

Fracture-Mechanics-Based Analytical and Finite Element Evaluation of the Virtual Crack Closure Integral for Filled-Crack VCCI

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Abstract

This paper presents a fracture-mechanics-based analytical framework for evaluating the virtual crack closure integral (VCCI) of structures containing filled cracks. Building on the formulation of Fowkes et al., closed-form expressions for the stress and displacement fields in the vicinity of a filled crack are derived by explicitly accounting for the presence of a wedge-shaped filling material. Using these fields, the virtual crack closure integral is analytically evaluated for the filled-crack problem. Finite element simulations are also conducted using ANSYS to evaluate the J-integral at the crack tip, which is equivalent to the VCCI. Comparisons between the analytical and numerical results for different filling material properties and crack depths show excellent agreement, validating the proposed formulation. The influence of the wedge angle on VCCI is examined. The applicability of the stress intensity factor to filled cracks is also discussed, and its limitations and relationship to energy-based fracture parameters are clarified. In addition, the effects of Poisson's ratio of both the filling material and the surrounding matrix are investigated. The proposed model provides an efficient tool for evaluating fracture parameters in filled-crack systems and can be readily incorporated into simplified static and dynamic structural analyses, thereby supporting the development of more reliable and resilient structures and contributing to infrastructure sustainability.

Keywords: Fracture Mechanics; Cracked beam; Four-point bending; J-integral; Stress intensity factor.

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1. Introduction

The fracture strengths of brittle materials such as glass are typically only a small fraction of their theoretical strength. This discrepancy arises primarily from the lack of any significant possibility of plastic deformation and the consequent high sensitivity of such materials to even extremely small (surface) flaws. Numerous techniques have been investigated to enhance the strength of glass, focusing on the elimination, reduction, healing, or mitigation of surface defects. Among these approaches, crack repair using filling materials has attracted considerable attention as an effective method for restoring structural integrity and extending service life. Cracks generate highly concentrated stresses near the crack tip, governing crack initiation and propagation in brittle materials. Early analytical descriptions of crack-tip stress fields in linear elastic materials were developed by Westergaard [1], Sneddon [2], Irwin [3], and Williams [4]. Among these, Irwin's formulation, based on Westergaard's stress function approach, remains one of the most widely adopted frameworks for characterizing the near-tip stress field. Irwin considered a straight crack of length $2a$ embedded in an infinite elastic plate subjected to remote loading and expressed the stresses in polar coordinates (r, θ) , with the crack tip taken as the origin.

For linear elastic materials, the asymptotic stress field near the crack tip, see Figure 1, is dominated by a singular term proportional to $r^{-1/2}$. In the case of mode I (opening mode) loading, the stress components near the crack tip may be expressed as

$$\sigma_{ij}(r, \theta) = \frac{K_I}{\sqrt{2\pi r}} f_{ij}(\theta) + \mathcal{O}(\sqrt{r}), \quad (1)$$

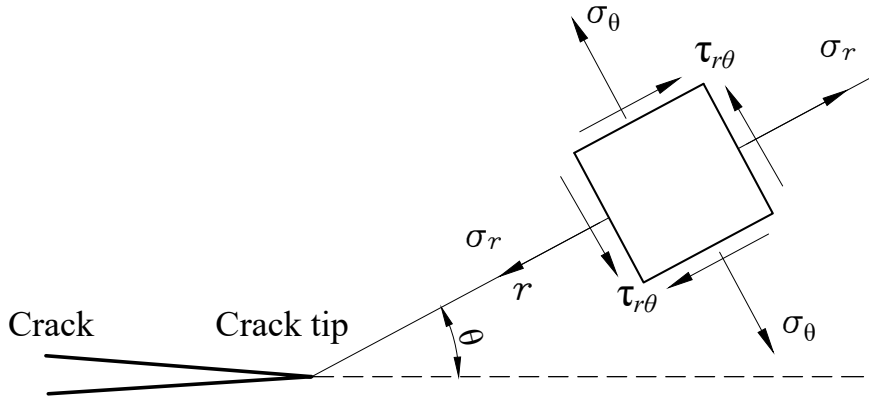


Figure 1: Stress field near crack tip in polar coordinates

where K_I denotes the mode I stress intensity factor, $f_{ij}(\theta)$ are known angular functions, and $\mathcal{O}(\sqrt{r})$ represents higher-order terms that become negligible as $r \rightarrow 0$. This expression is valid for both plane stress and plane strain conditions, with material properties influencing the associated displacement fields.

A key feature of this formulation is that the stress intensity factor K_I is coordinate-independent, depending only on the applied loading, crack geometry, and boundary conditions, while the polar coordinates r and θ describe the spatial variation of stresses around the crack tip [5]. The solution is asymptotic in nature and accurately represents the stress field only in the near-tip region; stresses far from the crack tip are governed by the remote boundary conditions of the structure.

Crack-tip deformation and the associated stress fields are commonly classified into three fundamental modes: opening (mode I), sliding (mode II), and tearing (mode III). Each mode is characterized by a corresponding stress intensity factor, denoted by K_I , K_{II} , and K_{III} , respectively. The superposition of these modes is sufficient to describe general mixed-mode crack-tip behavior.

The stress intensity factor plays a central role in fracture mechanics, as it uniquely characterizes the intensity of the singular stress field at the crack tip, with crack growth occurring when the stress intensity factor becomes equal to the material fracture toughness. For many practical configurations, stress intensity factors are expressed in the form $K = Y\sigma\sqrt{\pi a}$, where σ is the nominal applied stress, a is a characteristic crack dimension, and Y is a dimensionless geometry correction factor accounting for crack shape, size, and loading configuration. Closed-form solutions for Y are available for simple geometries, while numerical methods, most notably the finite element method, are commonly employed for more complex configurations. Comprehensive collections of stress intensity factor solutions for a wide range of geometries and loading conditions are available [6].

Irwin [3] first proposed the crack closure integral concept in the 1950s, using an energy conservation argument to describe crack extension. Zeng, et al [7] proposed an efficient fracture analysis framework by integrating the Virtual Crack Closure Technique (VCCT) into the Cell-Based Smoothed Finite Element Method (CS-FEM). This approach utilizes the strain smoothing technique over sub-divided smoothing cells, allowing for the calculation of stress intensity factors using only nodal forces and displacement openings without the need for domain integration. The method demonstrates significant insensitivity to mesh distortion and provides high-accuracy predictions for crack growth trajectories in mixed-mode fracture problems.

Repairing surface cracks is important in many engineering applications. A common example is a small stone impacting a car windshield, which can initiate a crack. If such a crack is left unrepaired, it may propagate rapidly, eventually necessitating complete replacement of the windshield. Timely crack repair can arrest crack propagation and significantly extend structural service life.

Crack repair typically involves coating the glass surface thereby filling the crack with the coating material, which can be broadly classified into two main categories: soft or hard, with the former tending to have a low Young's modulus compared to the latter. This topic has been extensively investigated experimentally in the literature. Representative studies on hard coatings include [8], while examples of studies on soft coatings can be found in [9].

Elsayed and Hand[10] investigated experimentally the filling of glass surface cracks via application of an epoxy coating to a glass surface. They concluded that under proper filling procedure the epoxy coating can completely repair cracked samples and increase its strength to levels of uncracked glass. Guner and Ozturk [11] experimentally analyzed the influence of curing time (1–28 days) on the tensile properties of two thin spray-on liners (TSLs). They found that increasing curing time improves tensile strength and Young's modulus while reducing elongation. Their study also demonstrated that axial extensometer measurements provide more accurate strain and modulus estimations than grip-based LVDT measurements. Denisyuk [12] presented an analytical wedge model containing a filling material with different mechanical properties. This work was extended by Fowkes, et al [13] who investigated the effect of filling the crack with elastic material. They assumed that the crack has a wedge shape with a very small wedge angle and produced a theoretical solution for this case. In addition, their model treats the cracked material and the filling material as two adjoining wedges. The large wedge represents the matrix material and the small wedge represents the filling material with $\omega_1 \rightarrow 2\pi$ (matrix) and $\omega_2 \rightarrow 0$ (filling material). Their model showed that the filling material modifies the crack-tip stress singularity. Consequently, the singularity changes from the classical order $1/\sqrt{r}$ to $1/r^\lambda$, where r is the distance from the crack tip and λ is obtained from the associated eigenvalue equation. They used the same Airy function for both materials

$$\phi_i = r^{\lambda+1} [A_{1i}c^+ + A_{2i}c^- + A_{3i}s^+ + A_{4i}s^-], \quad i = 1, 2 \quad (2)$$

where $A_{1i}, \dots$ were coefficients to be determined for both the cracked material ($i = 1$) and for the filling material ($i = 2$).

Teixeira de Freitas and Tiago[14] derived singular and regular solutions for crack plate repair problems with Mindlin–Reissner plate bending using harmonic, biharmonic and regularized Helmholtz displacement potentials, enabling the analysis of wedge and crack problems. They provided eigenproblems for homogeneous wedge boundary conditions and extended the model to biomaterial wedges and filled cracks, showing that weak elastic fillers can reduce stress singularities and stabilize crack growth. They also defined stress intensity factors for filled cracks and showed that they recovered classical results for open cracks in plate bending.

Existing studies mainly focus on bimaterial wedge formulations for crack repair problems, where the crack is represented by a wedge with a very small opening angle. However, the virtual crack closure integral (VCCI) and the interpretation of the stress intensity factor for filled cracks remain insufficiently investigated. This limitation is particularly important in applications involving beam vibration and buckling analyses [15–17], as well as fault diagnosis in structures containing filled cracks.

The present work introduces a physically consistent representation of filled cracks suitable for static and dynamic structural analyses. The proposed formulation is supported by both numerical validation using [18] and experimental validation through comparison with previously published work.

The remainder of the paper is structured as follows: Section 2 details the mathematical model for the filled crack, while Section 3 describes the development of the finite element model. Section 4 presents a comprehensive discussion of both the analytical and finite element results. Finally, the conclusion summarizes the most significant findings and provides concluding remarks.

2. Mathematical models

The mathematical model presented in this section outlines the procedure for evaluating the stress and displacement fields in the vicinity of the filled wedge tip, as illustrated in Figure 2. Subsequently, the model addresses the calculation of the VCCI and the determination of stress intensity factors (SIF) under Mode I loading condition. In the following both geometric and material parameters are indexed by the subscript ($i = 1, 2$), where $i = 1$ corresponds to the matrix and $i = 2$ to the filled crack (wedge); the elastic properties of each phase are characterized by Young’s modulus E_i , shear modulus G_i , and Poisson’s ratio ν_i , related through $E_i = 2G_i(1 + \nu_i)$. As illustrated in Figure 2, the geometry consists of a filled wedge occupying an angular domain ω_2 , while the surrounding matrix spans the complementary angle $\omega_1 = 2\pi - \omega_2$.

