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Stochastic social optimum traffic assignment

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Abstract

This paper formulates a Stochastic Social Optimum (SSO) that relates to the Stochastic User Equilibrium (SUE) in the same way as the Social Optimum (SO) relates to the User Equilibrium (UE) in a deterministic environment. At the SSO solution, the total of the users' perceived costs is minimised. The formulation and analysis is carried out in a general utility-maximising framework, with the probit and logit models being special cases. Conditions for the SSO flow pattern are derived, from which it can be seen that the marginal social costs play the same role in the SSO as the standard costs play in SUE. In particular, it is shown that the SSO solution can be obtained through the use of an algorithm for SUE, but with the marginal costs replacing the standard costs in the stochastic loading and that optimal tolls are the differences between the marginal social costs and the standard costs. For the case of the logit model an explicit path-based objective function is obtained which is of a pleasing symmetrical form when compared with the objective functions for SUE, SO and UE. Additionally, a link-based objective function for the general utility-maximising case is formulated for SSO, which is similar in form to the SUE objective function of Sheffi and Powell.

Keywords: Traffic assignment; Stochastic user equilibrium; Probit model; Logit model; Optimal tolls; Marginal social costs

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1. Introduction

In deterministic traffic assignment, there are two different solutions: the User Equilibrium (UE) and the Social (or System) Optimum (SO), corresponding to Wardrop's first and second equilibrium principles (Wardrop, 1952). The UE flow pattern is how we believe things will be, with drivers choosing their routes selfishly, whilst the SO flow pattern is how the traffic engineer might *like* things to be, in that the total network travel cost is minimised under SO. It is well-known that the SO solution can be found by using the marginal social cost-flow functions $m(x)$ in place of the unit link cost-flow functions $t(x)$ in an algorithm to produce the UE solution. It is also known that we can make the congestion-minimising SO flow pattern into a UE solution by imposing the toll $(m_a(x_a^*) - t_a(x_a^*))$ on link a, where $\boldsymbol{x}^*$ is the SO solution.

Here, we aim to formulate the same principles but in a stochastic environment. The Stochastic User Equilibrium (SUE) solution corresponds to the UE solution with drivers choosing the route which minimises their personal perceived travel cost, and so we seek to define a Stochastic Social Optimum (SSO) which relates to the SUE solution in the same way as the SO solution relates to the UE solution. The SSO solution therefore is that flow pattern which minimises the total of the travel costs perceived by drivers. Just as the SO solution generally requires some drivers travelling on paths which are not the minimum cost paths for that OD pair, so the SSO solution generally requires some drivers to be assigned to paths that are not their minimum perceived cost path. As will be seen later, Yang (1999) has characterised the SSO solution as that which maximises consumer surplus.

We also investigate whether there are similar results for (i) finding the SSO solution by use of an algorithm to produce the SUE solution, and (ii) whether there is a corresponding result about the tolls required to make the SSO solution into a SUE solution.

2. Notation and assumptions

For convenience, we set out here the notation for the principal variables and parameters used in the analysis to follow in the rest of the paper. This notation largely follows that of Sheffi (1985).

x_a	flow on link a
t_a	$t_a(x_a)$ = cost of travel along link a, a function of x_a only
q_{rs}	demand between OD pair rs
h_k^{rs}	flow on path k between OD pair rs
c_k^{rs}	mean perceived travel cost on path k between OD pair rs
m_a	$m_a(x_a)$ = marginal social travel cost on link $a = t_a + x_a \frac{dt_a}{dx_a}$
δ_{ak}^{rs}	1 if link a is on path k between OD pair rs, and 0 otherwise
S_{rs}	expected minimum perceived travel cost between OD pair rs
τ_a	value of the toll charged on link a

In addition to the separability of the link performance function $t_a(x_a)$, it is assumed throughout the paper that this function is positive, strictly increasing, and convex. Under these conditions, as Sheffi (1985) shows, the UE and SO solutions are unique. The demands q_{rs} are assumed to be continuous and therefore infinitely divisible, so that in calculating expected perceived travel costs, a limiting Weak Law of Large Numbers applies.

3. Defining the SSO

In stochastic assignment different drivers have different perceptions of the costs on the links and paths, and we use a distribution of perceived costs to describe these differences. Whereas the SO flow pattern is that which minimises the total network travel cost, the SSO is defined as that flow pattern that minimises the total of the *perceived* travel costs in the network.

To illustrate the concepts, let us first consider the case of a two-path network between a single O–D pair with a fixed demand q. The flows on the paths are denoted by h_1 and h_2 ($h_1 + h_2 = q$), a driver's perceived values of the path costs are denoted by u_1 and u_2 and the probability density function of the drivers' perceived costs is $f(u_1, u_2)$. Firstly, given path flows of h_1 and h_2 we need to allocate the drivers to the paths so as to minimise the total perceived cost. Generally, this will require some drivers to be assigned to paths that are not their minimum perceived cost paths. See Fig. 1: drivers whose perceived costs lie within the region R_1 (above the line BC) will be assigned to path 1; those whose perceived costs lie below BC will be assigned to path 2. The boundary between R_1 and R_2 is the line BC with equation $u_2 = u_1 + d_2$ where the value of d_2 is such that the probability mass contained within R_1 is $p_1 = h_1/q$. Note that those drivers whose perceived costs fall between the lines BC and OA (the $u_1 = u_2$ line) are those who, for the benefit of the population as a whole, are assigned to their non-minimum cost path.

