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# Application of lower-bound stress fields to stability assessment of soil backfilled masonry arch bridges

QI DANG\*, C. C. SMITH\*, M. GILBERT\*

Limit analysis is a convenient approach for the estimation of the ultimate collapse state of geotechnical constructions, including pipes, tunnels and retaining walls. A soil-filled masonry arch bridge is a combined geotechnical/masonry structure where rapid limit analysis can play a valuable role in assessing the ultimate load-carrying capacity for multiple load cases. However, currently available rapid analysis techniques for masonry arches only model masonry elements directly, with the soil backfill indirectly modelled leading to reduced clarity in the predictions of the bridge load-carrying capacity. To address this, a soil backfill model based on lower-bound stress field theory is proposed in this paper. The results, compared with upper-bound solutions obtained using the discontinuity layout optimization (DLO) numerical limit analysis procedure, show good matches of upper and lower bounds typically to within 10%. The proposed methodology has applications beyond that of arch bridges such as to tunnels and buried pipes.

**KEYWORDS:** lower bound; stress fields; soil backfilled masonry arch bridges; limit state design/analysis; bearing capacity

## INTRODUCTION

Limit analysis and specifically the lower-bound approach has been used historically for many geotechnical problems (Chen 1975). However it can be challenging to generate analytical stress fields for some problem types such as surface loads acting on buried pipes, tunnels or soil backfilled arches. To tackle this challenge, a combination of lower-bound stress fields involving the superposition of ‘Boussinesq’ and simple self weight stress fields is proposed in this paper and will be illustrated through the application to soil backfilled masonry arch bridges.

There are approximately more than one million masonry arch spans in service around the world (Sarhosis *et al.* 2016). Most of them were built over 120 years ago and backfilled with soil (Heyman 1982). With increasing traffic demands it is necessary to guarantee their safety and estimate their load-carrying capacity reliably. Although it is possible to model the soil backfill explicitly in a numerical limit analysis model, e.g. via use of finite element limit analysis (Cavicchi & Gambarotta 2005, 2007; Gilbert *et al.* 2007) or discontinuity layout optimization (Smith & Gilbert 2007; Gilbert *et al.* 2010), additional computational costs are incurred. This is particularly relevant where the arch bridge assessment involves consideration of large sets of multiple load cases and/or requires a reliability assessment, where the required number of individual model runs could run into the order of thousands for one bridge. Thus a method of rapid analysis becomes attractive.

This paper addresses a plane strain lower bound limit analysis approach, which requires that a complete stress field

must be found, everywhere satisfying equilibrium and the stress boundary conditions, and nowhere violating yield, Chen (1975).

Modelling of bare masonry arch structures is straightforward and rapid (due to the small number of blocks) using rigid-block limit analysis and has been previously applied to obtain an exact plastic solution (Gilbert & Melbourne 1994). The mathematical formulation of rigid-block analysis and relevant masonry stress fields is summarized in Supplementary Material A. The solution automatically provides the critical mechanism and the multiplier, or adequacy factor  $\lambda$  on the live load required to cause collapse. An initial unit live load is typically adopted such that  $\lambda$  equals the collapse load.

The development of lower-bound stress fields theory in this paper is focused on soil backfill material and extends significantly that presented by Dang *et al.* (2019).

## SOIL STRESS FIELDS

### Introduction

In the lower-bound limit analysis of masonry arch bridges, the challenge is to find a simple but sufficiently accurate stress field in the soil backfill due to both an applied surface load and the backfill dead weight. When a vehicle traverses a masonry arch bridge, each wheel of the vehicle can be idealised as a strip uniform load on the road (or rail) surface. For simplicity, a single load only is considered in this paper, however the method can be extended to multiple load positions.

If a single load is applied at around the quarter span of a masonry arch bridge, the collapse mechanism is typically a four hinged mechanism as shown in Fig. 1. The quarter span is often considered to be the critical loading position and is useful in

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illustrating the proposed analysis process which can then be generalised.

Considering Fig. 1, it is convenient to split the backfill into four different zones. The active and passive zones are defined as those bodies of soil adjacent to the parts of the arch ring that are moving away or into the fill respectively. The surface layer is defined as a horizontal layer located between the crown of the arch ring and the soil surface of thickness  $D$ . Since stress fields close to the load boundary typically cannot avoid yield, then the surface layer is further subdivided into (a) a shallow ‘cemented’ layer of thickness  $S_C$ , representing e.g. a bound road surface, and (b) an uncemented layer of thickness  $S_U$ , below consisting of the main soil backfill. This could be further subdivided as necessary.

### Solution strategy

The lower bound theorem states that any statically admissible stress field provides a lower bound estimate of the true collapse load. To achieve the optimal lower bound (i.e. find the largest load that can be carried) it is necessary to choose this stress field so as to minimize the effect of the disturbing actions and maximize the effect of the restoring actions acting on the arch. In this study, lateral earth coefficients are varied to define such a statically admissible stress field. In broad terms, the disturbing actions comprise the active earth pressures due to the combination of backfill dead weight and surface load and the restoring actions comprise the passive pressures due to backfill dead weight.

It is therefore necessary to find expressions for each term, ideally such that there are one or more degrees of freedom to maximise or minimise these actions. The solution consists of the combination of two stress fields. The first is that due to a surface load acting on a weightless soil. This is combined with a stress field for a soil possessing self weight with horizontal and vertical principal stress directions.

Utilization of the simple self weight stress field described above means that the solution stress field in the soil will be discretised into horizontal layers such that the self weight stress field in each layer is the same at all horizontal locations and yield is not violated anywhere in the layer for the combined stress field. To enforce the no-yield condition for each layer, critical locations will be at infinity (to ensure no-yield for areas unaffected by the surface load), local to the loaded area (to ensure no-yield due to the stresses induced by the surface load) and at the layer/arch interface (to ensure no-yield where the interface shear strength is less than that of the fill) as shown in Fig 2.

For simplicity in the following derivations no distinction will be made between effective stress and undrained parameters and general stress symbols  $\tau$ ,  $\sigma$ ,  $c$ ,  $\phi$ , representing shear stress, normal stress, cohesion intercept, and angle of shearing resistance respectively, will be used.

### Stress fields due to surface load

There are various options when seeking limit analysis solutions. Any suitable purely mathematical function that generates a 2D equilibrium stress field due to a surface load may be utilised. In this work, a ‘Boussinesq’ type stress field is employed as the mathematical function (the elastic material nature of the original solution is completely incidental to a lower bound solution). The additional requirement is to demonstrate ‘no yield’.

To provide an additional degree of freedom to this mathematical function, the anisotropic material solution to the Boussinesq stress field (Moroto, 1973) is adopted, where the additional mathematical degree of freedom  $n$ , is based on the degree of anisotropy in the original source equations i.e. the deformation modulus ratio  $n = E_x/E_z$ , where  $E_z$  and  $E_x$  are the elastic moduli in the vertical and horizontal direction respectively (Fig. 3).

For simplicity, in this work, the (live) surface load is represented by a uniform stress distribution  $q$  acting over a width  $B = 2b$ . However it is noted that stresses under rigid or semi-rigid surface loads may need to be modelled using a different stress distribution.

Based on the Moroto formulation, the incremental vertical stress  $\Delta\sigma_z$ , horizontal stress  $\Delta\sigma_x$  and shear stress  $\Delta\tau_{xz}$  due to the surface load  $q$  can be stated as follows:

$$\Delta\sigma_z = \frac{q}{\pi} \frac{1}{\eta_2 - \eta_1} (\eta_2 k_1 - \eta_1 k_2) \quad (1a)$$

$$\Delta\sigma_x = \frac{q}{\pi} \frac{\eta_1 \eta_2}{\eta_2 - \eta_1} (-\eta_1 k_1 + \eta_2 k_2) \quad (1b)$$

$$\Delta\tau_{xz} = \frac{q}{2\pi} \frac{\eta_1 \eta_2}{\eta_2 - \eta_1} k_3 \quad (1c)$$

where  $\eta_1 = \sqrt{n}$ ,  $\eta_2 = 1$ , and

$$k_1 = \tan^{-1} \left[ \frac{2b\eta_1 z}{(\eta_1 z)^2 + x^2 - b^2} \right] + i\pi \quad (2a)$$

$$k_2 = \tan^{-1} \left[ \frac{2b\eta_2 z}{(\eta_2 z)^2 + x^2 - b^2} \right] + i\pi \quad (2b)$$

$$k_3 = \ln \left[ \frac{(b+x)^2 + (\eta_1 z)^2}{(b+x)^2 + (\eta_2 z)^2} \cdot \frac{(b-x)^2 + (\eta_2 z)^2}{(b-x)^2 + (\eta_1 z)^2} \right] \quad (2c)$$

where  $i = 0$  unless  $k_1 < 0$  or  $k_2 < 0$ , in which case  $i = 1$ . If  $n$  is equal to one then equations (1a) ~ (1c) become the stress