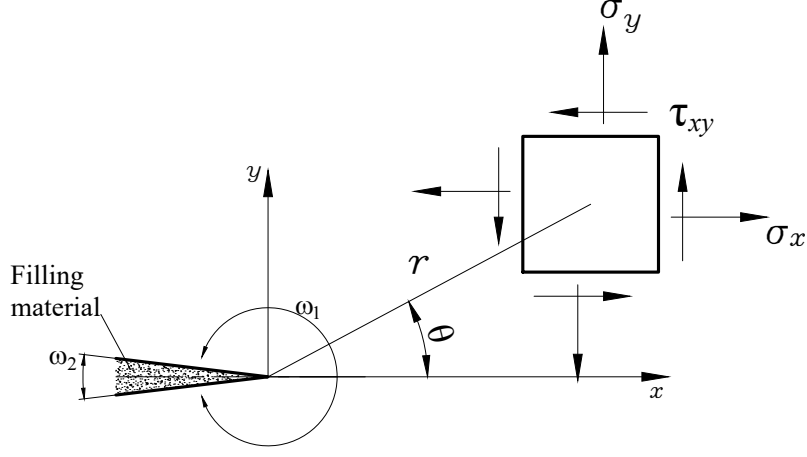


Figure 2: Wedge filled model spans an angle ω_2 adjoining with surrounding matrix spans ω_1

The near-tip stresses under bending stress are expressed using the classical Westergaard formulation [1], in terms of the complex stress function $Z(z)$ and its derivative $Z'(z)$ [6]:

$$\begin{aligned}\sigma_x &= \Re(Z) - y \Im(Z'), \\ \sigma_y &= \Re(Z) + y \Im(Z'), \\ \tau_{xy} &= -y \Re(Z').\end{aligned}\tag{3}$$

Under plane strain conditions, the corresponding in-plane displacements are given by

$$\begin{aligned}2G_1 u_x &= (1 - 2\nu_1) \Re(ZI) - y \Im(ZI), \\ 2G_1 u_y &= 2(1 - \nu_1) \Im(ZI) - y \Re(ZI),\end{aligned}\tag{4}$$

For a filled crack, the Westergaard function near the crack tip can be approximated as [13]

$$Z_{\text{tip}} \approx \frac{\sigma a^{(1-\lambda_I)}}{2^{(1-\lambda_I)} (r')^{(1-\lambda_I)}} e^{-i(1-\lambda_I)\theta'}\tag{5}$$

where σ is the far-field stress, a is the half-crack length, r' and θ' are the local polar coordinates at the crack tip, and λ_I is the crack-tip singularity exponent which is influenced by material mismatch.

To evaluate the crack opening mode, the radial, hoop, and shear stresses, as well as the radial and hoop displacements, are determined from the stress field presented in Eq. (2). Applying the boundary conditions at the interface between the filling and host materials—specifically, enforcing the continuity of hoop stress, shear stress, radial displacement, and angular displacement—leads to

an eigenvalue problem. The eigenvalue governing the crack opening mode is evaluated by setting the determinant of this matrix equation to zero, which [13] simplified to the form shown in Eq. (6). Solving this equation determines the crack tip singularity.

$$\begin{aligned}\Delta_{\text{sym}}(\lambda, \dots) = & (1 - g_r)\lambda \sin \omega_1 [(1 - g_r)\lambda \sin \omega_1 + (1 + g_r n_1) \sin(\lambda \omega_1) \\ & + (n_2 + g_r) \sin(\lambda(2\pi - \omega_1))] \\ & + (1 - g_r)(g_r n_1 - n_2) \sin^2(\lambda(\omega_1 - \pi)) \\ & + (1 + g_r n_1)(n_2 + g_r) \sin^2(\lambda\pi) = 0\end{aligned}\quad (6)$$

where $n_i = \frac{3-\nu_i}{1+\nu_i}$ for plane stress, and $n_i = 3 - 4\nu_i$ for plane strain and $g_r = G_2/G_1$. Fowkes, et al [13] presented a simplified version of Eq. (6) in the case of small ω_2 when the wedge becomes more crack-like:

$$\lambda \cot(\lambda\pi) = -\gamma_r \frac{(n_2 + 1)(n_1 + 1)}{4\pi(n_2 - 1)}, \quad (7)$$

where $\gamma_r = \omega_2 g_r$. Throughout the current manuscript the value of λ obtained from Eq. (6) will be denoted by λ_{wed} and a wedge angle defined from this value. The value λ obtained from Eq. (7) will be denoted by $\lambda_{\text{s-wed}}$.

Substituting Eq. (5) in Eqs. (3) and (4) the stress components can be written in compact vector form as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{\sigma a^{(1-\lambda_I)}}{2^{(2-\lambda)}(r')^{(1-\lambda)}} \begin{Bmatrix} (1 + \lambda_I) \cos((1 - \lambda_I)\theta') + (1 - \lambda_I) \cos((3 - \lambda_I)\theta') \\ (3 - \lambda_I) \cos((1 - \lambda_I)\theta') - (1 - \lambda_I) \cos((3 - \lambda_I)\theta') \\ (1 - \lambda_I) [\sin((3 - \lambda_I)\theta') - \sin((1 - \lambda_I)\theta')] \end{Bmatrix}. \quad (8)$$

The corresponding displacement vector is:

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \frac{1}{2G} \frac{\sigma a^{(1-\lambda_I)}}{2^{(1-\lambda_I)}} r^\lambda \begin{Bmatrix} \frac{1-2\nu}{\lambda_I} \cos(\lambda_I\theta) + \sin\theta \sin((1 - \lambda_I)\theta) \\ \frac{2(1-\nu)}{\lambda_I} \sin(\lambda_I\theta) - \sin\theta \cos((1 - \lambda_I)\theta) \end{Bmatrix}. \quad (9)$$

The VCCI associated with filled crack in Mode I is denoted by G_I and can be evaluated using:

$$G_I = \lim_{\Delta a \rightarrow 0} \frac{1}{2\Delta a} \int_0^{\Delta a} \sigma_y(r, 0) u_y(\Delta a - r, \pi) dr \quad (10)$$

where $\sigma_y(r, 0)$ is the normal stress ahead of the crack tip (at $\theta = 0$), Δa is the virtual crack extension length and $u_y(\Delta a - r, \pi)$ is the crack opening displacement on the crack face behind the new tip (at $\theta = \pi$).

The generalized Mode I stress field is expressed as

$$\sigma_y(r, 0) = \frac{K_{I\lambda}}{(2\pi r)^{1-\lambda_I}}. \quad (11)$$

where $K_{I\lambda} = \sigma(\pi a)^{1-\lambda_I}$.

From Eq. (9), the crack opening displacement on the crack face is

$$u_y(r, \pi) = \frac{K_{I\lambda}}{G_1(2\pi)^{1-\lambda}} \frac{(1-\nu_1)}{\lambda} \sin(\lambda\pi) r^{\lambda_I}. \quad (12)$$

Substituting into Eq. (10) yields

$$G_I = \frac{K_{I\lambda}^2(1-\nu_1)\sin(\lambda_I\pi)}{2G_1\lambda_I(2\pi)^{2-2\lambda_I}\Delta a} \int_0^{\Delta a} r^{\lambda_I-1}(\Delta a - r)^{\lambda_I} dr. \quad (13)$$

Introducing the change of variables $r = \Delta a t$, where t is a dimensionless parameter varying between 0 and 1, the integral evaluates to a beta function, resulting in:

$$G_I(\Delta a) = \left(\frac{1-\nu_1}{2G_1} \right) K_{I\lambda}^2(\Delta a)^{2\lambda_I-1} \Psi_I(\lambda_I) \quad (14)$$

with

$$\Psi_I(\lambda) = \frac{\sin(\lambda\pi) \Gamma(\lambda_I) \Gamma(1+\lambda_I)}{\lambda_I (2\pi)^{2-2\lambda} \Gamma(1+2\lambda_I)}. \quad (15)$$

where $\Gamma(x)$ is the Gamma function. For the case of four-point bending, the crack filled Mode I stress intensity factor, $K_{I\lambda}$, is given by

$$K_{I\lambda} = Y\sigma(\pi a)^{1-\lambda_I} \quad (16)$$

where

$$Y(a/h) = 1.122 - 1.40(a/h) + 7.33(a/h)^2 - 13.08(a/h)^3 + 14.0(a/h)^4 \quad (17)$$

and h is the beam thickness as shown in Figure 3 [19].

3. Finite Element Model

The finite element model was designed to verify the results obtained from the preceding analysis and is based on a four-point bending cracked model as shown in Figure 3. The model geometrical parameters are listed in Table 1. The crack depths considered in the model relate to our previous studies on filled cracks in glasses [10]. The filled crack is on the top surface. The model is symmetric; hence a half model can be used as shown in Figure 4.

Table 1: Cracked model dimensions

Parameter	Value
Beam thickness, h	1.04 mm
Beam width, b	25 mm
Load span length, $2c$	20 mm
Distance from load to support, ℓ	7 mm
Crack depth, a	0.202, 0.145, 0.096, and 0.063 mm

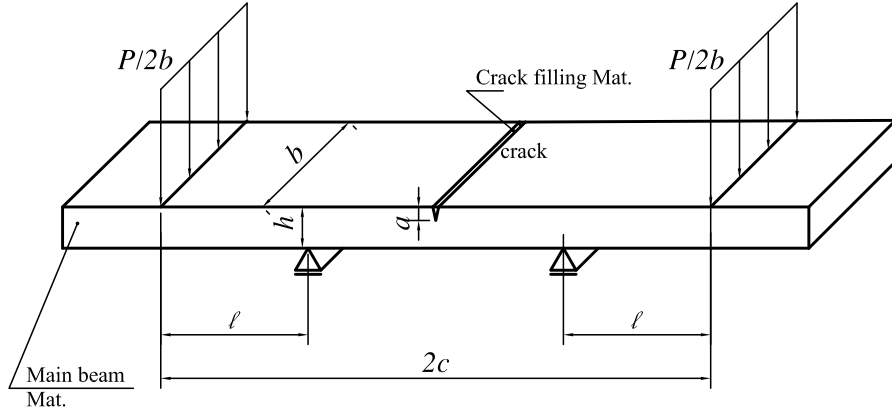


Figure 3: Four-point bending geometry with a filled crack on the centreline of the tensile surface

ANSYS [18] indicates that the crack faces should be coincident, the elements around the crack tip should be quadratic, and the mid-side node should be placed at the quarter position from the crack side. These modified elements are called singular elements as shown in Figure 4(c), which represent the singular elements for the two-dimensional element and three-dimensional models. This modification in the mid-side node of the singular element was the result of many studies aimed at reducing the importance of the mesh size near the crack tip [20]. The same result for stress intensity

values could be achieved without using singular elements but this could require smaller element sizes around the crack tip.