Therefore, d_2 must be found such that

$$\int_{R_1} f(u_1,u_2)\mathrm{d}u_1\,\mathrm{d}u_2 = p_1 = \frac{h_1}{q} \tag{1}$$

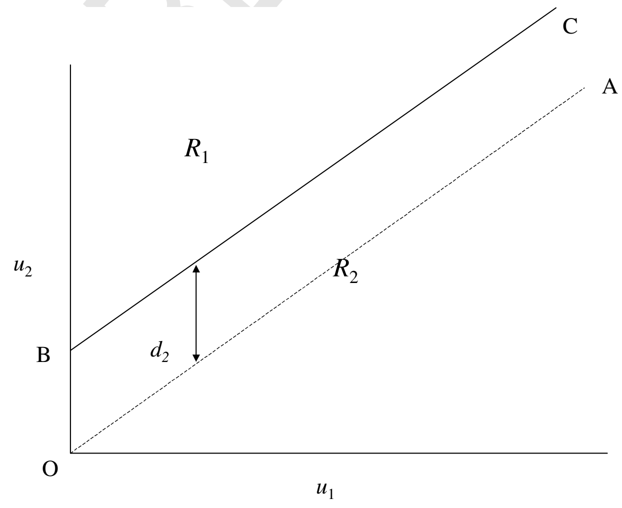

Fig. 1. Sample space of perceived costs u_1, u_2 divided into regions R_1 and R_2.

With this assignment, the total expected perceived cost is

$$z(h_1,h_2) = q\left(\int_{R_1} u_1 f(u_1,u_2)\mathrm{d}u_1\,\mathrm{d}u_2 + \int_{R_2} u_2 f(u_1,u_2)\mathrm{d}u_1\,\mathrm{d}u_2\right) \tag{2}$$

Note that the value of d_2 and hence the regions R_1 and R_2 depend on the path flows h_1, h_2 Also, the mean perceived path costs c_1 and c_2 are also functions of the path flows, through the cost-flow relations. However, we make the assumption throughout this paper that it is *only* the means c_1, c_2 that are affected by the path flows; the variances and covariances remain fixed. Therefore, the following condition holds for the bivariate density function of perceived path costs

$$f(u_1 + d_1, u_2 + d_2; c_1, c_2) = f(u_1, u_2; c_1 - d_1, c_2 - d_2) \quad \forall\, d_1, d_2 \tag{3}$$

The choice model is assumed to be a utility-maximising model, including both the logit and probit models. In the logit model, the perception errors are independent Gumbel variates, with fixed variances. In the probit model, the perception errors are multivariate Normal and we assume the (co)variances to be constant (possibly at values related to the free-flow mean costs, as suggested by Sheffi (1985)[p. 313] in connection with the Sheffi and Powell objective function for SUE).

Hence, from (2) and (3), the total perceived network cost, for flow pattern $\boldsymbol{h}$ is

$$\begin{aligned} z(h_1,h_2) &= q\left(\int_{u_1<u_2-d_2} u_1 f(u_1,u_2;c_1,c_2)\mathrm{d}u_1\,\mathrm{d}u_2 + \int_{u_1>u_2-d_2} u_2 f(u_1,u_2;c_1,c_2)\mathrm{d}u_1\,\mathrm{d}u_2\right) \\ &= q\left(\int_{u_1<u_2} u_1 f(u_1,u_2;c_1,c_2-d_2)\mathrm{d}u_1\,\mathrm{d}u_2 + \int_{u_1>u_2} (u_2+d_2) f(u_1,u_2;c_1,c_2-d_2)\mathrm{d}u_1\,\mathrm{d}u_2\right) \\ &= q\left(\int_{u_1<u_2} u_1 f(u_1,u_2;c_1,c_2-d_2)\mathrm{d}u_1\,\mathrm{d}u_2 + \int_{u_1>u_2} u_2 f(u_1,u_2;c_1,c_2-d_2)\mathrm{d}u_1\,\mathrm{d}u_2\right) \\ &\quad + q\,d_2 \int_{u_1>u_2} f(u_1,u_2;c_1,c_2-d_2)\mathrm{d}u_1\,\mathrm{d}u_2 \end{aligned}$$

Hence

$$z(h_1,h_2) = q(S(c_1,c_2-d_2) + d_2 p_2) = qS(c_1,c_2-d_2) + h_2 d_2 \tag{4}$$

where S denotes the "satisfaction" or composite travel cost, given for the logit case by the familiar "logsum" formula

$$S(c_1,c_2) = -\frac{1}{\theta}\log(\exp(-\theta c_1) + \exp(-\theta c_2))$$

The SSO flow pattern is then defined as that flow pattern h_1, h_2 that minimises the total perceived travel cost $z(h_1,h_2)$. Note that the decision rule for assigning users to paths can be expressed in the form: assign a user to path 1 if his perceived costs are such that $u_1 - d_1 < u_2 - d_2$ for any pair of values of d_1, d_2 that satisfy the condition in (1): that is, it is only the relative values of the d's that matters.

