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A BI-LEVEL PROGRAMMING APPROACH FOR TRIP MATRIX ESTIMATION AND TRAFFIC CONTROL PROBLEMS WITH STOCHASTIC USER EQUILIBRIUM LINK FLOWS

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Abstract — This paper deals with two mathematically similar problems in transport network analysis: trip matrix estimation and traffic signal optimisation on congested road networks. These two problems are formulated as bi-level programming problems with stochastic user equilibrium assignment as the second-level programming problem. We differentiate two types of solutions in the combined matrix estimation and stochastic user equilibrium assignment problem (or, the combined signal optimisation and stochastic user equilibrium assignment problem): one is the solution to the bi-level programming problem and the other the mutually consistent solution where the two sub-problems in the combined problem are solved simultaneously. In this paper, we shall concentrate on the bi-level programming approach although we shall also consider mutually consistent solutions so as to contrast the two types of solutions. The purpose of the paper is to present a solution algorithm for the two bi-level programming problems and to test the algorithm on several networks.

Keywords: Trip matrix estimation, Traffic signal optimisation, Stochastic user equilibrium assignment, Mathematical programming

1. INTRODUCTION

In this paper, we deal with two mathematically similar problems in transport network analysis: trip matrix estimation and traffic signal optimisation on congested road networks. These two problems are of great importance in transport planning, scheme appraisal and traffic management. A matrix estimation problem and signal optimisation problem have a common input: route choice proportions or, equivalently, link flows in the road network. These are the output of a traffic assignment model which, in turn, requires a trip matrix or signal settings as inputs. An equilibrium assignment (EA) model needs to be included in the matrix estimation and signal optimisation so as to achieve consistency in route choices and to model congestion effects in the network. This can either be a user equilibrium (UE) assignment model or a stochastic user equilibrium (SUE) assignment model. In the combined matrix estimation and EA problem, there are two linked optimisation problems: matrix estimation (ME) problem with fixed route choice proportions and the EA problem with a fixed trip matrix. Similarly in a combined signal optimisation and EA problem, there are also two linked optimisation problems: the signal optimisation (SO) problem with fixed link flows and the EA problem with fixed signal settings. In both combined problems, there is a mutual interaction between the two sub-problems.

In recent years there has been increasing interest in formulating the two combined problems as *bi-level programming* (BP) problems (Bard, 1988) in which the ME or the SO problem is at the upper level and the EA problem at the lower level. A BP problem has a hierarchical structure in which an upper-level and a lower-level decision maker must select their strategies so as to optimise their objective functions, respectively. But, the upper-level decision maker knows how the lower-level optimiser would react to a given upper-level decision and acts accordingly while the lower-level optimiser can act only according to given decisions of upper-level problem. On the other hand, a conventional approach to deal with the two combined problems is to treat the two sub-problems in a parallel way and to seek a mutually consistent solution. This "mutually consistent" problem falls into another type of mathematical programming problem, namely, the *equilibrium programming* (EP) problem (Garcia and Zangwill, 1981). In an EP problem, each of the two parties is continuously resolving his own sub-problem given the latest information on the actions of the other party.

The BP problem is different from the EP problem in that the upper-level decision maker knows *how* the lower-level decision maker makes a decision. Although he cannot intervene in the lower-level decision maker's decision, he can consider the lower-level decision maker's reaction in his own decision making. This is particularly important in the bi-level signal optimisation problem. In the EP problem, on the other hand, neither of the two optimisers knows how the other would react; each of them acts only according to the decision of the other. In this paper, we shall concentrate on the BP approach although we shall also consider mutual consistent solutions so as to contrast the two types of solutions.

The BP problem may be seen as a *single* programming problem with the upper-level variables being constrained by the lower-level solutions. In this sense, a BP problem is similar to a non-linear programming (NP) problem. However, in a BP problem, the evaluation of the upper-level objective function requires solving the lower-level optimisation problem whose functional form is generally unknown. A further complication is that a BP problem is in general non-convex. This implies the potential existence of local minimum solutions and so a global minimum may be difficult to find. The EP problem is similar to a multiple-objective NP problem in that there are two objectives. However, in a NP problem, there is only one decision maker who chooses *all* variables so as to optimise several objectives. In an EP problem, on the other hand, there are two decision makers, each having control of only one set of variables. (A more general EP problem can have more than two parties.) It is clear that most standard algorithms for NP problems may not be applicable to BP and EP problems.

The two types of programming problem may also be cast into the framework of game theory. Fisk (1984) discussed a range of combined problems in the framework of game theory, and the discussion was illustrated by the combined signal optimisation and UE assignment problem. In game theory, a mutually consistent solution corresponds to a *Nash non-cooperative game* and a BP problem to a *Stackelberg game* or *leader-follower game*. In fact, the Nash non-cooperative game is a special type of EP problem. Formulating the two combined problems as different types of mathematical programming problems or games can help to understand the nature of the problems. However, solution algorithms developed in these theories may not be applicable to general transport network problems because a combined problem in a transport network tends to be large, involving many road links and O-

D pairs. Most algorithms developed for the two combined problems in transport networks have been heuristic. These methods will be reviewed later in the paper.

Recently, the authors have developed algorithms for the solution of the combined ME and UE assignment problem, and of the combined SO and UE assignment problem (Zhang & Maher, 1998; Maher & Zhang, 1999). In this paper, we present a solution algorithm for the bi-level matrix estimation problem, and the bi-level signal optimisation problem, using the new algorithm for the logit-based SUE assignment model by Maher (1998). The two combined problems will be discussed separately in sections 2 and 3, each of which contains the problem formulation, the algorithm and the test results. The SUE algorithm will be described in subsection 2.3 before the proposed bi-level solution algorithm is described. The paper is summarised in the last section.

2. THE COMBINED MATRIX ESTIMATION AND SUE ASSIGNMENT PROBLEM

2.1. The problem formulation and the solution

The problem of trip matrix estimation has been considered by many researchers (*e.g.*, Cascetta and Nguyen, 1988). One of the most widely used formulations for matrix estimation is the minimisation of the weighted sum of squared distances between the observed and estimated traffic flows (Cascetta, 1984):

ME: $$\min_{\mathbf{t}} Z_{\mathrm{ME}}(\mathbf{t},\mathbf{v}) = (\bar{\mathbf{t}}-\mathbf{t})\mathbf{U}^{-1}(\bar{\mathbf{t}}-\mathbf{t}) + (\bar{\mathbf{v}}-\mathbf{v})\mathbf{W}^{-1}(\bar{\mathbf{v}}-\mathbf{v}) \qquad (1)$$

subject to $\mathbf{v}=\mathbf{Pt}$, $\mathbf{t}\geq 0$

where

Z_{ME} is the objective function for matrix estimation;

$\mathbf{t}$ is the vector containing the trip matrix to be estimated, $\mathbf{t}=(\ldots, t_i, \ldots)$, $i\in I$;

I is the set containing O-D pairs;

$\bar{\mathbf{t}}$ is the vector of the target matrix, $\bar{\mathbf{t}}=(\ldots, \bar{t}_i, \ldots)$, $i\in I$;

$\mathbf{v}$ is the vector containing the link flows to be estimated, $\mathbf{v}=(\ldots, v_a, \ldots)$, $a\in A$;

$\bar{\mathbf{v}}$ is the vector containing the observed link flows, $\bar{\mathbf{v}}=(\ldots, \bar{v}_a, \ldots)$, $a\in \bar{A}$;

$\bar{A}$ is the subset of links with observed link flows, $\bar{A}\subseteq A$;

A is the set of links in the network;

U and **W** are weighting matrices, or the variance-covariance matrices of target matrix and the observed link flows;

P is the matrix containing proportions of each O-D flow using each link, or link choice proportions.

