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Article:

Benjamini, I., Zakharov, G. and Zhukovskii, M. (2026) Majority dynamics on finite trees. Random Structures & Algorithms, 69 (2). e70092. ISSN: 1042-9832

<https://doi.org/10.1002/rsa.70092>

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RESEARCH ARTICLE OPEN ACCESS

Majority Dynamics on Finite Trees

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ABSTRACT

For an arbitrary finite tree T , we find the exact value of the worst-case stabilization time of majority dynamics on T . We also prove that for a perfect rooted cubic tree T with diameter D and uniformly random initial opinions, the dynamics stabilizes in time $\tau \in (D/4, D/3)$ with high probability.

1 | Introduction

Majority dynamics is a fundamental process on networks where each node interacts with its neighbors by changing its opinion to match the opinion of the majority. This simply described, and natural model has been of interest in biophysics McCulloch and Pitts (1990), social psychology Yin et al. (2019) and computer science Alistarh et al. (2017). Formally, this process is described as follows. Initially each individual $i \in V$ considered as a vertex of a graph G on a (not necessarily finite) set of vertices V has an *initial opinion* $\xi_0(i) \in \{-1, 1\}$. Then, at every time step $t \in \mathbb{N}$, they adopt the majority opinion of their neighbors, that is,

$$\xi_t(i) = \text{sign} \sum_{j \in N_G(i)} \xi_{t-1}(j),$$

where $N_G(i)$ is the set of neighbors of i in G . For convenience, we only consider graphs with odd degrees, so $\sum_{j \in N_G(i)} \xi_t(j)$ is never equal to 0.

It was proved by Goles and Olivos (1980) that the process stabilizes for every finite graph G . More formally, the following *period two property* holds true: For every vertex i and all large enough t , $\xi_{t+2}(i) = \xi_t(i)$. However, the question of estimating the time until stabilization

$$\tau(G; \xi_0) = \min\{t : \xi_{t+2}(i) = \xi_t(i) \text{ for all } i \in V(G)\}$$

is still open. In Poljak and Turzík (1986), it is proven that, for every G and every $\xi_0 \in \{-1, 1\}^{V(G)}$, $\tau(G; \xi_0) \leq |E(G)| - \frac{1}{2}|V(G)|$. It is known, however, that for most graphs and most assignments of initial opinions, τ is much smaller: It was proven in Fountoulakis et al. (2020) that if $p \gg n^{-1/2}$ then with high probability¹ $\tau := \tau(G(n, p); \xi_0) \leq 4$ for $\xi_0 \in$

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$\{-1, 1\}^{V(G)}$ chosen uniformly at random. Here, as usual, $G(n, p)$ denotes a binomial random graph on $[n] := \{1, \dots, n\}$ with every edge drawn independently with probability p . This was improved in Chakraborti et al. (2023): If $p \gg n^{-3/5} \log n$, then whp $\tau \leq 6$. Recently, the bound $\tau = O(1)$ whp was established for all $p \gg n^{-2/3}$ Jaffe (2025); Kim and Tran (2025). Moreover, in Tran and Vu (2025), it was proved that, for an arbitrary initial coloring with a sufficiently large gap between the two sides and for $p > (1 + \varepsilon) \log n/n$, $\tau = O(\log_{pn}(n))$ whp. In Benjamini et al. (2016); Tamuz and Tessler (2015) the stabilization phenomenon was also studied for infinite graphs. It was proven in Benjamini et al. (2016) that for any unimodular transitive graph G and any $\text{Aut}(G)$ -invariant distribution of initial opinions, the opinions of every vertex stabilize with probability 1, and the expected time until $\xi_t(i)$ stabilizes is at most $2d$, where d is the degree of G . Many natural classes of transitive infinite graphs are unimodular. In particular, the above is true for infinite regular trees T and independent and identically distributed $\xi_0(i)$, $i \in V(T)$. Our results imply a qualitative refinement of this result for regular trees of degree 3 (see Claim 1.4 below). The exact value of $\max_{\xi_0} \tau(G = C_n; \xi_0)$ for a cycle was obtained in Zehmakan (2023). For further background on majority dynamics and related processes, see the survey Mossel and Tamuz (2017).

In this paper, we study majority dynamics on *finite* trees. First of all, for a given tree T , we find the exact value of the worst-case stabilization time

$$\tau(T) := \max_{\xi_0 \in \{-1, 1\}^{V(T)}} \tau(T; \xi_0).$$

Note that, due to Poljak and Turzík (1986), $\tau(T) \leq \frac{1}{2}|V(T)| - 1$. As follows from the theorem below, this result is not the best possible for any T with $|V(T)| \geq 5$.

Theorem 1.1. *Let all vertices of a tree T have odd degrees and $|V(T)| \geq 5$. Let $\mathcal{Q}$ be a set of paths $Q = v_1 \cdots v_n$ such that*

- *at most $\frac{1}{2}(\deg_T v_n - 1)$ of neighbors of v_n in T are leaves,*
- *less than $\frac{1}{2}(\deg_T v_i - 1)$ of neighbors of v_i in T are leaves for $i \in [n - 1]$.*

For every $Q \in \mathcal{Q}$, let $t(Q)$ be defined as follows

$$t(Q) = \begin{cases} |V(Q)| + 1, & \text{if } v_n \text{ is adjacent to a leaf} \\ |V(Q)|, & \text{otherwise.} \end{cases}$$

Then $\tau(T) = \max_{Q \in \mathcal{Q}} t(Q)$.

In particular, for *perfect* trees (to satisfy the condition that all degrees are odd, we assume here that a root has the same degree as all non-leaf vertices) with diameter $D = D(T)$, we have that $\tau(T) = D - 3$.

A natural question to ask: Does τ drop significantly in the average case, that is, when ξ_0 is chosen uniformly at random? We prove that for perfect binary trees it remains linear in D , though the constant factor becomes less than 1. Note that, as above, the root of a perfect binary tree has degree 3 (see the definition of a perfect k -ary tree in Section 2).

Theorem 1.2. *Let $\xi_0 = \xi_0(T) \in \{-1, 1\}^{V(T)}$ be chosen uniformly at random on the vertices of a perfect binary tree $T = T(D)$ of diameter D . There exist $c_- > 1/4$ and $c_+ < 1/3$ such that whp, as $D \rightarrow \infty$,*

$$c_- \cdot D < \tau(T; \xi_0) < c_+ \cdot D.$$

For other k -ary trees, we are only able to prove an $\Omega(\sqrt{D})$ bound.

Theorem 1.3. *There exists $c_- > 0$ such that, for every even $k \geq 4$ and a uniformly random $\xi_0 \in \{-1, 1\}^{V(T)}$ on vertices of a perfect k -ary tree T , whp, as $D(T) \rightarrow \infty$,*

$$c_- \cdot \frac{\sqrt{\ln k}}{k} \sqrt{D(T)} < \tau(T; \xi_0) \leq D(T) - 3.$$

1.1 | Proof Strategy

We say that a vertex $v \in V(T)$ changes its opinion to $\xi \in \{-1, 1\}$ at step $t \geq 2$ if $\xi_{t-2}(v) \neq \xi$ and $\xi_t(v) = \xi$. The main observation that we use to prove the results is that for a vertex v of a tree, to change its opinion to $\xi \in \{-1, 1\}$ at some time step $t \geq 2$, there should exist a path of $t - 1$ vertices ending at v such that vertices from the path change their opinion to ξ one by one along the path. In particular, in Theorem 1.1 we construct, and in Theorems 1.2 and 1.3 we find such paths that let opinions pass through, to prove the lower bounds.

To prove the upper bound in Theorem 1.1, we will observe that only vertices v that have less than $\frac{1}{2}(\deg_T(v) - 1)$ leaf-neighbors (we will call them *active*) are able to convey an opinion to the next non-leaf vertex in a path with potentially good ‘conductivity’ properties. So, $\tau(T)$ is approximately the length of the longest path of active vertices. The lower bound is obtained via an explicit assignment of opinions that allows one to pass an opinion through the longest path of active vertices.

For random initial opinions and a fixed path P of active vertices, the main challenge is to extract independent events that allow one to get tight lower and upper bounds on the probability that initial opinions of vertices allow some opinion to pass through the path. These independent events are naturally associated with opinions of vertices that belong to disjoint sets. We define them using notions of *weak* and *strong stability*. The proof of the lower bound in Theorem 1.2 relies on the notion of strong stability: A vertex v is *strongly t -stable* if the initial opinions of the subtree T_v induced by the descendants of v guarantee t -stability of v whatever the opinions of all the other vertices. We will prove that a vertex is strongly t -stable with at least a constant positive probability. It is worth noting that this immediately implies the following refinement of the result from Benjamini et al. (2016) in the particular case of infinite regular trees of degree 3:

Claim 1.4. Let T be an infinite regular tree of degree 3 and let ξ_0 have uniform distribution on $\{-1, 1\}^{V(T)}$. Fix a root $v \in V(T)$. Then $\mathbb{P}(\xi_{2t}(v) = \xi_0(v) \text{ for all } t \in \mathbb{N}) > 0.5$.

The other two notions of stability we introduce in this work are *t -stability* and *$(\leq t)$ -stability*. A vertex v is *t -stable* (w.r.t. ξ_0) if it preserves the same opinion $\xi_t(v)$ at $t' \geq t$ with the same parity as t . A vertex v is *$(\leq t)$ -stable* if the initial opinions of the subtree T_v guarantee that v has opinion $\xi_t(v)$ at every step $t' \leq t$ with the same parity as t whatever the initial opinions of all the other vertices. Since the event that v is $(\leq t)$ -stable is described in terms of initial opinions of vertices of T_v only, we may define an event that (1) occurs when certain $O(d)$ vertices with non-overlapping pendant trees are strongly stable; (2) implies (assuming some other auxiliary technical properties) that vertices along P change their opinions one by one. This gives us that the probability that initial opinions of vertices allow some opinion to pass through P is at least $\exp(-O(d))$. It then remains to find $\exp(\Omega(d))$ such paths so that the approximation events are independent. This crucial event will be given by A1 – A5 in Lemma 5.2.

For the upper bound in Theorem 1.2, we introduce the notion of weak stability: In particular, v is *weakly 0-stable* if the initial opinions of the vertices of T_v guarantee the existence of opinions of all the other vertices so that v is 0-stable (the notion of weak t -stability is more complex and thus we omit it here; the definition for an arbitrary t is given in Section 4.1). As in the case of strong stability, the event that v is weakly 0-stable depends on initial opinions of vertices of T_v only (nevertheless, the event that v is weakly t -stable depends on opinions of vertices of T_v at time t). Using this notion, we prove that the probability that vertices along P change their opinion one by one is at most $\exp(-\Omega(d))$. Note that this fact is sufficient due to the union bound. The crucial observation that allows us to prove this fact is that a weakly t -stable vertex is ‘half-stable’ — when two weakly t -stable vertices have the same opinion at time t and are connected by a path consisting of vertices with the same opinion at time t , they stabilize immediately. Our intuition here is that, roughly speaking, there are typically many weakly t -stable vertices in the tree. Therefore, they frequently stabilize, and once they stabilize, an opinion cannot travel through them, preventing changes of any opinion after some fairly long time. A little more formally, we show that the fact that an opinion travels through some path P implies linearly in $|P|$ many equations of the form “ $\xi_t(v) = \xi$ ” for vertices v from the neighborhood of P . On the other hand, we describe a linear set of independent events that contradict these equations (we use Claims 4.11 and 4.12 to show that). Then, we show that $\mathbb{P}(v \in V \text{ is weakly 0-stable})$ and $\mathbb{P}(v \in V \text{ is weakly stable after the first change of its opinion})$ are bounded away from zero which, in turn, implies an exponential bound for the probability that neither from the set of contradicting events happens. This immediately implies the desired exponential upper bound on the probability that the travel of some opinion along P does not happen.

In the proof of the lower bound of Theorem 1.3, we introduce a new notion of $(\leq t)$ -stability. The proof is similar to the proof of Theorem 1.2. Roughly speaking, in the proof, $(\leq t)$ -stability plays the same role as strong t -stability in Theorem 1.2.

We replace t -stability by $(\leq t)$ -stability because, in contrast to the perfect binary tree, for an arbitrary perfect k -ary tree we did not manage to prove a positive constant lower bound on the probability of t -stability. However, our bound on the probability of $(\leq t)$ -stability for an arbitrary perfect k -ary tree is worse than the bound on strong t -stability for a perfect binary tree, which results in a weaker lower bound.

This paper is organized as follows. In Section 2, we introduce all the necessary definitions, including the properties of stability. Further, in Section 3 we prove Theorem 1.1. The average-case bounds are proven in the next two sections: Section 4 is devoted to the proof of Theorem 1.2, and Theorem 1.3 is proven in Section 5. Finally, in Section 6 we discuss our results as well as further questions and avenues for future research.

2 | Preliminaries

Let T be a tree and R be a vertex. The pair (T, R) is called a *rooted tree*, and R is called the *root*. For brevity, we will denote (T, R) simply by T and say that T is *R -rooted* or *rooted in R* . We only consider trees with odd degrees of vertices. A vertex u is a *descendant* of a vertex v in T if the path between u and R contains v . A descendant of v which is also its neighbor is called a *child* of v . For every vertex v of T , we denote by T_v the subtree of T induced by all descendants of v and v itself, rooted in v . The *height* of T is the maximum distance between its root and its leaf. Let us denote the height of T_v by $h_T(v)$ and call it the *height of v* .

We call a rooted tree T *k -ary* if all vertices other than leaves have $k + 1$ neighbors; for consistency with the definition of the majority dynamics, we let the root of a k -ary tree have exactly the same degree as other non-leaf vertices. We call a k -ary tree *perfect* if all its leaves are at the same distance from the root. A (perfect) 2-ary tree is also called a (*perfect*) *binary tree*. In particular, the root of a (perfect) binary tree has degree 3.

Let us now run the majority dynamics on a rooted tree T with an arbitrary initial vector of opinions ξ_0 . Let $t \in \mathbb{Z}_{\geq 0}$. Let us call a vertex u *t -stable*, if for every $s \geq t$ with the same parity as t , $\xi_s(u) = \xi_{s+2}(u)$ (here we assume that 0 is even). The vertex is *t -stationary*, if it is t -stable and $(t + 1)$ -stable.

3 | Proof of Theorem 1.1

In Section 3.1, we prove the upper bound $\tau(T) \leq \max_{Q \in \mathcal{Q}} t(Q)$. In Section 3.2, we constructively prove the lower bound $\tau(T) \geq \max_{Q \in \mathcal{Q}} t(Q)$ by assigning the initial opinions in a proper way.