3.1. A numerical example

To illustrate these ideas, consider a simple example with two parallel paths between a single O–D pair. We assume that the perceived path costs are independent and Gumbel distributed, with means c_1 and c_2 and a value of the sensitivity parameter θ of 0.1. The two paths have BPR-style cost-flow functions so that the mean path costs are given by, $c_1 = 10 + 0.02h_1$ and $c_2 = 15 + 0.005h_2$ The demand $q = 1000$.

Since u_1 and u_2 are Gumbel distributed with means c_1 and c_2 the proportion p_1 of drivers for whom $u_1 < u_2 - d_2$ is the same as the proportion for whom $u_1 < u_2$ when the means are c_1 and $c_2 - d_2$; that is

$$p_1 = \frac{\exp(-\theta c_1)}{\exp(-\theta c_1) + \exp(-\theta(c_2 - d_2))}$$

so that the value of d_2 required to give the correct probability mass $p_1 = h_1/q$ is

$$d_2 = -\frac{1}{\theta}\log\left(\frac{h_1}{h_2}\right) - c_1 + c_2$$

Hence the SSO objective function in this two-path logit case is

$$z(h_1,h_2) = -\frac{q}{\theta}\log(\exp(-\theta c_1(h_1)) + \exp(-\theta(c_2(h_2) - d_2)) - \left(\frac{1}{\theta}\log\left(\frac{h_1}{h_2}\right) - c_1(h_1) + c_2(h_2)\right)h_2$$

For this example, we can plot the value of this SSO objective function against h_1. For comparison, we also show in Fig. 2 the plots of the UE, SO and SUE objective functions against h_1. The posi-

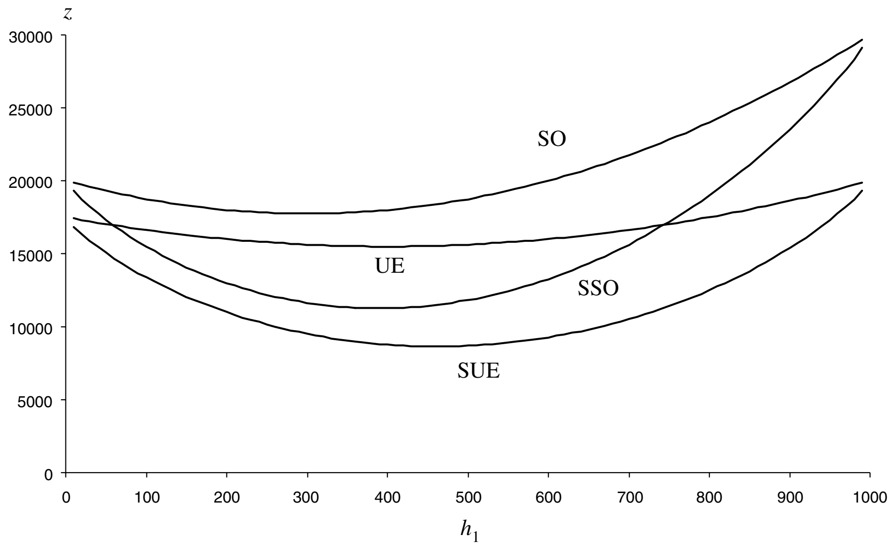

Fig. 2. Plots of z_{SSO}, z_{SO}, z_{UE} and z_{SUE} against h_1, the flow on path 1.

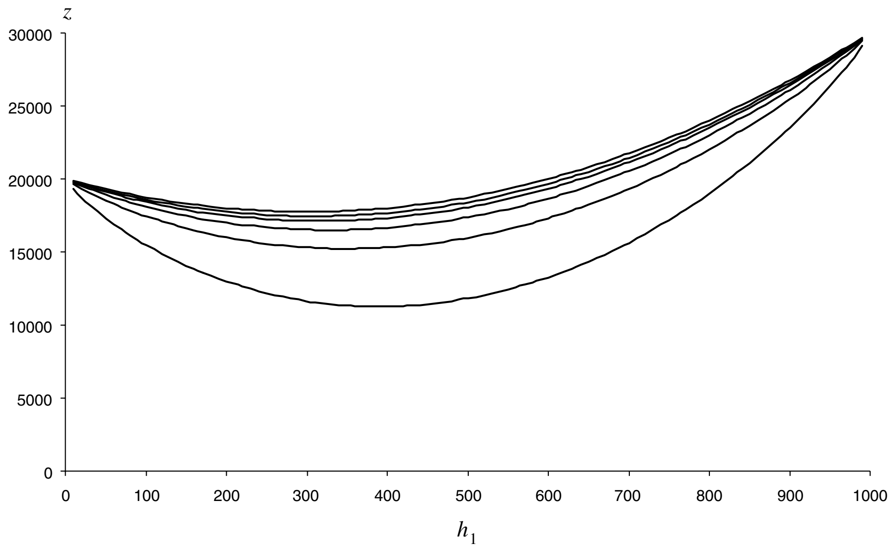

Fig. 3. Plots of z_{SSO} for values of θ = 0.1, 0.25, 0.5, 1 and 2, and z_{SO} against h_1 the flow on path 1 (from bottom to top).

tions of the minima show that the solution for UE is $\boldsymbol{h} = (400, 600)$, that for SO is $\boldsymbol{h} = (300, 700)$, that for SUE is $\boldsymbol{h} = (462, 538)$ and that for SSO is $\boldsymbol{h} = (390, 610)$ (We note in passing that it can be verified that, in this case, this SSO solution is the same as the SUE solution that is obtained by replacing the unit cost-flow functions by the marginal social cost-flow functions.). In Fig. 3, we plot the SSO objective function for several values of the sensitivity parameter θ (=0.1, 0.25, 0.5, 1 and 2) and it can be seen that as θ increases, the plot of the SSO objective function steadily approaches that of the SO objective function, as would be expected, since the degree of variation in the perceived costs is steadily reducing towards zero.