All vectors in this paper are column vectors. Note that the second term of the objective function is defined only for those links with traffic counts; it is not necessary to have all the links in the network observed. In the matrix estimation problem, **t** is the set of decision variables, **v** varies with **t**, and **P** is assumed to be given. The solution to this problem, **t***, is the generalised least squares (GLS) estimator and is given by (Cascetta, 1984)

$$\mathbf{t}^* = (\mathbf{U}^{-1} + \mathbf{P}^T\mathbf{W}^{-1}\mathbf{P})^{-1}(\mathbf{U}^{-1}\bar{\mathbf{t}} + \mathbf{P}^T\mathbf{W}^{-1}\bar{\mathbf{v}}) \tag{2}$$

The link flows and link choice proportions are determined from a trip assignment model. We shall use the optimisation formulation for SUE assignment proposed by Powell and Sheffi (1982):

SUE:
$$\underset{\mathbf{v}}{\mathrm{Min}}\ Z_{\mathrm{SUE}}(\mathbf{v},\mathbf{t}) = -\sum_i t_i S_i(\mathbf{v}) + \sum_{a\in A} v_a c_a(v_a) - \sum_{a\in A}\int_0^{v_a} c_a(x)\mathrm{d}x \tag{3}$$

where c_a (v_a) is the cost-flow function for link a and S_i is the value of the satisfaction function of O-D pair i arising from a stochastic loading based on link flow **v**. In the SUE assignment problem, it is possible to find a SUE solution of link choice proportions together with link flows. Although we have included the trip matrix **t** in the objective function, **t** is fixed in the SUE assignment problem. In this section, we will use **V**(**t**) and **P**(**t**) to denote SUE solutions of link flows and link choice proportions. Then we have **V**(**t**)=**P**(**t**)**t**. It worth mentioning that SUE assignment does not in general have an explicit functional form of **V**(**t**) or **P**(**t**), not even for a simple network of one O-D pair joined by two links with linear cost functions.

The bi-level trip matrix estimation problem is one in which the matrix estimator tries to minimise the matrix estimation error while the link flow pattern adjusts itself accordingly to a SUE. The problem may be written as

$$\operatorname*{Min}_{\mathbf{t}\in D_{\mathbf{t}}} Z_{\mathrm{ME}}(\mathbf{t},\mathbf{P}(\mathbf{t})\mathbf{t}) \tag{4a}$$

or

$$\operatorname*{Min}_{\mathbf{t}\in D_{\mathbf{t}}} Z_{\mathrm{ME}}(\mathbf{t},\mathbf{V}(\mathbf{t})) \tag{4b}$$

where $D_{\mathbf{t}}$ is the feasible regions for $\mathbf{t}$, $\mathbf{V}(\mathbf{t})=\mathbf{P}(\mathbf{t})\mathbf{t}$, and $\mathbf{V}(\mathbf{t})$ is the lower-level SUE assignment problem. The EP formulation of the problem, on the other hand, is

$$\operatorname*{Min}_{\mathbf{t}\in D_{\mathbf{t}}} Z_{\mathrm{ME}}(\mathbf{t},\mathbf{P}\mathbf{t}) \tag{5a}$$

$$\operatorname*{Min}_{\mathbf{v}\in D_{\mathbf{v}}} Z_{\mathrm{SUE}}(\mathbf{v},\mathbf{t}) \tag{5b}$$

where $D_{\mathbf{v}}$ is the feasible regions for $\mathbf{v}$. Note the difference between (4a) and (5a): the former is solved with *variable* **P** while the latter with *fixed* **P**. It is clear that the mutually consistent solution is also a feasible solution of the bi-level problem. The two solutions are in general different and the bi-level solution has a smaller value of the ME objective function than that of the mutually consistent solution. By definition, among all the solutions that satisfy SUE conditions, the bi-level solution has the minimum ME objective function value.

2.2. Previous solution algorithms for the combined problem

Hall *et at.* (1980) considered a combined ME and UE problem, in which the observed link flows are assumed to be error-free. An iterative algorithm for solving the problem was proposed, in which the two sub-problems are solved alternatively until convergence is achieved. This iterative estimation-assignment (IEA) algorithm has been widely used. However, it has been demonstrated (Fisk, 1988) that this algorithm may or may not converge, depending on whether the coupling between the two sub-problems is weak or not. In addition, when it does converge, it will converge to the mutually consistent solution. Yang *et al*. (1992) and Yang (1995) considered a more general bi-level trip matrix estimation problem with UE assignment at the lower level. They proposed two heuristic algorithms which also involve alternate optimisation of the upper- and lower-level problems. The first algorithm is essentially the same as that by Hall *et at.* (1980) mentioned above, though the algorithm is developed for solving the more general problem. The second algorithm involves calculating the gradient using a sensitivity analysis method (Tobin and Friesz, 1988) to obtain the partial derivatives of UE link flows with respect to O-D flows. It has been shown (Maher and Zhang, 1999) that, at least in a two-link network, the first algorithm converges to the mutually consistent solution while the second algorithm to the bi-level solution. In general, however, conditions for the convergence of both algorithms remain to be proved. An example in which the IEA algorithm diverges in a two-link network example was also shown in Maher and Zhang (1999).

Below, we describe a solution algorithm for the bi-level problem. But before that we describe the algorithm for logit-based SUE assignment, which is a basic building block for the bi-level solution algorithm.