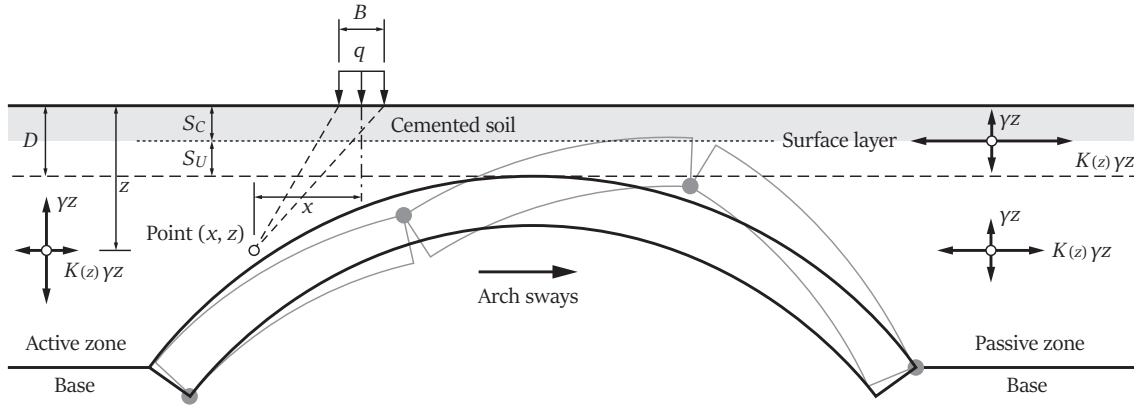


Fig. 1. Soil stress fields in a single-span masonry arch bridge. Indicated principal stress directions are for simple self weight cases.

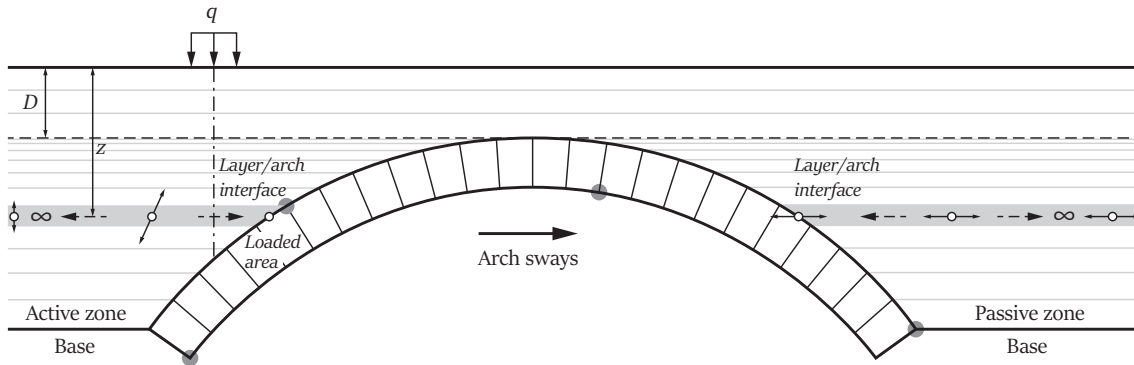


Fig. 2. Lower bound solution is generated using soil layers around a single-span masonry arch bridge ensuring yield is not violated at three critical locations in each layer (denoted by the small open circles within the grey layer in the figure).

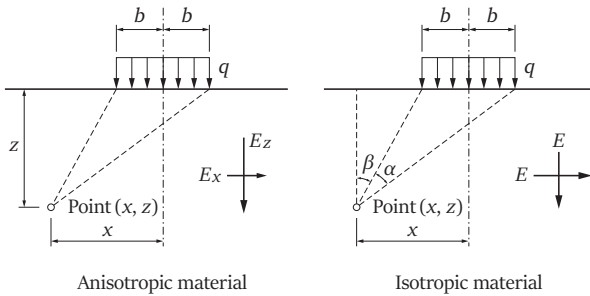


Fig. 3. Cross section through long flexible strip on infinite elastic bed carrying surcharge  $q$

where

$$\beta = \tan^{-1}[(x - b)/z], \quad \alpha = \tan^{-1}[(x + b)/z] - \beta$$

as illustrated in Fig. 3.

The contours of mean incremental vertical stress for various values of  $n$  are presented in Supplementary Material B. A larger value of  $n$  leads to greater load spreading, but may also change the strength mobilisation. However strength mobilisation will also be significantly affected by soil self weight.

In order to obtain the overall stresses including the effect of the soil unit weight  $\gamma$ , it necessary to combine the incremental stresses due to the surface load and due to those produced by the weight of the supporting material. This will also reduce the mobilised soil strength in general.

distributions derived by **Boussinesq (1885)**:

$$\Delta\sigma_z = \frac{q}{\pi} [\alpha + \sin \alpha \cos(\alpha + 2\beta)] \quad (3a)$$

$$\Delta\sigma_x = \frac{q}{\pi} [\alpha - \sin \alpha \cos(\alpha + 2\beta)] \quad (3b)$$

$$\Delta\tau_{xz} = \frac{q}{\pi} [\sin \alpha \sin(\alpha + 2\beta)] \quad (3c)$$

#### Stress fields due to backfill self-weight

The simplest stress fields that include soil self weight are those where the principal stress orientations are horizontal and vertical as shown in Fig. 1. The equations for stress fields due

to the soil self-weight can be expressed as

$$\sigma_{z,sw} = \gamma z \quad (4a)$$

$$\sigma_{x,sw} = K(z) \sigma_{z,sw} \quad (4b)$$

$$\tau_{xz,sw} = \tau_{zx,sw} = 0 \quad (4c)$$

where  $K(z)$  is a lateral earth pressure coefficient for the soil and may vary with depth  $z$  without violating equilibrium, and  $\gamma$  is the soil unit weight. Note that in this work  $K$  is simply a multiplier on the vertical stress and encompasses the effects of both friction and cohesion.

#### Combined stress fields

The ‘Boussinesq’ stress fields and soil self-weight stresses can be combined to generate a stress field that is in equilibrium with the applied loads and the soil self weight (Bolton 2013). Adopting the sign convention shown in Fig. 4. The combined vertical stress  $\sigma_z$ , horizontal stress  $\sigma_x$  and shear stress  $\tau_{xz}$ , can be then written as

$$\sigma_z = \Delta\sigma_z + \sigma_{z,sw} \quad (5a)$$

$$\sigma_x = \Delta\sigma_x + \sigma_{x,sw} \quad (5b)$$

$$\tau_{xz} = \tau_{zx} = \Delta\tau_{xz} = \Delta\tau_{zx} \quad (5c)$$

The effect of the value of  $n$  on mobilised strength is illustrated in Supplementary Material C. A higher value of  $n$  generally leads to increased strength mobilisation.

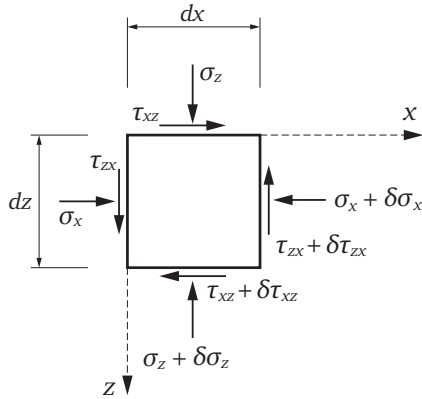


Fig. 4. Sign convention for stresses

#### YIELD CONSTRAINTS ON SOIL STRESS FIELDS

##### General equations

In accordance with lower-bound theorems, a combined stress field must not yield. In this paper the conventional Mohr-Coulomb yield condition will be adopted (Terzaghi 1942)

which can be expressed as

$$\frac{(\sigma_1 + \sigma_3)}{2} \sin \phi = \frac{(\sigma_1 - \sigma_3)}{2} - c \cos \phi \quad (6)$$

where

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_z) + \sqrt{\frac{1}{4}(\sigma_x - \sigma_z)^2 + \tau_{xz}^2} \quad (7a)$$

$$\sigma_3 = \frac{1}{2}(\sigma_x + \sigma_z) - \sqrt{\frac{1}{4}(\sigma_x - \sigma_z)^2 + \tau_{xz}^2} \quad (7b)$$

As previously stated, for the masonry arch problem, and for any given depth, the yield condition must be checked at three locations on each side of the arch (two if above the arch crown):

- At a critical location  $x$  in layer  $z = l$  (if within the soil body) determined by the combination of the surface and self weight stress fields
- At infinity  $\infty$  (laterally)
- At the soil/arch interface  $i$

#### Yield check for combined stress fields

Introducing parameters as follows

$$D_1 = \Delta\sigma_x + \Delta\sigma_z \quad (8a)$$

$$D_2 = \Delta\sigma_x - \Delta\sigma_z \quad (8b)$$

and substituting equations (5a) ~ (5c), (8a) and (8b) into equation (6), limiting values of the required lateral earth pressure coefficient that do not violate yield in the combined stress field can be derived (Appendix C) to give:

$$K(z) = \frac{-\chi \pm \sqrt{\chi^2 - 4\psi\kappa}}{2\psi} \quad (9)$$

where

$$\psi = \gamma^2 z^2 (\sin^2 \phi - 1) \quad (10a)$$

$$\chi = 2\gamma^2 z^2 (\sin^2 \phi + 1) + 2\gamma z (D_1 \sin^2 \phi - D_2 + c \sin 2\phi) \quad (10b)$$

$$\kappa = \gamma^2 z^2 (\sin^2 \phi - 1) + 2\gamma z (D_1 \sin^2 \phi + D_2 + c \sin 2\phi) + D_1^2 \sin^2 \phi + 2D_1 c \sin 2\phi - D_2^2 + 4c^2 \cos^2 \phi - 4\Delta\tau_{xz}^2 \quad (10c)$$

Equation (9) produces two solutions which give the lowest and highest allowable values of  $K$  at any given position that does not lead to yield violation, as illustrated using Mohr's

circles in Fig. 5. These will be termed  $K_{l,min}$  and  $K_{l,max}$ , where the subscript  $l$  denotes layer.

