc}\,\mathcal{LC}lass$; $\mathsf{Conc}\,\mathcal{R}el \subseteq \mathsf{Conc}\,\mathcal{LC}lass$ by $\{\boldsymbol{\lambda}, \mathsf{Bw}\} \subseteq \mathsf{Conc}\,\mathcal{LC}lass$; and $\mathsf{Conc}\,\mathcal{N}ewt \subseteq \mathsf{Conc}\,\mathcal{LC}lass$ by $\mathcal{N}ewt \stackrel{\text{def}}{=} \langle F^d, \cong_{\mathsf{S}}, \mathsf{Bw}, \mathsf{Rest}\rangle$ and $\{\cong_{\mathsf{S}}, \mathsf{Rest}, \mathsf{Bw}\} \subseteq \mathsf{Conc}\,\mathcal{LC}lass$. □

Demonstrating that Table 1—which shows which specific concepts are included in the various geometries considered above—is correct is only slightly more involved. We have already shown (Theorem 7.3(i)) that

$$\mathsf{Conc}\,\langle F^d, \boldsymbol{\lambda}, \mathsf{Bw}\rangle \stackrel{\text{def}}{=} \mathsf{Conc}\,\mathcal{R}el = \mathsf{Conc}\,\mathcal{M}ink \stackrel{\text{def}}{=} \mathsf{Conc}\,\langle F^d, \cong_\mu, \mathsf{Bw}\rangle\,.$$

Therefore, for any coordinate geometry $\mathcal{G}$ over $\mathfrak{F}$, we have that

$$\boldsymbol{\lambda} \in \mathsf{Conc}\,\mathcal{G} \iff \cong_\mu \in \mathsf{Conc}\,\mathcal{G}\,. \tag{18}$$

Similarly, by (16), we have

$$\boldsymbol{\delta} \in \mathsf{Conc}\,\mathcal{G} \iff \{\boldsymbol{\lambda}, \mathsf{S}\} \subseteq \mathsf{Conc}\,\mathcal{G}\,. \tag{19}$$

Furthermore, since S is definable in terms of $\cong_{\mathsf{S}}$ (as $\vec{p}\,\mathsf{S}\,\vec{q} \Leftrightarrow (\vec{p}, \vec{q}\,) \cong_{\mathsf{S}} (\vec{p}, \vec{q}\,)$), we have that

$$(\cong_{\mathsf{S}} \in \mathsf{Conc}\,\mathcal{G} \implies \mathsf{S} \in \mathsf{Conc}\,\mathcal{G}\,) \text{ and } (\mathsf{S} \notin \mathsf{Conc}\,\mathcal{G} \implies \cong_{\mathsf{S}} \notin \mathsf{Conc}\,\mathcal{G}\,). \tag{20}$$

We use these observations to prove the correctness of the table, one column at a time. For example, consider Table 1, column 1, which concerns $\mathcal{E}ucl \stackrel{\text{def}}{=} \langle F^d, \cong, \mathsf{Bw}\rangle$. We need to prove that $\cong$ is a concept—and that $\cong_\mu$, $\boldsymbol{\lambda}$, $\cong_{\mathsf{S}}$, S, Rest and $\boldsymbol{\delta}$ are not concepts—of $\mathcal{E}ucl$. Clearly, $\cong \in \mathsf{Conc}\,\mathcal{E}ucl$. To prove that the other relations are not concepts, it is enough to find an automorphism of $\mathcal{E}ucl$ that does not respect them. Let $E\colon F^d \to F^d$ be the linear transformation that takes the unit vectors $\vec{\mathrm{e}}_1$ and $\vec{\mathrm{e}}_2$, respectively, to $\frac{3}{5}\vec{\mathrm{e}}_1 + \frac{4}{5}\vec{\mathrm{e}}_2$ and $\frac{4}{5}\vec{\mathrm{e}}_1 - \frac{3}{5}\vec{\mathrm{e}}_2$, and keeps the rest of the unit vectors fixed. It can be proven that E is a Euclidean similarity, hence by Theorem 7.8(i), E is an automorphism of $\mathcal{E}ucl$. Transformation E respects neither Rest, S nor $\boldsymbol{\lambda}$ because $\vec{\mathrm{o}}\,\mathsf{Rest}\,\vec{\mathrm{e}}_1$, $\vec{\mathrm{o}}\,\mathsf{S}\,\vec{\mathrm{e}}_2$ and $\vec{\mathrm{o}}\,\boldsymbol{\lambda}\,(\vec{\mathrm{e}}_1 + \vec{\mathrm{e}}_2)$ all hold, but none of $E(\vec{\mathrm{o}}\,)\,\mathsf{Rest}\,E(\vec{\mathrm{e}}_1)$, $E(\vec{\mathrm{o}}\,)\,\mathsf{S}\,E(\vec{\mathrm{e}}_2)$, $E(\vec{\mathrm{o}}\,)\,\boldsymbol{\lambda}\,E(\vec{\mathrm{e}}_1 + \vec{\mathrm{e}}_2)$ do. Therefore, $\mathsf{Rest}, \mathsf{S}, \boldsymbol{\lambda} \notin \mathsf{Conc}\,\mathcal{E}ucl$. By these and (18)–(20), we have that $\cong_\mu, \cong_{\mathsf{S}}, \boldsymbol{\delta} \notin \mathsf{Conc}\,\mathcal{E}ucl$. □