2.3. Algorithm for logit-based SUE assignment

The solution to the SUE problem is an iterative process. At each iteration a search direction is found by carrying out a stochastic loading based on travel costs calculated from the current link flows, $\mathbf{v}^{(k)}$. Then the link flows are updated by

$$\mathbf{v}^{(k+1)} = \mathbf{v}^{(k)} + \lambda^{(k)} (\mathbf{u}^{(k)} - \mathbf{v}^{(k)})$$

where $\lambda^{(k)}$ is the step length taken along the search direction ($\mathbf{u}^{(k)} - \mathbf{v}^{(k)}$), and $\mathbf{u}^{(k)}$ is the auxiliary solution obtained from a stochastic network loading. Different algorithms differ in the way the step length is determined. In the method of successive average (MSA), a sequence of predetermined step lengths is used: $\lambda^{(k)}=1/(k+1)$ (Powell and Sheffi, 1982). The algorithm may be inefficient because the step size is not adaptive. In Maher (1998) , an approximate optimal step length is calculated at each iteration. A stochastic loading carried out with the current link flow $\mathbf{v}^{(k)}$ will produce not only the auxiliary flow pattern $\mathbf{u}^{(k)}$, but also the value of the satisfaction function $\{S_i(\mathbf{v}^{(k)})\}$ and hence the value of the objective function $Z_{\mathrm{SUE}}(\mathbf{v}^{(k)})$. Furthermore, the derivative of the objective function along the search direction at λ=0 can also be obtained:

$$\left.\frac{\mathrm{d}Z_{\mathrm{SUE}}(\mathbf{v}(\lambda))}{\mathrm{d}\lambda}\right|_{\lambda=0} = \sum_a (v_a^{(k)} - u_a^{(k)}) \left.\frac{\mathrm{d}c_a(v_a)}{\mathrm{d}v_a}\right|_{v_a=v_a^{(k)}} (u_a^{(k)} - v_a^{(k)})$$

Another stochastic loading at the auxiliary solution point will give rise to another pair of the objective function value $Z_{\mathrm{SUE}}(\mathbf{u}^{(k)})$ and the derivative:

$$\left.\frac{\mathrm{d}Z_{\mathrm{SUE}}(\mathbf{v}(\lambda))}{\mathrm{d}\lambda}\right|_{\lambda=1} = \sum_a (u_a^{(k)} - w_a^{(k)}) \left.\frac{\mathrm{d}c_a(v_a)}{\mathrm{d}v_a}\right|_{v_a=u_a^{(k)}} (u_a^{(k)} - v_a^{(k)})$$

where $\mathbf{w}^{(k)}=\{w_a^{(k)}\}$ is the auxiliary flow pattern from a stochastic loading based on link flow $\mathbf{u}^{(k)}$. Quadratic or cubic interpolation along the search direction can then be used to derive an estimate of the optimal step length (*i. e.* that at which Z_{SUE} is minimum or each component of the gradient is zero).

The logit stochastic loading algorithm considers only efficient links so as to avoid explicit route enumeration (Dial, 1971). However, when the stochastic loading is used as part of a SUE algorithm, the set of efficient links for each O-D pair may vary from iteration to iteration as link flows and costs vary. Consequently, the objective function is not necessarily continuous and convergence of the algorithms may be affected. One way round this difficulty is to make the choice of a set of efficient links based on some predetermined link flow pattern (such as free-flows) and to maintain the same set of efficient links throughout the iterative process (Leurent, 1997).

2.4. The proposed algorithm for the bi-level solution

Suppose we have a current solution, $[\mathbf{t}^{(n)},\mathbf{v}^{(n)}]$, where $\mathbf{v}^{(n)}=\mathbf{V}(\mathbf{t}^{(n)})$. At each iteration, the upper-level problem is firstly solved to get an auxiliary solution of the trip matrix, $\mathbf{t}^*$, assuming $\mathbf{v}=\mathbf{P}^{(n)}\mathbf{t}$. Then, a SUE assignment is performed to find the SUE link flows, $\mathbf{v}^*$, or $\mathbf{V}(\mathbf{t}^*)$ at $\mathbf{t}^*$. Thus, we have two points satisfying SUE conditions, $\mathbf{t}^{(n)}$ and $\mathbf{t}^*$. We then search for an optimal step length along $(\mathbf{t}^*-\mathbf{t}^{(n)})$ by a line search. In the bi-level problem (4a) or (4b), however, a line search directly based on the objective function requires repeated SUE assignment and is very inefficient. To overcome this difficulty, we linearise the SUE assignment map between the two points, $(\mathbf{t}^{(n)},\mathbf{v}^{(n)})$ and $(\mathbf{t}^*,\mathbf{v}^*)$, that is

$$\mathbf{V}(\mathbf{t}) \cong \mathbf{v}^{(n)}+\mathbf{Q}(\mathbf{t}-\mathbf{t}^{(n)})$$

where $\mathbf{Q}=[Q_{ai}]$ and $Q_{ai}=(v_a^*-v_a^{(n)})/(t_i^*-t_i^{(n)})$. Let

$$\mathbf{t}(\beta)=\mathbf{t}^{(n)}+\beta(\mathbf{t}^*-\mathbf{t}^{(n)}) \tag{6a}$$

We have

$$\mathbf{v}(\beta)=\mathbf{v}^{(n)}+\beta(\mathbf{v}^*-\mathbf{v}^{(n)}) \tag{6b}$$

Then an optimal step length β^* can be found by minimising $Z_{\mathrm{ME}}(\mathbf{t}(\beta),\mathbf{v}(\beta))$. This is a standard one-dimensional search and can be solved by, for example, the Newton method. The function $Z_{\mathrm{ME}}(\mathbf{t}(\beta),\mathbf{v}(\beta))$ and its derivatives with respect to β can be evaluated for any value of β. The first and the second derivatives of the $Z_{\mathrm{ME}}(\mathbf{t}(\beta),\mathbf{v}(\beta))$ with respect to β are

$$\frac{\mathrm{d}Z_{\mathrm{ME}}(\mathbf{t}(\beta),\mathbf{v}(\beta))}{\mathrm{d}\beta}=\nabla_{\mathbf{t}}F_{\mathbf{t}}(\mathbf{t}(\beta),\bar{\mathbf{t}})(\mathbf{t}^*-\mathbf{t}^{(n)})+\nabla_{\mathbf{v}}F_{\mathbf{v}}(\mathbf{v}(\beta),\bar{\mathbf{v}})(\mathbf{v}^*-\mathbf{v}^{(n)})$$

$$\begin{aligned}\frac{\mathrm{d}Z^2_{\mathrm{ME}}(\mathbf{t}(\beta),\mathbf{v}(\beta))}{\mathrm{d}\beta^2}&=(\mathbf{t}^*-\mathbf{t}^{(n)})^T\nabla^2_{\mathbf{t}}F_{\mathbf{t}}(\mathbf{t}(\beta),\bar{\mathbf{t}})(\mathbf{t}^*-\mathbf{t}^{(n)})\\&+(\mathbf{v}^*-\mathbf{v}^{(n)})^T\nabla^2_{\mathbf{v}}F_{\mathbf{v}}(\mathbf{v}(\beta),\bar{\mathbf{v}})(\mathbf{v}^*-\mathbf{v}^{(n)})\end{aligned}$$

The new solution of the trip matrix is then given by (6a) with β^*. However, $\mathbf{v}(\beta^*)$ obtained by (6b) is only an approximation to $\mathbf{V}(\mathbf{t}(\beta^*))$. Therefore, another SUE assignment is performed to find the exact SUE link flows, $\mathbf{V}(\mathbf{t}(\beta^*))$, for $\mathbf{t}(\beta^*)$.