3.1 | Upper Bound

Here we prove the upper bound in Theorem 1.1. We distinguish between three types of vertices. We call a vertex v of T *active*, if less than $\frac{1}{2}(\deg_T v - 1)$ of its neighbors are leaves. We call v *balky*, if exactly $\frac{1}{2}(\deg_T v - 1)$ of its neighbors are leaves. In particular, all leaves are balky. All the other vertices are *passive*. Let us observe the following crucial property of balky vertices.

Claim 3.1. If v is balky, u is a non-leaf neighbor of v , and, for some $s \in \mathbb{Z}_{\geq 0}$, $\xi_s(v) = \xi_{s+1}(u)$, then $\xi_{s+2}(v) = \xi_s(v)$.

Proof. Indeed, let $u_1, \dots, u_m$, $m = \frac{1}{2}(\deg_T v - 1)$, be all the leaves adjacent to v . Then, $\xi_s(v) = \xi_{s+1}(u_1) = \dots = \xi_{s+1}(u_m)$ and $\xi_{s+1}(u)$ is the same by the assumption of the claim. As u is not a leaf, we get that more than half of the neighbors of v share the same opinion $\xi_s(v)$ at time $s + 1$. Thus, $\xi_{s+2}(v) = \xi_s(v)$. $\square$

Now, note that all passive vertices are 0-stationary, so there is nothing to prove for them. Claim 3.2 formulated at the end of this section states that every active vertex u stabilizes in time no more than the number of vertices in the longest path Q of active vertices starting in u . In particular, it matches the upper bound in the statement of Theorem 1.1 since $Q \in \mathcal{Q}$. Due to Claim 3.1, a balky vertex v may *switch* at time $s \geq 2$ (i.e., $\xi_s(v) \neq \xi_{s-2}(v)$) only if $\xi_{s-1}(u) = \xi_s(v)$ for every non-leaf $u \in N_T(v)$. Therefore, v cannot cause a change of opinion of its non-leaf neighbor. This means that, as soon as all active neighbors of v stabilize, then non-leaf balky vertices stabilize in at most one additional step. Moreover, a balky non-leaf vertex can change its value only if it has an active neighbor, while a leaf can switch at time $s \geq 3$ if its neighbor is either active or balky. Therefore, for a balky vertex v with at least one active neighbor, the stabilization time is at most 1 greater

than the number of vertices in the longest path of active vertices starting at an active neighbor of v . Again, we shall note that this time is at most the upper bound stated in Theorem 1.1: For any path Q initiated at a balky vertex with all the other vertices being active, we have that $Q \in \mathcal{Q}$. Finally, at most one additional step is needed for leaves adjacent to balky vertices. Any path $v_{n+1}v_nv_{n-1} \cdots v_1$ with leaf v_{n+1} , balky vertex v_n , and active vertices $v_1, \dots, v_{n-1}$ has $t(Q) = |V(Q)| + 1$ vertices for $Q = v_1 \cdots v_n \in \mathcal{Q}$. Thus, leaves stabilize in time $\max_{Q \in \mathcal{Q}} t(Q)$ as well.

It remains to prove that, if any path consisting of active vertices initiated at an active vertex v has at most t vertices, then v is t -stable. Indeed, this would also imply that v is $(t + 1)$ -stable and, therefore, t -stationary by applying the same assertion to the process with the vector of initial opinions ξ_1 . The following claim completes the proof of the upper bound in Theorem 1.1.

Claim 3.2. Let all vertices of a tree T have odd degrees, and let v be an active vertex of T . Then v is t -stable, where t is the number of vertices in the longest path of active vertices starting at v .

Proof. Suppose towards a contradiction that there is a time step $t_1 \geq t$ such that $\xi := \xi_{t_1+2}(v) \neq \xi_{t_1}(v)$. Then the multisets of opinions of neighbors of v at time steps $t - 1$ and $t + 1$ should differ. In particular, there is a neighbor v_{t_1-1} of v such that $\xi = \xi_{t_1+1}(v_{t_1-1}) \neq \xi_{t_1-1}(v_{t_1-1})$. We can apply the same reasoning to v_{t_1-1} at time step $t_1 - 1$, so we find a neighbor v_{t_1-2} of v_{t_1-1} such that $\xi = \xi_{t_1}(v_{t_1-2}) \neq \xi_{t_1-2}(v_{t_1-2})$. In addition $v_{t_1-2} \neq v$ as their opinions differ at time step t_1 . Applying the same reasoning $t_1 - 2$ times more, we find a path $v = v_{t_1}v_{t_1-1} \cdots v_0$ such that, for every $t' \in \{0, \dots, t_1\}$, $\xi = \xi_{t'+2}(v_{t'}) \neq \xi_{t'}(v_{t'})$. Since $t_1 + 1 > t$, the length of this path is longer than the maximum length of a path of active vertices initiated at v . Thus, there exists $t' \in \{0, \dots, t_1 - 1\}$ such that $v_{t'}$ is not active. Then $v_{t'}$ is either balky or passive. Actually, it cannot be passive, since $\xi_{t'+2}(v_{t'}) \neq \xi_{t'}(v_{t'})$ while a passive vertex is 0-stationary. So it should be balky. However, $\xi_{t'}(v_{t'}) = \xi_{t'+1}(v_{t'+1}) \neq \xi$ and $v_{t'+1}$ is not a leaf—either it is adjacent to different vertices $v_{t'+2}$ and $v_{t'}$ or it coincides with v which is active and, thus, cannot be a leaf. So, by Claim 3.1, $\xi_{t'+2}(v_{t'}) = \xi_{t'}(v_{t'})$ —a contradiction, completing the proof. $\square$

3.2 | Lower Bound

Consider a path $Q = v_1 \cdots v_n \in \mathcal{Q}$ maximizing $t(Q)$. To prove the lower bound in Theorem 1.1, it is sufficient to present $\xi_0 : V(T) \rightarrow \{-1, 1\}$ such that v_n changes its opinion at time $n + 1$. Indeed, if $t(Q) = n$, then v_n is not $(t(Q) - 1)$ -stable. If $t(Q) = n + 1$, then v_n is adjacent to a leaf that, in turn, changes its opinion at time $n + 2$. Thus, this leaf is not $(t(Q) - 1)$ -stable, concluding the proof. It remains to prove the following claim.

Claim 3.3. Let $Q = v_1 \cdots v_n \in \mathcal{Q}$. Then there is $\xi_0 : V(T) \rightarrow \{-1, 1\}$ such that

- for every $i \in [n]$ holds $\xi_{i+1}(v_i) = -1$,
- for every $i \in [n]$ and $t \leq i$ holds $\xi_t(v_i) = 1$.

Proof. Let v_n be the root of T . Let us set $\xi_0(v_i) = 1$ for all $i \in [n]$ and for every child u of v_1 set $\xi_0(u) = 1$. For every $i \in [n] \setminus \{1\}$ and every leaf u adjacent to v_i , we set $\xi_0(u) = 1$. By the definition of $\mathcal{Q}$, every vertex v_i , $i \in [n] \setminus \{1\}$, has at least $\frac{1}{2}(\deg_T v_i - 1)$ children other than leaves and v_{i-1} . We then set $\xi_0(u) = -1$ for exactly $\frac{1}{2}(\deg_T v_i - 1)$ non-leaf children u of v_i other than v_{i-1} . All the other children of every v_i , $i \in [n] \setminus \{1\}$, get $\xi_0 = 1$. It only remains to assign initial opinions to vertices that are not in the (closed) 1-neighborhood of Q . For every $i \in [n] \setminus \{1\}$ and every non-leaf child u of v_i we assign the opinion $\xi_0(u)$ to every vertex of T_u . Lastly, every descendant of v_1 at distance at least 2 from v_1 gets initial opinion -1 .

First of all, note that $\xi_1(v_1) = \xi_1(v_2) = \cdots = \xi_1(v_n) = 1$ by the definition of ξ_0 . Next, note that, for every $i \in [n] \setminus \{1\}$, every non-leaf child of v_i is 0-stationary by the definition of ξ_0 . In addition, every leaf child of v_i has opinion 1 at time Step 1. So, $\xi_2(v_2) = \cdots = \xi_2(v_n) = 1$. However, every non-leaf child of v_1 has opinion -1 at time Step 1. So, $\xi_2(v_1) = -1$, as v_1 is active vertex.

It is sufficient for us to prove by induction that, for every $t \in [n]$, at time t we have the following: $\xi_{t+1}(v_t) = -1$ while $\xi_{t+1}(v_{t+1}) = \cdots = \xi_{t+1}(v_n) = 1$. The base of induction for $t = 1$ is already verified. We suppose that this is true for some $t \in [n - 1]$. Let $s > t + 1$ be the first moment when a certain v_τ , $\tau \geq t + 1$, changes its opinion (we let $s = \infty$ if all v_τ , $\tau \geq t + 1$, are 0-stationary). Then, $s - 1$ was the first time step when more than half of the neighbors of v_τ had opinion -1 . Note that all children of v_τ other than $v_{\tau-1}$ that are not leaves are 0-stationary. Also, all children u of v_τ that are leaves should have had $\xi_{s-1}(u) = 1$ since otherwise $\xi_{s-2}(v_\tau) = -1$, which contradicts the induction hypothesis. Thus, $s - 1$

was the first time step when at least one of the neighbors of v_t from Q , namely v_{t-1} or v_{t+1} , had opinion -1 . By the induction hypothesis, it might only be possible if $s = t + 2$ and $\tau = t + 1$. Hence, $\xi_{t+2}(v_{t'}) = 1$ for all $t' \geq t + 2$, completing the proof. $\square$

4 | Perfect Binary Trees: Proof of Theorem 1.2

In this section, we prove Theorem 1.2. It asserts both the lower and the upper bound for the typical stabilization time on a perfect binary tree. We prove these two bounds separately and, for convenience, state them as two lemmas in what follows. Before proceeding with their formulations and proofs, we introduce concepts of weak and strong stability.

The entire proof is organized as follows. In Section 4.1, we define weak and strong stability and discuss their basic properties. In particular, in Claim 4.6, we state lower bounds on probabilities of weak and strong stability for early time steps. In Section 4.2, we prove the lower bound from Theorem 1.2, and in Section 4.3, we prove the upper bound. Finally, in Section 4.4, we prove Claim 4.6.

4.1 | Weak and Strong Stability

For every positive integer h , we fix a perfect binary tree $T^{(h)}$ of depth h rooted in a vertex R . When the height is clear from the context, we omit the superscript and write simply T . For a uniformly random ξ_0 on $V(T)$, we let $\tau_T := \tau(T; \xi_0)$.

Let us now fix some $h \in \mathbb{Z}_{>0}$, $t \in \mathbb{Z}_{\geq 0}$, a vertex $v \in V(T = T^{(h)})$, $v \neq R$, and a vector of initial opinions $\xi_0 : V(T) \rightarrow \{-1, 1\}$. If v is not a leaf, then we call v *strongly t -stable* (w.r.t. ξ_0), if it is t -stable with the same opinion (i.e., $\xi_t(v) = \tilde{\xi}_t(v)$) for *any* other vector of initial opinions $\tilde{\xi}_0 : V(T) \rightarrow \{-1, 1\}$ such that $\tilde{\xi}_0|_{V(T_v)} = \xi_0|_{V(T_v)}$. Similarly, v (whether it is a leaf or not) is *weakly t -stable* (w.r.t. ξ_t), if v is 0-stable for *some* other vector of initial opinions $\tilde{\xi}_0 : V(T) \rightarrow \{-1, 1\}$ such that $\tilde{\xi}_0|_{V(T_v)} = \xi_t|_{V(T_v)}$.

Claim 4.1. *Let $u \in V(T)$ be a parent of a vertex $v \in V(T)$ and $t \in \mathbb{Z}_{\geq 0}$. The following three statements are equivalent.*

1. *The vertex v is weakly t -stable.*
2. *For the vector $\tilde{\xi}_0^* : V(T) \rightarrow \{-1, 1\}$ such that $\tilde{\xi}_0^*|_{V(T_v)} = \xi_t|_{V(T_v)}$ and $\tilde{\xi}_0^*(\{V(T) \setminus V(T_v)\}) = \{\xi_t(v)\}$, the vertex v is 0-stable w.r.t. $\tilde{\xi}_0^*$.*
3. *For every odd $k > 0$ and $\hat{\xi}_0 : V(T) \rightarrow \{-1, 1\}$ such that $\hat{\xi}_0|_{V(T_v)} = \xi_t|_{V(T_v)}$ and $\hat{\xi}_j(u) = \hat{\xi}_0(v)$ for every odd $j \in \{1, 3, \dots, k\}$, it holds that $\hat{\xi}_{k+1}(v) = \hat{\xi}_0(v)$.*

Proof. (1 $\implies$ 2) Let $\tilde{\xi}$ be any vector from the definition of weak t -stability. Let us note that for every $w \in V(T)$, if $\tilde{\xi}_0(w) = \xi_t(v)$ then $\tilde{\xi}_0^*(w) = \xi_t(v)$. So, by induction on k , we get that for every $k \in \mathbb{Z}_{\geq 0}$, if $\tilde{\xi}_k(w) = \xi_t(v)$, then $\tilde{\xi}_k^*(w) = \xi_t(v)$. Hence, if v is 0-stable w.r.t. $\tilde{\xi}_0$, then it is also 0-stable w.r.t. $\tilde{\xi}_0^*$.

(2 $\implies$ 3) Fix an odd $k > 0$, a vector $\hat{\xi}_0$ from the third definition of weak stability, and the vector $\tilde{\xi}_0^*$ from the second definition. Let us note that u is 1-stable w.r.t. $\tilde{\xi}_0^*$ with $\tilde{\xi}_1^*(u) = \tilde{\xi}_0^*(v) = \xi_t(v)$ since its child v is 0-stable and all vertices outside $V(T_v)$ have initial opinion $\tilde{\xi}_0^*(v)$. So, for every odd $j \in \{1, 3, \dots, k\}$,

$$\hat{\xi}_j(u) = \hat{\xi}_0(v) = \xi_t(v) = \tilde{\xi}_1^*(u) = \tilde{\xi}_j^*(u).$$

Therefore, for every $i \in \{0, 1, \dots, k\}$ and for every vertex $w \in V(T_v)$ such that $d_T(w, v)$ has the same parity as i , $\hat{\xi}_i(w) = \tilde{\xi}_i^*(w)$. Indeed, for $i = 0$, this is clearly true since $\tilde{\xi}_0^*|_{V(T_v)} = \hat{\xi}_0|_{V(T_v)}$. Assuming that $\hat{\xi}_{i-1}(w) = \tilde{\xi}_{i-1}^*(w)$ for every vertex $w \in V(T_v)$ such that $d_T(w, v)$ has the same parity as $i - 1$, we immediately get that $\hat{\xi}_i(w) = \tilde{\xi}_i^*(w)$ for every vertex $w \in V(T_v)$ such that $w \neq v$ and $d_T(w, v)$ has the same parity as i . Moreover, if i is odd, then we do not have to consider the vertex $w = v$ at time i . If i is even, then $\hat{\xi}_i(v) = \tilde{\xi}_i^*(v)$, since opinions in these two processes coincide for children of v at time $i - 1$, and the parent u has equal opinions $\hat{\xi}_{i-1}(u) = \tilde{\xi}_{i-1}^*(u)$ as we have just proved as $i - 1$ is odd. So, indeed, $\hat{\xi}_{k+1}(v) = \tilde{\xi}_{k+1}^*(v) = \tilde{\xi}_0^*(v) = \hat{\xi}_0(v)$.