7.5.1. Proof sketch of Theorem 7.4. Since the proofs of parts (i) and (ii) overlap to some extent, we will prove them together, considering each possible choice of $\mathcal{G}$ in turn.

Proof (i) *and* (ii): Here we consider only those cases where $\mathsf{R} \in \{\mathsf{Rest}, \mathsf{S}, \boldsymbol{\lambda}, \cong\}$ because the claims follow for the other relations easily by (16), Theorem 7.3(i) and the fact that S is definable in terms of $\cong_{\mathsf{S}}$. By Theorem 7.8(iii) (which says that $\mathsf{AffAut}\,\mathcal{LC}lass = \mathsf{TrivEuclSim}$) and by Corollary 4.6(ii), we have, for any FFD coordinate geometry $\mathcal{G}$ over $\mathfrak{F}$ and any relation R definable over $\mathfrak{F}$, that

$$\mathsf{Conc}\,\langle \mathcal{G}, \mathsf{R}\rangle = \mathsf{Conc}\,\mathcal{LC}lass \iff \mathsf{AffAut}\,\langle \mathcal{G}, \mathsf{R}\rangle = \mathsf{TrivEuclSim}. \tag{21}$$

Case $\mathcal{G} = \mathsf{Eucl}$. We will prove that $\langle \mathcal{E}ucl, \mathsf{S}\rangle$, $\langle \mathcal{E}ucl, \mathsf{Rest}\rangle$ and (if $d > 2$) $\langle \mathcal{E}ucl, \boldsymbol{\lambda}\rangle$ are all definitionally equivalent to $\mathcal{LC}lass$. By Theorem 7.8(i), $\mathsf{AffAut}\,\mathcal{E}ucl = \mathsf{EuclSim}$. By

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

this and Lemma 7.10(i), we have that

$$\begin{aligned}\mathsf{AffAut}\,\langle \mathcal{E}ucl, \mathsf{S}\rangle &= \mathsf{AffAut}\,\mathcal{E}ucl \cap \mathsf{Aut}\,\langle F^d, \mathsf{S}\rangle \\ &= \mathsf{EuclSim} \cap \mathsf{Aut}\,\langle F^d, \mathsf{S}\rangle = \mathsf{TrivEuclSim}\end{aligned} \tag{22}$$

and likewise

$$\mathsf{AffAut}\,\langle \mathcal{E}ucl, \mathsf{Rest}\rangle = \mathsf{EuclSim} \cap \mathsf{Aut}\,\langle F^d, \mathsf{Rest}\rangle = \mathsf{TrivEuclSim}. \tag{23}$$

Similarly, by Lemma 7.10(iii), by $\mathcal{R}el \stackrel{\text{def}}{=} \langle F^d, \boldsymbol{\lambda}, \mathsf{Bw}\rangle$ and by $\mathsf{AffAut}\,\mathcal{R}el = \mathsf{PoiSim}$ (Theorem 7.8(ii)), we have, when $d > 2$,

$$\begin{aligned}\mathsf{AffAut}\,\langle \mathcal{E}ucl, \boldsymbol{\lambda}\rangle &= \mathsf{AffAut}\,\mathcal{E}ucl \cap \mathsf{AffAut}\,\mathcal{R}el \\ &= \mathsf{EuclSim} \cap \mathsf{PoiSim} = \mathsf{TrivEuclSim}.\end{aligned} \tag{24}$$