The SUE solution itself is an iterative process. Therefore, there are two nested iterations in the algorithm for the bi-level solution: the outer iterations for the bi-level solution and the inner iterations for the SUE solution. We shall use free-flow costs to determine a set of efficient links and use this set throughout both inner iterations for SUE assignment and outer iterations for the bi-level solution. The bi-level algorithm can be outlined as follows.

Step 1: Determine a set of efficient links by carrying out a stochastic loading based on free-flow link costs.
Step 2: Initialise $\mathbf{t}^{(0)}$, $\mathbf{v}^{(0)}$, and $\mathbf{P}^{(0)}$; set n=0. The initial trip matrix can normally be set to be the target matrix. Assigning the target matrix to the network by SUE assignment gives $\mathbf{v}^{(0)}$ and $\mathbf{P}^{(0)}$.
Step 3: Determine a GLS estimation of $\mathbf{t}^*$ by equation (2), using $\mathbf{v}=\mathbf{P}^{(n)}\,\mathbf{t}$.
Step 4: Find $\mathbf{V}(\mathbf{t}^*)$ for $\mathbf{t}^*$ by carrying out a SUE assignment.
Step 5: Find β which minimises $Z_{\mathrm{ME}}(\mathbf{t}(\beta),\mathbf{v}(\beta))$ by, for example, the Newton method.
Step 6: Set $\mathbf{t}^{(n+1)}=\mathbf{t}^{(n)}+\beta(\mathbf{t}^*-\mathbf{t}^{(n)})$.
Step 7: Find $\mathbf{v}^{(n+1)} = \mathbf{V}(\mathbf{t}^{(n+1})$ as well as $\mathbf{P}^{(n+1)}(\mathbf{t}^{(n+1})$ by carrying out another SUE assignment.
Step 8: If the convergence criterion is met, stop; otherwise, set n:=n+1 and go to step 3.

The stopping criterion can be based on the maximum relative change in the elements of the estimated trip matrix at successive iterations:

$$\mathrm{Max}_i\ (|t_i^{(n+1)} - t_i^{(n)}|/\ t_i^{(n)}) \le \varepsilon$$

where ε is the error tolerance. This stopping criterion may not be a good indicator of an optimal solution. Another possible stopping criterion is the change of objective function values at successive iterations. However, due to the nonlinearity and nonconvexity of the problem, it is possible for different solutions to have similar objective function values. In the following numerical tests of the algorithm, the above stopping criterion is used to terminate the iterative process, and the changes in objective function values are also observed to make sure that the changes are also small.

Several comments need to be made about the algorithm. First, although we have considered the GLS method for matrix estimation, other methods, such as the entropy maximisation method, can also be used without changing the structure of the algorithm. Second, this algorithm needs two SUE assignments at each iteration. This is necessitated by the bi-level nature of the problem: the SUE condition must be satisfied at every solution. The algorithm can be made more efficient by starting the SUE assignment with the latest link flow pattern rather than the free-flow pattern. For example, at Step 4, the initial link flows for SUE assignment can be set to be $\mathbf{v}=\mathbf{P}^{(n)}\mathbf{t}^*$. Third, in this algorithm, the auxiliary solution does not necessarily point to a descent direction and the optimal step length is not limited to be positive. Fourth, the algorithm involves an approximation in the optimal step length calculation: the SUE assignment map is linearised over the interval between the current and the auxiliary solution. The interval is generally finite because the auxiliary solution does not in general become closer to the current solution with increasing iteration number. Thus there is no reason to expect that the linearisation will become more and more accurate as the algorithm converges. As a result, the algorithm may converge to some point in the neighbourhood of the true solution due to the approximation. This problem may be dealt with by reducing the interval between the current and the auxiliary solution of matrix by, for example, a MSA-type scheme so that the linearisation is made over a smaller and smaller interval. This can be implemented by replacing the auxiliary solution of matrix with $\mathbf{t}^{(n)}+(\mathbf{t}^*-\mathbf{t}^{(n)})/n$ at the end of step 3 of the bi-level algorithm. It can be expected that the algorithm will become less efficient with this modification. Numerical tests with the networks tried so far have shown that the first few iterations of the algorithm are most "cost effective"; the solution is close to the optimal one after only a few iterations. Therefore, if higher accuracy is desirable, we can introduce the modification after the first few iterations when the solution is close to optimal or when the ME objective function is not reduced at further iterations. The improvement of the algorithm can be observed by the reduction in the value of objective function of matrix estimation. This modification procedure will be demonstrated in the numerical calculations below.

2.4. Test results

There are two parts in this test. In the first part a simple example is used to test if the algorithm can identify the bi-level solution because in this example the bi-level solution can be found by direct search. The network has one O-D pair connected by two links The cost functions on the two links are

$$c_1 = 5 + v_1/1000$$

$$c_2 = 6.25 + v_2/1000$$

The target matrix is $\bar{t}_1$=2000 and the link count is $\bar{v}_2$=620, made on link 2. The value of "spread parameter" θ in the logit model used is 0.5, and an error tolerance ε is 0.001. These two values will be used for all numerical tests in this paper. The solutions by the bi-level algorithm, and the modified bi-level algorithm at the 20th iteration, together with the true bi-level solution and the mutually consistent solution are shown in Table 1. The modification to the bi-level solution was introduced after three initial iterations of the original algorithm when the iterations have passed the convergence test. The true solution was found by direct search in the trip matrix with an incremental size of 0.001 — a more detailed search showed that the objective function is rather flat and does not change much with changes in the matrix at the fourth decimal place. The mutually consistent solution was found by the IEA algorithm. It can be seen that the bi-level algorithm gives a good approximation to the true bi-level solution and that the modification improves the accuracy of the algorithm, though only marginally. It can also be seen that the bi-level solution has a smaller value of Z_{ME} than that of the mutually consistent solution, as expected.