(3 $\implies$ 1) Let us note that u is 1-stable w.r.t. the vector $\tilde{\xi}_0^*$ introduced in the second definition of weak stability with $\tilde{\xi}_1^*(u) = \tilde{\xi}_0^*(v)$, since u has degree 3 and all vertices outside $V(T_v)$ have initial opinion $\tilde{\xi}_0^*(v)$. So, the vector $\tilde{\xi}_0^*$ satisfies the

conditions in the third definition of weak stability; thus we set $\hat{\xi}_0 := \xi_0^*$. We know that $\hat{\xi}_{k+1}(v) = \hat{\xi}_0(v)$ for all odd k , that is v is 0-stable w.r.t. $\hat{\xi}_0$. Hence, $\hat{\xi}_0 := \xi_0^*$ witnesses the weak t -stability of v . $\square$

Claim 4.2. Let $u \in V(T)$ be a parent of a weakly t_1 -stable vertex $v \in V(T)$ for $t_1 \in \mathbb{Z}_{\geq 0}$. Let $t_2 > t_1$ have the same parity as t_1 . For every time step $t \in \{t_1 + 1, t_1 + 3, \dots, t_2 - 1\}$ let $\xi_t(u) = \xi_{t_1}(v)$. Then, $\xi_{t_2}(v) = \xi_{t_1}(v)$.

Proof. Let us apply the third equivalent definition from Claim 4.1 with $\hat{\xi}_0 := \xi_{t_1}$ and $k := t_2 - t_1 - 1$. Then, $\hat{\xi}_t = \xi_{t_1+t}$ for any $t > 0$. So, $\hat{\xi}_j(u) = \hat{\xi}_0(v)$ for every odd $j \in \{1, 3, \dots, k\}$. So it holds that $\hat{\xi}_{k+1}(v) = \hat{\xi}_0(v)$. Hence, $\xi_{t_2}(v) = \hat{\xi}_{k+1}(v) = \hat{\xi}_0(v) = \xi_{t_1}(v)$, as needed. $\square$

Claim 4.3. Let $t_1, t_2 \in \mathbb{Z}_{\geq 0}$ be such that $t_2 - t_1 \in 2\mathbb{Z}_{>0}$. Let $v \in V(T) \setminus \{R\}$ be weakly t_1 -stable, and, for every $t \in \{t_1, t_1 + 2, \dots, t_2\}$, let $\xi_t(v) = \xi_{t_1}(v)$, then v is weakly t_2 -stable.

Proof. It is sufficient for us to prove the statement when $t_2 = t_1 + 2$, as the former statement follows from applying the latter several times.

Let us consider the extension $\tilde{\xi}_0^*$ from the second definition of weak t_1 -stability from Claim 4.1: $\tilde{\xi}_0^*|_{V(T_v)} = \xi_{t_1}|_{V(T_v)}$ and $\tilde{\xi}_0^*|_{V(T) \setminus V(T_v)} \equiv \xi := \xi_{t_1}(v)$. Then v is 0-stable and, hence, 2-stable (w.r.t. $\tilde{\xi}^*$). Moreover, all vertices of T outside T_v share opinion ξ at time Step 2.

To apply the second definition from Claim 4.1 to show the weak $(t_1 + 2)$ -stability of v , it is sufficient to prove that $\tilde{\xi}_2^*|_{V(T_v)} = \xi_{t_1+2}|_{V(T_v)}$ for all vertices at even distance from v (including v itself). The latter is obvious for all vertices from T_v that are at positive even distance from v since their $\tilde{\xi}_2^*$ - and ξ_{t_1+2} -opinions are defined by equal vectors $\tilde{\xi}_0^*|_{T_v}$ and $\xi_{t_1}|_{T_v}$ respectively. We complete the proof by recalling that $\xi_{t_1+2}(v) = \tilde{\xi}_2^*(v) = \xi$, given by the conditions of the claim. $\square$

Note that both weak 0-stability and strong t -stability properties are defined by $\xi_0|_{V(T_v)}$ only. Now, letting ξ_0 to be chosen uniformly at random, we set

$$p_s(t, h, v) := \mathbb{P}(v \text{ is strongly } t\text{-stable in } T^{(h)}), \quad p_w(t, h, v) := \mathbb{P}(v \text{ is weakly } t\text{-stable in } T^{(h)}).$$

Remark 4.4. Note that, for any non-overlapping subsets $U_1, U_2 \subset V(T = T^{(h)})$ events defined by $\xi_0|_{U_1}$ and $\xi_0|_{U_2}$ are independent. In particular, if any vertex from a set $U \subset V(T)$ is not a descendant of any other vertex from U , then the events $\{v \text{ is strongly } t\text{-stable/weakly 0-stable in } T^{(h)}\}, v \in U$, are (mutually) independent.

Remark 4.5. Due to symmetry, a strongly t -stable vertex has equal probabilities for both opinions at time t , that is, the probability that $v \in V(T) \setminus \{R\}$ is strongly t -stable and $\xi_t(v) = -1$ equals to $p_s(t, h, v)/2$.

We also set

$$p_s(t) := \inf_{h \in \mathbb{Z}_{>0}, v \in V(T=T^{(h)}) \setminus R: h_T(v) > 0} p_s(t, h, v), \quad p_w(t) := \inf_{h \in \mathbb{Z}_{>0}, v \in V(T=T^{(h)}) \setminus R} p_w(t, h, v).$$

The key to most of our calculations in the proof of Theorem 1.2 is the following Claim 4.6.

Claim 4.6. We have the following bounds on $p_s(3), p_s(2), p_s(1), p_s(0)$, and $p_w(0)$:

$$p_s(3) > 0.4453, \quad p_s(2) > 0.5617, \quad p_s(1) \geq \frac{1}{2}p_w(0)^2, \quad p_s(0) > 0.5, \quad p_w(0) > 0.925.$$

We postpone the proof of Claim 4.6 to Section 4.4. In the subsequent section, we state and prove the lemma that asserts the lower bound in Theorem 1.2.

4.2 | Proof of the Lower Bound

Here we prove the following:

Lemma 4.7. Let T be an R -rooted perfect binary tree of height h . Let $\xi_0 \in \{-1, 1\}^{V(T)}$ be chosen uniformly at random. Then there exists $c_- > 0.5$ such that whp, as $h \rightarrow \infty$, there is at least one not $\lfloor c_- \cdot h \rfloor$ -stationary vertex.

Proof. Let us note that

$$0.25003 < \left(\log_2 \left(\frac{4}{0.2501} \right) \right)^{-1} < \left(\log_2 \left(\frac{4}{p_s(2)p_s(3)} \right) \right)^{-1}, \quad (1)$$

due to Claim 4.6. We fix an odd number d satisfying

$$0.25003 \cdot h < d < \left(\log_2 \left(\frac{4}{0.2501} \right) \right)^{-1} \cdot h. \quad (2)$$

Let $R^* \in V(T)$ be at distance d from a leaf, define $T^* := T_{R^*}$. First, we will define a class $\mathcal{F}$ of functions $\xi_0^* : V(T^*) \rightarrow \{-1, 1\}$ such that for each initial opinion ξ_0^* on $V(T^*)$ from $\mathcal{F}$ there exists a vertex $u \in V(T^*)$ which is not $(2d - 9)$ -stable for every extension of ξ_0^* to the entire $V(T)$. Secondly, we will show that this class is so large that whp, for a uniformly random $\xi_0 \in \{-1, 1\}^{V(T)}$, there exists R^* as above so that $\xi_0|_{V(T^*)} \in \mathcal{F}$ leading to the fact that $\tau_T \geq 2d - 8$ whp. The latter immediately implies the statement of Lemma 4.7 due to (2). $\square$

4.2.1 | Definition of the Desired Set $\mathcal{F}$

Let v_0, v_1 be the children of R^* , v_2, v_3 be the children of v_1 , and v_4, v_5 be the children of v_3 . For a vertex $v \in V(T^*)$ and $r := h_{T^*}(v) \bmod 2$, let us define the event

$$E(v) = \{v \text{ is strongly } (r + 2)\text{-stable and } \xi_{r+2}(v) = -1, \text{ for any extension } \xi_0 \text{ to the entire } V(T)\}.$$

Note that this event is entirely described by the values of ξ_0^* at $V(T_v)$. The following requirements on ξ_0^* define the set $\mathcal{F}$ (for an illustration, we refer to Figure 1).

1. $\xi_0^*|_{V(T_{R^*}) \setminus V(T_{v_3})}$ satisfies the following condition: For any of its extensions ξ_0 to the entire $V(T)$, v_1 is 0-stable with $\xi_0(v_1) = -1$.
2. there exists a path $P_4 \subset T_{v_4}$ from v_4 to some vertex u_4 at distance 2 from a leaf such that
 - $\xi_0^*(v) = 1$ for every $v \in V(P_4)$ at even distance from a leaf;
 - $E(u)$ holds for the child $u \notin V(P_4)$ of every vertex $v \in V(P_4) \setminus \{u_4\}$;
 - $\xi_0^*(u) = -1$ for all grandchildren u of u_4 .
3. there exists a path $P_5 \subset T_{v_5}$ from v_5 to some vertex u_5 at distance 2 from a leaf such that
 - $\xi_0^*(v) = 1$ for every $v \in V(P_5)$ at even distance from a leaf;
 - $E(u)$ holds for the child $u \notin V(P_5)$ of every vertex $v \in V(P_5) \setminus \{u_5\}$;
 - $\xi_0^*(u) = 1$ for all grandchildren u of u_5 .

Now we show that the elements ξ_0^* of $\mathcal{F}$ have the required property of the existence of not $(2d - 9)$ -stable vertices (i.e., if $\xi_0|_{T_{R^*}} = \xi_0^* \in \mathcal{F}$, then there exists a not $(2d - 9)$ -stable vertex). Namely, we will prove that:

- (1) first, sequentially from u_4 to v_4 , vertices along P_4 adopt opinion -1 for the first time (among those time steps that have the same parity as their heights),

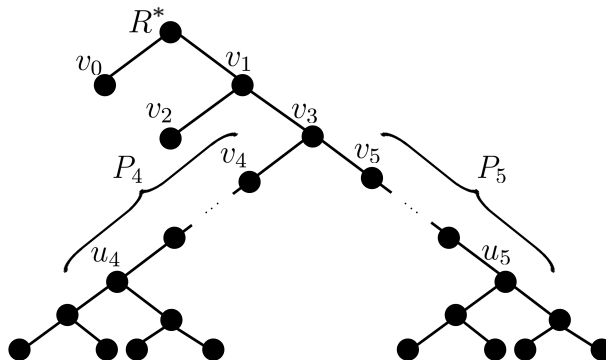


FIGURE 1 | Structure of the tree T^* .

- (2) then, v_3 changes its opinion for the first time (among those time steps that have the same parity as $h_T(v_3)$),
- (3) finally, after the above two ‘phases’, vertices of $P_5 \setminus \{u_5\}$ change their opinions for the first time (among those time steps that have the same parity as their heights) sequentially in the opposite direction from v_5 to u_5 .

This can be easily shown by induction, as we will see soon. Consider the set of vertices $S := V(P_4) \cup \{v_3\} \cup V(P_5) \setminus \{u_5\}$. The path induced by S in T defines on S the order with the least element u_4 . Let $t : S \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ map $v \in S$ to the first time step such that $\xi_{t(v)}(v) = -1$, among the time steps with the same parity as the height of v . Recall that d is odd. To show the existence of a not $(2d - 9)$ -stable vertex, it suffices to show by induction that, if $v \in S$ is a preceding neighbor of $u \in S$ and $t(v) < \infty$, then $t(v) = t(u) - 1$. Indeed, by the definition of $\mathcal{F}$, $t(u_4) = 2$. Therefore, we get (1), (2), (3), and eventually, for the parent u of u_5 , $t(u) = 2d - 7$, as required. So, let us finally show $t(v) = t(u) - 1$ for v and u as above.

First, let us prove the base of induction: $t(v) = t(u) - 1$ if $t(v) < \infty$, where u is the parent of u_5 and v is the parent of u . Indeed, u_5 is 0-stable since all its grandchildren are leaves and have initial opinion 1, and u_5 has opinion 1 by the definition of $\mathcal{F}$. So, each of these five vertices remains opinion 1 at every even time step. Let $w \neq u_5$ be the other child of u . Then, by definition, the event $E(w)$ holds. Hence, $\xi_t(w) = -1$ for all even $t \geq 0$. Therefore, for every odd $t \geq 1$, $\xi_t(u) = \xi_{t-1}(v)$. So, first, $\xi_0(v) = \xi_1(u) = 1$ and, second,

$$\begin{aligned} t(u) - 1 &= \min\{t > 1, t \text{ is odd} : \xi_t(u) = -1\} - 1 \\ &= \min\{t > 0, t \text{ is even} : \xi_t(v) = -1\} = t(v). \end{aligned}$$

Now, suppose that the induction hypothesis is true for some $v' \in S$. We have to prove that it is also true for the preceding neighbor u of v' . Indeed, let v be the preceding neighbor of u . By the definition of $\mathcal{F}$, for every $z \in \{v, u, v'\}$ holds $\xi_{(h_{T^*}(z) \bmod 2)}(z) = 1$, hence $t(u) \geq 2$. In addition, the neighbor w of u other than v and v' is $(r + 2)$ -stable with opinion $\xi_{r+2}(w) = -1$, where $r = (h_{T^*}(v') \bmod 2)$. Therefore, $\min(t(v), t(v')) \geq t(w)$. So, u changes its opinion to -1 exactly when at least one of its two neighbors v, v' changes its opinion to -1 , that is $t(u) = \min(t(v), t(v')) + 1$ (even for infinite $t(u)$). So, if $t(v) < \infty$, then $t(u) < \infty$. Also, by the induction step, $t(v') > t(u)$. Therefore, $t(u) - 1 = t(v)$, completing the inductive proof.

4.2.2 | The Existence of R^*

We have to show that the three requirements in the definition of $\mathcal{F}$ are met sufficiently frequently. Let us note that the first requirement is achieved if the following holds:

- $\xi_0(v_0) = \xi_0(v_1) = \xi_1(v_2) = -1$;
- v_0 is strongly 0-stable;
- v_2 is strongly 1-stable.