Therefore, by (21)–(23) we have that $\mathsf{Conc}\,\langle \mathcal{E}ucl, \mathsf{Rest}\rangle = \mathsf{Conc}\,\langle \mathcal{E}ucl, \mathsf{S}\rangle = \mathsf{Conc}\,\mathcal{LC}lass$. And by (21) and (24) we also have $\mathsf{Conc}\,\langle \mathcal{E}ucl, \boldsymbol{\lambda}\rangle = \mathsf{Conc}\,\mathcal{LC}lass$ when $d > 2$.

The rest follow from these, e.g., $\mathsf{Conc}\,\langle \mathcal{E}ucl, \cong_{\mathsf{S}}\rangle = \mathsf{Conc}\,\mathcal{LC}lass$ because $\cong_{\mathsf{S}} \in \mathsf{Conc}\,\mathcal{LC}lass$, S is definable in terms of $\cong_{\mathsf{S}}$ and $\mathsf{Conc}\,\langle \mathcal{E}ucl, \mathsf{S}\rangle = \mathsf{Conc}\,\mathcal{LC}lass$.

Case $\mathcal{G} = \mathcal{R}el$: We need to show that $\langle \mathcal{R}el, \mathsf{S}\rangle$, $\langle \mathcal{R}el, \mathsf{Rest}\rangle$ and (if $d > 2$) $\langle \mathcal{R}el, \cong\rangle$ are all definitionally equivalent to $\mathcal{LC}lass$. By Theorem 7.8(ii), we have that $\mathsf{AffAut}\,\mathcal{R}el = \mathsf{PoiSim}$. By this and Lemma 7.10(ii),

$$\mathsf{AffAut}\,\langle \mathcal{R}el, \mathsf{S}\rangle = \mathsf{PoiSim} \cap \mathsf{Aut}\,\langle F^d, \mathsf{S}\rangle = \mathsf{TrivEuclSim}, \text{ and} \tag{25}$$

$$\mathsf{AffAut}\,\langle \mathcal{R}el, \mathsf{Rest}\rangle = \mathsf{PoiSim} \cap \mathsf{Aut}\,\langle F^d, \mathsf{Rest}\rangle = \mathsf{TrivEuclSim}. \tag{26}$$

The proof is analogous to that just given. By (21), (25) and (26), we have that $\mathsf{Conc}\,\langle \mathcal{R}el, \mathsf{Rest}\rangle = \mathsf{Conc}\,\langle \mathcal{R}el, \mathsf{S}\rangle = \mathsf{Conc}\,\mathcal{LC}lass$.

From the definitions of $\mathcal{E}ucl$ and $\mathcal{R}el$, we have that $\langle \mathcal{R}el, \cong\rangle \doteq \langle \mathcal{E}ucl, \boldsymbol{\lambda}\rangle$. We have already seen that $\mathsf{Conc}\,\langle \mathcal{E}ucl, \boldsymbol{\lambda}\rangle = \mathsf{Conc}\,\mathcal{LC}lass$ if $d > 2$. Thus $\mathsf{Conc}\,\langle \mathcal{R}el, \cong\rangle = \mathsf{Conc}\,\mathcal{LC}lass$ if $d > 2$.

Case $\mathcal{G} = \mathcal{G}al$. We need to show that $\langle \mathcal{G}al, \cong\rangle$ and $\langle \mathcal{G}al, \boldsymbol{\lambda}\rangle$ are definitionally equivalent to $\mathcal{LC}lass$. This is straightforward because $\mathcal{LC}lass \stackrel{\text{def}}{=} \langle \mathcal{G}al, \boldsymbol{\lambda}\rangle$, $\mathcal{G}al \stackrel{\text{def}}{=} \langle F^d, \cong_{\mathsf{S}}, \mathsf{Bw}\rangle$ and $\mathcal{E}ucl \stackrel{\text{def}}{=} \langle F^d, \cong, \mathsf{Bw}\rangle$; whence it follows immediately that $\langle \mathcal{G}al, \cong\rangle \doteq \langle \mathcal{E}ucl, \cong_{\mathsf{S}}\rangle$. As we have already seen that $\mathsf{Conc}\,\langle \mathcal{E}ucl, \cong_{\mathsf{S}}\rangle = \mathsf{Conc}\,\mathcal{LC}lass$, the claim follows. □