{**Table 1** is about here}

The second part of the experiment is made to investigate the performance of the algorithm for different values of errors in the prior matrix and the observed links flows on two networks. In this test, the true trip matrix is supposed to be known. Assigning the true matrix to the network by SUE assignment gives the true link flows. The target matrix and the observed link flows are generated by (Yang *et al.*, 1992)

$$\bar{t}_i = \tilde{t}_i(1.0 - C_{\mathrm{vod}}\xi_i)$$
$$\bar{v}_a = \tilde{v}_a(1.0 - C_{\mathrm{vlk}}\zeta_a)$$

where $\tilde{t}_i$ and $\tilde{v}_a$ are the elements of the true matrix and link flows, ξ_i and ζ_a are randomly generated $N(0,1)$ variables, and C_{vod} and C_{vlk} are the coefficients of variation reflecting the random errors of the target matrix and observation errors in link flows respectively. The variance-covariance matrices, **U** and **W**, are assumed to be diagonal matrices with the variances (Yang *et al*., 1992)

$$\mathrm{Var}(\bar{t}_i)=(C_{vod}\ \tilde{t}_i)^2$$

$$\mathrm{Var}(\bar{v}_a)=(C_{vlk}\ \tilde{v}_a)^2$$

The BPR (Bureau of Public Roads) cost function will be used:

$$c_a(v_a)=c_a(0)\left[1+\alpha(\frac{v_a}{q_a})^{\gamma}\right]$$

where $[c_a(0)]$ is uncongested link costs, $[q_a]$ is link capacity, and α and γ are constants. The values α=1.0 and γ=4.0 will be used. We shall show the results calculated on two networks and then analyse the results.

The first network is the grid network shown in Figure 1. The network has 9 nodes and 24 links. There are 4 centroids (nodes 1, 3, 5, and 7) and 4 O-D pairs (1→5, 3→7, 5→1, 7→3). The true matrix is assumed to be: [72 60 72 60]. The uncongested link costs $[c_a(0)]$ and link capacities $[q_a]$ are listed in Table 2. Calculations with different values of C_{vod} and C_{vlk} are summarised in Table 3, including the solutions and the values of objective functions at the 20th iteration of the algorithm, the number of iterations as well as the c.p.u. times (in seconds) needed for the algorithm to converge at the given error tolerance. Also shown in the table is the mutually consistent solutions found by the IEA algorithm. In addition, convergence of the proposed algorithm in terms of objective function values with the largest values of C_{vlk} is shown in Figure 2. The second test was made on the well-known Sioux Falls network. Information in the data set includes the network characteristics (link-node topology and the parameters in the cost functions) and a demand trip matrix which is treated as the true matrix. The network has 76 links, 24 nodes. All nodes are both origins and destinations, and so there are 576 O-D pairs. The values of objective functions at the 20th iteration of the algorithm, the number of iterations as well as the c.p.u. times (in seconds) needed for the algorithm to converge together with the mutually consistent solutions found by the IEA algorithm are summarised Table 4. Convergence of the algorithm in terms of objective function values is shown in Figure 3.

{**Figure 1** is about here}

{**Table 2** is about here}

{**Table 3** is about here}

{**Figure 2** is about here}

{**Table 4** is about here}

{**Figure 3** is about here}

Several points can be observed from the results. First, the proposed algorithm converges after a few iterations and is quite efficient. Second, the two types of solutions are different and the value of the objective function of the matrix estimation problem is lower at the bi-level solution than that at the mutually consistent solution. Third, the larger the errors in the prior matrix and observed link flows, the more effort it takes for the algorithms to converge. Fourth, the computation time of the algorithms also depends on the size of a network. The calculations were made on a 300MHz Pentium II machine with 64.0 Mb RAM. Whilst an estimation takes about 2-3 minutes to converge on the Sioux Falls network, it takes only one or two seconds for the iterations to converge on the grid network at the same error tolerance. The main computational burden in the proposed algorithm is the solution of the ME problem and the SUE assignment problem; the former involves a matrix inversion and the latter is itself an iterative process. If there are many more O-D pairs than links, such as in the Sioux Falls network, the solution of the ME problem contributes more significantly to the c.p.u. time. On the other hand, if there are a lot more links than O-D pairs, such as in the grid network, SUE assignment contributes more significantly to c.p.u. time.

3. THE COMBINED SIGNAL OPTIMISATION AND SUE ASSIGNMENT PROBLEM

3.1. The problem formulation and previous algorithms

The combined signal optimisation and SUE assignment problem is mathematically similar to that of the combined matrix estimation and SUE assignment (Note that the trip matrix is assumed to be fixed in the signal optimisation problem). The most commonly used policy for signal optimisation is to minimise the total journey costs in the network:

SO: $$\underset{\mathbf{s}}{\text{Min}}\ Z_{\text{SO}}(\mathbf{s},\mathbf{v}) = \sum_{a \in A} v_a\ c_a(v_a, s_a)$$

subject to $$s_a^{\max} \geq s_a \geq s_a^{\min}, \quad a \in A$$

$$\sum_{a \in A_j} s_a = 1, \quad A_j \subset A$$

where s_a is the ratio of green for link a, $\mathbf{s}$=(…, s_a, …); $s_a^{\max}$ and $s_a^{\min}$ are maximum and minimum allowable green split for link a, $s_a^{\min}>0$, $s_a^{\max}<1$; A_j is the set of links heading for the jth signal controlled intersection. If link a is not controlled by a signal, then $s_a^{\max}$, s_a, and $s_a^{\min}$ will all be equal to 1. The set of link flows $\mathbf{v}$ is the output from a SUE assignment problem. Given a signal setting, $\mathbf{s}$, the SUE assignment problem (3) may be re-written as

SUE: $$\underset{\mathbf{v}}{\text{Min}}\ Z_{\text{SUE}}(\mathbf{v},\mathbf{s}) = -\sum_i q_i S_i(\mathbf{v}) + \sum_{a \in A} v_a c_a(v_a, s_a) - \sum_{a \in A} \int_0^{v_a} c_a(x, s_a)\mathrm{d}x$$

In this section, we will use $\mathbf{V}(\mathbf{s})$ to denote the SUE link flows for given $\mathbf{s}$. The bi-level solution, $\mathbf{s}^{\text{BL}}$ is defined as

$$\mathbf{s}^{\mathrm{BL}} = \mathrm{Arg}\min_{\mathbf{s}\in D_{\mathbf{s}}} Z_{\mathrm{SO}}(\mathbf{s}, \mathbf{V}(\mathbf{s}))$$

while a mutually consistent solution, [$\mathbf{s}^{\mathrm{MC}}$, $\mathbf{v}^{\mathrm{MC}}$], can be expressed as

$$\mathbf{s}^{\mathrm{MC}} = \mathrm{Arg}\min_{\mathbf{s}\in D_{\mathbf{s}}} Z_{\mathrm{SO}}(\mathbf{s}, \mathbf{v}^{\mathrm{MC}})$$

$$\mathbf{v}^{\mathrm{MC}} = \mathrm{Arg}\min_{\mathbf{v}\in D_{\mathbf{v}}} Z_{\mathrm{SUE}}(\mathbf{v}, \mathbf{s}^{\mathrm{MC}})$$

The comparison of the two types of solutions here is the same as that in the combined matrix estimation and SUE assignment problem. However, it is important to note that the bi-level solution has a smaller value of SO objective function which is the total cost in the network. Therefore, the system would perform better under bi-level approach.