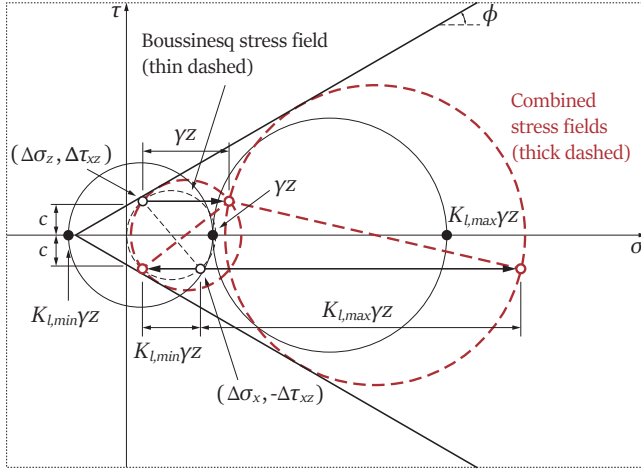


**Fig. 5. Mohr's circle diagrams illustrating combination of stress fields and definition of  $K_{l,min}$  and  $K_{l,max}$ .** The principal stress directions of the Boussinesq stress field are not horizontal and vertical, while those of the self weight field are. Note that in general it is possible that the Boussinesq stress field and/or the self weight stress field may violate yield on their own.

An example of the typical spatial variation of  $K_{l,min}$  and  $K_{l,max}$  relative to the applied load position is given in Fig. 6. For shallow depths it is seen that there is no self weight stress field that can provide stability at certain locations. This issue will be addressed later in the paper. It is also noted that  $K$ , as defined in this paper, can take on negative values in cases when there is a non-zero value of  $c$  and where it is assumed that tensile stresses can be sustained by the fill.

For a good lower bound, it is required to utilise  $K_{l,min}$  on the active side and  $K_{l,max}$  on the passive side. However in order to maintain the simplicity of the solution the value of  $K$  must be valid for any value of  $x$  within a given infinitesimally thick horizontal layer at depth  $z$ . Thus it is necessary to find the highest value of  $K_{l,min}$  on the active side and the lowest value of  $K_{l,max}$  on the passive side so that yield is not violated at any location  $x$ . For a layer that extends horizontally to its full extent, these values are shown in Fig. 6 using the circle and square symbols. Fig 7 illustrates the variation of the required single value of  $K_{l,min}$  and  $K_{l,max}$  for any given depth for an example arch geometry. It can be seen that this varies monotonically with depth.

#### Yield check at infinity

The range of values of  $K(z)$  is limited by the Rankine (1857) active and passive lateral earth pressure coefficients (representing stress states located a long way laterally from the arch) as follows:

$$K_{\infty,min} = K_a - K_{ac} \frac{c}{\sigma_{z,sw}} \leq K(z) \leq K_p + K_{pc} \frac{c}{\sigma_{z,sw}} = K_{\infty,max} \quad (11)$$

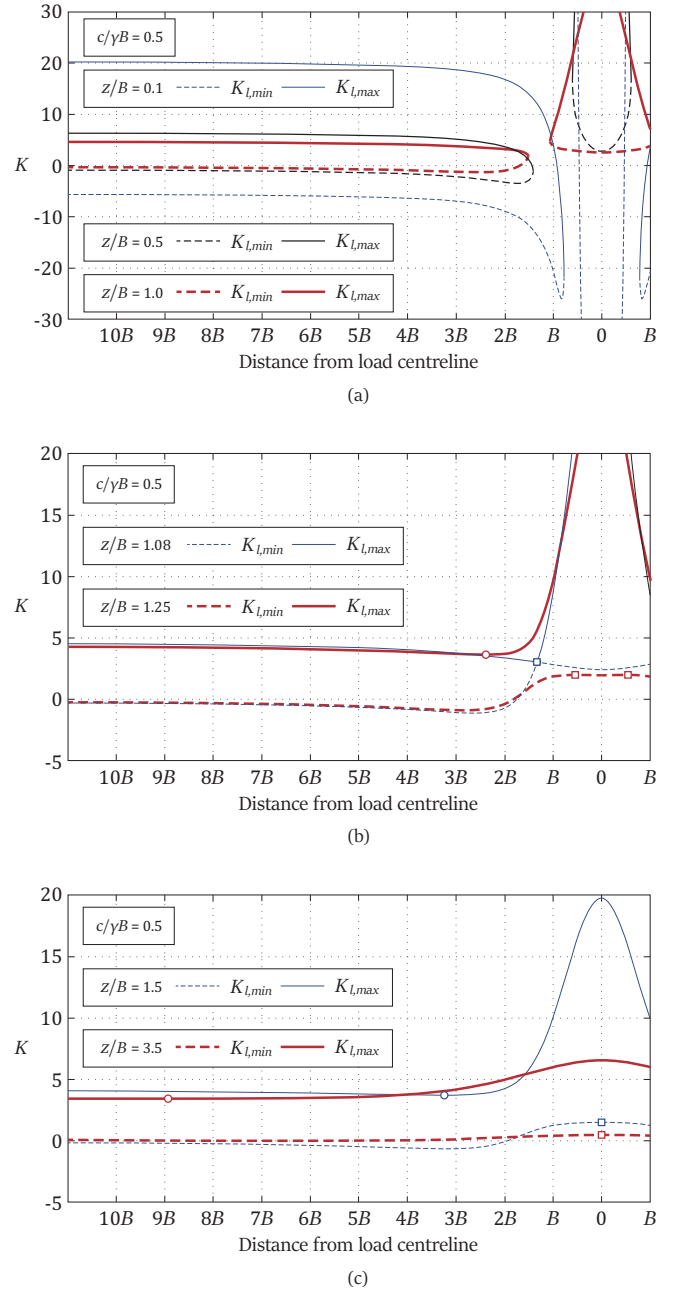


**Fig. 6. Lateral variation of  $K_{l,min}$  and  $K_{l,max}$  for  $n = 1$ ,  $q/\gamma B = 20$ ,  $c/\gamma B = 0.5$ ,  $\phi = 30^\circ$  at depths: (a)  $z/B = 0.1, 0.5, 1.0$ ; (b)  $z/B = 1.08, 1.25$ ; (c)  $z/B = 1.5, 3.5$ . The circle symbol denotes the lowest value of  $K_{l,max}$  and the square symbol the highest value of  $K_{l,min}$ .**

where

$$\begin{cases} K_a = \tan^2 \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \\ K_{ac} = 2\sqrt{K_a} \end{cases} \quad (12a)$$

$$\begin{cases} K_p = \tan^2 \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \\ K_{pc} = 2\sqrt{K_p} \end{cases} \quad (12b)$$

and

$$\begin{cases} K_p = \tan^2 \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \\ K_{pc} = 2\sqrt{K_p} \end{cases} \quad (13a)$$

$$\begin{cases} K_p = \tan^2 \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \\ K_{pc} = 2\sqrt{K_p} \end{cases} \quad (13b)$$

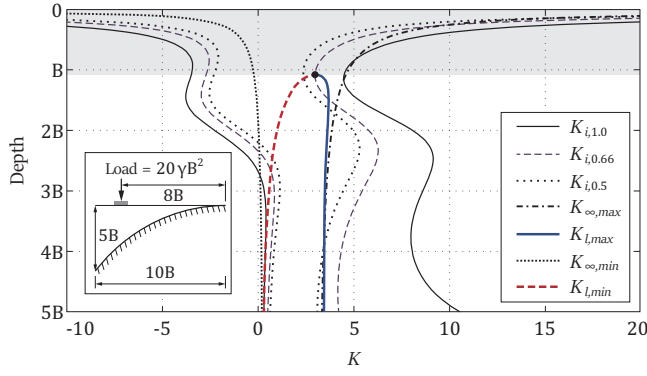


**Fig. 7. Limiting lateral earth pressure coefficients with  $n = 1$ ,  $ql/\gamma B = 20$ ,  $d/\gamma B = 0.5$ ,  $\phi = 30^\circ$  for an arch of span  $20B$ , rise  $5B$ , zero crown cover depth and load centre a lateral distance  $8B$  from the crown.**

An example plot of these values is shown in Fig. 7. In principle it is unnecessary to check these values if values of  $K_{l,min}$  and  $K_{l,max}$  are determined for all values of  $x$ . However it is efficient to check these latter values near to the loading position only and undertake a separate check of  $K_{\infty,min}$  and  $K_{\infty,max}$ .