Proof of (iii). We have to show that $\langle \mathcal{R}el, \cong\rangle$ and $\langle \mathcal{E}ucl, \boldsymbol{\lambda}\rangle$ are definitionally equivalent to one another; and that they are not definitionally equivalent to $\mathcal{LC}lass$ when $d = 2$. Note first that $\langle \mathcal{E}ucl, \boldsymbol{\lambda}\rangle$ and $\langle \mathcal{R}el, \cong\rangle$ are clearly definitionally equivalent, because $\mathcal{E}ucl \stackrel{\text{def}}{=} \langle F^d, \cong, \mathsf{Bw}\rangle$ and $\mathcal{R}el \stackrel{\text{def}}{=} \langle F^d, \boldsymbol{\lambda}, \mathsf{Bw}\rangle$. However, geometries $\langle \mathcal{E}ucl, \boldsymbol{\lambda}\rangle$ and $\mathcal{LC}lass$ are not definitionally equivalent if $d = 2$, because the linear transformation that interchanges the unit vectors $\vec{\mathrm{E}}_1$ and $\vec{\mathrm{E}}_2$ is an automorphism of $\langle \mathcal{E}ucl, \boldsymbol{\lambda}\rangle$, but it is not an automorphism of $\mathcal{LC}lass$ as it does not respect $\cong_{\mathsf{S}}$. □

Proof of (iv). We have to show that $\langle \mathcal{OA}ff, \boldsymbol{\delta}\rangle$ is definitionally equal to $\mathcal{LC}lass$. Recall that $\mathcal{OA}ff \stackrel{\text{def}}{=} \langle F^d, \mathsf{Bw}\rangle$ and $\mathcal{R}el \stackrel{\text{def}}{=} \langle F^d, \boldsymbol{\lambda}, \mathsf{Bw}\rangle$, and that $\mathsf{Conc}\,\langle F^d, \boldsymbol{\delta}\rangle = \mathsf{Conc}\,\langle F^d, \boldsymbol{\lambda}, \mathsf{S}\rangle$ by (16). We have already seen that $\mathsf{Conc}\,\langle \mathcal{R}el, \mathsf{S}\rangle = \mathsf{Conc}\,\mathcal{LC}lass$. Therefore, $\mathsf{Conc}\,\langle \mathcal{OA}ff, \boldsymbol{\delta}\rangle = \mathsf{Conc}\,\langle \mathcal{OA}ff, \boldsymbol{\lambda}, \mathsf{S}\rangle = \mathsf{Conc}\,\langle \mathcal{R}el, \mathsf{S}\rangle = \mathsf{Conc}\,\mathcal{LC}lass$. □

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

§8. Spacetimes with units. In a spirit similar to ours, but using a metric approach to define Galilean, Newtonian and Minkowski spacetimes instead of a Tarskian first-order language, Barrett [4] compares the "amount of structure" in each model. Barrett's results seemingly contradict our own because for Barrett, the automorphism group of Newtonian spacetime is a subgroup of that of Minkowski spacetime, whereas in our work it is not. This contradiction is only apparent; it arises because the metric approach gives more rigid models of these spacetimes (it fixes the units of measurements) while our Tarskian approach gives unit-free models: certain scalings, intuitively corresponding to changing the units of measurement, are automorphisms of the spacetimes considered by us but not those of [4].

Adding units to our own geometries can be achieved by defining additional relations telling us how to identify when a length is 'one unit' long, and using these to extend our existing constructions. We define the binary relations u of *Euclidean unit*, u_μ of *relativistic unit*, u_S of *spatial unit*, and u_T of *temporal unit* on F^d as

$$\begin{aligned}
\vec{p}\,\mathsf{u}\,\vec{q} &\overset{\text{def}}{\iff} \mathsf{d}^2(\vec{p},\vec{q}\,)=1,\\
\vec{p}\,\mathsf{u}_\mu\,\vec{q} &\overset{\text{def}}{\iff} \mathsf{d}_\mu^2(\vec{p},\vec{q}\,)=1,\\
\vec{p}\,\mathsf{u}_\mathsf{S}\,\vec{q} &\overset{\text{def}}{\iff} \vec{p}\,\mathsf{u}\,\vec{q}\ \wedge\ \vec{p}\,\mathsf{S}\,\vec{q},\\
\vec{p}\,\mathsf{u}_\mathsf{T}\,\vec{q} &\overset{\text{def}}{\iff} |p_1-q_1|=1.
\end{aligned}$$