We now aim to extend the formulation in (4) to a more general case, with many (generally overlapping) paths between an O–D pair and, subsequently, to multiple O–D pairs. Now, whereas in the SUE case, the user chooses the path for which the perceived cost is minimum, for the SSO case we extend the decision rule for the two-path case to one that states that a driver should be assigned to path j if its "augmented cost" $u_j - d_j$ is smaller than the augmented costs $u_k - d_k$ for all other paths k. The values of the d_j must then be set such that the proportion assigned by this process to path j is h_j/q. To justify this form of decision rule, we consider in the next section a discrete version of the problem, before returning to the continuous case thereafter.

4. A discrete version of the problem

We consider here a discrete version of the problem: that is, we assume that there is a (large) number N of users each of which has the same J paths to choose from. The users have randomly-drawn and independent sets of values of the perceived path costs u_{ij} which we take to be set out in an $N \times J$ matrix. These costs can be thought of being made up a mean value μ_j which depends on the flow(s) on that path and a perception error e_{ij} that is drawn from some distribution

(Gumbel or Normal) with zero mean and constant variances (so that as the mean of any path changes through congestion effects so the perceived costs for all users on that path change by the same amount).

The problem is how to assign users to paths so as to minimise the total perceived cost, whilst ensuring that the numbers assigned to each path are as given. That is, given that we are to assign a total of n_j ($j = 1,\ldots,J$) users to path j ($n_1 + n_2 + \cdots + n_J = N$), we are to find the optimal values of the variables y_{ij} (where $y_{ij} = 1$ if user i is to be assigned to path j, and zero otherwise) so as to minimise the total perceived cost $z = \sum_{ij} y_{ij} u_{ij}$. As each user is to be assigned to just one path we must have $\sum_j y_{ij} = 1$ and since we must satisfy the constraint on the numbers assigned to each path, we must have $\sum_i y_{ij} = n_j$. This is a special case of the "classical transportation problem" (special in that the row totals are all 1).

It is well-known (see, for example, Taha, 1976) that, for such a problem, a basic solution consists of exactly $(N + J - 1)$ of the NJ cells being used. Of these it is clear that exactly N will take the value 1 (one per row). The other $J - 1$ must be zeroes (but still be basic). These zero-valued basic cells must therefore appear in at most $(J - 1)$ rows (they could all be in a single row, or at the other extreme could each be in a different row). The optimal solution is a basic solution and the standard solution algorithm iterates through a sequence of basic solutions until the optimum is reached, with the value of the objective function z reducing at each iteration.

It is known that the following conditions hold for any basic solution at any iteration. For each basic cell (whether its y_{ij} value is 0 or 1)

$$u_{ij} = \alpha_i + \beta_j$$

and for each non-basic cell a negative value of

$$v_{ij} = u_{ij} - \alpha_i - \beta_j$$

indicates that if this cell were to be brought into the basis (in exchange for one of the current basics) the z value would reduce. The condition for a basic solution to be optimal is that, for all the non-basic cells, the v_{ij} are $\geqslant 0$ (an equality indicates the existence of an equally-optimal solution). The values of the β_j are determined from the (at most) $(J - 1)$ rows that contain the zero-valued basics. Once they have been found, it is trivial to determine the values of the α_i for all other rows.

With an optimal assignment of users to paths, then, a user i is assigned to that path j for which $y_{ij} = 1$. Therefore $\alpha_i = u_{ij} - \beta_j$ and for any other path k, $v_{ik} \geqslant 0$ so that $u_{ik} - \alpha_i - \beta_k \geqslant 0$. Hence $u_{ik} - \beta_k \geqslant u_{ij} - \beta_j$ for all other paths k in that row. That is, at the optimal solution, the β_j values are such that each user i is assigned to that path that has minimal value of $(u_{ij} - \beta_j)$ and the total number assigned to path j is the required value n_j ($j = 1,\ldots,J$).

The β_j play the role, then, of the d_j in the continuous case (where the form of the decision rule was previously *assumed* by extrapolation from the two-path case). Since we can make the discrete case as close as we like to the underlying continuous case, by making the number of rows (users) N as large as we like, we deduce that the same result applies in the continuous case: that is, to find the optimal assignment of users to paths such that the proportions so assigned should be constrained to take the values h_j/q, we need to determine values d_j such that each user is assigned to that path j for which his value of $(u_j - d_j)$ is minimum.