The traffic signal optimisation problem is a special case of the more general optimal network design problem, in which the number of phases, the cycle time, and the offsets of traffic signals are determined. The optimal network design problem has been considered by many researchers (See *e.g.*, Davis, 1994; Friesz *et. al.*, 1992; Harker and Friesz, 1984; Suwansirikul *et. al.*, 1987 among others). In this paper, we consider signal optimisation for isolated intersections. Thus, given a set of link flows, the SO problem is reduced to several sub-problems of determining the optimal green split for each signal controlled intersection. Each of them may be solved by any standard one-dimensional optimisation algorithm, such as the Newton method.

An iterative algorithm in which the SO and UE problems are solved alternately has been used for the solution of the combined SO and UE problem (Van Vuren and Van Vliet, 1992; Smith and Van Vuren, 1993). As in the matrix estimation problem, this iterative optimisation-assignment (IOA) procedure may converge to the mutually consistent solution but convergence is not guaranteed (Fisk, 1984, 1988). Several types of algorithms have been proposed for the solution of the bi-level signal optimisation problem with UE assignment (Sheffi and Powell, 1983; Heydecker and Khoo, 1990; Yang and Yagar, 1995). See Maher and Zhang (1999) for a review for these algorithms. However, these algorithms require repeated UE assignment for direction finding and/or for line search. Using MSA instead of a line search can avoid repeated UE assignment, but will slow down the convergence of the algorithms. Cascetta *et al*. (1998) considered a combined signal optimisation and SUE assignment problem. They proposed two methods for direction finding. The first one is the opposite gradient direction identified by numerical differentiation. This needs several SUE assignments, as in the method by Sheffi and Powell (1983). The second method is the use of the solution of the SO problem with fixed link flows as a direction. This, however, does not necessarily provide a descent direction. The step lengths calculation is a modified MSA algorithm in which step size is reduced only when the objective function is not reduced.

3.2. The proposed algorithm and the test results

The algorithm proposed here is similar to that for trip matrix estimation, with the trip matrix estimation being replaced by signal optimisation with fixed link flows. The algorithm will not be repeated here but the method of line search is described briefly. The optimal step length is found by solving

$$\underset{\beta}{\text{Min}}\, Z_{\text{SO}}(\mathbf{s}(\beta), \mathbf{v}(\beta))$$

subject to $\beta \in [\beta^{\min}, \beta^{\max}]$

where

$$\mathbf{s}(\beta) = \mathbf{s}^{(n)} + \beta(\mathbf{s}^* - \mathbf{s}^{(n)})$$
$$\mathbf{v}(\beta) = \mathbf{v}^{(n)} + \beta(\mathbf{v}^* - \mathbf{v}^{(n)})$$

and $[\beta^{\min}, \beta^{\max}]$ is determined from constraints on signal parameters and link flows. The line search can be solved by the bisection method and no stochastic loading is needed. The first derivative of the objective function with respect to β needed in the bisection method is given by

$$\frac{\mathrm{d}}{\mathrm{d}\beta} Z_{\text{SO}}(\mathbf{s}(\beta), \mathbf{v}(\beta)) = \sum_{a \in A} (c_a + v_a \frac{\partial c_a}{\partial v_a})(v_a{}^* - v_a{}^{(n)}) + \sum_{a \in C} v_a \frac{\partial c_a}{\partial s_a}(s_a{}^* - s_a{}^{(n)})$$

The algorithm is tested on two networks. The cost function used is a combination of the BPR function (for link travel time) and the signal delay formula by Doherty (1977), that is

$$c_a(v_a) = c_a(0)\left[1 + \alpha(\frac{v_a}{q_a})^{\gamma}\right] + d_a$$

Here d_a is signal delay for link a and is given by

$$d_a = \frac{T}{2}(1 - s_a)^2 + \frac{1980}{q_a s_a}\frac{v_a}{q_a s_a - v_a}, \quad v_a/(q_a\, s_a) \le 0.95$$

$$d_a = \frac{T}{2}(1 - s_a)^2 - \frac{198.55 \times 3600}{q_a s_a} + \frac{220 \times 3600 v_a}{(q_a s_a)^2}, \quad v_a/(q_a\, s_a) > 0.95$$

where T is the cycle time and is fixed at 90 seconds in the test. The values of α=1.0 and γ=4.0 are used in the BPR function.

The first test was made on a simple three-link network shown in Figure 4. The network has two O-D pairs, with demand $t_1 = t_2 = 100$. O-D pair 1 is connected by link 1 and link 2. O-D pair 2 is connected by link 3. There is a signal at the intersection of links 1 and 3. The uncongested link costs [$c_a(0)$] and link capacities [q_a] are

$$[c_a(0)] = [1\ \ 2\ \ 1]$$
$$[q_a] = [200\ \ 100\ \ 200]$$

Direct search (by exhaustive trial of all possible solutions of signal settings, with increment size of 0.0001) has shown that in this example, there is only one optimal bi-level solution and that the solution is s_1=0.3070. Three initial signal splits for s_5 are used in the test: 0.3, 0.5, and 0.7; and the algorithm converges to the same solution. The solutions by the bi-level algorithm, and the modified bi-level algorithm (modification introduced at the fourth iteration) at the 20th iteration, together with the true bi-level solution found by direct search and the mutually consistent solution found by the IOA algorithm are summarised in Table 5. It can be seen that the bi-level algorithm converges almost to the true bi-level solution and that the modification improves the solution marginally. In this example, link 2 is twice as long as link 1, although its capacity is comparable to that of link 1 (considering signal control). More drivers would naturally use link 1 at low demand. However, if the signal optimiser knows drivers' route choice behaviour, as in the bi-level problem, he can reduce the green split on link 1 and thus divert more traffic to link 2. Therefore, we have in Table 5 $s_1^{BL}<s_1^{MC}$; $v_1^{BL}<v_1^{MC}$; and the total cost, Z_{SO}, in the bi-level solution is lower than that in the mutually consistent solution.

Another similar test was carried out on the same grid network as shown in Figure 1 used for the matrix estimation problem. A traffic signal is added at node 9 and the capacities on all links controlled by the signal (links 5, 10, 15, 20) is doubled. The convergence of the green splits and the objective function values with the three initial values of s_5 are shown in Figure 5. It can be seen that the bi-level algorithm is very efficient. In fact, in just a few iterations, the algorithm converges to $[s_5, s_{10}]$ = [0.5506, 0.4494] with Z_{SO}=15058.3954, which is the same as the true bi-level solution found by direct search in the signal split with increment size of 0.0001. Therefore, in this case, the modification to the bi-level algorithm is not necessary.