Now we expand our spacetimes and Euclidean geometry by adding in the appropriate units:

$$\begin{aligned}
\mathcal{E}uclUn &\overset{\text{def}}{=} \langle \mathcal{E}ucl, \mathsf{u}\rangle,\\
\mathcal{M}inkUn &\overset{\text{def}}{=} \langle \mathcal{M}ink, \mathsf{u}_\mu\rangle,\\
\mathcal{G}alUn &\overset{\text{def}}{=} \langle \mathcal{G}al, \mathsf{u}_\mathsf{S}, \mathsf{u}_\mathsf{T}\rangle,\\
\mathcal{N}ewtUn &\overset{\text{def}}{=} \langle \mathcal{N}ewt, \mathsf{u}_\mathsf{S}, \mathsf{u}_\mathsf{T}\rangle,\\
\mathcal{LC}lassUn &\overset{\text{def}}{=} \langle \mathcal{LC}lass, \mathsf{u}_\mathsf{S}, \mathsf{u}_\mathsf{T}\rangle.
\end{aligned}$$

Results similar to those given above for unit-free geometries can be derived using our approach. For example, it can be proven that the affine automorphisms of both $\mathcal{N}ewtUn$ and $\mathcal{LC}lassUn$ are the Euclidean isometries that respect Rest, from which it follows that $\mathsf{Conc}\,\mathcal{N}ewtUn = \mathsf{Conc}\,\mathcal{LC}lassUn$ (ensuring agreement with Barrett), while $\mathsf{Conc}\,\mathcal{N}ewt \subsetneq \mathsf{Conc}\,\mathcal{LC}lass$. Likewise the affine automorphisms of $\mathcal{E}uclUn$ are the Euclidean isometries, and the affine automorphisms of $\mathcal{M}inkUn$ are the Poincaré transformations. These results are summarised in Figure 5.

§9. Concluding remarks. We have introduced FFD coordinate geometries over fields and ordered fields, and have shown that the definable relations (concepts) of an FFD coordinate geometry are exactly those relations that are closed under the automorphisms of the geometry and are definable over the (ordered) field; moreover they are exactly the relations closed under the affine automorphism and are definable over the (ordered) field. These results of §4, and in particular Corollary 4.6, yield a simple but powerful strategy for comparing the conceptual content of different geometries: *If we wish to compare the sets of concepts of FFD coordinate geometries*

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

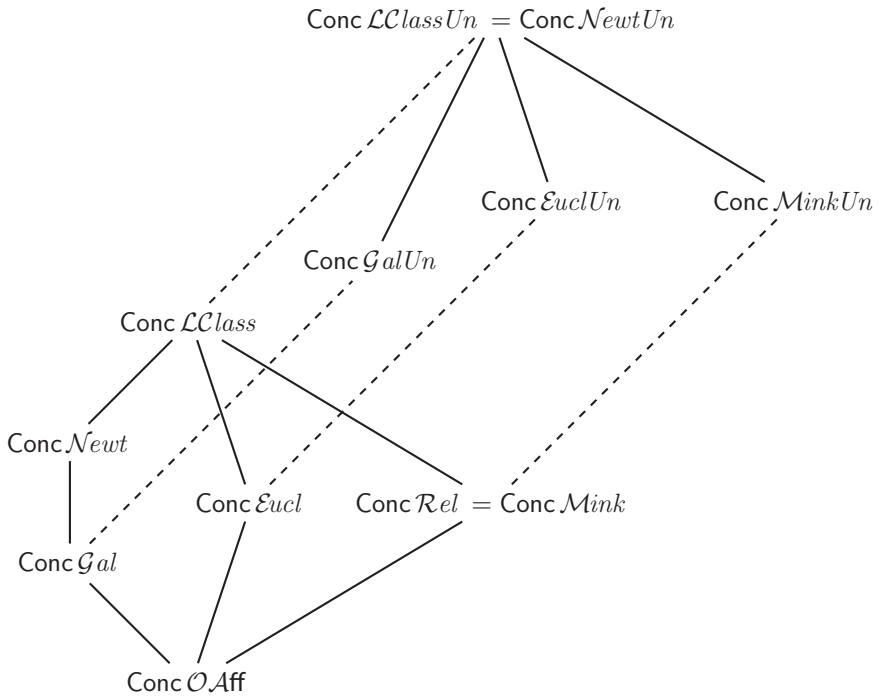


Figure 5. Extension of Figure 3 to include historically significant geometries in which units are defined.

over the same field and of the same dimension, it is enough to compare their affine automorphism groups.