5. The general case

For a single O–D pair, then, with a demand q and with given path flows h_j, we must assign users to paths so as to minimise their total perceived travel cost, by seeking to partition the whole space of perceived costs $\boldsymbol{u}$ into mutually exclusive and exhaustive regions $\{R_j\}$ in an optimal manner. From the previous section we have seen that the region R_j is that within which $u_j - d_j < u_k - d_k$ for all other k. Therefore,

$$\int_{R_j} f(u_1,u_2,\ldots)\mathrm{d}u_1\,\mathrm{d}u_2\ldots = p_j = \frac{h_j}{q} \tag{5}$$

With this assignment, by extension of the expression in (2) the total perceived cost is

$$z(h_1,h_2,\ldots) = q\sum_j \int_{R_j} u_j f(u_1,u_2\ldots)\mathrm{d}u_1\,\mathrm{d}u_2\ldots$$

which, setting $w_j = u_j - d_j$ and denoting by C_j the set of perceived path costs for which path j is the optimum

$$\begin{aligned} z(h_1,h_2,\ldots) &= q\sum_j \int_{C_j} (w_j + d_j) f(w_1,w_2\ldots;c_1-d_1,c_2-d_2,\ldots)\mathrm{d}w_1\,\mathrm{d}w_2 \\ &= q\sum_j \int_{C_j} w_j f(w_1,w_2,\ldots;c_1-d_1,c_2-d_2,\ldots)\mathrm{d}w_1\,\mathrm{d}w_2\ldots + q\sum_j d_j p_j \\ &= qS(c_1-d_1,c_2-d_2,\ldots) + \sum_j \mathrm{d}_j h_j \end{aligned}$$

This is for a single O–D pair. Suppose we now have multiple O–D pairs, identified by rs, and with path flows denoted by h_j^{rs}. The mean travel cost on path j between O–D pairs rs is denoted by c_j^{rs}. Then the objective function is the total perceived travel cost, taken over all rs

$$z_{\mathrm{SSO}}(\mathbf{h}) = \sum_{rs} q_{rs} S^{rs}(\mathbf{c}^{rs} - \mathbf{d}^{rs}) + \sum_{rs}\sum_j d_j^{rs} h_j^{rs} \tag{6}$$

which is to be minimised w.r.t. the h_j^{rs} subject to $q_{rs} = \sum_j h_j^{rs}\ \forall rs$. The path travel costs c_j^{rs} are potentially functions of *all* path flows (that is, from other O–D pairs as well as from rs), whilst the d_j^{rs} that are required to satisfy the constraints on the proportions assigned to each path are functions of the path flows for that rs pair only. Hence the Lagrangian is

$$L(\mathbf{h}) = \sum_{rs} q_{rs} S^{rs}(\mathbf{c}^{rs}(\mathbf{h}) - \mathbf{d}^{rs}(\mathbf{h}^{rs})) + \sum_{rs}\sum_j d_j^{rs} h_j^{rs} + \sum_{rs} \lambda_{rs}\Big(q_{rs} - \sum_j h_j^{rs}\Big) \tag{7}$$

Then, differentiating (7) partially w.r.t. a typical path flow, and using the well-known result (see, for example, Sheffi, 1985) that $\frac{\partial S^{rs}}{\partial \mu_j^{rs}} = p_j^{rs}$, we get

$$\frac{\partial L}{\partial h_j^{rs}} = \sum_{od} q_{od} \sum_k p_k^{rs} \frac{\partial c_k^{od}}{\partial h_j^{rs}} - q_{rs} \sum_k p_k^{rs} \frac{\partial d_k^{rs}}{\partial h_j^{rs}} + d_j^{rs} + \sum_k \frac{\partial d_k^{rs}}{\partial h_j^{rs}} h_k^{rs} - \lambda_{rs} \tag{8}$$

Since for all feasible flow patterns h_j^{rs}, the choice of the d_j^{rs} ensures that $q_{rs}p_k^{rs} = h_k^{rs}$, the second and fourth terms in the above expression cancel, and by setting the derivative above to zero, we obtain the following condition at the SSO solution

$$d_j^{rs} = -\sum_{od} q_{od} \sum_k p_k^{rs} \frac{\partial c_k^{od}}{\partial h_j^{rs}} + \lambda_{rs} \tag{9}$$

Now the travel cost on a path is the sum of the costs on the links that make up that path. Hence $c_k^{od} = \sum_a \delta_{ak}^{od} t_a$ where $\delta_{ak}^{od} = 1$ if link a is part of path k between O–D pair od, and zero otherwise. Also, we note that it is only the relative values of the d_j^{rs} for any rs pair that matter, so that the term λ_{rs} on the right hand side can be dropped. Therefore, at the SSO solution

$$d_j^{rs} = -\sum_{od} q_{od} \sum_k p_k^{od} \sum_a \delta_{ak}^{od} \frac{\mathrm{d}t_a}{\mathrm{d}x_a} \delta_{aj}^{rs} = -\sum_a \left(\sum_k \sum_{od} \delta_{ak}^{od} q_{od} p_k^{od} \right) \delta_{aj}^{rs} \frac{\mathrm{d}t_a}{\mathrm{d}x_a} = -\sum_a x_a \delta_{aj}^{rs} \frac{\mathrm{d}t_a}{\mathrm{d}x_a}$$

Hence, the condition at the SSO solution is that the mean augmented travel cost on path j between O–D pair rs is given by

$$c_j^{rs} - d_j^{rs} = c_j^{rs} + \sum_a x_a \delta_{aj}^{rs} \frac{\mathrm{d}t_a}{\mathrm{d}x_a} \tag{10}$$

which is the marginal social cost for that path.