A spider-web mesh configuration was adopted around the crack tip, with the first circumferential ring consisting of singular quarter-point elements to accurately capture the stress singularity [18]. In the present two-dimensional model developed in ANSYS APDL, singular elements were generated at the crack-tip key point. The radius of the first ring of elements was selected as approximately $a/50$, where a denotes the crack depth [18]. In the circumferential direction, 36 elements were distributed around the crack tip (i.e., one element every 5°), which was found to provide stable and mesh-independent results.

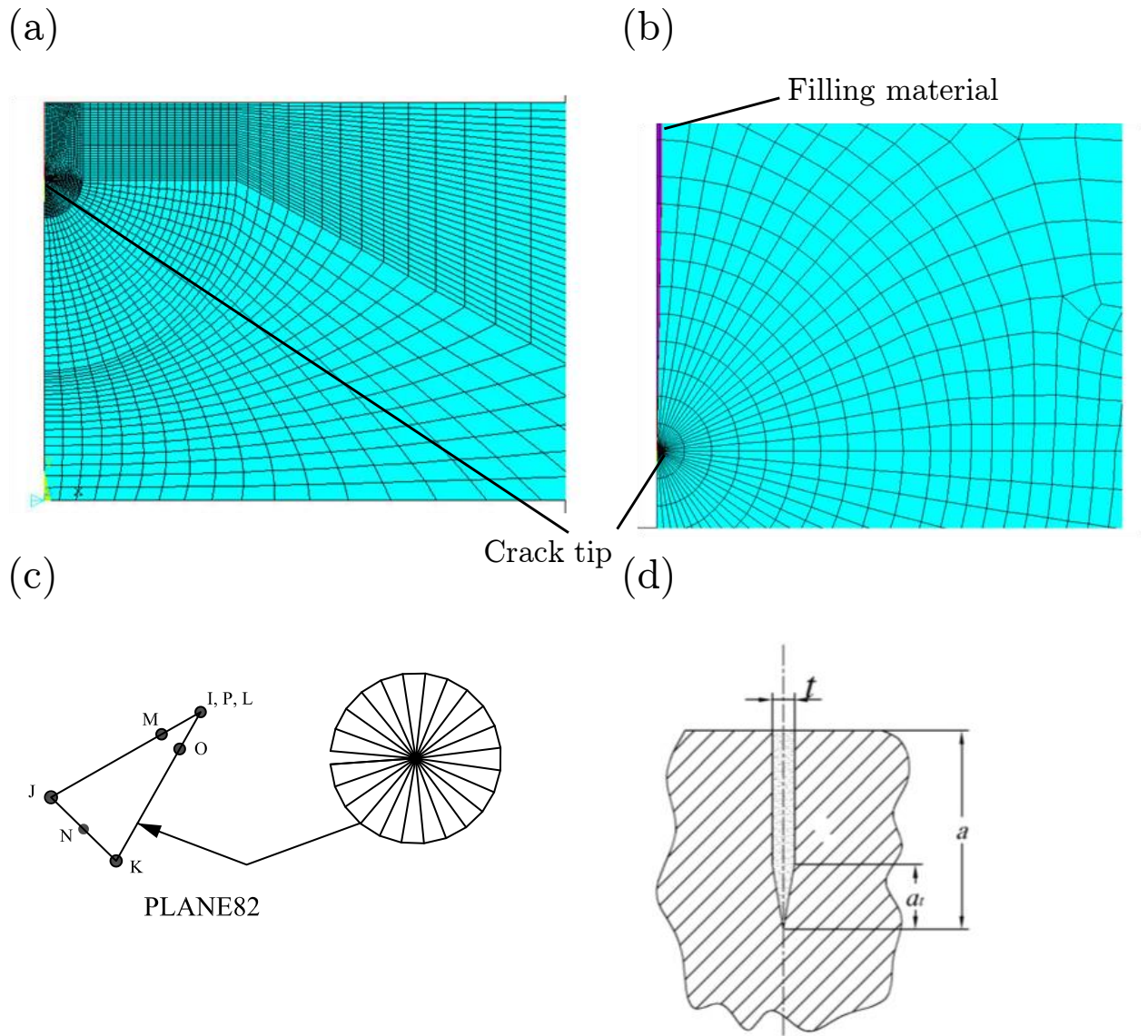


Figure 4: FE mesh for model: (a) 2-D overall mesh, (b) 2-D enlargement of the near crack, (c) crack tip singular element, (d) crack geometric profile in the two-dimensional finite element model

It was assumed that the crack was equivalent to a through-thickness edge crack and thus half a lengthwise cross section could be modelled with the symmetric boundary passing through the crack as shown in Figure 4(a). The adopted coating models are perfect interface models and in these models the coating was assumed to be perfectly glued to the internal faces of the crack, i.e., the glass and coating elements shared the same nodes at the interface. The material filling the crack was modelled as a linear elastic medium.

In all crack filled models, approximately 1% of the filling material was removed in the vicinity of the crack tip to retain the singularity field at the crack tip. The element types used for these models are ANSYS PLANE82 and PLANE183 elements, both of which have eight nodes. They have compatible displacement shapes and they are suitable for curved boundaries. The first row of elements around the crack tip should be singular, as shown in Figure 4(c). Undistorted isosceles triangular elements were used at the crack tip as shown in Figure 4(c). Nearly 15,000 elements were used in forming the overall mesh.

The J -integral was evaluated using the domain integral method within ANSYS APDL, where a refined mesh consisting of quarter-point singular elements was implemented at the crack tip to capture the stress-strain singularity. Six concentric integration contours are automatically defined surrounding the crack front to calculate the energy release rate. The use of ANSYS APDL allowed for automated parametric studies via loops to efficiently investigate the influence of various model parameters.

4. Results and Discussion

This section begins with the verification of the present analytical formulation through comparison with previously published results for the evaluation of the crack-tip singularity exponent. Following the validation stage, a detailed investigation is conducted to examine the influence of the crack-tip singularity exponent on the stress field distribution in the vicinity of the filled crack tip. Particular attention is given to the evolution of the stress field and the modification of the crack-tip stress field resulting from the presence of filling materials with different stiffness characteristics.

Subsequently, the effect of the virtual crack extension length on the evaluation of the Virtual Crack Closure Integral (VCCI) is investigated in order to establish the appropriate numerical conditions for accurate energy release rate estimation. The VCCI values obtained from the proposed analytical formulation are then evaluated and compared with the corresponding results obtained from ANSYS

finite element (FE) simulations to further validate the accuracy and applicability of the developed analytical model.

The analysis is conducted for four different crack depths, namely 0.202, 0.145, 0.096, and 0.063 mm which correspond to crack ratios of $a/h = 0.194, 0.139, 0.092, 0.059$. These values were selected in accordance with the experimental and theoretical investigations reported in references [10, 19], see Appendix A. In experimental studies, controlled defects were intentionally introduced into the glass beam specimens using Vickers indentation, thereby generating well-defined surface cracks. Four different indentation loads (10, 5, 2.5, and 1 kg) were applied to obtain cracks of varying depths giving the a/h values used here. As well as analysing different crack depths, the influence of the wedge angle is examined, followed by an investigation of the effects of both the filling material and the surrounding matrix Poisson's ratio. All results were obtained for a cracked glass beam, with an elastic modulus of 74 GPa, subjected to four-point bending.

4.1. Model Verification by Comparison with Fowkes, et al [13] in Evaluation of λ_I

In this section, the crack-tip singularity exponent, λ_I , is evaluated using both Eq. (6) and Eq. (7), and the obtained results are compared with those reported by [13], as illustrated in Figure 5. Two material combinations are considered in the analysis. For both cases, the Poisson's ratio of material 1 is fixed at $\nu_1 = 0.25$. In the first case, represented by the blue circular markers, the Poisson's ratio of material 2 is taken as $\nu_2 = 0.3$, whereas in the second case, represented by the red circular markers, $\nu_2 = 0.4$. The scattered data shown in Figure 5 were extracted from Figure 2 reported by [13] for validation purposes.

The horizontal axis in Figure 5 represents the scaled shear modulus ratio, defined as $\gamma_r = \varepsilon g_r$, where $\omega_2 = 2\pi\varepsilon$. The vertical axis corresponds to the crack-tip singularity exponent λ . The comparison demonstrates that the predictions obtained from Eq. (7) are in excellent agreement with the results reported by [13] over the investigated range.

In addition, Eq. (6) is employed to evaluate the eigenvalue for several wedge angles, namely $\omega_2 = 0.2, 0.4, 1, 2, 3$, and 4. An important observation is that the calculated values of λ also exhibit very good agreement for all investigated wedge angles. Furthermore, the results indicate that when λ is plotted against the scaled shear modulus ratio, the curves corresponding to different wedge angles become nearly coincident within the considered range of small wedge angles. This behavior suggests that both equations provide consistent and reliable predictions and can therefore be effectively used

to characterize the crack-tip singularity response in the small wedge-angle regime throughout the remainder of the manuscript.