Indeed, this implies that -1 is the common opinion of both v_0 and v_1 at every even time step and both R^* and v_2 at every odd time step. The probability of the latter event is at least $\frac{1}{8}p_s(1)p_s(0) \geq \frac{1}{16}p_w(0)^2p_s(0) > 0$ due to Remarks 4.4 and 4.5, and Claim 4.6.

In what follows, we prove coinciding bounds for probabilities of the second and third requirements in the definition of $\mathcal{F}$. Since the proofs are literally the same (one is obtained from another by replacing P_4, v_4, u_4 with P_5, v_5, u_5 respectively, and by changing the number of the requirement), we provide the proof of the bound for the second requirement only.

There are exactly 2^{d-5} different paths from v_4 to a grandparent $u_4 \in V(T_{v_4})$ of a leaf. Fix one such path P . For every $i \in \{2, \dots, d-4\}$, we denote by w_i the unique vertex at distance 1 from P with $h_T(w_i) = i$. The probability that the requirement 2 from the definition of $\mathcal{F}$ holds for this P playing the role of P_4 (we denote this event by $B(P)$) is

$$q := 2^{-(d-3)/2-4} \cdot \prod_{i=2}^{d-4} \frac{p_s((i \bmod 2) + 2, h, w_i)}{2},$$

due to Remarks 4.4 and 4.5.

By the definitions,

$$q \geq 2^{-(d-3)/2-4} \cdot \left(\frac{p_s(2) \cdot p_s(3)}{4} \right)^{(d-5)/2}.$$

Let us consider two paths P, P' in T_{v_4} from v_4 to grandparents of leaves and let assume that they fulfill the requirement 2 and separate at some vertex v (i.e., v is the closest to leaves vertex such that $v \in V(P) \cap V(P')$), then if $d(v, v_4) \leq d-7$ then the respective events $\mathcal{B}(P)$ and $\mathcal{B}(P')$ are disjoint. Indeed, let $w \in P, w' \in P'$ be two children of v . Since P satisfies the second requirement, as we proved above, the vertex w is not $((d(v, v_4) \bmod 2) + 2)$ -stable while the vertex w' is $((d(v, v_4) \bmod 2) + 2)$ -stable (the latter follows immediately from the second requirement as $E(w')$ holds). At the same time, the opposite is also true since we have to replace w and w' in the second requirement for the path P' —a contradiction.

So, if P and P' differ by at least one of their $d-7$ vertices closest to v_4 , excluding v_4 , then events $\mathcal{B}(P)$ and $\mathcal{B}(P')$ do not intersect. Therefore, probability that $\mathcal{B}(P)$ holds for some P is at least

$$2^{d-7} \cdot q \geq 2^{-(d-3)/2-6} \cdot (p_s(2) \cdot p_s(3))^{(d-5)/2}.$$

Let us notice, that the three requirements in the definition of $\mathcal{F}$ are defined by initial opinions of vertices from three non-overlapping sets. In particular, the second and the third requirements are supported by vertices from $V(T_{v_4})$ and $V(T_{v_5})$, respectively. Thus, due to Remark 4.4, the three events that a uniformly random ξ_0^* satisfies each of the requirements are independent. We conclude the asymptotic lower bound

$$\mathbb{P}(\xi_0^* \in \mathcal{F}) = \Omega\left(\left(\frac{p_s(2)p_s(3)}{2}\right)^d\right).$$

The events $\xi_0^* := \xi_0|_{T_{R^*}} \in \mathcal{F}$ are independent over all R^* with $h_T(R^*) = d$ by Remark 4.4. Due to (1) and (2),

$$3 \cdot 2^{h-d-1} \cdot \mathbb{P}(\xi_0^* \in \mathcal{F}) = \Omega(2^{h-2d}(p_s(2)p_s(3))^d) = \omega(1), \quad (3)$$

and then whp the event $\xi_0^* \in \mathcal{F}$ holds for at least one R^* , as needed.

The proof of the lower bound in Theorem 1.2 is complete, and we now switch to the upper bound, which is separately stated in the subsequent section.

4.3 | Proof of the Upper Bound

This section is devoted to the proof of the upper bound in Theorem 1.2, which is, in somewhat different terms, stated below.

Lemma 4.8. *Let T be an R -rooted perfect binary tree of height h . Let $\xi_0 \in \{-1, 1\}^{V(T)}$ be chosen uniformly at random. Then there exists $c_+ < 2/3$ such that whp, as $h \rightarrow \infty$, all vertices are $\lceil c_+ \cdot h \rceil$ -stationary.*

We postpone the proof of Lemma 4.8 to the end of this section since it uses several auxiliary stability properties that are stated and proved in Section 4.3.1. We complete the proof of Lemma 4.8 in Section 4.3.2.

Let us first briefly overview the proof. We note that a vertex v changes its opinion at a time step t if there is a path P ending in v such that the opinion “travels” along this path. We show that, for a fixed path P , the probability that an opinion “travels” along P is exponentially small in terms of the length of the path and then apply the union bound over all paths of the same length, concluding the proof of the upper bound.

To get the exponentially small probability bound of the described event, we observe specific properties of the neighborhood of P that prevent the travel of an opinion along P (due to Claims 4.11 and 4.12 stated in the next section). Probabilities of these preventing properties are estimated via Claims 4.13 and 4.14 stated in the next section.

4.3.1 | Auxiliary Claims

In this section, we prove six claims that we will use in the proof of Lemma 4.8 in Section 4.3.2. Claims 4.9 and 4.10 describe sufficient conditions for the weak t -stability to be inherited from some descendants of a vertex. Claims 4.11 and 4.12 describe sufficient conditions for stabilization of opinions of a path between two weakly t -stable vertices. In particular, Claim 4.11 deals with the case when these weakly t -stable vertices have the same opinion, and Claim 4.12 deals with the case when these vertices have different opinions, requiring an additional property of being 1-close to stability. Claims 4.13 and 4.14 describe bounds on probabilities of some specific events.

Claim 4.9. Let $t \in \mathbb{Z}_{\geq 0}$ and let $u, v \in V(T) \setminus \{R\}$ be such that

- u is a grandchild of v ;
- u is weakly t -stable;
- $\xi_t(v) = \xi_t(u)$.

Then v is weakly t -stable.

Proof. Consider the vector of initial opinions $\tilde{\xi}_0$ defined as

- $\tilde{\xi}_0(w) = \xi_t(w)$ for every $w \in V(T_v)$;
- $\tilde{\xi}_0(w) = \xi_t(v) =: \xi$ for every $w \in V(T) \setminus V(T_v)$.

Let us show that $\tilde{\xi}_0$ is the desired vector from the definition of t -stability, that is, that v is 0-stable for this vector of initial opinions. So, further in the proof, we consider majority dynamics on T with the vector of initial opinions $\tilde{\xi}_0$. Clearly, the parent of v has opinion ξ at every time step since it has degree 3 (note that R has degree 3 by the definition of a binary tree). Let us prove that both u and v have opinion ξ at every even time step. Assume the contrary and let $t_1 > 0$ be the first even time step such that this does not hold. Therefore, for every odd time step $t_0 < t_1$ the parent of u has opinion ξ as it is adjacent to both u and v . So, $\tilde{\xi}_{t_1}(u) = \xi$ by the third equivalent definition of weak t -stability from Claim 4.1. Moreover, $\tilde{\xi}_{t_1}(v) = \xi$ since the parent of v has opinion ξ at time $t_1 - 1$ as we mentioned above—a contradiction. $\square$

Claim 4.10. Let $t \in \mathbb{Z}_{\geq 0}$ and let $u, v \in V(T) \setminus \{R\}$ be such that

- u is a child of v ;
- u is weakly t -stable;
- $\xi_{t+1}(v) = \xi_t(u)$.

Then v is weakly $(t + 1)$ -stable.

Proof. Consider the vector of initial opinions $\tilde{\xi}_0$ defined as

- $\tilde{\xi}_0(w) = \xi_{t+1}(w)$ for every $w \in V(T_v)$;
- $\tilde{\xi}_0(w) = \xi_{t+1}(v) =: \xi$ for every $w \in V(T) \setminus V(T_v)$.

As in the proof of the previous claim, we consider majority dynamics on T with the vector of initial opinions $\tilde{\xi}_0$, and so it remains to prove that v is 0-stable in this process. We note that the parent of v has opinion ξ always. Let us show that u has opinion ξ at every odd time step. This will clearly lead to 0-stability of v . Assume the contrary: Let $t_1 > 0$ be the first odd moment such that $\tilde{\xi}_{t_1}(u) \neq \xi$. Therefore, for every even time step $t_0 < t_1$ vertex v has opinion ξ . So, we can conclude a contradiction: $\tilde{\xi}_{t_1}(u) = \xi$ by the third equivalent definition of weak t -stability from Claim 4.1—let us elaborate on that below. Consider the following vector of initial opinions $\hat{\xi}$:

- $\hat{\xi}_0(w) = \xi_t(w)$ for every $w \in V(T_v)$;
- $\hat{\xi}_0(w) = \xi_t(v) =: \xi$ for every $w \in V(T) \setminus V(T_v)$.

Let us show that $\hat{\xi}_1 = \tilde{\xi}_0$. For every $w \in V(T_v) \setminus \{v\}$ it is clear that $\hat{\xi}_1(w) = \xi_{t+1}(w) = \tilde{\xi}_0(w)$. Next, for every $w \in V(T) \setminus V(T_v)$, we have $\hat{\xi}_1(w) = \xi = \tilde{\xi}_0(w)$. Last, for every neighbor w of v in T we either have $\hat{\xi}_0(w) = \xi_t(w)$ or $\hat{\xi}_0(w) = \xi$. So, as $\xi_{t+1}(v) = \xi$, it also holds that $\hat{\xi}_1(v) = \xi$, which, in turn, also equals $\tilde{\xi}_0(v)$. Thus, $\hat{\xi}_1 = \tilde{\xi}_0$. We get $\hat{\xi}_{t+1}(u) = \tilde{\xi}_{t_1}(u) = \xi$ —a contradiction with the third equivalent definition from Claim 4.1. $\square$

Claim 4.11. Let $t \in \mathbb{Z}_{\geq 0}, \xi \in \{-1, 1\}, d \in 2\mathbb{Z}_{>0}$, and a path $P = v_0 v_1 \cdots v_d$ in T be such that

- $T_{v_0} \cap T_{v_d} = \emptyset$ (i.e., the path is not monotone);
- $\xi_t(v_i) = \xi$ for every even $i \in \{0, \dots, d\}$;
- v_0 and v_d are weakly t -stable.

Then, v_i is t -stable for every even i .

Proof. Let us prove Claim 4.11 by contradiction. Let $t_1 > t$ be the first time step with the same parity as t such that, for some even $i \in \{0, \dots, d\}$, $\xi_{t_1}(v_i) \neq \xi$. If $i \notin \{0, d\}$ then, $\xi_{t_1-2}(v_{i-2}) = \xi_{t_1-2}(v_i) = \xi_{t_1-2}(v_{i+2}) = \xi$, hence $\xi_{t_1-1}(v_{i-1}) = \xi_{t_1-1}(v_{i+1}) = \xi$, and hence $\xi_t(v_i) = \xi$ —a contradiction. If $i \in \{0, d\}$, then the only neighbor of v_i from $V(P)$ is the parent of v_i in T . Clearly, this neighbor has opinion ξ at every time step $t_0 \in \{t+1, t+3, \dots, t_1-1\}$. Therefore, by Claim 4.2, we get $\xi_{t_1}(v_i) = \xi$ —a contradiction. $\square$

Let us say that a non-leaf vertex $v \in V(T)$, $v \neq R$, is *1-close to stability* (w.r.t. ξ_0) if the following property holds. For all $\tilde{\xi}_0 : V(T) \rightarrow \{-1, 1\}$ such that $\tilde{\xi}_0|_{V(T_v)} = \xi_0|_{V(T_v)}$ and $\tilde{\xi}_t(v) \neq \tilde{\xi}_0(v)$ for some even $t > 0$, the vertex v is weakly t -stable w.r.t. $\tilde{\xi}_t$, where t is the first even moment when $\tilde{\xi}_t(v) \neq \tilde{\xi}_0(v)$.

Claim 4.12. Let $t \in 2\mathbb{Z}_{\geq 0}, d \in 2\mathbb{Z}_{>0}$, even $\ell \in [d]$, and a path $P = v_0 v_1 \cdots v_d$ in T be such that

- $T_{v_0} \cap T_{v_d} = \emptyset$ (i.e., the path is not monotone);
- $\xi_t(v_i) = \xi_t(v_0)$ for every even $i \in [\ell - 2]$;
- $\xi_t(v_i) = \xi_t(v_d)$ for every even $i \in [d - 2] \setminus [\ell - 2]$;
- $\xi_s(v_0) = \xi_0(v_0) \neq \xi_0(v_d) = \xi_s(v_d)$ for every even $s \in [t]$;
- v_0 and v_d are weakly 0-stable;
- v_0 and v_d are 1-close to stability.

Then, for every even $t' \geq t$ and every even $i \in \{2, \dots, d - 2\}$, $\xi_{t'}(v_i) \in \{\xi_{t'}(v_{i-2}), \xi_{t'}(v_{i+2})\}$. In addition, for every even $t' \geq t$, v_d is either weakly t' -stable or t' -stable.

Proof. Let t_1 be the first even time step such that either $\xi_{t_1}(v_0) \neq \xi_0(v_0)$ or $\xi_{t_1}(v_d) \neq \xi_0(v_d)$. From the fourth condition of the claim, it follows that $t_1 > t$. We shall prove that

A1 for every time step $t' \in \{t, t+1, \dots, t_1-1\}$, there exists $\ell(t') \in [d+1]$ with the same parity as t' , such that, for every $i \in \{0, 1, \dots, d\}$ with the same parity as t' ,

$$\xi_{t'}(v_i) = \xi_t(v_0) \quad \text{if and only if} \quad i < \ell(t');$$

A2 all v_i , over even $i \in \{0, 1, \dots, d\}$, share the same opinion at time step t_1 and are t_1 -stable.

Let us note that Claim 4.12 indeed follows from the latter statement. First, we have that $\xi_{t'}(v_i) \in \{\xi_{t'}(v_{i-2}), \xi_{t'}(v_{i+2})\}$ for every even $i \in \{2, \dots, d - 2\}$ both for all even $t' \in \{t, t+2, \dots, t_1-2\}$ due to A1 and all even $t' \geq t_1$ due to A2. In addition, the vertex v_d has not changed its opinion up to time $t_1 - 2$, so v_d is weakly t' -stable for all even $t' \in \{t, t+2, \dots, t_1-2\}$ due to Claim 4.3 and the fifth requirement of the current claim. Finally, from A2, for every even $t' \geq t_1$, the vertex v_d is t' -stable.