This condition in (10), taken together with the conditions $q_{rs}p_j^{rs} = h_j^{rs}$ (for all j, r, s), implies that, at the SSO solution $\mathbf{h}_{\mathrm{SSO}}^*$ when the marginal social path costs are used as the mean path costs in a stochastic loading to provide the path choice proportions $\{p_j^{rs}\}$ and hence the auxiliary flow pattern, this auxiliary flow pattern is $\mathbf{h}_{\mathrm{SSO}}^*$.

By comparison, we note that at the SUE flow pattern $\mathbf{h}_{\mathrm{SUE}}^*$ when the standard path costs c_j^{rs} are used as the mean path costs in a stochastic loading to provide the path choice proportions and the auxiliary flow pattern, this auxiliary flow pattern is $\mathbf{h}_{\mathrm{SUE}}^*$.

This suggests that whereas in finding the SUE solution by an iterative process such as the Method of Successive Averages (MSA), it is the standard path costs c_j^{rs} that are used to calculate the auxiliary flow pattern at each iteration, it is the *marginal social costs* that should be used instead to find the SSO solution.

The finding in (10) shows that there is the same relationship between the SSO and SUE solutions as there is between the SO and UE solutions. Just as an algorithm for solving the UE problem can be used to find the SO problem, by replacing the standard costs by the marginal social costs, so an algorithm for solving the SUE problem can be used to find the SSO solution by replacing the standard path costs by the marginal social costs.

Furthermore, it should be noted that at the SSO solution the difference between the marginal social path cost and standard path cost consists of a sum over the links a that make up that path, and that the contribution from a link is the same for all O–D pairs. Therefore, the result for the deterministic case about optimal tolls holds also for the stochastic case; that is, if the tolls $\tau_a = x_a \frac{\mathrm{d}t_a}{\mathrm{d}x_a}$ are applied (with values of the x_a taken at the SSO solution), the resulting SUE solution is the SSO flow pattern. Of course, this marginal social cost (MSC) toll set is not the *only* toll set capable of making the SSO solution into a SUE solution. Any (non-negative) toll set $\{\tau_a - \Delta_a\}$ that satisfies the constraints

$$\sum_a \delta_{ak}^{rs} \Delta_a - \gamma_{rs} = 0 \quad \forall k,r,s \quad \text{and} \quad \Delta_a \leqslant \tau_a \quad \forall a \tag{11}$$

will maintain the same relative values of the mean path costs between any O–D pair rs, and provide non-negative tolls. The values of the Δ_a and the γ_{rs} can then be found according to any chosen criterion, such as that of minimum revenue, by solving the linear programming problem

$$\text{Maximise } z = \sum x_a^* \Delta_a \tag{12}$$

(where the x_a^* are the SSO link flows) subject to the constraints in (11).

6. The logit case

In the logit case, we can additionally derive a closed-form expression for the SSO objective function $z_{\text{SSO}}(\mathbf{h})$, for a given set of path flows $\{h_j^{rs}\}$.

$$p_j^{rs} = \frac{\exp(-\theta(c_j^{rs} - d_j^{rs}))}{\sum_k \exp(-\theta(c_k^{rs} - d_k^{rs}))}$$

so that the values of any pair of d_j^{rs} and d_k^{rs} can be found from

$$\log\left(\frac{h_j^{rs}}{h_k^{rs}}\right) = -\theta(c_j^{rs} - d_j^{rs} - c_k^{rs} + d_k^{rs})$$

so that

$$d_j^{rs} - d_k^{rs} = c_j^{rs} - c_k^{rs} + \frac{1}{\theta}\log\left(\frac{h_j^{rs}}{h_k^{rs}}\right)$$

Without any loss of generality, any one of the d_j^{rs} may be set to zero. Here we shall set $d_1^{rs} = 0$ so that for all other j ($\neq 1$) the necessary value of d_j^{rs} is given by

$$d_j^{rs} = c_j^{rs} - c_1^{rs} + \frac{1}{\theta}\log\left(\frac{h_j^{rs}}{h_1^{rs}}\right)$$

so that the objective function for any one O–D pair rs is

$$z_{\text{SSO}}(\mathbf{h}^{rs}) = -\frac{q_{rs}}{\theta}\log\left(\sum_j \exp(-\theta(c_j^{rs} - d_j^{rs}))\right) + \sum_{j\neq 1} h_j^{rs}\left[c_j^{rs} - c_1^{rs} + \frac{1}{\theta}\log\left(\frac{h_j^{rs}}{h_1^{rs}}\right)\right] \tag{13}$$

After substitution for the d_j^{rs} followed by some simplification, and then summing over all O–D pairs, the expression for the SSO objective function in the logit case is

$$z_{\text{SSO}}(\mathbf{h}) = \sum_{rs}\sum_j h_j^{rs} c_j^{rs}(\mathbf{h}) - \sum_{rs}\frac{q_{rs}}{\theta}\log q_{rs} + \frac{1}{\theta}\sum_{rs}\sum_j h_j^{rs}\log h_j^{rs} \tag{14}$$

The middle term can be omitted, as it is constant, so we can write

$$z_{\mathrm{SSO}}(\mathbf{h}) = \sum_{rs}\sum_{j} h_j^{rs} c_j^{rs}(\mathbf{h}) + \frac{1}{\theta}\sum_{rs}\sum_{j} h_j^{rs}\log h_j^{rs} \tag{15}$$