{**Figure 4** is about here}

{**Table 5** is about here}

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4. SUMMARY

The problem of combined trip matrix estimation and SUE assignment, and that of traffic signal optimisation and SUE assignment have been addressed in this paper. Two types of solutions are identified and compared. An algorithm for the bi-level solution of the two problems has been described. At each iteration, the algorithms use standard routines of matrix estimation (or signal optimisation) and SUE assignment to find a search direction. Then a line search is made by linearising the SUE assignment model, which does not need repeated SUE assignments.

The algorithm was tested on simple two- or three- link networks, a 3×3 grid network with 24 links and 4 O-D pairs, and the Sioux Falls network with 24 nodes, 76 links, and 528 O-D pairs. It was shown to be convergent and efficient in terms of the number of iterations and c.p.u. times. In the two- or three- link network examples in which the true bi-level solution can be found by direct search, it was shown that the bi-level algorithm converges almost exactly to the true bi-level solution. The errors are caused by the linearisation of the SUE map. A modification to the algorithm is proposed and has been shown to be effective.

The algorithm presented here is heuristic in nature. It has not been possible to prove theoretically that the algorithm is convergent. In addition, it is not guaranteed that the algorithm identifies the global optimal even when it does converge. Fletcher (1987) argued that the existence of convergence proof for any algorithm is not a guarantee of good performance in practice; and the development of an algorithm also relies on experimentation. The algorithm presented here has been used to solve the bi-level ME problem with UE assignment or logit-based SUE assignment at the lower level, and the bi-level SO problem, again with UE assignment or logit-based SUE assignment at the lower level. The networks tested so far include simple two- or three-link networks, 3×3 grid networks, the Sioux Falls network and the Headingley network. The last two networks are used in the congested ME problems only. The Headingley network has 73 nodes, 188 links, and 240 O-D pairs. Test on this network can be found in Zhang *et. al.*, 1999. In all the tests so far it has been found that the algorithm is convergent. In those cases where the (global) optimal solution can be found by direct search (two- or three-link networks or the grid network with one traffic signal), it was found that the algorithm is able to identify or to give a good approximation of the optimal solutions. Further tests of the algorithm on more general networks will be carried out.

Because of its simplicity, the logit assignment model has been most widely used. However, it has well-known weaknesses. For example, it does not take account of overlapping or correlated routes. Cascetta *et al*. (1996) have proposed a modified logit model which allows for overlapping routes, but the model requires complete route enumeration. On the other hand, probit-based assignment model does not suffer from these weakness. Recently, Maher and Hughes (1997) have developed a probit-based SUE assignment algorithm which does not require route enumeration. Further work of the current research is to adapt the bi-level algorithm for use with probit-based SUE assignment.

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TABLES

Table 1. Solutions of the matrix estimation problem on the two-link network

	$t_1^{(20)}$	$v_1^{(20)}$	$P_1^{(20)}$	$Z_{ME}^{(20)}$	$Z_{SUE}^{(20)}$
Bi-level algorithm	1937.1100	1170.4515	0.604226	25463.8575	-9022.1208
Modified bi-level algorithm	1937.1157	1170.4548	0.604226	25463.8574	-9022.1506
True bi-level solution	1937.1160	1170.4550	0.604226	25463.8574	-9022.1507
Mutually consistent solution	1941.2442	1172.8129	0.604155	25484.0922	-9043.4699

Table 2. Uncongested link travel costs and link capacities on the grid network.

Link No.	Start node	End node	$c_a(0)$	q_a	Link No.	Start node	End node	$c_a(0)$	q_a
1	1	2	10	80	13	6	5	10	80
2	1	8	10	80	14	6	7	15	80
3	2	1	15	80	15	6	9	15	40
4	2	3	15	80	16	7	6	10	80
5	2	9	15	40	17	7	8	10	80
6	3	2	10	80	18	8	1	15	80
7	3	4	10	80	19	8	7	10	80
8	4	3	10	80	20	8	9	15	40
9	4	5	15	80	21	9	2	15	80
10	4	9	15	40	22	9	4	10	80
11	5	4	15	80	23	9	6	15	80
12	5	6	10	80	24	9	8	10	80

Table 3. Performance of the matrix estimation algorithm on the grid network with ε=0.001.

C_{vlk}	C_{vod}	Bi-level solution algorithm							Mutually consistent solution				
		$t_1^{(20)}$	$t_2^{(20)}$	$t_3^{(20)}$	$t_4^{(20)}$	$Z_{ME}^{(20)}$	c.p.u.	N	$t_1^{(20)}$	$t_2^{(20)}$	$t_3^{(20)}$	$t_4^{(20)}$	$Z_{ME}^{(20)}$
0.05	0.05	72.4248	62.1171	73.3805	59.8124	21.7269	0.22	2	74.86	60.93	73.84	58.30	24.02
0.05	0.10	72.7429	63.0385	73.8583	59.4134	22.0442	0.33	3	78.59	57.65	77.15	54.69	25.94
0.05	0.15	73.5010	62.8052	73.7669	58.7810	23.3130	0.44	5	81.94	54.43	80.31	51.42	26.46
0.10	0.10	72.8381	64.2430	74.7682	59.6426	21.7169	0.33	3	77.69	61.87	75.66	56.59	24.15
0.10	0.20	73.5056	66.0323	75.7283	58.7798	22.1468	0.33	3	85.57	54.88	82.65	48.91	26.41
0.10	0.30	74.3838	66.5703	74.3789	58.8879	23.2844	0.55	5	92.35	48.40	89.03	42.26	26.90
0.15	0.15	73.2404	66.3779	76.1611	59.4935	21.7083	0.33	3	80.48	62.81	77.45	54.85	24.31
0.15	0.30	73.9014	70.1670	76.4926	59.6675	21.9938	0.33	3	93.16	51.46	88.73	42.35	26.98
0.15	0.45	75.6174	69.4799	75.3692	58.2051	23.4984	0.66	6	102.3	42.71	97.32	33.35	27.03

Table 4. Performance of the matrix estimation algorithm on the Sioux Falls network with ε=0.001.