Let us prove A1 by induction on $t' \in \{t, t+1, \dots, t_1-1\}$. The base of induction $t' = t$ follows from the second, the third, and the fourth requirements in the statement of the claim by taking $\ell(t') := \ell$. For the induction step from $t' - 1$ to t' ,

let us note that a vertex sandwiched between two vertices with the same opinion will adopt that opinion at the next step. Also, for even time steps t' , we assumed that $\xi_{t'}(v_0) = \xi_t(v_0)$ and $\xi_{t'}(v_d) = \xi_t(v_d)$. So, $v_{\ell(t'-1)-1}$ is the only vertex whose opinion is not determined by the previous step. Hence, depending on its opinion $\ell(t')$ either equals $\ell(t' - 1) - 1$ or $\ell(t' - 1) + 1$. Finally, let us prove A2. Without loss of generality, let v_0 change its opinion at time step t_1 . By Claim 4.3, v_0 is weakly $(t_1 - 2)$ -stable. By weak $(t_1 - 2)$ -stability and Claim 4.2, if $\xi_{t_1-1}(v_1) = \xi_{t_1-2}(v_0)$, then $\xi_{t_1}(v_0) = \xi_{t_1-2}(v_0)$. Hence, $\xi_{t_1-1}(v_1) \neq \xi_{t_1-2}(v_0)$, which means that $\ell(t_1 - 1) = 1$. Hence, v_d does not change its opinion at time step t_1 because $\ell(t_1 - 1) \neq d + 1$. Due to Claim 4.3, v_d is weakly t_1 -stable, and v_0 is weakly t_1 -stable since it is 1-close to stability. Lastly, as $\ell(t_1 - 1) = 1$, for every even $i \in [d]$ holds $\xi_{t_1}(v_i) = \xi_{t_1}(v_d) = \xi_{t_1}(v_0)$. So, the statement follows from application of Claim 4.11 to $v_0 v_1 \cdots v_d$. $\square$

Claim 4.13. Let $\xi_0 \in \{-1, 1\}^{V(T)}$ be uniformly random. For any non-leaf vertex $v \in V(T)$, $v \neq R$,

$$\mathbb{P}(v \text{ is 1-close to stability}) \geq 0.9957.$$

Proof. Let us note that if $h_T(v) = 1$, then v is strongly 0-stable and, hence, v is 1-close to stability. So, we may assume that $h_T(v) \geq 2$. Let us first describe the proof strategy. For a child u of v , we will define three specific events expressed only in terms of initial opinions on $V(T_u)$. We will show that if v is not 1-close to stability, then for some child of v at least one of the three events holds. Then, we will bound probabilities of these three events and use these bounds to derive the statement of Claim 4.13. $\square$

Let $\xi_0^* : V(T) \rightarrow \{-1, 1\}$ be any vector of initial opinions such that v is not 1-close to stability w.r.t. ξ_0^* . Let $\xi := \xi_0^*(v)$. Then, there exists a vector of initial opinions $\tilde{\xi}_0^* : V(T) \rightarrow \{-1, 1\}$ and a time step $t_1 \in \mathbb{Z}_{\geq 0}$ such that the following holds:

- $\tilde{\xi}_0^*|_{V(T_v)} = \xi_0^*|_{V(T_v)}$;
- t_1 is the first even time step such that $\tilde{\xi}_{t_1}^*(v) \neq \xi$;
- v is not weakly t_1 -stable.

Let us prove that for some child u of v , one of the following disjoint properties holds.

- A1 At least one of the children of u has initial opinion ξ . For every child v_1 of u with initial opinion ξ , there exists an even time step $t_2 < t_1$ such that $\tilde{\xi}_{t_2}^*(v_1) \neq \xi$ for the first time and v_1 is not t_2 -stable.
- A2 Both children of u have initial opinion opposite to ξ and are not weakly 0-stable.
- A3 Both children of u have initial opinion opposite to ξ . One child v_1 of u is weakly 0-stable. There exists an even time step $t_2 < t_1$ such that the other child v_2 of u has opinion ξ at time step t_2 for the first time and v_2 is not weakly t_2 -stable.

Suppose, towards a contradiction, that A1, A2, and A3 do not hold. Let us prove that v is weakly t_2 -stable, which will lead to a contradiction.

Note that $t_1 \geq 2$. Let u be a child of v such that $\tilde{\xi}_{t_1-1}^*(u) \neq \xi$. Then, both children of u have opinion opposite to ξ at time step $t_1 - 2$.

Suppose at least one child of u has opinion ξ at time step zero. For a child v' of u , let $t_2(v') \leq t_1 - 2$ be the first even time step such that $\tilde{\xi}_{t_2}^*(v') \neq \xi$. Since, by assumption, A1 does not hold for u , there exist a child v_1 of u such that v_1 is $t_2(v_1)$ -stable and has initial opinion ξ . So, $\tilde{\xi}_{t_1}^*(v) = \tilde{\xi}_{t_1}^*(v_1)$ and v_1 is t_1 -stable. Hence, v is weakly t_1 -stable—a contradiction.

Now let both children of u have initial opinions opposite to ξ . If both of them are weakly 0-stable, then both of them are 0-stable due to Claim 4.11. So, v is weakly t_1 -stable—a contradiction. Due to the negation of A2, at least one of the children of u has to be weakly 0-stable. So it remains to consider the case when only one child v_1 of u is weakly 0-stable. Let $t_2 > 0$ be the first even time step such that the other child v_2 has opinion ξ which is opposite to $\tilde{\xi}_0^*(v_2)$. If $t_2 < t_1$, then v_2 is weakly t_2 -stable, due to the negation of A3. Since $\tilde{\xi}_{t_2}^*(v) = \xi$, we get that $\tilde{\xi}_{t_2+1}^*(u) = \xi$ as well. Due to weak t_2 -stability and Claim 4.2, $\tilde{\xi}_{t_2+2}^*(v_2) = \xi$. Proceeding by induction, we get that for every even time step $t_2 \leq t < t_1$, $\tilde{\xi}_t^*(v_2) = \xi$. Hence, $\tilde{\xi}_{t_1-1}^*(u) = \xi$ —a contradiction. Lastly, if $t_2 \geq t_1$, then, in the same way as above, applying inductively Claim 4.2,

we get $\tilde{\xi}_0^*(v_1) = \tilde{\xi}_2^*(v_1) = \dots = \tilde{\xi}_{t_2}^*(v_1)$. So, v_1 is weakly t_1 -stable by Claim 4.3. Due to Claim 4.9, v is weakly t_1 -stable—a contradiction.

Let us fix an opinion $\xi \in \{-1, 1\}$. Everywhere below we assume that the event $\xi_0(v) = \xi$ holds deterministically. Now, fix any child u of v , and, for every $i \in \{1, 2, 3\}$, denote by p_i the probability of the following event Q_i : There exists an extension $\tilde{\xi}_0$ of $\xi_0|_{V(T_u)}$ to the whole tree and time step t_1 such that

- t_1 is the first even time step when $\tilde{\xi}_{t_1}(v) \neq \tilde{\xi}_0(v) = \xi$;
- the property A_i holds with $\tilde{\xi}_i^*$ replaced by $\tilde{\xi}_i$.

The rest of the proof is devoted to estimating probabilities p_1, p_2, p_3 since, due to Remark 4.4 and the union bound, it is sufficient for us to show $(1 - p_1 - p_2 - p_3)^2 > 0.9957$.

Let us start with p_1 . First, note that at least one child of u has initial opinion ξ with probability $3/4$. Let us now assume that, for some $\xi_0|_{V(T_u)}$, the considered event holds, and that an extension $\tilde{\xi}_0|_{V(T)} \rightarrow \{-1, 1\}$ certifies the validity of this event. Let v_1 be a child of u with initial opinion $\tilde{\xi}_0(v_1) = \xi$. Then v_1 has to change its opinion for the first time at some even time step $t_2 < t_1$. Hence, v_1 is not weakly 0-stable by Claim 4.2.

Let us prove that $h_T(v_1) \geq t_2$. First, let us note that for every $t > 1$, if a vertex changed its opinion to the opposite of ξ at time step $t + 1$, then one of its neighbors changed its opinion to the opposite of ξ at time step t . Now, $\tilde{\xi}_{t_2-1}(u) = \xi$ since $\tilde{\xi}_{t_2-2}(v) = \tilde{\xi}_{t_2-2}(v_1) = \xi$, so at least one of the children of v_1 should change opinion at time step $t_2 - 1$. Thus, we may apply a similar argument to the child of v_1 and then proceed by induction: For every $i \geq 1$, we have a vertex v_i and its child v_{i+1} such that $\tilde{\xi}_{t_2-i-1}(v_i) = \xi$ and v_{i+1} changed its opinion to the opposite to ξ at time step $t_2 - i$ (in particular, $\tilde{\xi}_{t_2-i-2}(v_{i+1}) = \xi$), then there is a neighbor v_{i+2} of v_{i+1} that changed its opinion at time $t_2 - i - 1$. Let us notice that v_{i+2} is actually a child of v_{i+1} , since, by induction hypothesis, the parent of v_{i+1} , that is v_i , changed its opinion to the opposite of ξ at time step $t_2 - i + 1$ and, hence, $\tilde{\xi}_{t_2-i-1}(v_i) = \xi$. Eventually we get a descending path of vertices of length $t_2 - 2$ starting at v_1 , where, in particular, for the last vertex in the path v_{t_2-1} the following holds: $\tilde{\xi}_0(v_{t_2-1}) = \xi \neq \tilde{\xi}_2(v_{t_2-1})$. So, $h_T(v_{t_2-1}) \neq 1$, as that would imply that v' is 0-stable. Also, $h_T(v_{t_2-1}) \neq 0$, as that would mean that $\tilde{\xi}_2(v_{t_2-1})$ equals the opinion of its parent at time Step 1, which is ξ —a contradiction. So, $h_T(v') \geq 2$, as needed.

Let us describe the way we bound the probability p_1 . Let us specify one of the two children of u and call it *canonical*. For an even time step $\tau > 0$, let $Q_1^*(\tau)$ denote the following event: There exists an extension $\tilde{\xi}_0$ of $\xi_0|_{V(T_u)}$ to the whole tree such that

- $\tilde{\xi}_t(v) = \tilde{\xi}_0(v) = \xi$ for all even $t \leq \tau$;
- at least one of the children of u has initial opinion ξ ;
- $\tilde{\xi}_\tau(v_1) \neq \xi$ for the first time, where v_1 is a (random) child of u defined as follows: if both children of u have initial opinion $\xi_0(u) = \xi$, then v_1 is the canonical child of u , otherwise, v_1 is the only child of u with the initial opinion ξ .

Let us prove that for every time step $t \leq \tau$ and every descendant v' of v_1 , which distance from v_1 has the same parity as t , the opinion $\tilde{\xi}_t(v')$ is described by initial opinions of vertices from $V(T_u)$ only, given that the event $Q_1^*(\tau)$ holds. More formally, let us consider two different vectors of initial opinions $\tilde{\xi}_0^1$ and $\tilde{\xi}_0^2$ that agree with ξ_0 at $V(T_u)$ and certify the validity of the events $Q_1^*(\tau_1)$ and $Q_1^*(\tau_2)$ for some time steps τ_1 and τ_2 respectively. Let τ be the minimum of τ_1 and τ_2 . Let us prove by induction on t that for every time step $t \leq \tau$ and every descendant v' of v_1 , which distance from v_1 has the same parity as t , the equality $\tilde{\xi}_t^1(v') = \tilde{\xi}_t^2(v')$ holds. In particular, $\tau_1 = \tau_2$, as $\tilde{\xi}_\tau^1(v_1) = \tilde{\xi}_\tau^2(v_1) \neq \xi$ for the first time, and then the statement follows as needed. The base of induction for $t = 0$ holds since $\tilde{\xi}_0^1|_{V(T_{v_1})} = \tilde{\xi}_0^2|_{V(T_{v_1})} = \xi_0|_{V(T_{v_1})}$. The induction step follows from the definition of majority dynamics for every vertex, except for v_1 at any even time step t . Finally, for an even time step t the equality $\tilde{\xi}_t^1(v_1) = \tilde{\xi}_t^2(v_1)$ holds as both children of v_1 have same opinions at time step $t - 1$ in both processes and $\tilde{\xi}_{t-1}^1(u) = \tilde{\xi}_{t-1}^2(u) = \xi$, since $\tilde{\xi}_{t-2}^1(v_1) = \tilde{\xi}_{t-2}^2(v_1) = \tilde{\xi}_{t-2}^1(v) = \tilde{\xi}_{t-2}^2(v) = \xi$.

Let $V(\tau)$ be the set of vertices of T_u at distance at most $\tau + 1$ from u . Let us first prove the following:

$$\{\exists \tau Q_1^*(\tau)\} = \bigcup_{\tau \in 2\mathbb{Z}_{>0}} \bigcup_{\chi \in \{-1, 1\}^{V(\tau)}} [Q_1^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}].$$

In other words, we have to prove that, for distinct pairs (τ_1, χ_1) and (τ_2, χ_2) , the two events on the right-hand side of the latter equality are disjoint. Suppose, for some ξ_0 , both events hold for some distinct (τ_1, χ_1) and (τ_2, χ_2) simultaneously. First, we have already proved that τ such that $Q_1^*(\tau)$ holds is uniquely defined for $\xi_0|_{V(T_u)}$. Therefore, $\tau_1 = \tau_2 =: \tau$. But then, the equality $\chi_1 = \chi_2$ follows, since both vectors equal $\xi_0|_{V(\tau)}$.