This insight is strongly connected to the famous Erlangen program of Felix Klein's because it directly states that to understand the concepts of these geometries it is enough to understand their affine automorphisms. In line with this observation, we have shown the connection between the concept-sets of Galilean, Newtonian, Late Classical, Relativistic and Minkowski spacetimes and ordered affine and Euclidean geometries illustrated by Figures 3 and 4.

9.1. Open questions.

QUESTION 1. *Do the results of §4 hold for all field-definable coordinate geometries*?

The concept-sets of field-definable coordinate geometries form a complete lattice under $\subseteq$. This is so because complete semilattices are complete lattices; and it is a complete join-semilattice where the supremum of any subset is the geometry obtained by taking all of the associated relations simultaneously.

QUESTION 2. *Do the concept-sets of FFD coordinate geometries over* $\mathfrak{F}$ *form a lattice under* $\subseteq$?

A positive answer to Question 2 could be obtained by answering Question 3 positively.

QUESTION 3. *Let* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *be two FFD coordinate geometries over* $\mathfrak{F}$. *Is there an FFD coordinate geometry* $\mathcal{G}_3$ *such that* $\mathsf{Conc}\,\mathcal{G}_1 \cap \mathsf{Conc}\,\mathcal{G}_2 = \mathsf{Conc}\,\mathcal{G}_3$?

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

QUESTION 4. The status of $\mathcal{OAff}$. *By Theorem* 7.4, *we know that* $\mathcal{LC}$*lass is the conceptually smallest geometry that contains* $\mathsf{Conc}\,\mathcal{Gal} \cup \mathsf{Conc}\,\mathcal{Rel}$, $\mathsf{Conc}\,\mathcal{Eucl} \cup \mathsf{Conc}\,\mathcal{Gal}$ *or* $\mathsf{Conc}\,\mathcal{Eucl} \cup \mathsf{Conc}\,\mathcal{Rel}$. *What about if we take the intersections of incomparable geometric concept sets do we always obtain* $\mathsf{Conc}\,\mathcal{OAff}$, *or can we find concepts showing this not to be the case*? *For example, is there a common concept of Newtonian spacetime and Relativistic spacetime which is not a concept of ordered affine geometry*? *In all, we have five related questions of this kind*:

(i) *Is* $\mathsf{Conc}\,\mathcal{OAff}$ *a proper subset of* $\mathsf{Conc}\,\mathcal{Newt} \cap \mathsf{Conc}\,\mathcal{Rel}$?
(ii) *Is* $\mathsf{Conc}\,\mathcal{OAff}$ *a proper subset of* $\mathsf{Conc}\,\mathcal{Gal} \cap \mathsf{Conc}\,\mathcal{Rel}$?
(iii) *Is* $\mathsf{Conc}\,\mathcal{OAff}$ *a proper subset of* $\mathsf{Conc}\,\mathcal{Eucl} \cap \mathsf{Conc}\,\mathcal{Gal}$?
(iv) *Is* $\mathsf{Conc}\,\mathcal{OAff}$ *a proper subset of* $\mathsf{Conc}\,\mathcal{Eucl} \cap \mathsf{Conc}\,\mathcal{Newt}$?
(v) *Is* $\mathsf{Conc}\,\mathcal{OAff}$ *a proper subset of* $\mathsf{Conc}\,\mathcal{Eucl} \cap \mathsf{Conc}\,\mathcal{Rel}$?

Notice that, because $\mathsf{Conc}\,\mathcal{Gal} \subseteq \mathsf{Conc}\,\mathcal{Newt}$, *problems* (*i*) *and* (*ii*) *are not independent, and similarly for problems* (*iii*) *and* (*iv*).

Funding. This research is supported by the Hungarian National Research, Development and Innovation Office (NKFIH) (Grant Nos. FK-134732 and TKP2021-NVA-16).