#### Yield check at soil arch interface

To account for an arch/soil interface friction that is lower than that of the soil, an additional constraint on  $K$  is required.

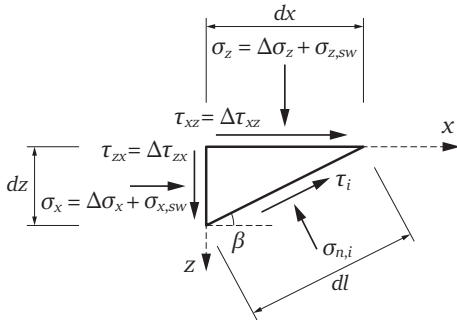


**Fig. 8. Diagram illustrating equilibrium for soil-arch interface stress fields.**

If the inclination of soil-arch interface to the horizontal is  $\beta$  as shown in Fig. 8 then the shear and normal stresses on the interface due to the surface load are given by:

$$\begin{cases} \sigma_{n,i} = \sigma_x \sin^2 \beta + \sigma_z \cos^2 \beta + 2\tau_{xz} \sin \beta \cos \beta & (14a) \\ \tau_i = \tau_{xz}(\sin^2 \beta - \cos^2 \beta) + (\sigma_z - \sigma_x) \sin \beta \cos \beta & (14b) \end{cases}$$

It then remains to show that:

$$\begin{cases} \tau_i \leq c_w + \sigma_{n,i} \tan \delta & (15a) \end{cases}$$

$$\begin{cases} \tau_i \geq -c_w - \sigma_{n,i} \tan \delta & (15b) \end{cases}$$

where  $c_w$  is the soil-arch interface cohesion intercept and  $\delta$  is the soil-arch interface angle of shearing resistance.

Substitution of equations (4a) ~ (5c), (14a) and (14b) into equations (15a) and (15b), allows the limiting lateral earth pressure coefficient on the soil-arch interface to be obtained:

$$K_{i,min} = \frac{\Delta\tau_{xz} \varepsilon_1 + (\Delta\sigma_z + \gamma z) \nu_1 - \Delta\sigma_x \eta_1 - c_w}{\gamma z \eta_1} \quad (16)$$

where

$$\varepsilon_1 = \sin^2 \beta - \cos^2 \beta - 2 \sin \beta \cos \beta \tan \delta \quad (17a)$$

$$\nu_1 = \sin \beta \cos \beta - \cos^2 \beta \tan \delta \quad (17b)$$

$$\eta_1 = \sin \beta \cos \beta + \sin^2 \beta \tan \delta \quad (17c)$$

and

$$K_{i,max} = \frac{\Delta\tau_{xz} \varepsilon_2 + (\Delta\sigma_z + \gamma z) \nu_2 - \Delta\sigma_x \eta_2 + c_w}{\gamma z \eta_2} \quad (18)$$

where

$$\varepsilon_2 = \sin^2 \beta - \cos^2 \beta + 2 \sin \beta \cos \beta \tan \delta \quad (19a)$$

$$\nu_2 = \sin \beta \cos \beta + \cos^2 \beta \tan \delta \quad (19b)$$

$$\eta_2 = \sin \beta \cos \beta - \sin^2 \beta \tan \delta \quad (19c)$$

This constraint will vary for different arch geometries. An example of the influence of the additional constraint for soil-structure interface friction multiplier values of  $\alpha = 0.5, 0.66$  and  $1.0$  is shown in Fig. 7 (where  $\tan \delta = \alpha \tan \phi$  and  $c_w = \alpha c$ ).

#### Cemented layer

Where equation (9) has no solution (i.e the discriminant of equation (9) is less than zero), it means that no suitable value of  $K$  can avoid yield for the layer in question, as is seen in Fig. 6(a) for shallow layers and for distances from the load centroid of  $\sim B$ . In place of computing  $K$  only, it is necessary to calculate the cohesion  $c_C$  required to avoid yield in that layer and the corresponding value of  $K$ . For simplicity all analysis layers that require a larger cohesion will be considered as a single cemented layer. The required strength for this layer will be the largest value of  $c_C$  found in the component analysis layers.

The required value of cohesion  $c_C$  of the layer can be found as the value required to make the discriminant in equation (9) equal to zero:

$$\chi^2 - 4\psi\kappa = 0 \quad (20)$$

Substituting equations (10a) ~ (10c) into equation (20), the expression of the required cohesion  $c_C$  can also be derived as follows:

$$c_C = \frac{-\rho \pm \sqrt{\rho^2 - 4\omega\xi}}{2\omega} \quad (21)$$

where

$$\omega = \sin^2 2\phi + 4 \cos^2 \phi - 4 \cos^2 \phi \sin^2 \phi \quad (22a)$$

$$\rho = 4\gamma z \sin 2\phi + 2 \sin^2 \phi (D_1 - D_2) \quad (22b)$$

$$\xi = 4\gamma^2 z^2 \sin^2 \phi + 2\gamma z [D_1 (\sin^4 \phi + \sin^2 \phi + \sin 2\phi - \sin^2 \phi \sin 2\phi) - 2D_2 \sin^2 \phi] + \sin^2 \phi (D_1 - D_2)^2 + 4\Delta\tau_{xz}^2 (\sin^2 \phi - 1) \quad (22c)$$

The derivation of equation (21) is given in Appendix A. The equation gives two solutions, the higher of which gives a valid yield constraint.

The corresponding value of  $K$  for the layer in questions is given simply by:

$$K(z) = \frac{-\chi}{2\psi} \quad (23)$$

which is the value of equation (9) when the discriminant is equal to zero.

#### DETERMINATION OF THE LOWER BOUND STRESS FIELD

Pragmatically it is useful to divide the soil body into horizontal layers of finite thickness coinciding with the extrados of each voussoir and determine an appropriate value of  $K_{min}$  or  $K_{max}$  for each layer in the active or passive zones that maintains a lower bound. This can be determined from the most critical value from the three sets  $K_l$ ,  $K_\infty$  and  $K_i$ . As illustrated in Figure 6(c), it can be seen that for  $K_{min}$  the critical (highest) value will either be where the arch intersects the layer (or on the load centreline if there is no intersection) or it will be at  $x = \pm\infty$ . For  $K_{max}$  the critical (lowest) value is at an intermediate minimum point. Due to the complexity of the Moroto equations it is necessary to identify this point using a standard rapid 1D min/max search algorithm or at the arch/layer intersection.

For each layer it is necessary to check values at the upper and lower boundaries of the layer and use the most critical case. It can be seen from Fig. 7 that since  $K_{l,min}$  and  $K_{l,max}$  vary monotonically with depth, then the value of  $K_{l,min}$  or  $K_{l,max}$  that will be valid across the finite height of a discrete layer will either be that value at the top or the bottom of the layer. In the absence of the arch the critical value of  $K_{min}$  will be at the base of the layer and the critical value of  $K_{max}$  will be at the top of the layer, except for the value corresponding to  $K_i$  which may be intermediate.

#### APPLICATION EXAMPLE

##### Introduction

The use of the proposed lower-bound stress fields can be illustrated by considering an example of a soil-filled masonry arch bridge, tested at the University of Salford (Smith *et al.* 2006). The bridge was tested under effectively plane-strain conditions and the load was applied at the quarter span of the bridge. Fig. 9 shows the geometry of the bridge and Table 1 shows the material properties reported by Smith *et al.*. The load-carrying capacity of the bridge was reported to be 149 kN (Gilbert *et al.* 2010).

**Table 1. Material parameters of Salford arch bridge**

| Masonry properties                                | Values |
|---------------------------------------------------|--------|
| Crushing strength (N/mm <sup>2</sup> )            | 25     |
| Coefficient of friction                           | 0.64   |
| Unit weight (kN/m <sup>3</sup> )                  | 23.7   |
| Soil properties                                   | Values |
| Cohesion, $c$ (kN/m <sup>2</sup> )                | 3.3    |
| Internal angle of friction, $\phi$ (°)            | 54.5   |
| Passive side mobilisation coefficient, $m_p$      | 0.33   |
| Soil-arch interface strength multiplier, $\alpha$ | 0.66   |
| Unit weight, $\gamma$ (kN/m <sup>3</sup> )        | 19.1   |

Note that, taking on board experimental observations the soil strength on the passive side was scaled down by a mobilization factor in the numerical models. Gilbert *et al.* (2007) indicated that soil strains in the passive regions of a masonry arch bridge are relatively small when the collapse load is reached. Thus if peak soil strength is used in the model, bridge load-carrying capacity will be over-predicted. In order to address this issue a mobilization factor of  $m_p = 0.33$  was therefore applied to the soil strength parameters  $c$  and  $\tan \phi$  to give the operational parameters  $c_p$  and  $\tan \phi_p$  in passive regions, where  $c_p = m_p c$  and  $\tan \phi_p = m_p \tan \phi$ . Following Smith *et al.* (2004), a soil-arch interface strength multiplier of  $\alpha = 0.66$  was also applied.