Next, let us note that the event Q_1 implies the event $\{\exists \tau Q_1^*(\tau)\}$. So,

$$\mathbb{P}(Q_1) = \mathbb{P}(Q_1 \cap \{\exists \tau Q_1^*(\tau)\}).$$

Thus,

$$\begin{aligned} \mathbb{P}(Q_1) &= \mathbb{P}\left(Q_1 \cap \bigsqcup_{\tau \in 2\mathbb{Z}_{>0}} \bigsqcup_{\chi \in \{-1,1\}^{V(\tau)}} [Q_1^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}]\right) \\ &= \sum_{\tau \in 2\mathbb{Z}_{>0}} \sum_{\chi \in \{-1,1\}^{V(\tau)}} \mathbb{P}(Q_1 \cap Q_1^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}) \\ &= \sum_{\tau \in 2\mathbb{Z}_{>0}} \sum_{\chi \in \{-1,1\}^{V(\tau)}} \mathbb{P}(Q_1|Q_1^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}) \cdot \mathbb{P}(Q_1^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}). \end{aligned}$$

Below, for all pairs (τ, χ) , we give the same upper bound q_1 on $\mathbb{P}(Q_1|Q_1^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\})$. We determine the exact value of q_1 later. As soon as this bound is determined, we may conclude that

$$\mathbb{P}(Q_1) \leq q_1 \cdot \sum_{\tau \in 2\mathbb{Z}_{>0}} \sum_{\chi \in \{-1,1\}^{V(\tau)}} \mathbb{P}(Q_1^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}) = q_1 \cdot \mathbb{P}(\exists \tau Q_1^*(\tau)).$$

Let us note that the event $\{\exists \tau Q_1^*(\tau)\}$ implies the event that at least one of the children of u has initial opinion ξ and the vertex v_1 (determined in the definition of $Q_1^*(\tau)$) is not weakly 0-stable. Indeed, consider the random variable τ such that $Q_1^*(\tau)$ holds and an extension $\tilde{\xi}_0$ whose existence is claimed by the event $Q_1^*(\tau)$. For any odd time step $t < \tau$, the vertex u has opinion $\tilde{\xi}_t(u) = \xi$, so the fact that v_1 changes its opinion at time step τ contradicts the third equivalent definition of the weak 0-stability. So, $\mathbb{P}(\exists \tau Q_1^*(\tau))$ does not exceed the probability of the event it implies, which is at most $3/4 \cdot (1 - p_w(0))$. Thus, finally, we may conclude that

$$p_1 \leq q_1 \cdot 3/4 \cdot (1 - p_w(0)). \quad (4)$$

Let us now prove that $q_1 \leq 1 - (1 - (1 - p_w(0))^2)^2$. The main advantage for us is that the condition $Q_1^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}$ is defined by the initial opinions of vertices from $V(\tau)$, hence any event independent from these initial opinions is also independent from the condition.

For the rest of the proof, let us fix some even $\tau > 0$ and some $\chi \in V(\tau)$. We also assume that the event $Q_1^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}$ holds and that $\mathbb{P}(Q_1|Q_1^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}) \neq 0$.

In what follows, we assume that Q_1 holds and fix an extension $\tilde{\xi}_0$ whose existence is claimed by this event. Let t_2 be the time step from the definition of Q_1 (with respect to $\tilde{\xi}_0$). We know that the event $Q_1^*(t_2)$ holds and its validity is certified by the extension $\tilde{\xi}_0$. From this, we conclude that the random variable τ such that $Q_1^*(\tau)$ holds equals t_2 from the definition of Q_1 .

Let us prove that all four grandchildren of v_1 have the opinion $\tilde{\xi}_{t_2-2}(\cdot)$ opposite to ξ at time step $t_2 - 2$ for the first time (among even time steps). First of all we note that $\tilde{\xi}_{t_2-1}(u) = \xi$ since $\tilde{\xi}_{t_2-2}(v_1) = \tilde{\xi}_{t_2-2}(v) = \xi$. Let us also recall that $\tilde{\xi}_{t_2}(v_1) \neq \xi$. It may only happen when both children of v_1 have an opinion opposite to ξ at time step $t_2 - 1$. At the same time, we have that $\tilde{\xi}_{t_2-2}(v_1) = \xi$. We immediately get that, indeed, all four grandchildren of v_1 have the other opinion at time $t_2 - 2$. This step is indeed the first time when it happens, since otherwise v_1 would change its opinion earlier than at t_2 .

Let us note that if a vertex has an opinion opposite to ξ at time step $t + 1$, then at least one of its children has an opinion opposite to ξ at time step t . So, for every $i \in [4]$, every grandchild u_i of v_1 has a descendant $w_i \in V(T_{u_i})$ at distance $t_2 - 2$ such that for every $j \in \{0, 1, \dots, t_2 - 2\}$ and every vertex w in the path between w_i and u_i such that $d(w_i, w) = j$ has $\tilde{\xi}_j(w) \neq \xi$. Let us note that the dynamics on T_{v_1} on $[0, t_2]$ does not depend on the choice of initial opinions outside of T_u in the following sense. For all extensions of $\xi_0|_{V(T_u)}$ that have the property described in $Q_1^*(\tau)$, for all vertices $r \in V(T_{v_1})$,

and all time steps $t' \in [0, t_2]$ that have the same parity as $d(v, r)$, the opinion of r at time t' does not depend on the choice of extension of $\xi_0|_{V(T_u)}$. So, we may choose these vertices w_i independently of the extension of $\xi_0|_{V(T_u)}$ to the entire tree. In addition, for the vertex w in the path between u_i and w_i at distance i from w_i , the initial opinion $\tilde{\xi}_i(w)$ is uniquely determined by the opinions of the vertices at the distance from w not exceeding i . So, we may choose a set $\{w_1, w_2, w_3, w_4\}$ such that each w_i relates to u_i in the described way by knowing the initial opinions of $V(t_2)$ only. Let us fix any such set.

Note that events that w_i are weakly 0-stable are independent of each other and of the fixed condition $Q_1^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}$. So, the probability that

- at least one of w_1 and w_2 is weakly 0-stable and
- at least one of w_3 and w_4 is weakly 0-stable

is at least $(1 - (1 - p_w(0))^2)^2$. Due to Claim 4.10, the weak 0-stability of w_i together with the fact that opinions of vertices along the path from w_i to u_i remain the same (as time grows simultaneously with moving forward towards u_i) imply the weak $(t_2 - 2)$ -stability of u_i . Hence, the former event implies that

- at least one of u_1 and u_2 is weakly $(t_2 - 2)$ -stable and
- at least one of u_3 and u_4 is weakly $(t_2 - 2)$ -stable.

If u_i is weakly $(t_2 - 2)$ -stable, then it is also weakly t_2 -stable with the same opinion — here we apply the fact that its parent has the same opinion at time step $t_2 - 1$, together with Claims 4.2 and 4.3. So,

- at least one of u_1 and u_2 is weakly t_2 -stable with opinion opposite to ξ and
- at least one of u_3 and u_4 is weakly t_2 -stable with opinion opposite to ξ .

Due to Claim 4.11, this leads to t_2 -stability and hence t_1 -stability of v_1 — a contradiction. Altogether, our upper bound on p_1 is as follows

$$p_1 \leq \frac{3}{4} \cdot (1 - p_w(0)) \cdot \left(1 - (1 - (1 - p_w(0))^2)^2\right) \stackrel{\text{Claim 4.6}}{\leq} 0.00064. \quad (5)$$

Next, let us bound p_2 . Both children of u have initial opinions opposite to ξ with probability $1/4$. Next, both of them are not weakly 0-stable with probability at most $(1 - p_w(0))^2$ since these two events are independent. Moreover, they are independent of the initial opinions of these two children. So,

$$p_2 \leq \frac{1}{4} \cdot (1 - p_w(0))^2 \leq 0.00141. \quad (6)$$

Lastly, we bound p_3 . The proof strategy here is analogous to the proof of the bound for p_1 . We outline the differences below.

There are two ways to choose which child of u plays the role of v_1 (and then the other child plays the role of v_2). In the same way as in the proof for p_1 , we get that $h_T(v_2) \geq t_2$.

Let $Q_3^*(\tau)$ denote the following event: There exists an extension $\tilde{\xi}_0$ of $\xi_0|_{V(T_u)}$ to the whole tree such that

- $\tilde{\xi}_t(v) = \tilde{\xi}_0(v) = \xi$ for all even $t \leq \tau$;
- both children of u have initial opinion opposite to ξ ;
- some (random) child v_1 of u is weakly 0-stable;
- the other child v_2 of u has opinion ξ at time step τ for the first time.

Let us note that the vertex v_2 from the definition of the event $Q_3^*(\tau)$ is not weakly 0-stable, since otherwise it is even 0-stable by Claim 4.11 and hence it does not change its opinion at time step τ . So, labels v_1 and v_2 are uniquely defined by $\xi_0|_{V(T_u)} \in Q_3^*(\tau)$.

Let $V(\tau)$ be the union of $V(T_{v_1})$, the set $\{u\}$, and the set of vertices of T_{v_2} at distance at most τ from v_2 . Then we may prove the following in the same way as for p_1 :

$$\{\exists \tau \ Q_3^*(\tau)\} = \bigsqcup_{\tau \in 2\mathbb{Z}_{>0}} \bigsqcup_{\chi \in \{-1,1\}^{V(\tau)}} [Q_3^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}].$$

Next, let us note that the event Q_3 implies the event $\{\exists \tau \ Q_3^*(\tau)\}$. So, as for p_1 ,

$$\mathbb{P}(Q_3) = \sum_{\tau \in 2\mathbb{Z}_{>0}} \sum_{\chi \in \{-1,1\}^{V(\tau)}} \mathbb{P}(Q_3|Q_3^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}) \cdot \mathbb{P}(Q_3^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}).$$

Below, for all pairs (τ, χ) , we give the same upper bound q_3 on $\mathbb{P}(Q_3|Q_3^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\})$. We determine the exact value of q_3 later. As soon as this bound is determined, we may conclude that

$$\mathbb{P}(Q_3) \leq q_3 \cdot \mathbb{P}(\exists \tau \ Q_3^*(\tau)).$$

As was mentioned above, the event $\{\exists \tau \ Q_3^*(\tau)\}$ distinguishes between the two children of u and implies that one of them (v_2) is not weakly 0-stable. In addition, it says that $\xi_0(v_1) = \xi_0(v_2) \neq \xi$. So,

$$\mathbb{P}(\exists \tau \ Q_3^*(\tau)) \leq 1/4 \cdot 2 \cdot (1 - p_w(0)).$$

Let us finally prove that $q_3 \leq (1 - p_w(0))^4$. In what follows, we condition all events on $Q_3^*(\tau) \cap \{\xi_0|_{V(\tau)} = \chi\}$. In particular, under this condition, the random variable t_2 from the definition of the event Q_3 gets the value τ since Q_3 implies $Q_3^*(t_2)$.

By induction over odd $t \leq t_2 - 1$ and by Claim 4.2, we have that $\tilde{\xi}_{t+1}(v_1) = \tilde{\xi}_t(u) \neq \xi$ for all odd $t \leq t_2 - 1$. So, $\tilde{\xi}_{t_2-1}(u) \neq \xi$. Since $\tilde{\xi}_{t_2}(v_2) = \xi$, we get that all 4 grandchildren of v_2 have opinion ξ at time step $t_2 - 2$.

Applying the same argument as for p_1 , we may assign to every grandchild u_i , $i \in [4]$, of v_2 a canonical descendant w_i located at distance $t_2 - 2$ from u_i . Here, we again get that each w_i is weakly 0-stable independently of all the other events. Due to Claim 4.10, v_2 is weakly t_2 -stable if at least one of w_i is weakly 0-stable, which contradicts Q_3 .

We conclude that

$$p_3 \leq \frac{1}{4} \cdot 2 \cdot (1 - p_w(0)) \cdot (1 - p_w(0))^4 \stackrel{\text{Claim 4.6}}{\leq} 0.00001. \quad (7)$$

From Equations (5), (6), and (7), we get the required lower bound $(1 - p_1 - p_2 - p_3)^2 > 0.9957$, completing the proof.

Claim 4.14. For all large enough even integer t , the following holds. Fix any positive integer h and a perfect binary tree of depth h . Let $\xi_0 \in \{-1, 1\}^{V(T)}$ be chosen uniformly at random. Then, for any non-leaf vertex $v \in V(T)$, $v \neq R$, the probability that

- there exists an extension $\tilde{\xi}_0$ of $\xi_0|_{V(T_v)}$ to the whole tree T such that $\tilde{\xi}_t(v) = 1$;
- v is not weakly 0-stable w.r.t. ξ_0

is less than 0.038.

Proof. Let $\varepsilon > 0$ be small enough and let $t = t(\varepsilon)$ be a large even integer. Let T be a perfect binary tree. Consider the following events:

- $Q_1 = \{v \text{ is not weakly 0-stable}\};$
- $Q_2 = \{\xi_0(v) = 1 \text{ and, for every extension } \tilde{\xi}_0 \text{ of } \xi_0|_{V(T_v)} \text{ to the whole tree, } \tilde{\xi}_t(v) = -1\}.$

Then the probability of the desired event is at most

$$\mathbb{P}(Q_1 \wedge \neg Q_2) = (1 - \mathbb{P}(Q_2|Q_1))\mathbb{P}(Q_1) \leq (1 - \mathbb{P}(Q_2|Q_1))(1 - p_w(0)).$$

It remains to bound from below $\mathbb{P}(Q_2|Q_1)$.

Consider $\tilde{\xi}_0 : V(T) \rightarrow \{-1, 1\}$ such that

- $\tilde{\xi}_0(w) = \xi_0(w)$ for every $w \in V(T_v)$;
- $\tilde{\xi}_0(w) = 1$ for every $w \in V(T) \setminus V(T_v)$.

If v is not weakly 0-stable, then, due to the definition of weak 0-stability, there exists an even time step τ such that $\tilde{\xi}_\tau(v) = -1$ for the first time. If such a time step does not exist, let $\tau = \infty$. Consider the event $Q^* = Q^*(h(T_v), t) = \{\tau < t\}$. Let us observe that $\mathbb{P}(Q^*|Q_1) = 1$ when $h(T_v) < t$. Let us show this by contradiction: Suppose $\tau \geq t > h(T_v)$. The parent of v has opinion 1 at time step $\tau - 1$ due to the definition of $\tilde{\xi}_0$, so v can change its opinion to -1 at time step τ only if it has a child that changes opinion to -1 at time step $\tau - 1$. Next, we proceed by induction: Apply the same argument to this child, then to a specific grandchild, etc. So, since $h(T_v) < \tau$, there is a descendant u of v with $d(u, v) = h(T_v) - 1$ such that it changes opinion to -1 at time step $\tau - h(T_v) + 1 \geq 2$. However, this is impossible, since at time step $\tau - h(T_v) + 1$ vertex u has the same opinion as both its leaf-children at time step $\tau - h(T_v)$ that is inherited from u at time step $\tau - h(T_v) - 1 \geq 0$. So, $\tilde{\xi}_{\tau-h(T_v)+1}(u) = \tilde{\xi}_{\tau-h(T_v)-1}(u)$ —a contradiction.