BIBLIOGRAPHY

[1] Alscher, D. (2016). *Theorien der Reellen Zahlen Und Interpretierbarkeit*, Logos, 25. De Gruyter.

[2] Andréka, H., Madarász, J. X., Németi, I., Németi, P., & Székely, G. (2011). Vienna circle and logical analysis of relativity theory. In Máté, A., Rédei, M., & F. Stadler, editors. *The Vienna circle in hungary (Der Wiener Kreis und Ungarn)*. Vienna: Springer Verlag, pp. 247–268.

[3] Barrett, T. W. (2015). On the structure of classical mechanics. *The British Journal for the Philosophy of Science*, **66**(4), 801–828.

[4] ———. (2015). Spacetime structure. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, **51**, 37–43.

[5] ———. (2022). How to count structure. *Noûs*, **56**(2), 295–322.

[6] Barrett, T. W., Manchak, J. B., & Weatherall, J. O. (2023). On automorphism criteria for comparing amounts of mathematical structure. *Synthese*, **201**(6), 191.

[7] Berger, M. (1987). *Geometry I*, Universitext. Berlin: Springer.

[8] Büchi, J. R. (1990). Relatively categorical and normal theories. In Lane, S. M., & Siefkes, D., editors. *The Collected Works of J. Richard Büchi*. New York: Springer, pp. 277–286.

[9] Cocco, L., & Babic, J. (2021). A system of axioms for Minkowski spacetime. *Journal of Philosophical Logic*, **50**(1), 149–185.

[10] Formica, G., & Friend, M. (2021). In the footsteps of Hilbert: The Andréka–Németi group's logical foundations of theories in physics. In Madarász, J., & Székely, G., editors. *Hajnal Andréka and István Németi on Unity of Science: From Computing to Relativity Theory through Algebraic Logic*. Switzerland, AG: Springer International Publishing, pp. 383–408.

[11] Friend, M. (2015). On the epistemological significance of the Hungarian project. *Synthese*, **192**(7), 2035–2051.

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

[12] Goldblatt, R. (1987). *Orthogonality and Spacetime Geometry*. Berlin: Springer–Verlag.

[13] Grimson, R., Kuijpers, B., & Othman, W. (2012). Quantifier elimination for elementary geometry and elementary affine geometry. *Mathematical Logic Quarterly*, **58**(6), 399–416.

[14] Gupta, H. N. (1969). On some axioms in foundations of Cartesian spaces. *Canadian Mathematical Bulletin*, **12**(6), 831–836.

[15] Henkin, L., Monk, J. D., & Tarski, A. (1971). *Cylindric Algebras Part I*. Amsterdam, London: North-Holland.

[16] Hilbert, D. (1899). *Grundlagen der Geometrie*. Leipzig: Verlag von B. G. Teubner.

[17] Hodges, W. (1993). *Model Theory*. Cambridge: Cambridge University Press.

[18] Ketland, J. (2023). Axiomatization of Galilean Spacetime. *Dialectica* **77**(2): 155–204. https://doi.org/10.48106/dial.v77.i2.02.

[19] Khaled, M., & Székely, G. (2021). Algebras of concepts and their networks. In Allahviranloo, T., Salahshour, S., & Arica, N., editors. *Progress in Intelligent Decision Science*. Cham: Springer International Publishing, pp. 611–622.

[20] ———. (2025). Conceptual distance and algebras of concepts. *The Review of Symbolic Logic*, **18**(1), 300–315.

[21] Khaled, M., Székely, G., Lefever, K., & Friend, M. (2020). Distances between formal theories. *The Review of Symbolic Logic*, **13**(3), 633–654.

[22] Lefever, K. (2017). *Using logical interpretation and definitional equivalence to compare classical kinematics and special relativity theory*, PhD Thesis, Vrije Universiteit Brussel.

[23] Lefever, K., & Székely, G. (2018). Comparing classical and relativistic kinematics in first-order-logic. *Logique et Analyse*, **61**(241), 57–117.

[24] Lester, J. A. (1977). Cone preserving mappings for quadratic cones over arbitrary fields. *Canadian Journal of Mathematics*, **29**(6), 1247–1253.

[25] Madarász, J., Stannett, M., & Székely, G. (2025). Definable coordinate geometries over fields, part 1: Theory. Preprint, arxiv:2507.10279.