The results obtained using the proposed lower-bound calculations were calibrated against upper-bound (mechanisms) solutions obtained using the discontinuity layout optimization (DLO) upper-bound limit analysis technique (Smith & Gilbert 2007), incorporated in the LimitState:GEO commercial software (LimitState 2021).

##### Upper bound analysis

The DLO models were modelled as described in Gilbert *et al.* (2010) using the dimensions in Fig. 9 and material properties listed in Table 1. Nodes were distributed in a square grid in solid bodies with a spacing of  $5/B$  following the nodal density study

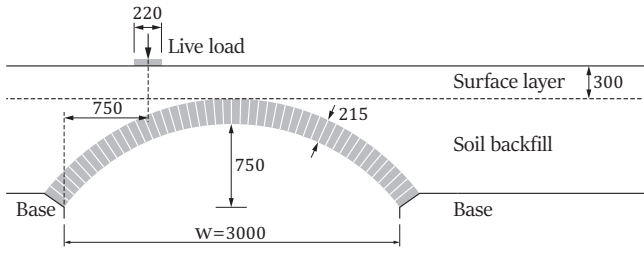


Fig. 9. Geometry of arch test (all dimensions in mm)

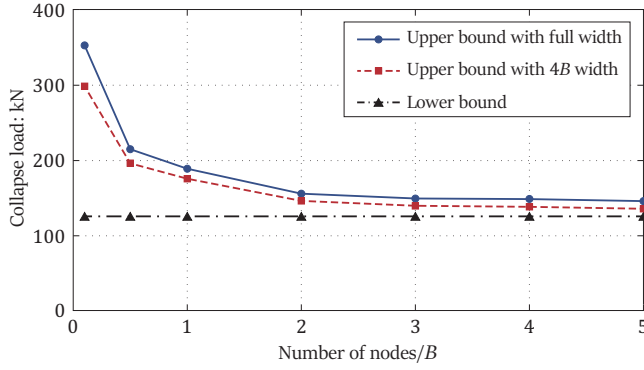


Fig. 10. Influence of nodal density in load-carrying capacity of masonry arch bridges

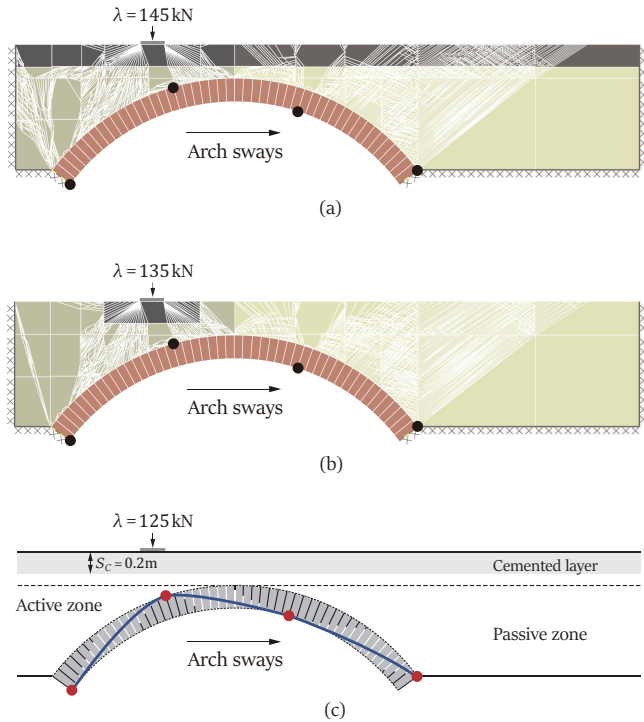


Fig. 11. Failure mechanisms: (a) upper bound model for case (i); (b) upper bound model for case (ii); (c) lower bound model

in Figure 10. The nodal spacing on boundaries was selected as half this value following the default in the software.

Since a ‘cemented’ layer was modelled in the lower-bound analysis, it must also appear in the upper-bound analysis for proper comparison. However while a full width cemented layer would be expected in practice, the lower bound solution only

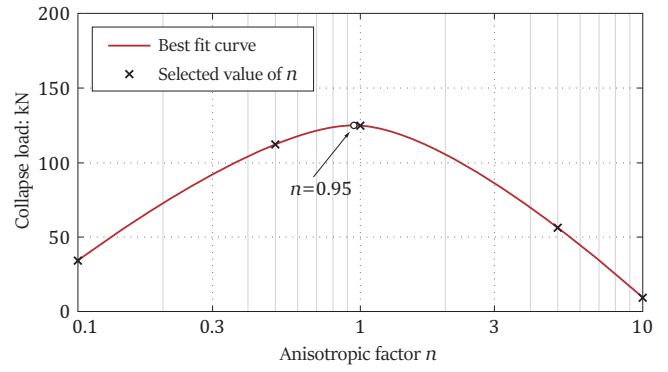


Fig. 12. Influence of anisotropic factor  $n$  in load-carrying capacity of masonry arch bridges.

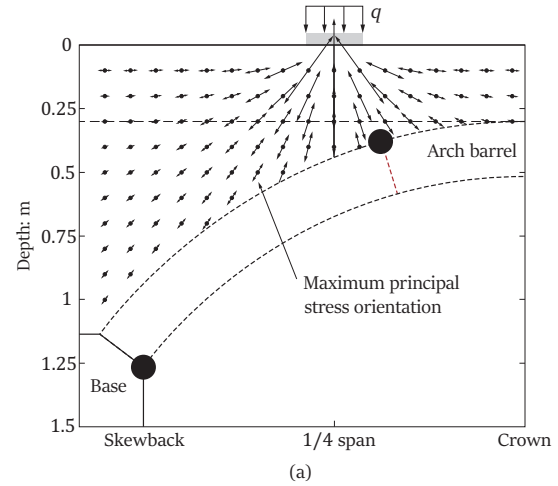
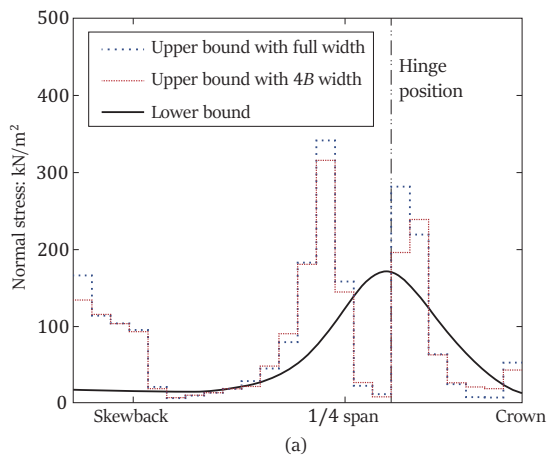
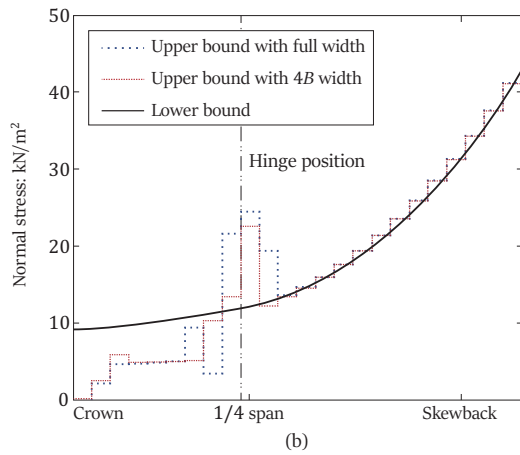


Fig. 13. Maximum principal stress in soil stress fields: (a) active zone; (b) passive zone

utilises this layer for a short length below the load, while the upper bound would be influenced by the full width. To evaluate the sensitivity of the analyses to this, two upper-bound models were analysed: (i) cemented layer extends full width of model; (ii) cemented layer extends only for a width  $4B$  below the load as shown in Figures 11(a) and 11(b). The passive mobilisation factor of 0.33 was also applied to the cemented layer for case (i). The optimal lower bound cemented layer properties are

**Table 2. Material parameters of Salford arch bridge**

| Parameters                                        | Default value | Varied in Fig. 15 |
|---------------------------------------------------|---------------|-------------------|
| Cohesion, $c$ (kN/m <sup>2</sup> )                | 3.3           | (a)               |
| Internal angle of friction, $\phi$ (degree)       | 54.5          | (b)               |
| Unit weight, $\gamma$ (kN/m <sup>3</sup> )        | 19.1          | (c)               |
| ‘Active side’ mobilisation factor, $m_a$          | 1.0           | (d)               |
| ‘Passive side’ mobilisation factor, $m_p$         | 0.33          | (e)               |
| Soil-arch interface strength multiplier, $\alpha$ | 0.66          | (f)               |
| Masonry crushing strength (MN/m <sup>2</sup> )    | 25            | (g)               |
| Loading position, $x$                             | 1/4 span      | (h)               |

**Fig. 14. Normal stress distributions on the extrados of Salford arch bridge: (a) active zone; (b) passive zone**

given in the next section. It is noted that a cemented layer was not required in the test, but that the load plate did undergo some ‘bedding in’ during initial loading which would correspond to local failure in the numerical model. However the magnitude of the bedding in was not significant.