C_{vlk}	C_{vod}	Bi-level solution algorithm			Mutually consistent solution
		$Z_{ME}^{(20)}$	c.p.u.	N	$Z_{ME}^{(20)}$
0.05	0.05	39.4834	57.58	2	39.7423
0.05	0.10	41.8604	84.34	3	42.9385
0.05	0.15	44.3356	85.16	3	46.4643
0.10	0.10	39.4407	67.09	2	39.6997
0.10	0.20	41.7600	84.56	3	42.8199
0.10	0.30	44.1281	86.54	3	46.2565
0.15	0.15	39.4084	88.46	3	39.6667
0.15	0.30	41.6537	86.43	3	42.7068
0.15	0.45	43.9314	89.01	3	46.1082

Table 5. Solutions of the signal optimisation problem on the three-link network.

	s_1	s_3	v_1	v_2	Z_{SO}	Z_{SUE}
Bi-level algorithm	0.3104	0.6896	44.1310	55.8690	416.8505	117.6111
Modified Bi-level algorithm	0.3074	0.6926	43.8272	56.1728	416.8194	117.1417
True bi-level solution	0.3070	0.6930	43.7952	56.2048	416.8189	117.0786
Mutually consistent solution	0.3412	0.6588	46.9890	53.0110	420.9068	121.0235

FIGURES

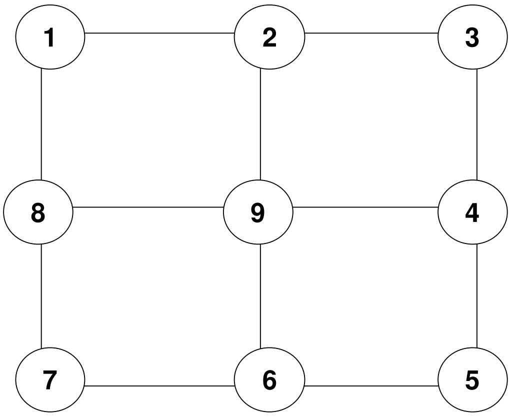


Figure 1. The grid network. All links are two-directional.

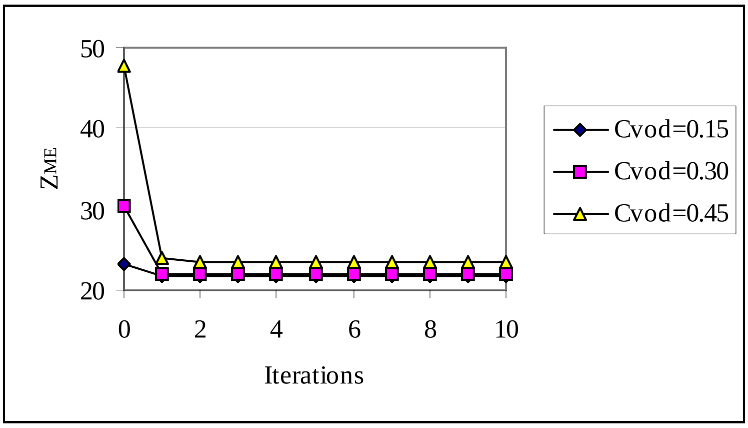


Figure 2. Convergence of the matrix estimation algorithm on the grid network, with C_{vlk}= 0.15.

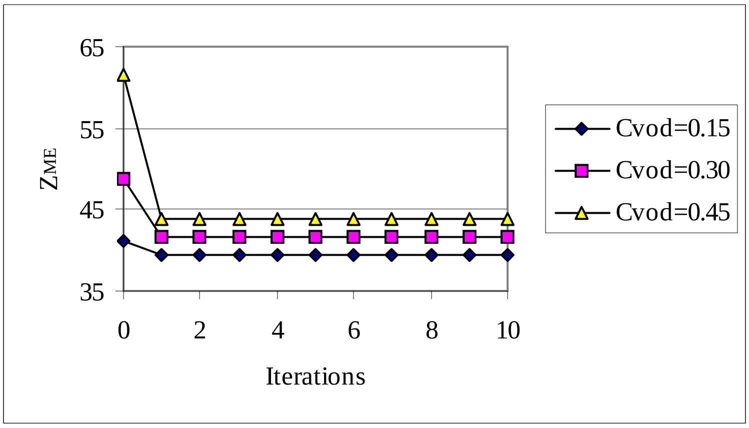


Figure 3. Convergence of the matrix estimation algorithm on the Sioux Falls network, with $C_{v\text{lk}}$= 0.15.

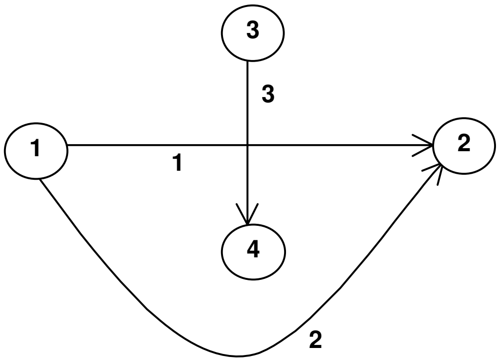
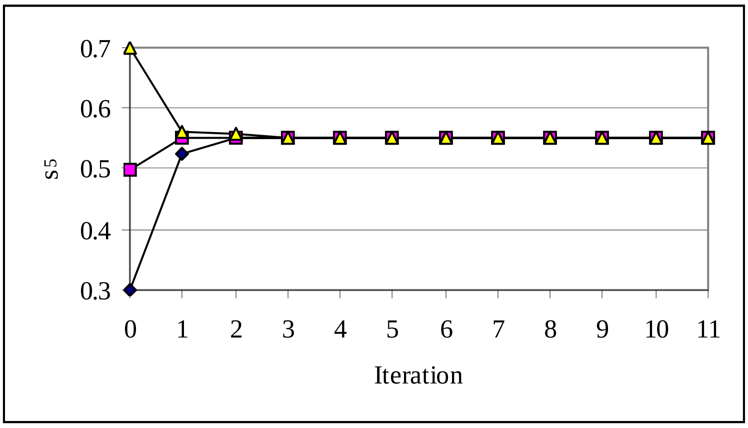
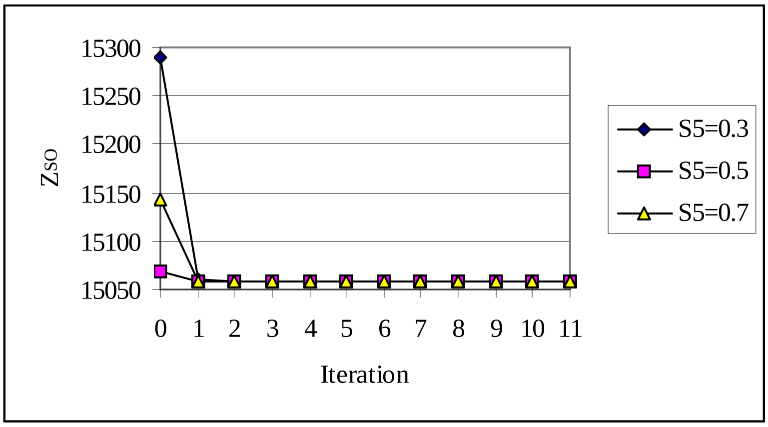


Figure 4. The three-link network.



(a)



(b)

Figure 5. Convergence of the algorithm for signal optimisation on the grid network from different initial conditions. (a) The green splits, (b) Values of signal optimisation objective function.