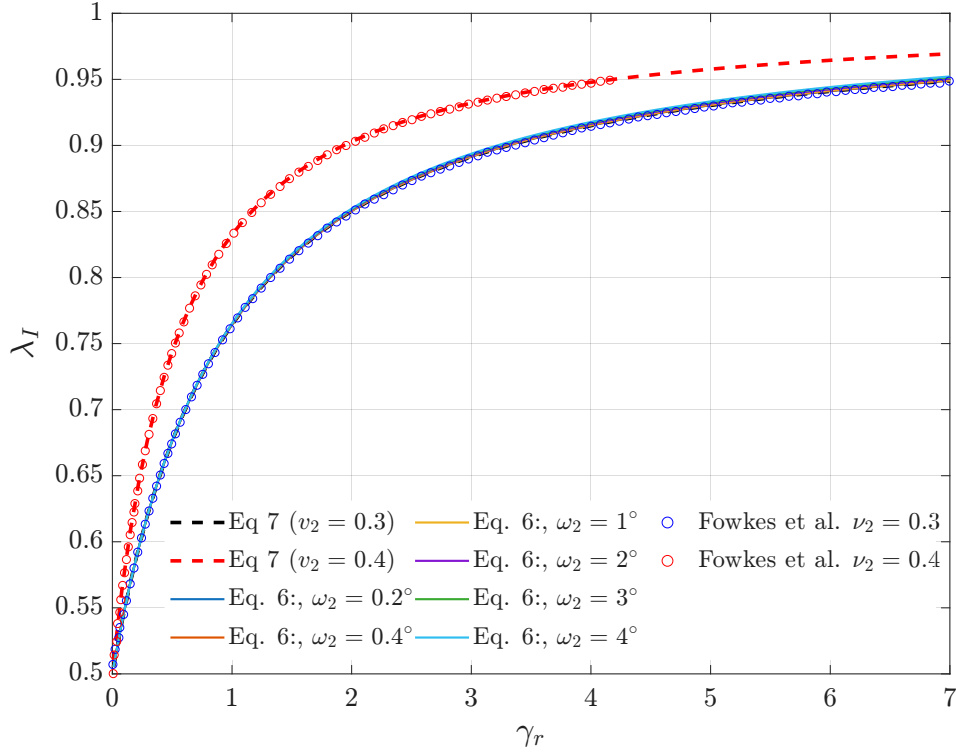


Figure 5: Variation of the crack-tip singularity exponent with the normalized shear modulus ratio, γ_r . The upper curve corresponds to the material combination $\nu_1 = 0.25$ and $\nu_2 = 0.4$, while the lower curve represents the case of $\nu_1 = 0.25$ and $\nu_2 = 0.3$. The circles denote reference data extracted from Fowkes et al. for validation purposes. The results associated with different wedge angles are presented only for the lower-case configuration corresponding to $\nu_2 = 0.4$ as the different wedge angles lead to identical results.

4.2. Effect of Changing the Crack Tip Singularity on the Stress Field

In this section, the effect of crack tip singularity on the stress field in the vicinity of the crack tip is closely investigated. The investigation of the influence of the crack-tip singularity exponent on the stress field of a filled crack, using Eq. (8), is essential for understanding the role of the filling material in reducing stress concentration near the crack tip. In the present study, six representative values of the crack-tip singularity exponent are considered, namely $\lambda = 0.5, 0.6, 0.7, 0.8, 0.9$, and 1.0 . These values cover the transition from the classical unfilled crack condition to a nearly healed crack state. The case of $\lambda = 0.5$ corresponds to the classical unfilled crack, which exhibits the well-known square-root stress singularity. As the value of λ increases, the effect of the filling material becomes more pronounced, indicating a progressive reduction in the crack-tip stress singularity due to the increased stiffness of the filling material. The limiting case of $\lambda = 1$ corresponds to a fully healed or non-singular crack-tip condition in which the stress concentration is nearly eliminated.

The stress field components σ_x , σ_y , and τ_{xy} in the vicinity of a crack with half-length $a = 0.202$ mm are evaluated under a normalized remote applied stress of $\sigma_\infty = 1$, as illustrated in Figure 6 and Figure 7. Figure 6 presents the stress distributions for $\lambda = 0.5, 0.6$, and 0.7 , whereas Figure 7 shows the corresponding results for $\lambda = 0.8, 0.9$, and 1.0 . To ensure a meaningful comparison between all investigated cases, the contour maps are plotted using identical spatial dimensions and unified stress scales. This representation enables a direct assessment of the influence of the singularity exponent on both the magnitude and spatial variation of the stress field surrounding the crack tip.

The results presented in Figure 6 demonstrate that the highest stress concentration occurs for the classical crack case corresponding to $\lambda = 0.5$. In this case, the stress contours are highly localized near the crack tip, reflecting the strongly singular nature of the stress field. As the singularity exponent increases from $\lambda = 0.5$ to 0.6 and 0.7 , a noticeable reduction in the stress components σ_x , σ_y , and τ_{xy} in the vicinity of the crack tip is observed. The stress contours become progressively smoother and less localized, indicating that the filling material effectively redistributes the stresses and alleviates the severity of the crack-tip singularity.

The results shown in Figure 7 further confirm this trend. For $\lambda = 0.8$ and 0.9 , the values of the stress components σ_x , σ_y , and τ_{xy} in the vicinity of the crack tip are substantially reduced compared with the lower values of λ . The stress field becomes more diffuse and spatially distributed, indicating a significant weakening of the singular behavior. In the limiting case of $\lambda = 1.0$, the stress field exhibits an almost non-singular distribution, with the stress contours appearing smooth and nearly uniform around the crack-tip region and approaching the applied far-field stress. This behavior provides strong evidence of crack healing or crack-bridging action induced by the high-stiffness filling material.

Overall, the obtained results demonstrate that increasing the crack-tip singularity exponent, through the incorporation of a stiffer filling material, effectively suppresses the crack-tip stress concentration and transforms the crack response from a strongly singular state toward a nearly non-singular condition. This behavior highlights the important role of filling materials in improving the structural integrity and fracture resistance of cracked components.

4.3. Effect of the Virtual Crack Tip Increment on the Evaluation of VCCI

In this section, the Virtual Crack Closure Integral (VCCI) formulated in Eq. (14) is evaluated to establish its connection to classical fracture mechanics. The results indicate that the calculated

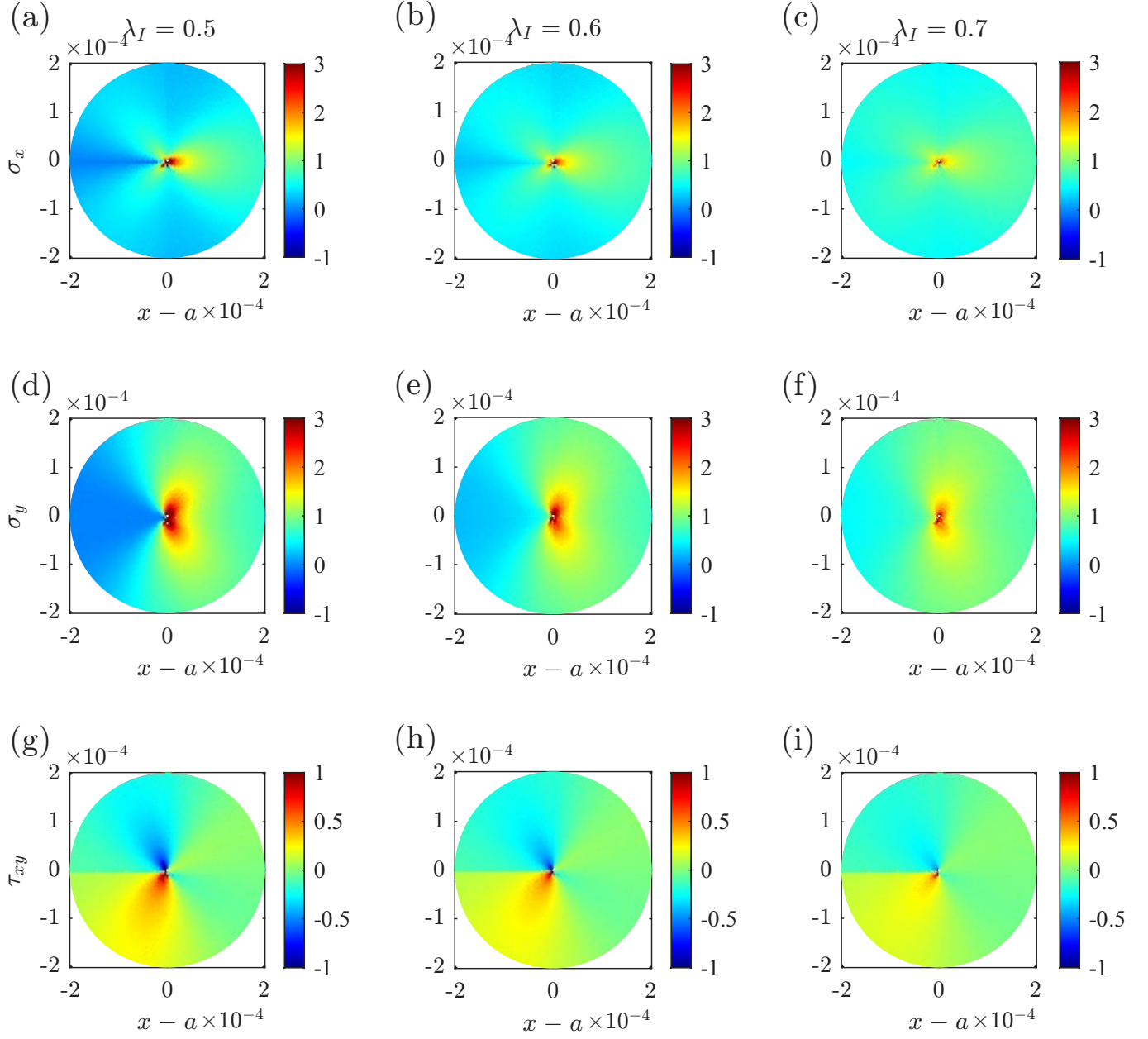


Figure 6: Contour maps of the normalized stress field around a crack tip with crack half-length $a = 0.202$ mm subjected to a remote applied stress $\sigma_\infty = 1$. The stress distributions are presented for three values of the singularity exponent, $\lambda = 0.5, 0.6$, and 0.7 , corresponding to Columns 1, 2, and 3, respectively. The first row illustrates the distribution of the normal stress component σ_x , the second row shows the normal stress component σ_y , and the third row presents the shear stress component τ_{xy} .