#### Lower bound analysis

The masonry arch model was discretized as a series of rigid blocks separated by contacts, using the same properties as the upper bound model but without modelling masonry crushing. Post analysis checks of the stresses on the masonry block interfaces indicated that the crushing strength was not reached.

Since the parameters of the cemented layer and the values of  $K$  are a function of the applied load it is necessary to use an iterative process to determine the optimum lower-bound solution. The details of this iterative process are shown in Appendix B.

A range of solutions were undertaken based on initial assumed values of  $n = 0.1, 0.5, 1.0, 5.0, 10.0$ . The results are given in Fig 12. It can be seen that  $n$  has a significant effect on the results and for this parameter set a value of  $n = 0.95$  gives the best lower bound collapse load. For simplicity, this value of  $n$  was utilised in the subsequent analyses, but could form part of the iterative analysis process.

The best lower-bound solution of the bridge was found to be 125 kN, with values of  $S_C$  and  $c_C$  of 0.2 m and 15 kN/m<sup>2</sup> respectively. The corresponding failure mechanism is shown in Fig. 11(c). The upper bound solutions using the above values of  $S_C$  and  $c_C$  were 145 kN and 135 kN for cases (i), full width cemented layer and (ii), 4B width cemented layer. The gap between the lower-bound and the upper-bound for case (ii) is approximately 8%.

The hinging locations predicted by the lower bound analysis are given in Fig. 11(c) and correspond identically with the upper bound analyses.

In Fig. 13(a), the final modelled maximum principal stress orientations were approximately vertical on the active side close to the load, while on the passive side, Fig. 13(b) indicates that maximum principal stress orientations were closely horizontal, due to the negligible effect of the Boussinesq stress fields in this zone.

To compare the lower and upper-bound solutions in more detail it is instructive to compare the stress distributions on the bridge extrados. Fig. 14 shows the normal stress distributions.

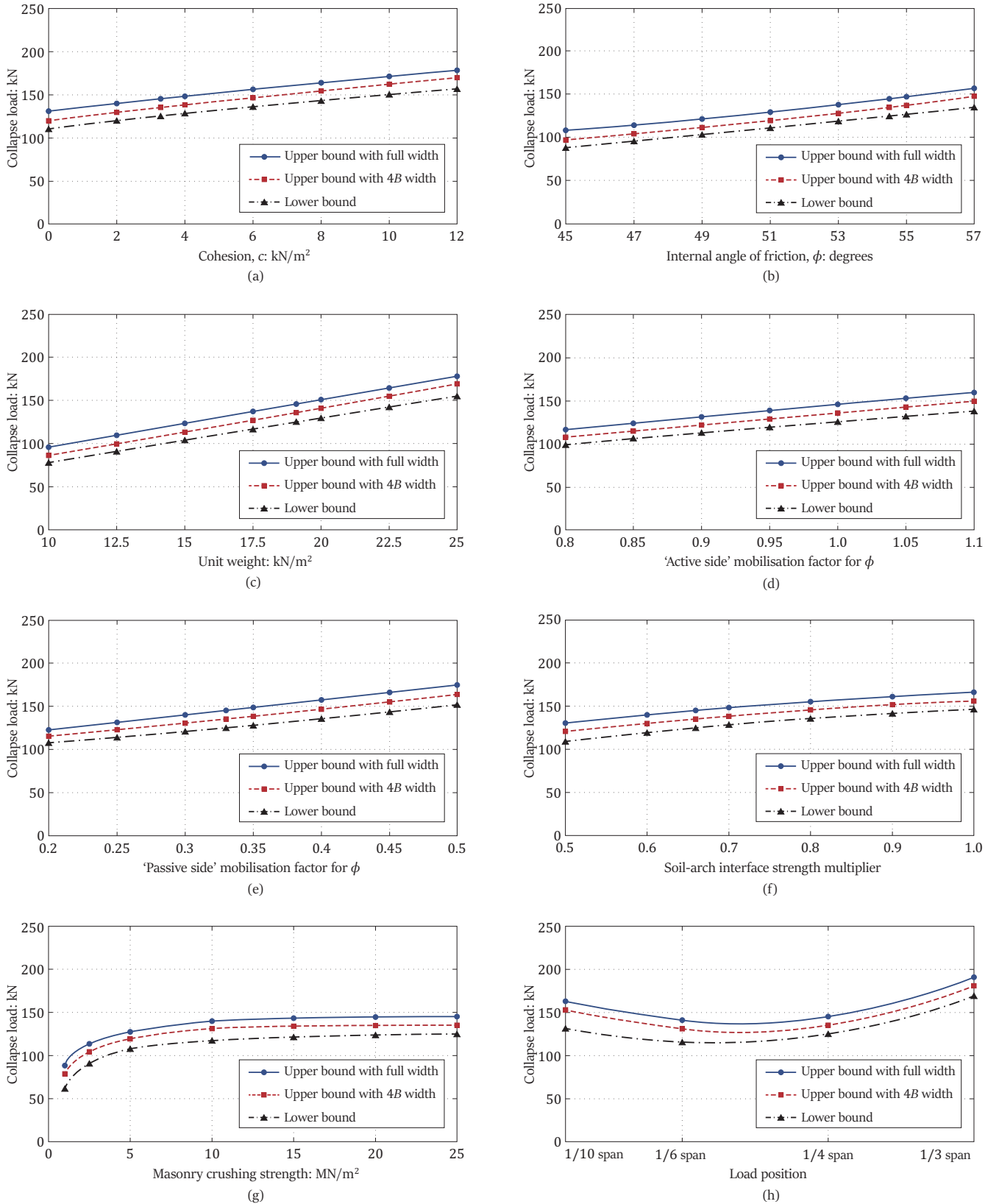


**Fig. 15. Parametric studies: influence of Table 2 parameters on computed collapse load**

A good match can be seen from the third plastic joint to the fourth joint on the passive side, since the 'Boussinesq' stress fields are weak on this arch segment. However, from the first

joint to the third joint, the specified 'Boussinesq' stress fields show a larger difference from the DLO model. At the skewback the DLO predicted stresses have a significant degree of freedom

since there is no work done at this point. At the active side quarter span, the DLO solution shows a double peak due to soil arching at the second hinge. This is not possible to capture with the simple 'Boussinesq' type stress field in the lower bound solution, but it is inferred that the averaged equivalent effect can give a good result.

### PARAMETRIC STUDIES

In order to investigate the sensitivity of the analyses to input parameters, eight parameters, cohesion  $c$ , friction angle  $\phi$ , unit weight  $\gamma$ , active and passive mobilisation factor, soil-arch interface strength multiplier, masonry crushing strength and loading position, were varied in turn while the other parameters were kept fixed. The results in Fig. 15, show that the proposed lower bound solutions closely match the upper bound solutions across a wide range of parameters with a range of approximately 6% ~ 22%. The greatest sensitivities were to loading positions close to the bridge abutment, and to masonry crushing strength values close to weak strength.

In all analyses the surface cemented layer did not extend below arch crown level and typically extended between 0.15 m and 0.29 m below the surface.

### Solution speed

In order to compare solution speed, high precision DLO solutions were determined using the extrapolation method described in Gilbert *et al.* (2014) to generate good approximations to baseline 'true' solutions. Solutions were then determined using two practical levels of nodal density of  $5/B$  and  $2.5/B$ . Fig 16 shows the average time taken for solutions at these two nodal densities and for the lower bound approach which was programmed in MATLAB. The lower bound solutions were on average 5% below the 'true' solution. It can be seen that the lower bounds were solved typically  $50\times$  faster than the 5% accuracy upper bound solutions and  $2\times$  faster than the 10% accuracy upper bound solutions. It is anticipated that the lower bound solution times could be further reduced, through refinement in coding and iteration heuristics.

### DISCUSSION

The results presented in the example bridge model parametric study showed a good match between the upper and lower bound solutions for engineering purposes, and is a promising approach for other problem types such as buried pipes and tunnels.