This is identical to the objective function derived by Yang (1999) that uses the expected indirect utility received by a randomly sampled individual as the benefit measure (consumer surplus). For comparative purposes, the expressions for the SUE objective functions in the case of logit loading, and expressed in terms of a mixture of path flows $\boldsymbol{h}$, link flows $\boldsymbol{x}$ and link costs $\boldsymbol{t}$, is (Fisk, 1980):

$$z_{\mathrm{SUE}}(\mathbf{h}) = \sum_{a}\int_0^{x_a} t_a(\omega)\mathrm{d}\omega + \frac{1}{\theta}\sum_{rs}\sum_{j} h_j^{rs}\log h_j^{rs} \tag{16}$$

For completeness, the objective functions for SO and UE are

$$z_{\mathrm{SO}}(\mathbf{h}) = \sum_{rs}\sum_{j} h_j^{rs} c_j^{rs}(\mathbf{h}) = \sum_{a} x_a t_a(x_a) \tag{17}$$

$$z_{\mathrm{UE}}(\mathbf{x}) = \sum_{a}\int_0^{x_a} t_a(\omega)\mathrm{d}\omega \tag{18}$$

There is a clear connection or symmetry between the four objective functions (15)–(18).

7. A link-based objective function for the general case

As it has been established in Section 5 that at the SSO solution, a stochastic loading using the marginal path costs produces an auxiliary flow pattern that is identical to the current flow pattern, this enables us to write down a link-based objective function for the general, utility-maximising case.

Sheffi and Powell (1982) showed that the general SUE problem is equivalent to the unconstrained minimisation, with respect to the link flows $\mathbf{x}$ of the following objective function

$$z_{\mathrm{SUE}}(\mathbf{x}) = -\sum_{a}\int_0^{x_a} t_a(u)\mathrm{d}u + \sum_{a} x_a t_a(x_a) - \sum_{rs} q_{rs} S_{rs}[\mathbf{t}(\mathbf{x})] \tag{19}$$

where t_a is the travel cost on link a, and S_{rs} is the "satisfaction" function, the expected value of the minimum perceived travel cost for users travelling between OD pair rs. Sheffi and Powell (1982) shows that, under the conditions set out earlier for the link performance functions, the uniqueness of the SUE solution is guaranteed. Sheffi (1985) shows that the derivative of this function with respect to a link flow is given by

$$\frac{\partial z_{\mathrm{SUE}}}{\partial x_a} = (x_a - y_a)\frac{\mathrm{d}t_a(x_a)}{\mathrm{d}x_a}$$

It follows that at the SUE solution, the auxiliary flows $\{y_a\}$ are equal to the current flows $\{x_a\}$. That is, when a stochastic loading is carried out using mean link costs based on the current link flows, the resulting auxiliary flow pattern is identical to the current flow pattern

$$y_a(\mathbf{t}(\mathbf{x})) = x_a \quad \forall a$$

In Section 4, it was shown that at the SSO solution, if the marginal costs are used in place of the unit costs, the auxiliary flow pattern produced from a stochastic loading is equal to the current solution. It follows therefore that we need merely replace the unit link costs t_a in the Sheffi and Powell objective function for SUE by thea marginal link costs $m_a = t_a + x_a(\mathrm{d}t_a/\mathrm{d}x_a)$ in order to give an objective function for SSO

$$z_{\mathrm{SSO}}(\mathbf{x}) = -\sum_a \int_0^{x_a} m_a(u)\mathrm{d}u + \sum_a x_a m_a(x_a) - \sum_{rs} q_{rs} S_{rs}[\mathbf{m}(\mathbf{x})] \tag{20}$$

At the SSO solution, we therefore have

$$y_a(\mathbf{m}(\mathbf{x})) = x_a \quad \forall a$$

Because of the correspondence between the SSO and SUE objective functions, the uniqueness of the SSO solution is guaranteed by the conditions placed on the link performance functions.

8. An illustrative example

To illustrate and confirm the results of section 4 consider the five-link network shown in Fig. 4 (that has the same topology as that in Yang, 1999). There is a single O–D pair from node A to node D (with a demand of 1000 vph), and three paths: 1–4, 1–3–5, and 2–5. The link cost-flow functions are

$$t_1 = 5 + 0.01x_1$$
$$t_2 = 10 + 0.01x_2$$
$$t_3 = 3.5 + 0.005x_3$$
$$t_4 = 8 + 0.01x_4$$
$$t_5 = 5 + 0.01x_5$$

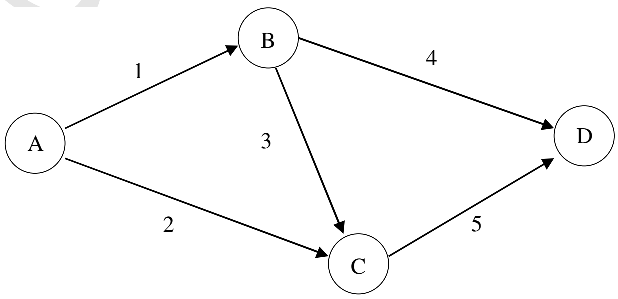

Fig. 4. Five-link, three-path network.