[26] ———. (2025). Definable coordinate geometries over fields, part 2: Applications. Preprint, arxiv:2507.10289.

[27] ———. (2025). On Andréka's conjecture that special relativity is the only possible conceptual reduct of classical kinematics. Preprint, arxiv:2507.21180.

[28] Misner, C. W., Thorne, K. S., & Wheeler, J. A. (1973). *Gravitation*. New York: W. H. Freeman and Company.

[29] Monk, J. D. (2000). An introduction to cylindric set algebras. *Logic Journal of the IGPL*, **8**(4), 451–496.

[30] Muchnik, A. A., & Semenov, A. L. (2020). Lattice of definability in the order of rational numbers. *Mathematical Notes*, **108**(1), 94–107.

[31] North, J. (2009). The "structure" of physics. *Journal of Philosophy*, **106**(2), 57–88.

[32] ———. (2021). *Physics, Structure and Reality*. Oxford: Oxford University Press.

[33] Pambuccian, V. (1989). Simple axiom systems for Euclidean geometry. *Mathematical the Chronicle*, **18**, 63–74.

[34] ———. (2024). The simplicity degree of Tarski's Euclidean geometry of ruler and dividers is 5. *Results in Mathematics*, **79**(4), 167.

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press

[35] Papadimitriou, C. H., & Yannakakis, M. (1999). On the complexity of database queries. *Journal of Computer and System Sciences*, **58**(3), 407–427.

[36] Schwabhäuser, W., Szmielew, W., & Tarski, A. (1983). *Metamathematische Methoden in der Geometrie*. Berlin: Springer-Verlag.

[37] Semenov, A., Soprunov, S., & Uspensky, V. (2014). The lattice of definability. Origins, recent developments, and further directions. In Hirsch, E. A., Kuznetsov, S. O., Pin, J.-É., & Vereshchagin, N. K., editors. *Computer Science—Theory and Applications*. Cham: Springer International Publishing, pp. 23–38.

[38] Semenov, A. L., & Soprunov, S. F. (2015). A combinatorial version of the Svenonius theorem on definability. *Logic Journal of the IGPL*, **23**(6), 966–975.

[39] ———. (2021). Lattice of definability (of reducts) for integers with successor. *Izvestiya: Mathematics*, **85**, 1257–1269.

[40] ———. (2022). Automorphisms and definability (of reducts) for upward complete structures. *Mathematics*, **10**(20), 3478.

[41] ———. (2024). On a lattice of relational spaces (reducts) for the order of integers. Preprint, arXiv:2411.18181.

[42] Swanson, N., & Halvorson, H. (2012). On North's "the structure of physics". Manuscript.

[43] Szczerba, L., & Tarski, A. (1979). Metamathematical discussion of some affine geometries. *Fundamenta Mathematicae*, **104**(3), 155–192.

[44] Tarrida, A. R. (2011). *Affine Maps, Euclidean Motions and Quadrics*, Springer Undergraduate Mathematics Series. London, Dordrecht, Heidelberg, New York: Springer.

[45] Tarski, A. (1959). What is elementary geometry? In Henkin, L., Suppes, P., & Tarski, A., editors. *The Axiomatic Method*, Studies in Logic and the Foundations of Mathematics, 27. Amsterdam: Elsevier, pp. 16–29.

[46] ———. (1998). A decision method for elementary algebra and geometry. In Caviness, B. F., & Johnson, J. R., editors. *Quantifier Elimination and Cylindrical Algebraic Decomposition*. Vienna: Springer, pp. 24–84.

[47] Tarski, A., & Givant, S. (1999). Tarski's system of geometry. *Bulletin of Symbolic Logic*, **5**(2), 175–214.

[48] Wallace, D. (2019). Who's afraid of coordinate systems? An essay on representation of spacetime structure. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, **67**(C), 125–136.

[49] Wilhelm, I. (2021). Comparing the structures of mathematical objects. *Synthese*, **199**(3), 6357–6369.

HUN-REN ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS
HUNGARY
E-mail: madarasz.judit@renyi.hu

UNIVERSITY OF MANCHESTER
UK
E-mail: mike.stannett@gmail.com

HUN-REN ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS &
UNIVERSITY OF PUBLIC SERVICE
HUNGARY
E-mail: szekely.gergely@renyi.hu

https://doi.org/10.1017/S175502032610118X Published online by Cambridge University Press