While FE and other lower bound analysis approaches may give a more accurate result, the analyses will typically be too slow for analysing what can often be hundreds of load cases for any given arch. Current masonry arch software such as LimitState:RING (LimitState 2025) adopt a similar limit analysis approach to the arch barrel analysis, and use a similar approach to estimating active or passive backfill stress distributions on the arch but do not impose rigorous lower

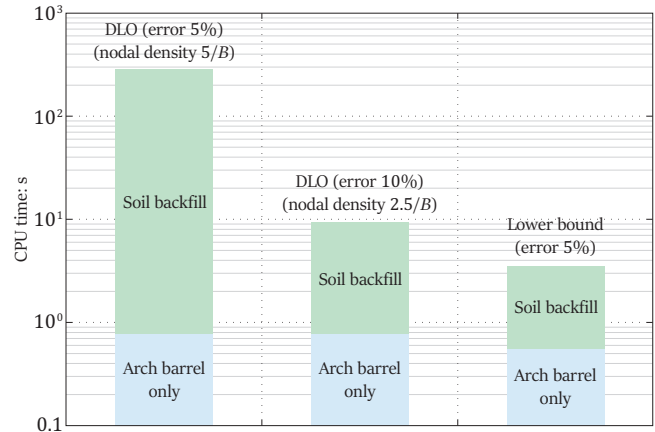


**Fig. 16. Average CPU time comparison of upper and lower bound (to obtain the solutions a 2.2 GHz Intel Core PC equipped with 8 GB of memory, running under 64 bit Windows 10 Professional. Mosek version 7.0, a commercially available interior point linear programming solver was used.**

bound criteria, thus giving an overall limit equilibrium type analysis.

Incorporating the proposed lower bound stress fields into software such as LimitState:RING, would therefore allow the software to furnish true lower bound solutions at minimal additional computational cost.

Extension of the methodology presented in this paper to multiple applied loads is straightforward by computing the active and passive stress fields that can accommodate the relevant superposed Boussinesq stress fields. The rigid-block limit analysis formulation can then automatically select the appropriate active or passive field coefficients depending on whether the relative soil/arch movement is positive or negative and thus model any arch failure mechanism.

### CONCLUSIONS

It has been demonstrated that lower-bound stress fields can be used in conjunction with rigid block limit analysis to provide rapid lower bound estimates of the load carrying capacity of a masonry arch bridge (approximately 50 times faster than corresponding high accuracy upper bound DLO analyses). The lower bound results obtained for a case study bridge model were found to be close to corresponding upper bound solutions obtained using the discontinuity layout optimization (DLO) limit analysis procedure.

The method provides a promising approach for other problem types such as buried pipes and tunnels.

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Professor Matthew Gilbert sadly passed away after the initial submission of the manuscript. His valuable contribution to this work is respectfully acknowledged.

## NOTATION

|           |                                                                          |
|-----------|--------------------------------------------------------------------------|
| $b$       | half-width of surface load                                               |
| $c$       | cohesion intercept                                                       |
| $c_w$     | soil/arch interface cohesion intercept                                   |
| $c_C$     | cohesion intercept for cemented layer                                    |
| $d$       | compressive length on block contact                                      |
| $m_a$     | active side strength mobilization factor                                 |
| $m_p$     | passive side strength mobilization factor                                |
| $q$       | pressure applied by surface load                                         |
| $t$       | thickness of arch barrel                                                 |
| $x$       | distance from load centre line                                           |
| $z$       | depth from soil surface                                                  |
| $B$       | width of surface load                                                    |
| $D$       | thickness of surface layer                                               |
| $K$       | lateral earth pressure coefficient*                                      |
| $M$       | moment                                                                   |
| $N$       | normal force                                                             |
| $S$       | shear force                                                              |
| $S_C$     | thickness of cemented part of surface layer                              |
| $S_U$     | thickness of uncemented part of surface layer                            |
| $\alpha$  | soil/arch interface coefficient (e.g. $\tan \delta = \alpha \tan \phi$ ) |
| $\delta$  | soil/arch interface angle of friction                                    |
| $\phi$    | internal angle of friction                                               |
| $\sigma$  | normal stress                                                            |
| $\gamma$  | unit weight                                                              |
| $\lambda$ | adequacy factor                                                          |
| $\tau$    | shear stress                                                             |
| $\zeta$   | yield factor                                                             |

\*Note definition in equation (4b).

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## APPENDIX A: LATERAL EARTH PRESSURE COEFFICIENT

To automatically compute required lateral earth pressure coefficient  $K(z)$ , it can be derived by substituting equations (5a) ~ (5c), (8a) and (8b) into equation (6), then

$$\begin{aligned} & \frac{1}{2}[D_1 + (K + 1)\gamma z] \sin \phi + c \cos \phi \\ & = \sqrt{\frac{1}{4}[D_2 + (K - 1)\gamma z]^2 + \Delta \tau_{xz}^2} \end{aligned} \quad (24)$$

Equation (24) can be rewritten as

$$\begin{aligned} & [D_1^2 + 2D_1(K + 1)\gamma z + (K + 1)^2\gamma^2 z^2] \sin^2 \phi \\ & + 4[D_1 + (K + 1)\gamma z]c \sin \phi \cos \phi + 4c^2 \cos^2 \phi \\ & = [D_2^2 + 2D_2(K - 1)\gamma z + (K - 1)^2\gamma^2 z^2] + 4\Delta \tau_{xz}^2 \end{aligned} \quad (25)$$

expanding,

$$\begin{aligned}
& D_1^2 \sin^2 \phi + K(2D_1 \gamma z \sin^2 \phi) + 2D_1 \gamma z \sin^2 \phi \\
& + K^2(\gamma^2 z^2 \sin^2 \phi) + K(2\gamma^2 z^2 \sin^2 \phi) \\
& + \gamma^2 z^2 \sin^2 \phi + 2D_1 c \sin 2\phi + K(2\gamma z c \sin 2\phi) \\
& + 2\gamma z c \sin 2\phi + 4c^2 \cos^2 \phi = D_2^2 + K(2D_2 \gamma z) \\
& - 2D_2 \gamma z + K^2(\gamma^2 z^2) - K(2\gamma^2 z^2) + \gamma^2 z^2 + 4\Delta\tau_{xz}^2
\end{aligned} \quad (26)$$

Combining similar terms, equation (26) can be written as

$$\begin{aligned}
& K^2(\gamma^2 z^2 \sin^2 \phi - \gamma^2 z^2) + K(2D_1 \gamma z \sin^2 \phi + 2\gamma^2 z^2 \sin^2 \phi \\
& + 2\gamma z c \sin 2\phi - 2D_2 \gamma z + 2\gamma^2 z^2) + (D_1^2 \sin^2 \phi \\
& + 2D_1 \gamma z \sin^2 \phi + \gamma^2 z^2 \sin^2 \phi + 2D_1 c \sin 2\phi - \gamma^2 z^2 \\
& + 2\gamma z c \sin 2\phi + 4c^2 \cos^2 \phi - D_2^2 + 2D_2 \gamma z - 4\Delta\tau_{xz}^2) = 0
\end{aligned} \quad (27)$$

The coefficients of equation (27) in  $K^2$ ,  $K$  and 1 can be given as

$$\psi = \gamma^2 z^2 (\sin^2 \phi - 1) \quad (28a)$$

$$\begin{aligned}
\chi &= 2\gamma^2 z^2 (\sin^2 \phi + 1) + 2\gamma z (D_1 \sin^2 \phi - D_2 \\
&+ c \sin 2\phi)
\end{aligned} \quad (28b)$$

$$\begin{aligned}
\kappa &= \gamma^2 z^2 (\sin^2 \phi - 1) + 2\gamma z (D_1 \sin^2 \phi + D_2 \\
&+ c \sin 2\phi) + D_1^2 \sin^2 \phi + 2D_1 c \sin 2\phi \\
&- D_2^2 + 4c^2 \cos^2 \phi - 4\Delta\tau_{xz}^2
\end{aligned} \quad (28c)$$