We consider the probit case: that is, Normally distributed perception errors with link variances being fixed and equal to β times the mean free-flow cost. In carrying out the stochastic loading at any iteration, the algorithm of Donnolly (1973) is used for calculating probabilities from the bivariate Normal distribution. For more general, larger networks, a variety of approximate methods can be used to calculate the p_j^{rs} values (see Rosa and Maher, 2002).

The SUE and SSO solutions are found by using respectively the standard path costs c_j^{rs} or the marginal social path costs $c_j^{rs} + \sum_a x_a \delta_{aj}^{rs} \frac{\mathrm{d}t_a}{\mathrm{d}x_a}$ at each iteration when calculating the path choice proportions p_j^{rs} to produce the auxiliary path flow pattern. The results in Table 1 have been obtained, to a high degree of convergence (measured directly by the difference between the current and auxiliary solutions), by an iterative process that optimises the step length along the search direction $(\mathbf{y}–\mathbf{x})$ at each iteration (see, for example, Maher and Hughes (1997)).

The results are shown in Table 1. It is clear that, as the variance-to-mean ratio β is steadily reduced in value, the SUE solution moves towards the UE solution and the SSO solution moves towards the SO solution, as would be expected.

For a value of $\beta = 1$, then, the SSO link flows are $\mathbf{x}_{\mathrm{SSO}} = (578.340, 421.660, 119.275, 459.066, 540.934)$ and hence the MSC tolls are $\boldsymbol{\tau} = (5.783, 4.217, 0.596, 4.591, 5.409)$. By setting up the linear programming problem specified in (11) and (12), the minimal revenue toll set can be found

$$\text{Maximise} \quad z = 578.34\Delta_1 + 421.66\Delta_2 + 119.28\Delta_3 + 459.07\Delta_4 + 540.93\Delta_5$$

$$\text{subject to :}$$

$$\Delta_1 + \Delta_4 - \gamma = 0$$
$$\Delta_1 + \Delta_3 + \Delta_5 - \gamma = 0$$
$$\Delta_2 + \Delta_5 - \gamma = 0$$

and $\Delta_1 \leqslant 5.783$, $\Delta_2 \leqslant 4.217$, $\Delta_3 \leqslant 0.596$, $\Delta_4 \leqslant 4.591$, $\Delta_5 \leqslant 5.409$

Table 1
Path flows assigned by SUE and SSO for various β values

	h_1	h_2	h_3
SUE $\beta = 1$	463.318	144.990	391.692
SUE $\beta = 0.1$	500.046	89.525	410.429
SUE $\beta = 0.01$	519.977	56.640	423.383
SUE $\beta = 0.001$	528.571	41.773	429.656
SUE $\beta = 0.0001$	531.750	36.155	432.094
SUE $\beta = 0.00001$	532.824	34.244	432.933
UE	533.333	33.333	433.333
SSO $\beta = 1$	471.275	99.277	429.448
SSO $\beta = 0.1$	496.446	54.406	449.148
SSO $\beta = 0.01$	508.804	31.535	459.660
SSO $\beta = 0.001$	513.897	21.936	464.167
SSO $\beta = 0.0001$	515.749	18.416	465.835
SSO $\beta = 0.00001$	516.372	17.230	466.399
SO	516.667	16.667	466.667

An optimal solution is found to be $\Delta = (5.035, 4.217, -0.818, 4.591, 5.409)$ so that the minimal revenue tolls $\boldsymbol{\tau}-\Delta = (0.748, 0, 1.414, 0, 0)$, raising a total revenue of 601 compared with a revenue of 10,227 raised from the MSC tolls.

9. Summary

In this paper, we have formulated the SSO (Stochastic Social Optimum) traffic assignment problem to complement the well-known UE, SO and SUE problems and investigated the relationships and similarities between them. The formulation is for a general utility-maximisation framework, which includes as special cases both logit and probit. Under SSO assignment, the total perceived travel cost is minimised with, generally, some users being assigned to paths that are not their personal minimum perceived cost paths. The analysis was developed in two stages. The first stage involved the optimal assignment of users to paths for any given set of path flows **h**, with the idea of augmented path costs **c**–**d** in which the d_j^{rs} values were such that the flow pattern produced by the stochastic loading matched the required flow pattern, and produced an expression $z_{\mathrm{SSO}}(\mathbf{h})$ for the minimum total perceived cost. The second stage then investigated the conditions for the minimisation of $z_{\mathrm{SSO}}(\mathbf{h})$with respect to the path flows**h**. It was shown that under a general utility-maximising framework that includes the two most important cases of logit and probit loading, the augmented path costs at the SSO solution were the marginal social costs, and hence the relationship of the SSO solution to the SUE is the same as that of SO to UE. In particular, the SSO solution can be found by means of an SUE algorithm, by replacing the standard path costs by the marginal social path costs in the stochastic loading; and the toll set that is optimal in the stochastic case has the same form as that which is optimal in the deterministic case, but evaluated at the SSO flow values instead of the SO flow values. Additionally, for the logit case, an expression for the objective function $z_{\mathrm{SSO}}(\mathbf{h})$ has been derived which has a pleasing symmetry with those for UE, SO and SUE. Finally, a link-based objective function has been formulated for the general utility-maximising case (that includes probit as well as logit), which is similar in form to the SUE objective function of Sheffi and Powell.

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