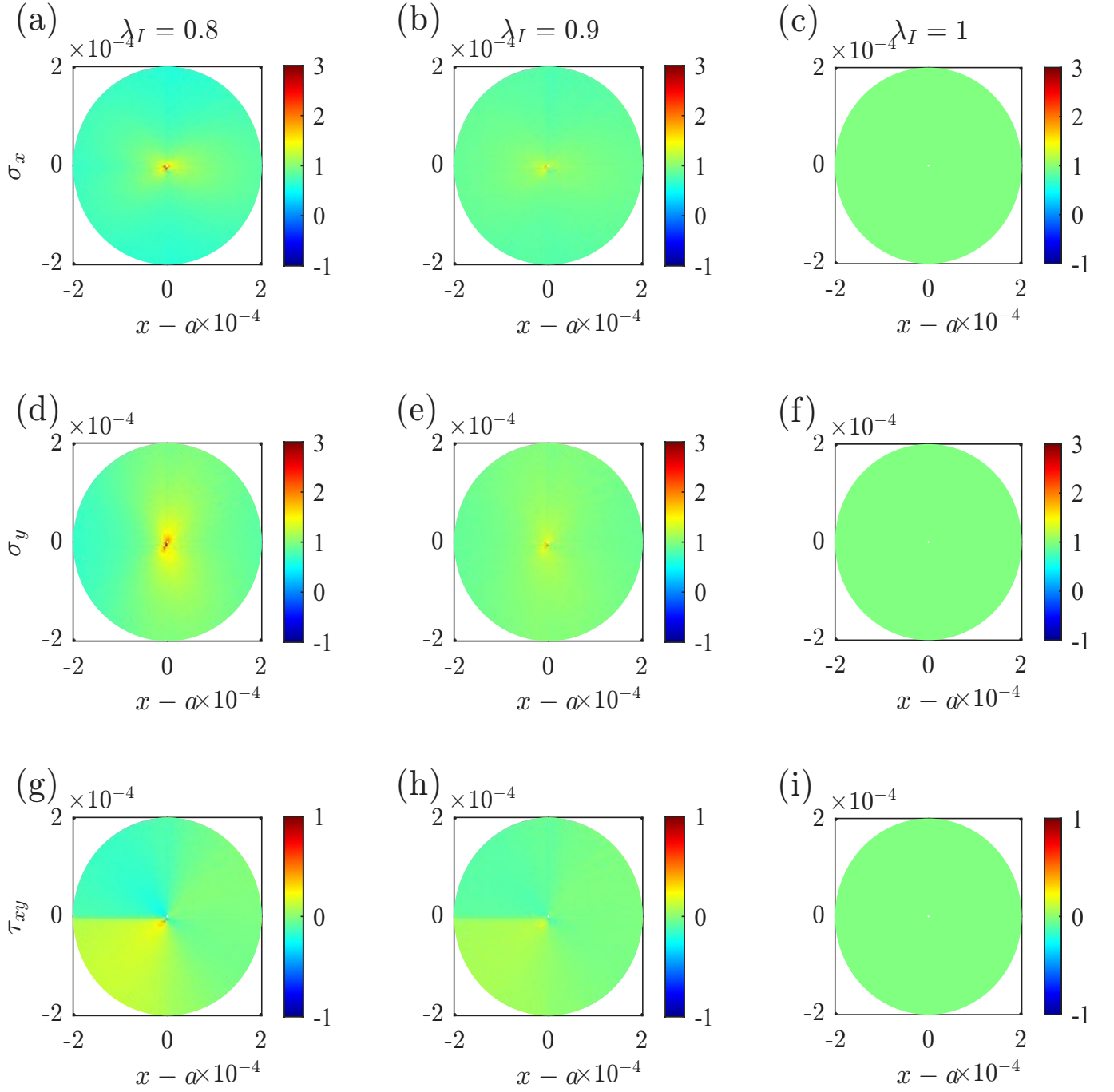


Figure 7: Contour maps of the normalized stress field around a crack tip with crack half-length $a = 0.202$ mm subjected to a remote applied stress $\sigma_\infty = 1$. The stress distributions are presented for three values of the singularity exponent, $\lambda = 0.8, 0.9$, and 1 , corresponding to Columns 1, 2, and 3, respectively. The first row illustrates the distribution of the normal stress component σ_x , the second row shows the normal stress component σ_y , and the third row presents the shear stress component τ_{xy} .

energy release rate depends on the selected virtual crack-tip increment, Δa . According to Eq. (14), when $\lambda = 0.5$, the exponent $2\lambda - 1$ becomes zero, and consequently the VCCI becomes independent of Δa . This case corresponds to the classical fracture mechanics square-root singularity associated with an unfilled crack (Eq. (1)). As $\lambda_I \rightarrow 1$ the evaluated VCCI also approaches zero, corresponding to the limiting case of a fully healed or non-singular crack. This behavior aligns perfectly with the stress-field results presented in the previous section, where the crack-tip singularity progressively vanishes as λ approaches unity.

For the intermediate range $0.5 < \lambda < 1$, the analytical expression predicts that the VCCI tends toward zero as $\Delta a \rightarrow 0$. However, this behavior appears to contradict both the stress-distribution results presented previously and the experimental observations, as well as the physical expectation that even a filling material with relatively low stiffness should still provide a finite degree of crack healing and reduction in the energy release rate. It should also be noted that the present analytical model is developed based on idealized assumptions, including complete crack filling and perfect bonding between the filling material and the surrounding matrix along the crack interfaces. In practical crack-repair applications, achieving perfect filling and ideal interfacial adhesion is unlikely, and therefore some discrepancy between the ideal analytical behavior and the actual structural response is expected.

To further investigate this issue, the VCCI is evaluated for several small values of the virtual crack increment, namely $\Delta a = 10^{-6}, 10^{-7}, 10^{-8}$, and 10^{-9} m, the obtained results are compared with the corresponding two-dimensional finite element results obtained using ANSYS.

The results presented in Figure 8 indicate that, for low values of the filling or coating material elastic modulus, the evaluated VCCI remains nearly constant regardless of the selected value of Δa . This observation suggests that, within the low-stiffness regime, the VCCI evaluation is practically insensitive to the magnitude of the virtual crack extension, and the response tends to resemble that of an unfilled crack. As the elastic modulus of the filling material gradually increases, the VCCI correspondingly decreases and eventually approaches zero, indicating progressive crack closure and suppression of the energy release via crack growth. Within this transition region, a slight dependence on Δa can be observed, where smaller values of Δa tend to produce a more pronounced reduction in the calculated VCCI values.

Nevertheless, for filling materials with elastic modulus values greater than approximately 1 GPa,

the influence of Δa becomes negligible, as the VCCI approaches zero for all considered values of Δa , representing an effectively healed crack condition. This observation is particularly significant because most practical structural adhesives and coating materials are expected to possess elastic moduli exceeding this threshold. Consequently, the sensitivity of the VCCI formulation to the choice of virtual crack increment becomes minimal for most engineering applications, thereby confirming the robustness and practical applicability of the proposed analytical formulation.

Comparison of the present analytical predictions with the corresponding two-dimensional finite element results demonstrates that the value of $\Delta a = 1 \times 10^{-7}$ m provides excellent agreement with the ANSYS results through the full range of elastic moduli. Based on this, a value of $\Delta a = 1 \times 10^{-7}$ m was adopted throughout the present analysis.

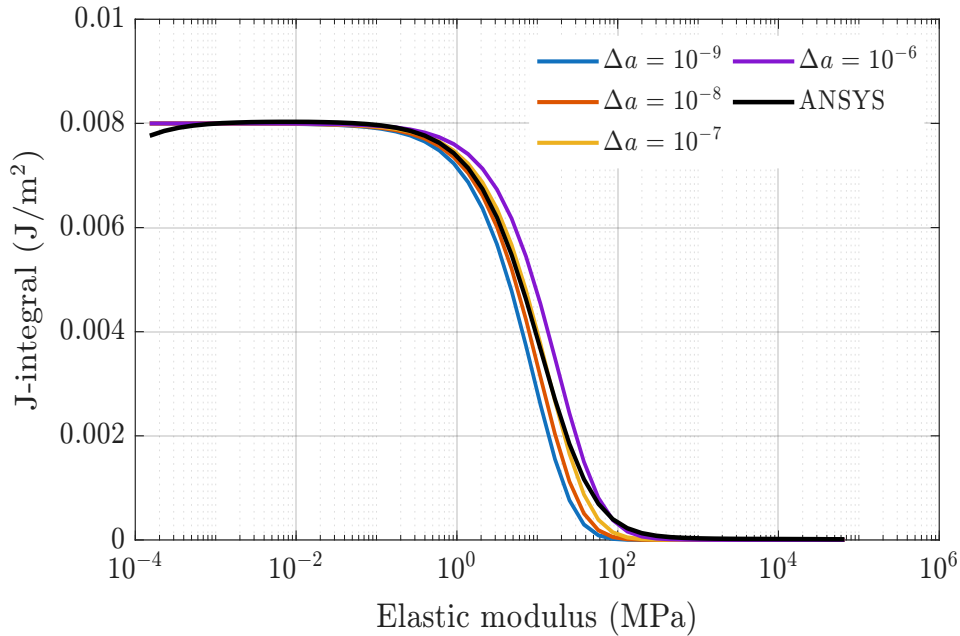


Figure 8: Effect of coating elastic modulus on the VCCI for a filled crack: for crack depth 0.202 mm and different values of virtual crack tip extension Δa as stated in Eq. (14).

4.4. Effect of Changing the Coating Elastic Modulus

The effect of the coating (filling material) elastic modulus was investigated using both the proposed analytical solution for the VCCI and the finite element method. The analysis was based on the evaluation of the J -integral for a cracked beam subjected to four-point bending. The elastic modulus of the coating material varied over a wide range, from 10^{-3} to 10^5 MPa, while the matrix material properties were kept constant and correspond to those of glass.

The results for an applied load of 20 N are presented in Figure 9 for the four crack depths considered. All results in this section are obtained for a wedge angle of 1° . In Figure 9, the J -integral

is evaluated using ANSYS as well as VCCI using two analytical approaches: the first accounts for the actual wedge angle used in the FE model, while the second is based on the small-wedge-angle approximation (VCCI-s-wed) with VCCI-wed being evaluated using Eq. (6) and VCCI-s-wed using Eq. (7).