Let $p(t)$ be the infimum of $\mathbb{P}(Q^*|Q_1)$ over all possible $h(T_v)$. We proved that $\mathbb{P}(Q^*|Q_1) = 1$ when $h(T_v) < t$. On the other hand, for all v and T such that $h(T_v) \geq t$, $\mathbb{P}(Q^*)$ does not depend on $h(T_v)$ and equals $1 - p_w(0, h, u)$ for any $h > t$ and any vertex u with height t . So, $\mathbb{P}(Q^*) = 1 - p_w(0, t + 1, u) =: 1 - p_w(0, t)$. Note that $p_w(0, t)$ is non-increasing and $\lim_{t \rightarrow \infty} p_w(0, t) = p_w(0)$. Let $t(\epsilon)$ be so large that $1 - p_w(0, t) > (1 - \epsilon)(1 - p_w(0))$ for all $t \geq t(\epsilon)$. We also have that $Q^* \subseteq Q_1$. So, since $\mathbb{P}(Q_1) \leq 1 - p_w(0)$, we get that, for all $t \geq t(\epsilon)$,

$$p(t) = \inf_{v, T} \mathbb{P}(Q^*|Q_1) = \inf_{v: h(T_v) \geq t} \frac{\mathbb{P}(Q^*)}{\mathbb{P}(Q_1)} \geq \frac{\mathbb{P}(Q^*)}{1 - p_w(0)} \geq 1 - \epsilon.$$

Now,

$$\begin{aligned} \mathbb{P}(Q_2|Q_1) &\geq \mathbb{P}(Q_2 \wedge Q^*|Q_1) = \sum_{s < t} \mathbb{P}(Q_2 \wedge \{\tau = s\}|Q_1) \\ &= \sum_{s < t} \mathbb{P}(Q_2|\{\tau = s\} \wedge Q_1) \mathbb{P}(\tau = s|Q_1) \\ &\geq p(t) \cdot \min_{\text{even } s < t} \mathbb{P}(Q_2|\{\tau = s\} \wedge Q_1). \end{aligned}$$

Let us assume that $Q_1 \wedge \{\tau = s\}$ holds for some $s < t$. For every grandchild w of v , we have that $\tilde{\xi}_{s-2}(w) = -1$. In the usual way (and in exactly the same way as in the proof of Claim 4.13), we get that every grandchild w of v has a (canonical) descendant $w' = w'(w)$ such that

- $d(w, w') = s - 2$, and
- every vertex from the path between w and w' at distance j from w' has opinion -1 at time step j .

We note that the four vertices $w'(w)$ are uniquely defined by the initial opinions of vertices from the tree T_v pruned at depth s .

Since validity of the event $Q_1 \wedge \{\tau = s\}$ as well as the vertices w' are defined by the pruned tree, the events that w' are weakly 0-stable are independent of each other and of the event $Q_1 \wedge \{\tau = s\}$. Moreover, weak 0-stability of any w' implies weak $(s - 2)$ -stability of the corresponding w , due to Claim 4.10. Also, if at least two non-sibling grandchildren of v are weakly $(s - 2)$ -stable (recall that all grandchildren of v have the same opinion -1 at the time step $s - 2$), then both children of v have opinion -1 at time step $s - 1$ and both are weakly $(s - 1)$ -stable by Claim 4.10. Hence, both children are $(s - 1)$ -stable by Claim 4.11. As both children of v have the same opinion -1 at time step $s - 1$, so v is s -stable with $\xi_s(v) = -1$. Thus, $\xi_t(v) = -1$, as $s < t$.

We conclude that, for every s ,

$$\mathbb{P}(Q_2|\{\tau = s\} \wedge Q_1) \geq 0.5(1 - (1 - p_w(0))^2)^2.$$

Summing up, we get that the probability of the desired event from the statement of Claim 4.14 is at most

$$\left(1 - \frac{1}{2}(1 - \varepsilon)(1 - (1 - p_w(0))^2)^2\right)(1 - p_w(0)) < 0.038,$$

due to the choice of a small enough ε and due to Claim 4.6. $\square$

4.3.2 | Proof of Lemma 4.8

Let $p_+ = 0.01$, $p_- = 0.0513$. Let us choose $c_+ < 2/3$ large enough so that

$$2^{h+d/2}(p_+ + p_-)^{d/2} = o(1) \quad (8)$$

for the biggest even number $d < \lceil c_+ h \rceil$.

Let us note that if, for some vertex v , $\xi_d(v) \neq \xi_{d-2}(v)$ then the opinion $\xi_d(v)$ reached the vertex v by traveling across some path $v_0 v_1 \cdots v_d =: v$, that is, $\xi_i(v_i) = \xi_d(v_d)$ for every $i \in \{0, \dots, d\}$ and $\xi_{i-2}(v_i) \neq \xi_d(v_d)$ for every $i \in \{2, \dots, d\}$. Let us recall that, for some constant $C_1 > 0$, the number of paths of length d is less than $2^{h+d/2} \cdot C_1$ (see, e.g., (de Jong et al. 2016, Lemma 5.7)). We plan to prove that for some constant $C_2 > 0$ and any path $v_0 v_1 \cdots v_d$ the above-mentioned traveling happens with probability less than $C_2(p_+ + p_-)^{d/2}$. This will conclude the proof of Lemma 4.8 due to (8).

Consider a path $P = v_0 v_1 \cdots v_d := v$. Let us denote by $\mathcal{E}(P)$ the following event:

$$\mathcal{E}(P) = \{\xi_i(v_i) = 1 \text{ for every } i \in \{0, 1, \dots, d\} \text{ and } \xi_{i-2}(v_i) = -1 \text{ for every } i \in \{2, \dots, d\}\}.$$

For every $i \in [d-1]$, denote by u_i the neighbor of v_i in T that does not belong to $V(P)$. Let t_0 be a large enough even integer constant so that the conclusion of Claim 4.14 holds for all $t \geq t_0$. For every $i \in [d-1]$, we color the children of u_i in two colors—red and blue. For $2i \in [d-4]$, let A_{2i} be the set of all vertices that are either descendants² of the blue child of u_{2i} , or descendants of u_{2i+1} , or descendants of the red child of u_{2i+2} . Let $I \subset \{t_0 + 6, t_0 + 8, \dots, d-4\}$ be the maximum set of even integers that satisfy the following condition: For every $i \in I$,

- either v_i is a grandchild of v_{i+2} or v_{i+2} is a grandchild of v_i ;
- $\min(h_T(v_i), h_T(v_{i+2})) \geq t_0 + 6$.

The fact that $i \geq t_0 + 6$ allows us to apply Claim 4.14 at time steps at least $i-4$. Also, let us note that, as the sets $A_i, A_{i'}$, for $i \neq i'$, are disjoint, the probabilities of events defined by $\xi_0|_{A_i}$ and $\xi_0|_{A_{i'}}$ are independent due to Remark 4.4. Lastly, let us note that $||I| - d/2| = O(1)$ (Figure 2).

We denote the number of vertices $u \in V(P)$ at even distance from v_0 such that $\xi_0(u)$ equals 1 and -1 by m_+ and m_- , respectively. Fix an even $i \in I$. Denote the number of vertices $u \in \{v_i, v_{i+2}\}$ such that $\xi_0(u)$ equals 1 and -1 by $k_+(i)$ and

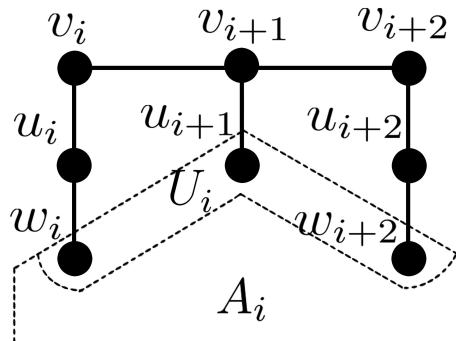


FIGURE 2 | The structure of sets A_i and U_i .

$k_-(i)$, respectively. Let the random variable $\Xi_i := \Xi_i(\xi_0|_{V(P)})$ be the set of vectors of initial opinions of vertices from A_i such that $\mathbb{P}(\mathcal{E}(P)|\xi_0|_{A_i} \in \Xi_i, \xi_0|_{V(P)} > 0)$. In the remaining part of the proof, we show that

$$\mathbb{P}(\xi_0|_{A_i} \in \Xi_i | \xi_0|_{V(P)}) \leq 2\sqrt{p_+^{k_+(i)} p_-^{k_-(i)}} =: q(i). \quad (9)$$

This inequality completes the proof of Lemma 4.8 since, due to Remark 4.4,

$$\begin{aligned} \mathbb{P}(\mathcal{E}(P)|\xi_0|_{V(P)}) &\leq \mathbb{P}\left(\bigcap_{i \in I} \xi_0|_{A_i} \in \Xi_i | \xi_0|_{V(P)}\right) \\ &\leq \prod_{i \in I} q(i) = O(2^{d/2}) \prod_{i \in I} \sqrt{p_+^{k_+(i)} p_-^{k_-(i)}} = O(2^{d/2} p_+^{m_+} p_-^{m_-}). \end{aligned}$$

The latter equality holds because each (but at most a bounded number) i such that $\xi_0(v_i) = 1$ (or -1) contributes an additive 1 to both $k_+(i-2)$ and $k_+(i)$ (or $k_-(i-2)$ and $k_-(i)$) for $i-2, i \in I$. So, since $||I| - d/2| = O(1)$, the product

$$\prod_{i \in I} \sqrt{p_+^{k_+(i)} p_-^{k_-(i)}} = \sqrt{p_+^{\sum_{i \in I} k_+(i)} p_-^{\sum_{i \in I} k_-(i)}}$$

differs from the $p_+^{m_+} p_-^{m_-}$ by a multiplicative constant.

Let $P^{even} = \{v_0, v_2, \dots, v_d\}$ be the set of vertices in P at even distance from v_0 . Since all initial values on vertices along P are equally likely, we conclude that

$$\begin{aligned} \mathbb{P}(\mathcal{E}(P)) &= \sum_{\mathbf{x} \in \{-1,1\}^{V(P^{even})}} O\left(2^{d/2} p_+^{m_+(\mathbf{x})} p_-^{m_-(\mathbf{x})}\right) \mathbb{P}(\xi_0|_{V(P^{even})} = \mathbf{x}) \\ &= O\left(\sum_{\mathbf{x} \in \{-1,1\}^{V(P^{even})}} p_+^{m_+(\mathbf{x})} p_-^{m_-(\mathbf{x})}\right) \\ &= O\left(\sum_{i=0}^{(d+1)/2} p_+^i p_-^{(d+1)/2-i}\right) = O((p_+ + p_-)^{d/2}), \end{aligned}$$

as needed.

From now on, we assume that the vector $\xi_0|_{V(P)}$ is fixed and deterministic; we also fix an index $i \in I$, and work only with the corresponding subset A_i . Our aim is to prove the inequality (9). We denote the children of u_i and u_{i+2} that belong to A_i by w_i and w_{i+2} , respectively. Let

$$U_i := \{w_i, u_{i+1}, w_{i+2}\} \subset A_i.$$

Let us define the event

$$\mathcal{E}_1(P) := \{\xi_{i-2}(v_i) = \xi_i(v_{i+2}) = -1, \xi_i(v_i) = \xi_{i+2}(v_{i+2}) = \xi_{i-2}(w_i) = \xi_i(u_{i+1}) = \xi_i(w_{i+2}) = 1\}$$

and prove that it follows from $\mathcal{E}(P)$. Indeed, the only non-trivial equalities that are not included in the definition of $\mathcal{E}(P)$ are $\xi_{i-2}(w_i) = \xi_i(u_{i+1}) = \xi_i(w_{i+2}) = 1$. Since $\xi_i(v_i) = 1 \neq \xi_{i-1}(v_{i+1})$, we get that $\xi_{i-1}(u_i) = 1$. It implies $\xi_{i-2}(w_i) = 1$, since $\xi_{i-2}(v_i) = -1$. So, we get $\xi_{i-1}(u_i) = \xi_{i-2}(w_i) = 1$. The remaining equalities $\xi_i(u_{i+1}) = \xi_i(w_{i+2}) = 1$ can be proved in literally the same way by replacing, first, i with $i+1$ and, then, i with $i+2$.

Thus, the event $\xi_0|_{A_i} \notin \Xi_i$ implies $\xi_0|_{A_i} \notin \Xi'_i$, where Ξ'_i is the set of vectors of initial opinions of vertices from A_i such that $\mathbb{P}(\mathcal{E}_1(P)|\xi_0|_{A_i} \in \Xi'_i) > 0$ (recall that the initial opinions along $V(P)$ are fixed), and so it is sufficient to get an upper bound on the probability of the latter. In other words, we will bound from below the probability of the set of initial opinions of vertices from A_i so that, whatever the initial opinions of all other vertices, $\mathcal{E}_1(P)$ never holds. Let us now describe one of the main instruments that we will use to get such a lower bound. Let

$$\mathcal{Q} \in \mathcal{Q} := \{(w_i u_i v_i v_{i+1} u_{i+1}), (u_{i+1} v_{i+1} v_{i+2} u_{i+2} w_{i+2}), (w_i u_i v_i v_{i+1} v_{i+2} u_{i+2} w_{i+2})\}$$

be a path. Let us show that

if Q satisfies either the requirements of Claim 4.11 or the requirements of Claim 4.12
with some even $t \leq i - 2$, then $\mathcal{E}_1(P)$ does not hold. (10)

First, let $Q = (w_i u_i v_i v_{i+1} v_{i+2} u_{i+2} w_{i+2})$. If Q satisfies the requirements of Claim 4.11 with some even $t \leq i - 2$, then the conclusion of this claim— t -stability of $w_i, v_i, v_{i+2}, w_{i+2}$ —contradicts to the inequality $\xi_i(v_{i+2}) \neq \xi_i(v_i)$, thus $\mathcal{E}_1(P)$ does not hold. If Q satisfies the requirements of Claim 4.12 for some even $t \leq i - 2$, then it asserts that either $\xi_i(v_{i+2}) = \xi_i(v_i)$ or $\xi_i(v_{i+2}) = \xi_i(w_{i+2})$. This is again not possible for $\xi_0 \in \mathcal{E}_1(P)$. Now, let $Q = (u_{i+1} v_{i+1} v_{i+2} u_{i+2} w_{i+2})$. In the same way as above, the conclusion of Claim 4.11 for $t \leq i - 2$ contradicts to the property $\xi_i(v_{i+2}) \neq \xi_i(u_{i+1})$ required by $\mathcal{E}_1(P)$; and the conclusion of Claim 4.12 implies either $\xi_i(v_{i+2}) = \xi_i(u_{i+1})$ or $\xi_i(v_{i+2}) = \xi_i(w_{i+2})$, that contradicts to $\mathcal{E}_1(P)$ as well. Now, if the conclusion of Claim 4.11 holds for $Q = (w_i u_i v_i v_{i+1} u_{i+1})$ then v_i is t -stable, which contradicts to $\xi_{i-2}(v_i) = \xi_i(v_i)$. Also, if the conclusion of Claim 4.12 holds for Q , then (1) $\xi_{i-2}(v_i)$ equals at least one of $\xi_{i-2}(w_i), \xi_{i-2}(u_{i+1})$, (2) u_{i+1} is either $(i - 2)$ -stable or weakly $(i - 2)$ -stable. On the other hand, if $\mathcal{E}_1(P)$ holds as well, then $-1 = \xi_{i-2}(v_i) \neq \xi_{i-2}(w_i)$. Therefore, $\xi_{i-2}(v_i) = \xi_{i-2}(u_{i+1}) = -1$ and then $\xi_{i-1}(v_{i+1}) = -1$. By Claim 4.2, since u_{i+1} is either $(i - 2)$ -stable or weakly $(i - 2)$ -stable, we get $\xi_i(u_{i+1}) = -1$ —contradiction with $\mathcal{E}_1(P)$.