Equation (27) can then be solved and the lateral earth pressure coefficient obtained at any depth using the below expression:

$$K(z) = \frac{-\chi \pm \sqrt{\chi^2 - 4\psi\kappa}}{2\psi} \quad (29)$$

#### Cemented layer

In many cases there will be a depth  $z$  for which equation (29) has no solution, i.e. where  $\chi^2 < 4\psi\kappa$ . It is thus necessary to find the cohesion  $c_C$  required such that  $\chi^2 = 4\psi\kappa$ . In this circumstance,  $K = -\chi/2\psi$  and  $c_C$  can be determined by substituting equations (10a) ~ (10c) into equation (20). Hence:

$$\begin{aligned}
& [2\gamma^2 z^2 (\sin^2 \phi + 1) + 2\gamma z (D_1 \sin^2 \phi - D_2 + c \sin 2\phi)]^2 \\
& = 4\gamma^2 z^2 (\sin^2 \phi - 1) [\gamma^2 z^2 (\sin^2 \phi - 1) + 2\gamma z (D_1 \sin^2 \phi \\
& + D_2 + c \sin 2\phi) + D_1^2 \sin^2 \phi + 2D_1 c \sin 2\phi - D_2^2 \\
& + 4c^2 \cos^2 \phi - 4\Delta\tau_{xz}^2]
\end{aligned} \quad (30)$$

and then

$$\begin{aligned}
& c^2 (\sin^2 2\phi) + c(2\gamma z \sin^2 \phi \sin 2\phi + 2\gamma z \sin 2\phi \\
& + 2D_1 \sin^2 \phi \sin 2\phi - 2D_2 \sin 2\phi) + (2\gamma^2 z^2 \sin^2 \phi \\
& + 2\gamma z D_1 \sin^4 \phi - 2\gamma z D_2 \sin^2 \phi + 2\gamma z D_1 \sin^2 \phi \\
& - 2D_1 D_2 \sin^2 \phi) = c^2 (4 \sin^2 \phi \sin 2\phi - 4 \cos^2 \phi) \\
& + c(2\gamma z \sin^2 \phi \sin 2\phi + 2D_1 \sin^2 \phi \sin 2\phi - 2\gamma z \sin 2\phi \\
& - 2D_2 \sin 2\phi) + (2\gamma z D_1 \sin^2 \phi \sin 2\phi + 2\gamma z D_2 \sin^2 \phi \\
& - 2\gamma^2 z^2 \sin^2 \phi - 2\gamma z D_1 \sin 2\phi - D_1^2 \sin^2 \phi - D_2^2 \sin^2 \phi \\
& - 4\Delta\tau_{xz}^2 \sin^2 \phi + 4\Delta\tau_{xz}^2)
\end{aligned} \quad (31)$$

The quadratic coefficients of equation (31) in terms of  $c$  are thus:

$$\omega = \sin^2 2\phi + 4 \cos^2 \phi - 4 \cos^2 \phi \sin^2 \phi \quad (32a)$$

$$\rho = 4\gamma z \sin 2\phi + 2 \sin^2 \phi (D_1 - D_2) \quad (32b)$$

$$\begin{aligned}
\xi &= 4\gamma^2 z^2 \sin^2 \phi + 2\gamma z [D_1 (\sin^4 \phi + \sin^2 \phi \\
&+ \sin 2\phi - \sin^2 \phi \sin 2\phi) - 2D_2 \sin^2 \phi] \\
&+ \sin^2 \phi (D_1 - D_2)^2 + 4\Delta\tau_{xz}^2 (\sin^2 \phi - 1)
\end{aligned} \quad (32c)$$

Solution of equation (31) provides the required cohesion at a given depth:

$$c = \frac{-\rho \pm \sqrt{\rho^2 - 4\omega\xi}}{2\omega} \quad (33)$$

#### APPENDIX B: ITERATIVE PROCEDURE

Since the proposed methodology requires advance knowledge of  $K$ ,  $c_C$  and  $S_C$ , it is necessary to iterate to a final solution.

The flow chart used in the analysis is shown in Fig. 17. Initially an anisotropic ratio  $n$  is selected in input data. Trial and error has indicated that a value of  $n = 0.95$  yields good lower bound solutions. If desired a better solution can be found more rigorously using numerical optimization and a search for the maximum value.

After determination of  $n$ , initial estimates of the operational active and passive lateral earth pressure coefficients are made and linear programming (LP) optimization is then used to find the critical collapse load  $q_{j=0}$ . Inputting  $q_{j=0}$  into equation (9) allows  $K_{j=1}(z)$  to be obtained where  $K_j$  is the lowest or highest value of the three values  $K_l$ ,  $K_\infty$ ,  $K_i$  as appropriate for the active or passive layer in question (above the arch crown the active or passive status is not relevant and only a single value of  $K$  for the whole layer is required).  $K_j$  is then used to calculate a new load  $q_j$ . The difference between  $q_{j+1}$  and  $q_j$  is computed. If the difference is larger than the required tolerance, then the computation programme will implement a

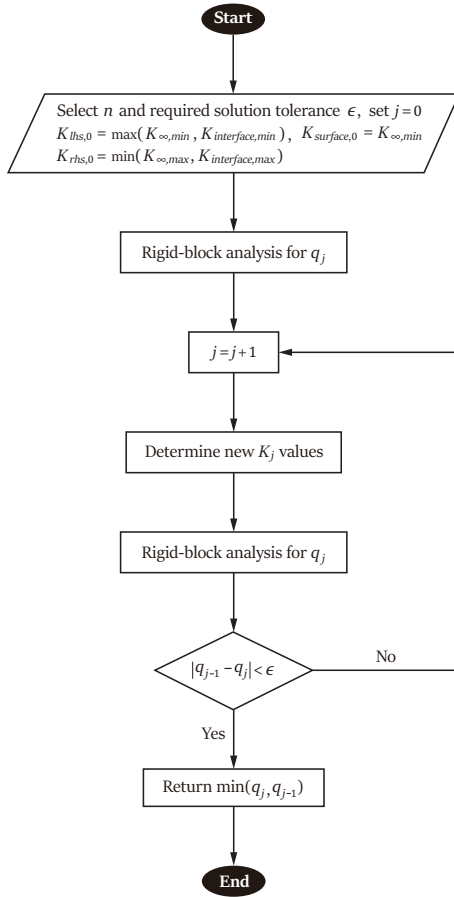


**Fig. 17. Iterative process for determining lower bound collapse load. Cemented layer parameters are determined at the end of the process.**

loop to obtain a better lower bound solution, otherwise it returns the lower of  $q_j$  and  $q_{j-1}$ .

If  $q_j$  is higher than  $q_{j-1}$  must provide a valid solution, based on values of  $K$ ,  $c_C$  and  $S_C$  since  $q_{j-1}$  will by definition not violate the soil stress field and will also not violate yield in the arch as it must increase in value to do so. However if  $q_j$  is lower then it may violate the yield conditions by a small amount related to the tolerance selected and this will need reporting. Alternatively a new value of  $q = (1 - p\epsilon)q_j$  where  $p \approx 2$  can be input on the basis that the next value of  $q$  is likely to be higher thus, if confirmed, giving a lower bound.

It has been found in all the analyses undertaken in this paper that the final solution is approached from below and thus  $q_{j-1}$  may be returned directly as a valid lower bound.

Fig. 18 shows the algorithm convergence for the example of Salford masonry arch bridge. In this figure, layers A and B are examples illustrating the change in  $K_{lhs}$  for each iteration and layers C and D examples for  $K_{rhs}$ . Layer E is an example for a surface layer, here denoted  $K_{surface}$ . In an actual analysis these values would be representative of a finite thickness layer. It is actually unnecessary to calculate  $K_{surface}$ , but rather confirm that the cemented layer remains below the required thickness. It is clear to see that the algorithm rapidly converges.

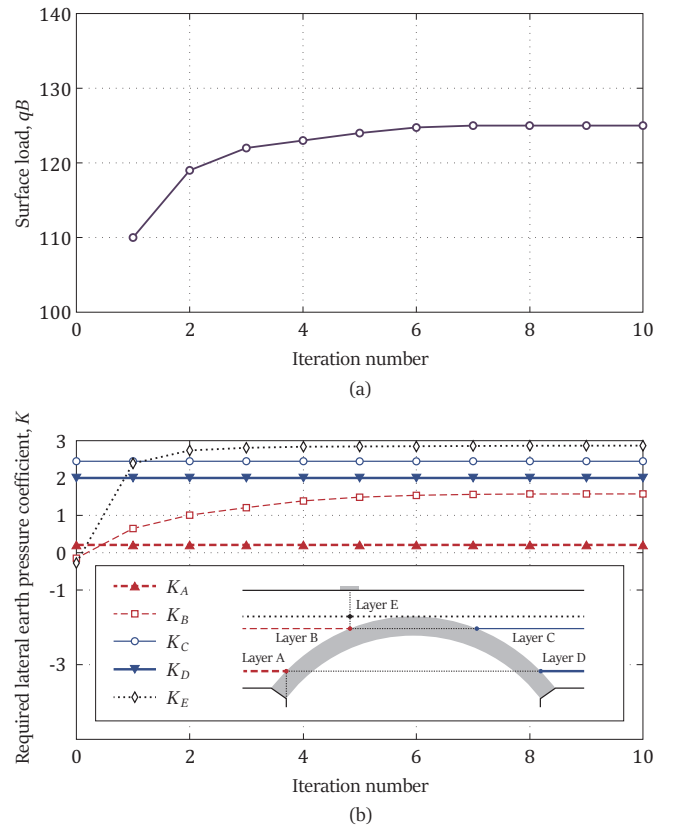


**Fig. 18. Algorithm convergence for Salford masonry arch bridge: (a) surface load  $q_B$ ; (b) required lateral earth pressure coefficient  $K$  at various locations.**