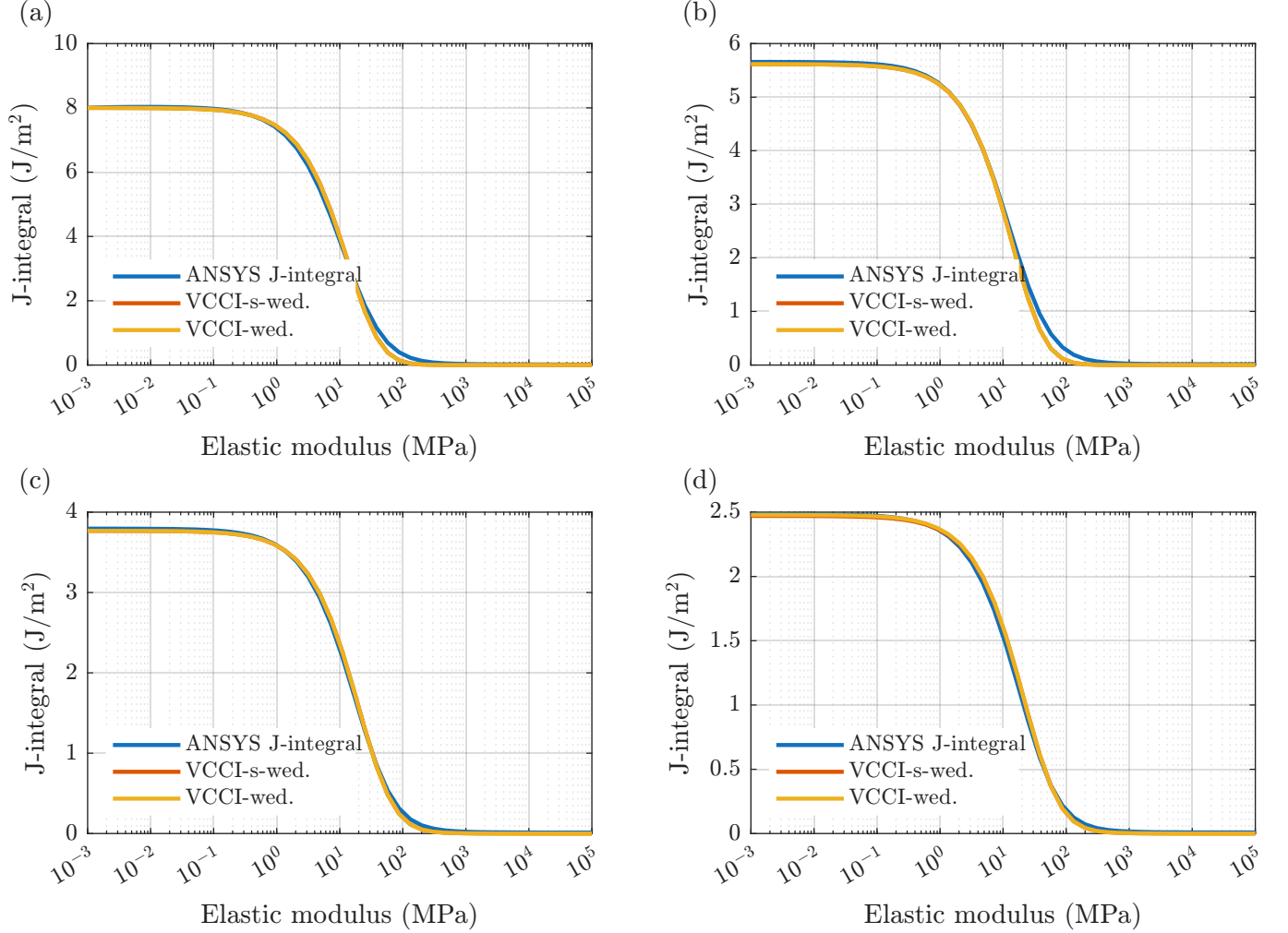


Figure 9: Effect of coating elastic modulus on the VCCI for a filled crack: (a) crack depth 0.202 mm, (b) 0.145 mm, (c) 0.096 mm, and (d) 0.063 mm. The main beam is glass with an elastic modulus of 74 GPa and a thickness of 1.04 mm.

The results in Figure 9 show that, for all four crack depths, the analytical predictions are in good agreement with the finite element results. Since the same external load is applied in all cases, the VCCI increases with increasing crack depth, as expected. Furthermore, the results demonstrate that increasing the elastic modulus of the coating material leads to a significant reduction in the VCCI, indicating an effective crack-healing mechanism.

The results also indicate that a filling material with a relatively low elastic modulus, on the order of 1 GPa (typical of polymeric coatings), is sufficient to repair the crack under the assumption of ideal bonding.

It is important to note that, for a filled crack, when evaluating the VCCI, the virtual crack opening displacement Δa (see Eq. (14)) is taken as 10^{-7} , as discussed in Section 4.3.

4.5. Effect of Changing Wedge Angle

The influence of the wedge angle was examined using both the analytical model and finite element simulations under four-point bending conditions. The analytical solution of Eq. (6), which governs the crack-tip singularity exponent, is presented in Figure 10.

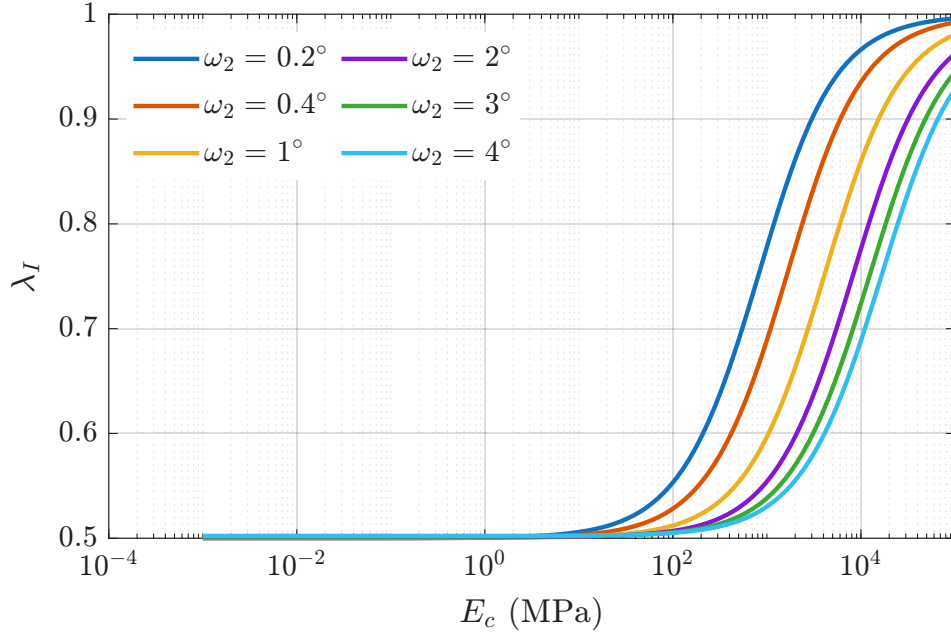


Figure 10: Effect of filling material elastic modulus on the crack-tip singularity exponent for different wedge angles.

Figure 10 shows that increasing the elastic modulus of the filling material leads to an increase in the crack-tip singularity exponent, thereby reducing the severity of the crack-tip singularity. These results should not be confused with Figure 5 as the x -axis is different in the two figures.

The effect of wedge angle on the VCCI is illustrated in Figure 11(a)–(f), where results obtained using both the analytical approach and the finite element method are compared for several wedge angles.

The results indicate that, for very low filling material elastic moduli, the wedge angle has a negligible effect on the VCCI. However, as the elastic modulus of the filling material increases, the influence of the wedge angle becomes increasingly significant. In particular, for larger wedge angles, a higher elastic modulus of the filling material is required to achieve effective crack healing. Figure 12 further elucidates this trend by presenting the J -integral as a function of the elastic modulus of the filling material for several wedge angles, ranging from 0.2° to 4° , as determined using the VCCI-

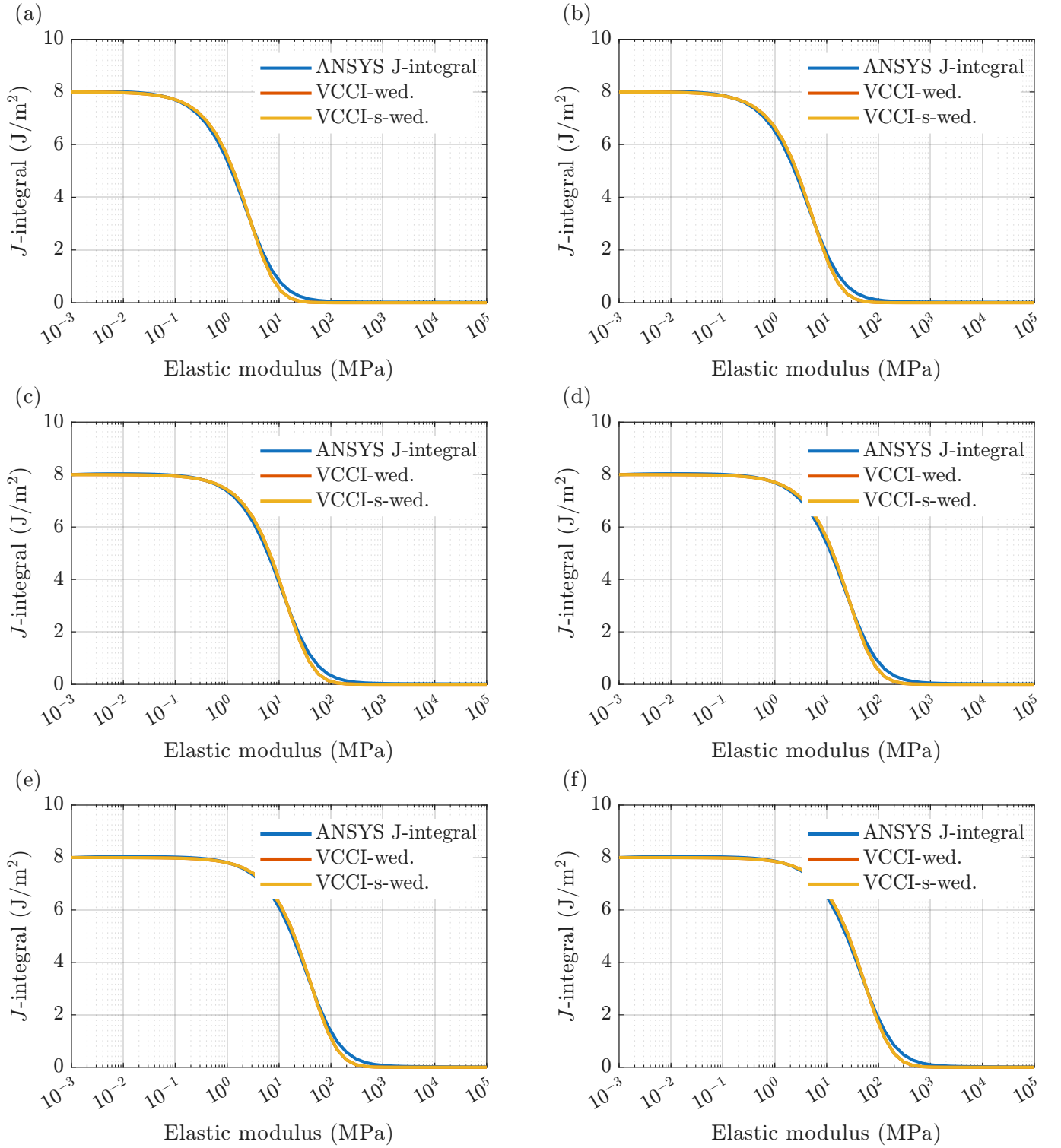


Figure 11: Effect of filling material elastic modulus on the wedge VCCI for different wedge angles: (a) 0.2° , (b) 0.4° , (c) 1° , (d) 2° , (e) 3° , and (f) 4° . The values are obtained using ANSYS and the present analytical approach.

wedge method. Examination of the vertical gridline corresponding to an elastic modulus of 100 MPa reveals that a coating with this modulus is effective at mitigating small wedge angles. However, its effectiveness diminishes substantially with increasing wedge angle, as evidenced by the rise in the J -integral value, indicating that it is insufficient for larger wedge angles. Moving up to an elastic modulus of 3000 MPa, which represents a reasonable value for epoxy materials, it can be seen that the wedge angle has almost no effect on the strengthening.

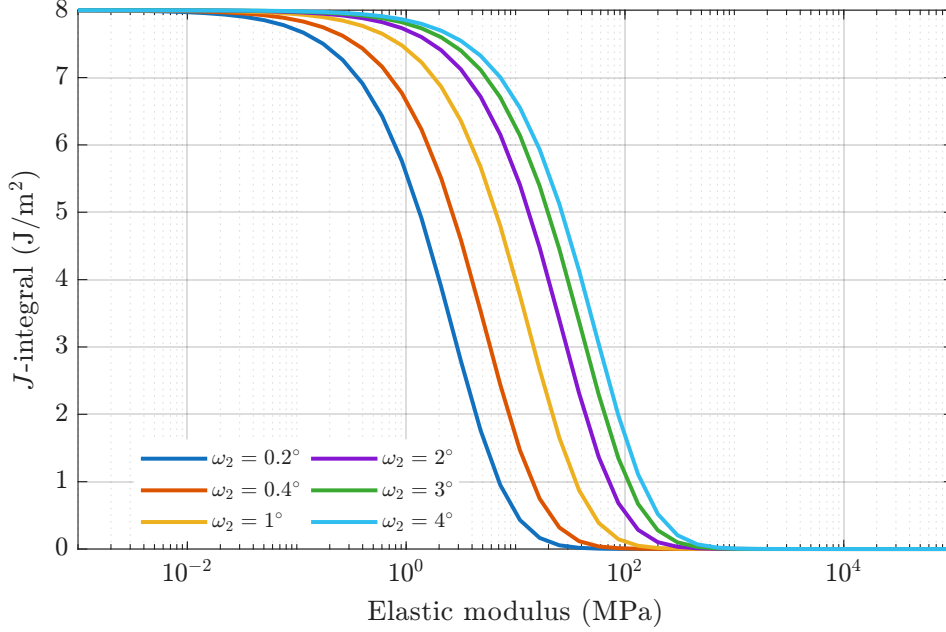


Figure 12: Effect of filling material elastic modulus on the wedge VCCI for several wedge angles, obtained using the present analytical approach.

4.6. Effect of Changing the Coating on the Stress Intensity Factor

In this section, the applicability of the stress intensity factor (SIF) $K_{I\lambda}$ concept to filled cracks is examined. A crack depth of 0.202 mm is considered, corresponding to a crack-depth ratio of $a/h = 0.194$. The SIF is evaluated using ANSYS by removing the first three finite elements in the vicinity of the crack tip, while the analytical SIF $K_{I\lambda}$ is calculated using Eq. (16).

The filling material elastic modulus varies from 0.001 to 100,000 MPa, and the results of the corresponding SIF based on ANSYS and the present filled crack model are presented in Figure 13.

The finite element results show that the stress intensity factor decreases as the elastic modulus of the filling material increases, which is consistent with the trends observed for the J -integral. In contrast, the analytical solution predicts an increase in K_I with increasing filling material stiffness. This discrepancy arises because increasing the filling material elastic modulus alters the crack-tip singularity exponent, leading to an increase in the term $(\pi a)^{1-\lambda_I}$.

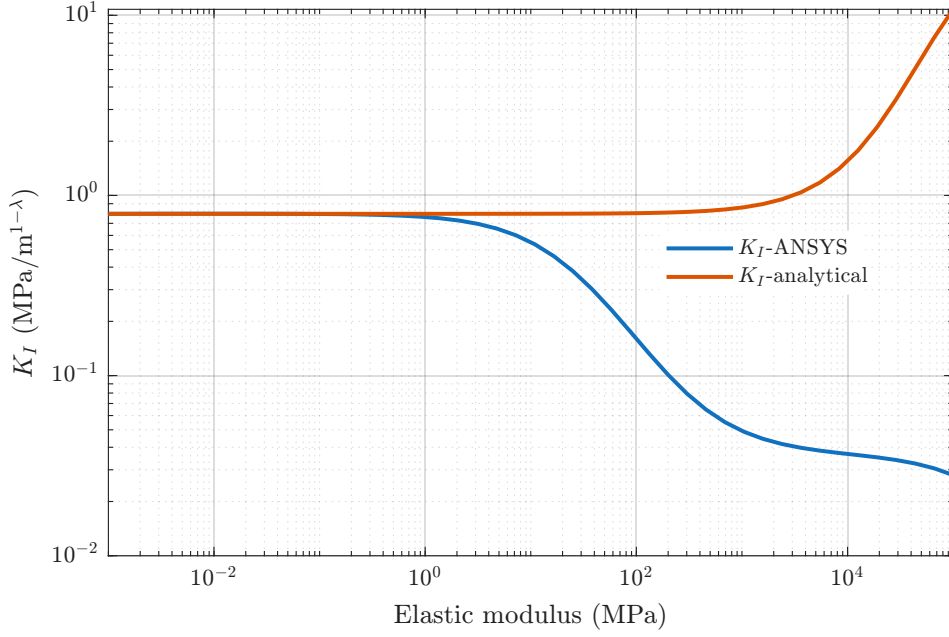


Figure 13: Variation of the stress intensity factor for a filled crack with filling material elastic modulus, obtained using ANSYS and the analytical method.

These findings indicate that the classical concept of the stress intensity factor becomes inadequate and potentially misleading for filled cracks. In such cases, energy-based fracture parameters, such as the J -integral or VCCI, provide a more physically meaningful description of crack behavior. Alternatively, an equivalent SIF analysis could be performed by neglecting the filling material in the immediate vicinity of the crack tip, consistent with the FE modeling strategy used in ANSYS.

4.7. Effect of Changing the Coating and Matrix Poisson's Ratios

The proposed analytical model is further employed to investigate the combined effects of the Poisson's ratios of both the filling material and the matrix on the VCCI as shown in Figure 14, for the case of a crack depth of 0.145 mm, corresponding to $a/h = 0.139$.

The surface plot indicates that, for a filling material elastic modulus of 1 GPa, higher Poisson's ratios for both the filling material and the matrix lead to improved crack-healing performance, as reflected by lower VCCI values. Conversely, lower Poisson's ratios result in less effective crack repair. Although each material possesses a fixed Poisson's ratio in practice, this theoretical analysis provides valuable insight into why certain filling materials are more effective than others in crack-repair applications.

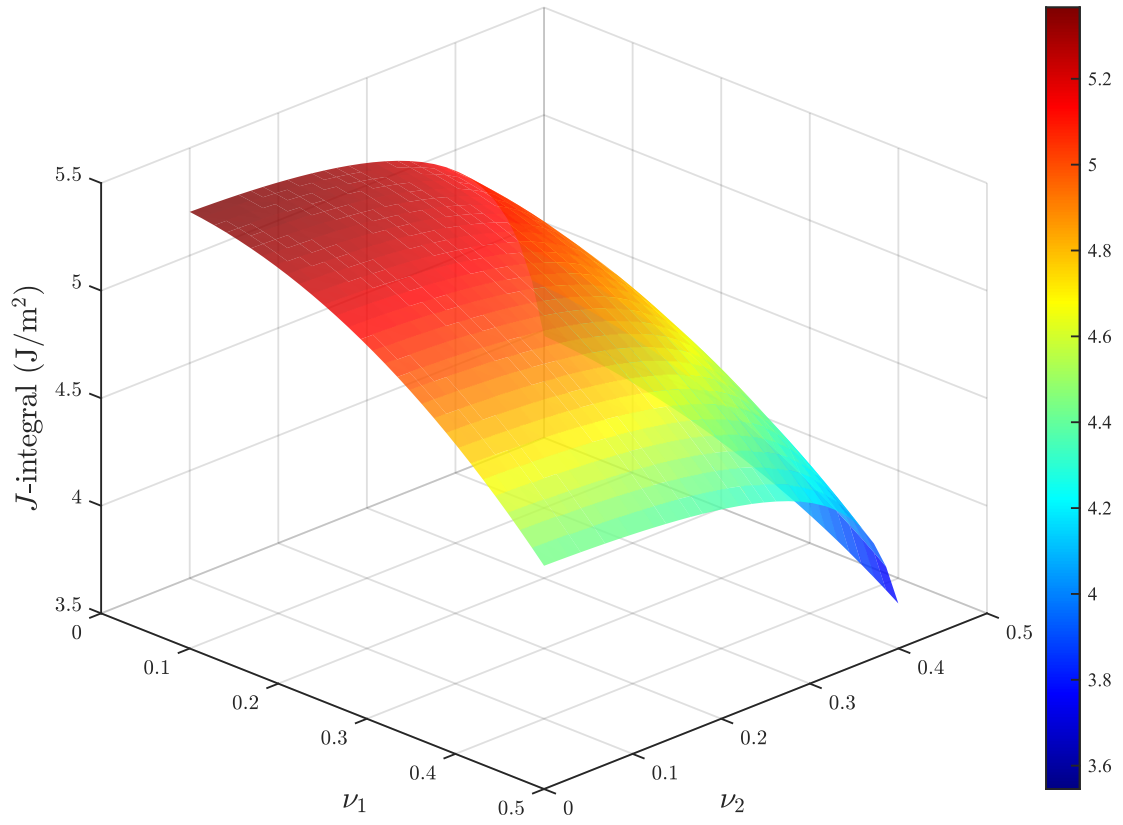


Figure 14: Effect of filling material and matrix material Poisson's ratio on the virtual crack closure integral for crack depth = 0.145 mm.

5. Conclusion

This study presents a novel and practical analytical framework for describing and quantifying crack repair using wedge-shaped filling materials. The main contribution lies in extending existing wedge-coating models and reformulating them into a simplified analytical approach that can be readily applied to mode I structural problems, including cracked beam buckling and vibration analyses.

The present results are validated through comparison with ANSYS finite element simulations. The findings are also discussed in light of previously published experimental results, confirming that proper crack filling using coating materials can effectively heal the crack. Furthermore, the analysis demonstrates that larger wedge angles require stiffer filling materials to achieve effective crack repair. It is also shown that, unlike unfilled cracks, the VCCI of filled cracks depends on the virtual crack opening.

Finally, the study highlights the limitations of the stress intensity factor concept for filled cracks and emphasizes that energy-based fracture parameters, such as the J-integral and VCCI, provide a more reliable and physically meaningful description of crack behavior in the presence of filling materials.

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Data availability statement

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflict of interest

The authors of this paper declare that they do not have any conflict of interest.

Acknowledgement

References

- [1] H. M. Westergaard, Bearing pressure and cracks, *Journal of Applied Mechanics* 6 (1939) 49–53.
- [2] I. N. Sneddon, The distribution of stress in the neighbourhood of a crack in an elastic solid, *Proceedings of the Royal Society of London* 187 (1946) 229–260.
- [3] G. R. Irwin, Analysis of stresses and strains near the end of a crack traversing a plate, *Journal of Applied Mechanics* 24 (3) (1957) 361–364.
- [4] M. L. Williams, On the stress distribution at the base of a stationary crack, *Journal of Applied Mechanics* 24 (1957) 109–114.
- [5] P. C. Paris, G. C. Sih, Stress analysis of cracks, Report (1964).
- [6] H. Tada, P. C. Paris, G. R. Irwin, *The Stress Analysis of Cracks Handbook*, Third Edition, ASME Press, 2000. doi:10.1115/1.801535.
URL <https://doi.org/10.1115/1.801535>
- [7] W. Zeng, G. R. Liu, C. Jiang, X. W. Dong, H. D. Chen, Y. Bao, Y. Jiang, An effective fracture analysis method based on the virtual crack closure-integral technique implemented in cs-fem, *Applied Mathematical Modelling* 40 (5) (2016) 3783–3800. doi:<https://doi.org/10.1016/j.apm.2015.11.001>.
URL <https://www.sciencedirect.com/science/article/pii/S0307904X15007064>
- [8] M. Chen, P. F. James, F. R. Jones, D. A. Dalton, B. H. Howard, S. Bedford, Strengthening of soda-lime-silica glass by gel-derived zinc borosilicate and sodium alumino-borosilicate glass coatings, *Journal of Non-Crystalline Solids* 139 (1992) 185–197.
URL <http://www.sciencedirect.com/science/article/B6TXM-4GXVY54-10/2/6e212d403c8efa3787b967b019e7de80>

- [9] M. P. M. Tas, W. G. J. H. M. van Sark, Experimental repair technique for glass defects of glass-glass photovoltaic modules – a techno-economic analysis, *Solar Energy Materials and Solar Cells* 257 (2023) 112397. doi:<https://doi.org/10.1016/j.solmat.2023.112397>.
URL <https://www.sciencedirect.com/science/article/pii/S0927024823002180>
- [10] T. El-Sayed, R. J. Hand, Modelling the strengthening of glass using epoxy based coatings, *Journal of the European Ceramic Society* 31 (15) (2011) 2783–2791. doi:<https://doi.org/10.1016/j.jeurceramsoc.2011.05.033>.
URL <https://www.sciencedirect.com/science/article/pii/S0955221911002780>
- [11] D. Guner, H. Ozturk, Experimental and numerical analysis of the effects of curing time on tensile mechanical properties of thin spray-on liners, *Rock Mechanics and Rock Engineering* 49 (8) (2016) 3205–3222. doi:10.1007/s00603-016-0997-x.
URL <https://doi.org/10.1007/s00603-016-0997-x>
- [12] I. T. Denisyuk, Singular stresses in an isotropic matrix containing an elastic wedge, *Soviet materials science : a transl. of Fiziko-khimicheskaya mekhanika materialov / Academy of Sciences of the Ukrainian SSR* 28 (4) (1992) 388–393. doi:10.1007/BF00723219.
URL <https://doi.org/10.1007/BF00723219>
- [13] N. Fowkes, J. A. Teixeira de Freitas, R. Stacey, Crack repair using an elastic filler, *Journal of the Mechanics and Physics of Solids* 56 (9) (2008) 2749–2758.
URL <http://www.sciencedirect.com/science/article/B6TXB-4SR71CP-1/2/203568ec176a1d9eccc6bce2ebb74db2>
- [14] J. A. Teixeira de Freitas, C. Tiago, Wedge and crack solutions for mindlin–reissner plates, *International Journal of Fracture* 221 (1) (2020) 1–23. doi:10.1007/s10704-019-00397-3.
URL <https://doi.org/10.1007/s10704-019-00397-3>
- [15] M. S. Taima, T. A. El-Sayed, M. B. Shehab, S. H. Farghaly, R. J. Hand, Vibration analysis of cracked beam based on reddy beam theory by finite element method, *Journal of Vibration and Control* (2022) 10775463221122122doi:10.1177/10775463221122122.
URL <https://doi.org/10.1177/10775463221122122>
- [16] A. A. Elkaashlawy, Y. K. Younes, H. H. El-Mongy, Dynamic behavior analysis of a rotating shaft with an elliptical breathing surface crack, *Journal of Vibration Engineering & Technologies* 11 (8) (2023) 4371–4385. doi:10.1007/s42417-022-00820-5.
URL <https://doi.org/10.1007/s42417-022-00820-5>
- [17] M. S. Taima, T. A. El-Sayed, M. I. Friswell, Thermal vibration analysis of cracked nanobeams submerged in elastic foundations by nonlocal continuum mechanics, *Thin-Walled Structures* 193 (2023) 111249. doi:<https://doi.org/10.1016/j.tws.2023.111249>.
URL <https://www.sciencedirect.com/science/article/pii/S0263823123007279>
- [18] ANSYS, ANSYS Structural analysis guide : ANSYS release 11, ANSYS Inc, Canonsburg, PA, 2007.
- [19] T. A. E. Elsayed, Modelling the strengthening of glass using epoxy based coatings, Thesis (2010).
- [20] T. L. Anderson, Fracture mechanics : fundamentals and applications, 3rd Edition, Taylor & Francis, Boca Raton, Florida, 2005.

Appendix A. Summary of previous experimental work

Microscope glass slides with dimensions of $26 \times 76 \times 1$ mm were cut into two specimens with final dimensions of $26 \times 38 \times 1$ mm. Controlled surface defects were introduced using Vickers indentation under four different indentation loads of 1, 2.5, 5, and 10 kg. The corresponding average crack lengths were approximately 0.061 ± 0.003 , 0.096 ± 0.005 , 0.142 ± 0.009 , and 0.202 ± 0.010 mm, respectively. The elastic modulus of the glass substrate was 74 GPa, while the elastic modulus of the epoxy coating layer was measured as 4.2 GPa using nanoindentation.

The flexural strength of the specimens was evaluated using a four-point bending test similar to the setup illustrated in Figure 3. The average flexural strength of the uncoated and unindented glass specimens was measured as 123 ± 13 MPa and was used as a reference value for evaluating the effectiveness of the coating treatment.

To investigate the effect of crack size on the mechanical performance, annealed cracked specimens were tested after introducing controlled defects using Vickers indentation. The average flexural strengths corresponding to indentation loads of 1, 2.5, 5, and 10 kg were found to be 63 ± 4 , 51 ± 2 , 45 ± 3 , and 39 ± 3 MPa, respectively. These results demonstrate the significant reduction in fracture strength with increasing crack size.

Extensive experimental investigations were conducted to optimize both the epoxy mixture composition and the coating application procedure; further details are provided in [10]. Figure A.1 presents the fracture stress results obtained using the optimum coating conditions for specimens containing different crack sizes. For specimens containing small cracks generated using 1 and 2.5 kg indentation loads, the coated samples exhibited average flexural strengths exceeding that of the as-received uncracked glass specimens. In the case of specimens indented using a 5 kg load, the flexural strength after coating was very close to that of the uncracked glass, while specimens containing cracks generated using a 10 kg indentation load exhibited slightly lower strength values.

These results indicate that the water-based epoxy coating was highly effective in healing small and moderate surface cracks and significantly improving the fracture resistance of the glass specimens. Fractographic examination further confirmed this behavior, as fracture initiation in several coated specimens no longer originated from the indentation site, as illustrated in Figure A.2.

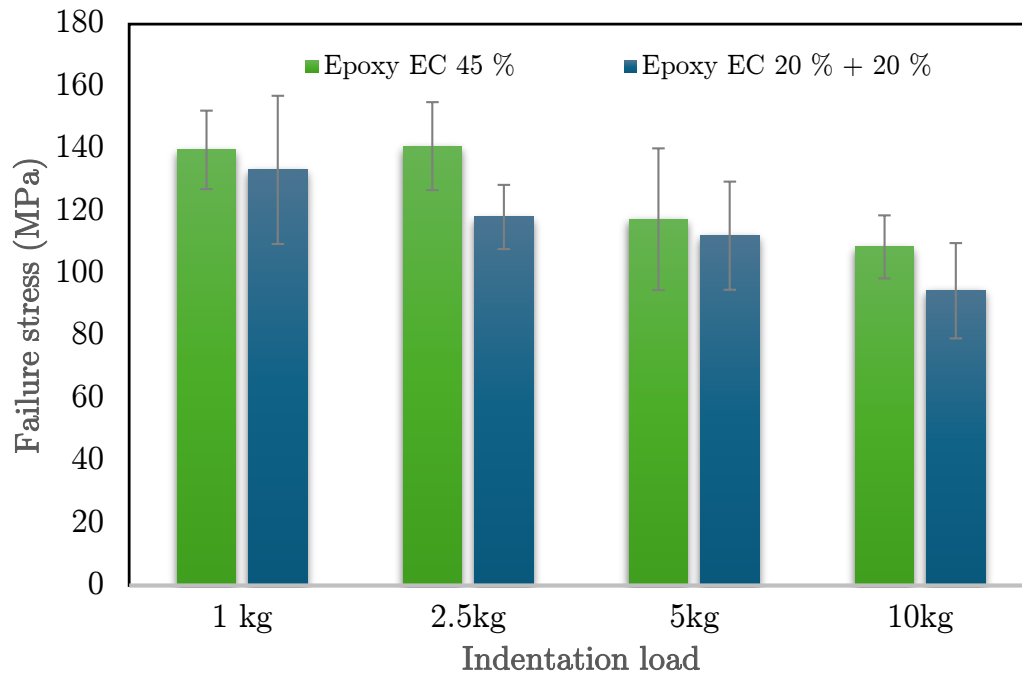


Figure A.1: The flexural fracture stress of coated glass specimens containing controlled Vickers indentation cracks produced under different indentation loads. The results correspond to the optimum epoxy coating conditions identified experimentally.



Figure A.2: Representative fracture surface of a glass specimen indented using a 1 kg Vickers load and coated twice using a 20% epoxy emulsion concentration. Fractographic observations show that fracture initiation did not originate from the indentation site, indicating effective healing of the controlled surface defect by the epoxy coating.