For the sake of convenience, let us list the conditions a path $Q \in \mathcal{Q}$ has to satisfy to meet the requirements of Claim 4.11 and Claim 4.12 applied to this path. We give the conditions in the following remark.

Remark 4.15. To apply Claim 4.11 at time step t we need to verify that

- at time t , every second vertex (starting from any end) of Q has the same opinion;
- both ends of Q are weakly t -stable.

To apply Claim 4.12 at time step t we need to verify that Q can be partitioned into two subpaths so that

- both ends of Q are weakly t -stable and have different opinions at time t ;
- at time t , every second vertex u of Q (starting from any end) has the same opinion as the end of Q that belongs to the same subpath as u .

Let us also give the following remark.

Remark 4.16. All the events that we use below to bound $\mathbb{P}(\xi_0|_{A_i} \in \Xi'_i)$ depend on initial opinions of A_i only. Therefore, it is convenient to have in mind the following distribution of initial opinions: (1) $\xi_0|_{V(P)}$ is fixed and deterministic; (2) $\xi_0|_{A_i}$ is uniformly random; (3) on all the other vertices, $\xi_0|_{V(T) \setminus (V(P) \cup A_i)}$ is a canonical vector, certifying $\mathcal{E}_1(P)$, if such a vector exists. In what follows, we denote this probability measure by $\mathbb{P}^*$. We have that $\mathbb{P}(\xi_0|_{A_i} \in \Xi'_i) = \mathbb{P}^*(\mathcal{E}_1(P))$. Therefore, if we manage to find an event $\mathcal{B}$ that depends only on initial opinions of A_i such that $\mathcal{E}_1(P)$ implies $\mathcal{B}$, then $\mathbb{P}(\xi_0|_{A_i} \in \Xi'_i) = \mathbb{P}^*(\mathcal{E}_1(P)) \leq \mathbb{P}^*(\mathcal{B}) = \mathbb{P}(\mathcal{B})$. Thus, we may use an upper bound for $\mathbb{P}(\mathcal{B})$ to bound the desired $\mathbb{P}(\xi_0|_{A_i} \in \Xi'_i)$.

4.3.2.1 | Proof of (9). First of all, let

$$q_1 = \mathbb{P}^*(\mathcal{E}_1(P) \wedge \text{not all vertices from } U_i \text{ are 1-close to stability}).$$

Note that $\mathcal{E}_1(P)$ implies that none of the vertices from U_i is strongly 0-stable with initial opinion -1 . Therefore, due to Claims 4.6 and 4.13,

$$\begin{aligned} q_1 &\leq \prod_{u \in U_i} \mathbb{P}^*(u \text{ is not strongly 0-stable with } \xi_0(i) = -1) \\ &\quad - \prod_{u \in U_i} \mathbb{P}^*(u \text{ is not strongly 0-stable with } \xi_0(i) = -1, \text{ but is 1-close to stability}) \end{aligned}$$

$$\begin{aligned}
&= \prod_{u \in U_i} \mathbb{P}(u \text{ is not strongly 0-stable with } \xi_0(i) = -1) \\
&\quad - \prod_{u \in U_i} \mathbb{P}(u \text{ is not strongly 0-stable with } \xi_0(i) = -1, \text{ but is 1-close to stability}) \\
&\leq \left(1 - \frac{p_s(0)}{2}\right)^3 - \left(1 - \frac{p_s(0)}{2} - (1 - 0.9957)\right)^3 < 0.0073.
\end{aligned}$$

Let

$$q_2 = \mathbb{P}^*(\mathcal{E}_1(P) \wedge \{\text{at least two vertices from } U_i \text{ are not weakly 0-stable}\}).$$

Since $i \geq t_0 + 6$, we can apply Claim 4.14 to any vertex from U_i : The probability that $\mathcal{E}_1(P)$ holds and a fixed vertex $u \in U_i$ is weakly 0-stable is at most 0.038. Thus, due to Claim 4.14,

$$q_2 \leq 3 \cdot 0.038^2 < 0.0044.$$

We conclude that

$$\mathbb{P}^*(\mathcal{E}_1(P)) = q_1 + q_2 + q_3 < q_3 + 0.0117,$$

where $q_3 = \mathbb{P}^*(\mathcal{E}_2(P))$ and

$$\begin{aligned}
\mathcal{E}_2(P) &= \mathcal{E}_1(P) \wedge \{\text{every vertex from } U_i \text{ is 1-close to stability}\} \\
&\quad \wedge \{\text{the number of not weakly 0-stable vertices in } U_i \text{ does not exceed 1}\}.
\end{aligned} \tag{11}$$

We bound q_3 separately for several different cases depending on the value of $\xi_0(v_i) + \xi_0(v_{i+2})$. For the case $\xi_0(v_i) = \xi_0(v_{i+2})$, we will need an auxiliary proposition.

Claim 4.17. Let $\xi_0(v_i) = \xi_0(v_{i+2}) =: \xi$. If $\mathcal{E}_1(P)$ holds and

$$\text{every vertex from } U_i \text{ is weakly 0-stable and 1-close to stability,} \tag{12}$$

then

$$\text{for every } u \in U_i \text{ and every even time step } t \leq i - 2, \xi_t(u) \neq \xi. \tag{13}$$

Proof. First, suppose that $\xi_0(u) = \xi$ holds for some $u \in U$. Let $Q \in \mathcal{Q}$ be a path such that one of its ends coincides with u . Depending on the opinion of the vertex at the other end of Q , we get the requirements of either Claim 4.11 or Claim 4.12. By (10), this contradicts $\mathcal{E}_1(P)$.

Next, suppose that $\xi_t(u) = \xi$ for some even $2 \leq t \leq i - 2$. The existence of such t and the fact that all $u \in U_i$ have $\xi_0(u) \neq \xi$ implies that some $u \in U_i$ changes its opinion at least once before the time $i - 2$. Let $t_1 \leq i - 2$ be the first even time step such that some $v \in U_i \sqcup \{v_i, v_{i+2}\}$ has opinion different from its initial opinion $\xi_0(v)$. Suppose that none of the vertices from U_i changed their opinion at time step t_1 . Then, both v and the two vertices from U_i at distance 2 from v share the opinion opposite to ξ at time step t_1 ; those three vertices belong to the single path $Q \in \mathcal{Q}$ and, by Remark 4.15, the requirements from Claim 4.11 are satisfied by Q at time step t_1 — contradiction with $\mathcal{E}_1(P)$ by (10). So, now we may assume that $v \in U_i$.

Recall that $t_1 \leq i - 2$ is the first time step such that some $v \in U_i \sqcup \{v_i, v_{i+2}\}$ has opinion different from its initial opinion $\xi_0(v)$. We have just proved that at least one vertex from U_i also changed its opinion at time step t_1 . So, we can assume that $v \in U_i$. Let us separately consider all three possible values of v .

1. First, assume $v = u_{i+1}$, or, in other words, $\xi_{t_1}(u_{i+1}) = \xi$. Let $t_2 > t_1$ be the first even time step that $\xi_{t_2}(u_{i+1}) \neq \xi$ (t_2 might be $+\infty$). Assume that, for some even time step $t \in \{t_1, t_1 + 2, \dots, t_2\}$, $\xi_t(w_i) = \xi$ for the first time. Our aim is to apply Claim 4.11 at time step t to the path between vertices w_i and u_{i+1} . For this, we have to show that

1. $\xi_t(v_i) = \xi_t(u_{i+1}) = \xi$ and
2. w_i, u_{i+1} are weakly t -stable.

We first note that w_i is weakly $(t-2)$ -stable, due to assumption (12) and Claim 4.3. So, as $\xi_t(w_i) = \xi \neq \xi_{t-2}(w_i)$, Claim 4.2 concludes $\xi_{t-1}(u_i) = \xi$ and hence $\xi_{t-2}(v_i) = \xi$. We get $\xi_{t-1}(v_{i+1}) = \xi$ —for $t > t_1$ it follows from $\xi_{t-2}(u_{i+1}) = \xi$ and for $t = t_1$ it follows from $\xi_{t-2}(v_{i+2}) = \xi$ (recall that t_1 is the first time moment when a vertex from $U_i \sqcup \{v_i, v_{i+2}\}$ changed its opinion). Therefore, $\xi_t(v_i) = \xi$, since $\xi_{t-1}(u_i) = \xi_{t-1}(v_{i+1}) = \xi$, as needed. Let us remind, that by assumption (12), u_{i+1} is 1-close to stability, so it is weakly t_1 -stable. Then, by Claim 4.3, it is weakly $(t-2)$ -stable. So, $\xi_t(u_{i+1}) = \xi$ by Claim 4.2, since $\xi_{t-1}(v_{i+1}) = \xi$. That completes verification of the first condition from Remark 4.15 that we had to check to apply Claim 4.11. We can also conclude that $t < t_2$. Moreover, w_i is weakly t -stable, since it is 1-close to stability by assumption (12), and u_{i+1} is weakly t -stable by Claim 4.3. Hence, by Remark 4.15, we can apply Claim 4.11 with time step t to the path Q from w_i to u_{i+1} . Then, if $t \leq i-2$, we get a contradiction with $\mathcal{E}_1(P)$ due to (10). If $i \geq t_1$, then $\xi_{i-2}(w_i) \neq \xi = \xi_i(u_{i+1})$, contradicting $\mathcal{E}_1(P)$. So $i < t_1$, completing the proof of (13).

2. Let $v = w_i$. By assumption (12) and Claim 4.3, w_i is weakly (t_1-2) -stable. Since $\xi_{t_1}(w_i) = \xi \neq \xi_{t_1-2}(w_i)$, we derive $\xi_{t_1-1}(u_i) = \xi$ due to Claim 4.2. Let us recall that t_1 is the first moment when some vertex from $U_i \sqcup \{v_i, v_{i+2}\}$ changes its opinion. Therefore, $\xi_{t_1-2}(v_i) = \xi_{t_1-2}(v_{i+2}) = \xi$ and hence $\xi_{t_1-1}(v_{i+1}) = \xi$. Since v_{i+1} and u_i are neighbors of v_i , we get $\xi_{t_1}(v_i) = \xi$. However, $\mathcal{E}_1(P)$ implies that, at time moment $i-2$, $\xi_{i-2}(w_i) \neq \xi_{i-2}(v_i)$, in contrast to time moment t_1 . Recalling that $t_1 \leq i-2$, we get that at least one of v_i, w_i should change its opinion at a time step after t_1 and not exceeding $i-2$. Note that w_i is weakly t_1 -stable since it is 1-close to stability by (12). Let t_2 be the first even time moment after t_1 when w_i changes its opinion again (it can be infinite). Assuming that $t_2 < \infty$, we get that w_i is weakly (t_2-2) -stable due to Claim 4.3. So, by Claim 4.2, $\xi_{t_2-1}(u_i) \neq \xi$. But then $\xi_{t_2-2}(v_i) \neq \xi$ as well since $\xi_{t_2-2}(w_i) = \xi$ and v_i, w_i are neighbors of u_i . Thus, v_i changes its opinion at some even step $t_3 \in [t_1+2, t_2-2]$. Due to the above conclusion, we also get that $t_3 \leq i-2$. The neighbor u_i of v_i at any odd time step between t_1+1 and t_3-1 has opinion ξ since vertices v_i, w_i agree on $[t_1, t_3-2]$. Thus, $\xi_{t_3-1}(v_{i+1}) \neq \xi$, implying $\xi_{t_3-2}(v_{i+2}) \neq \xi$ (recall that $\xi_{t_3-2}(v_i) = \xi$). Since t_1 is the first moment when some vertex from $U_i \sqcup \{v_i, v_{i+2}\}$ changes its opinion, we get that v_{i+2} changes opinion for the first time at some even time step $t_4 \in \{t_1, t_1+2, \dots, i-4\}$.

To summarize, we get the following dynamics of opinions of vertices w_i, v_i, v_{i+2} at even time moments from $[0, t_4]$: The vertex w_i changes its opinion once at time $t_1 \leq t_4$ to ξ , the vertex v_i holds the opinion ξ throughout the entire time interval, and the vertex v_{i+2} changes its opinion once to $-\xi$ at time t_4 . Let us now observe the behavior of opinions of vertices u_{i+1} and w_{i+2} within the same time interval. It will allow us to arrive to a contradiction with $\mathcal{E}_1(P)$ in a similar way to the case 1 (when $v = u_{i+1}$) due to Remark 4.15 and Claim 4.11. We consider four scenarios.

- Assume that u_{i+1} changes its opinion for the first time at some even time step $t \leq t_4$. Here, we may apply Claim 4.11 at time step t to the path Q between w_i and u_{i+1} : First, since $t \leq t_4$, as we have already proved above, w_i and v_i preserve opinion ξ up to time step t . This means that $\xi_t(w_i) = \xi_t(v_i) = \xi_t(u_{i+1})$ and that w_i is weakly t -stable by Claim 4.3 (since it is 1-close to stability by (12)). Finally, u_{i+1} is weakly t -stable as well since it is 1-close to stability by (12). Applying Claim 4.11 we get a contradiction with $\mathcal{E}_1(P)$ due to (10) and Remark 4.15.
- Assume that both u_{i+1}, w_{i+2} do not change their opinion in any even time step from $[0, t_4]$. As above, we apply Claim 4.11 to the path Q from u_{i+1} to w_{i+2} and get a contradiction with $\mathcal{E}_1(P)$ due to (10). Indeed, $\xi_{t_4}(w_{i+2}) = \xi_{t_4}(v_{i+2}) = \xi_{t_4}(u_{i+1}) \neq \xi$ and both u_{i+1} and w_{i+2} are weakly t_4 -stable due to Claim 4.3 and weak 0-stability that is assumed in Equation (12), so, by Remark 4.15, we can apply Claim 4.11.
- Assume that u_{i+1} does not change its opinion in any even time step from $[0, t_4]$, but w_{i+2} changes its opinion at some time step $t_1 \leq t < t_4$. Here we have that $\xi_t(w_i) = \xi_t(v_i) = \xi_t(v_{i+2}) = \xi_t(w_{i+2}) = \xi$ and both w_i and w_{i+2} are weakly t -stable due to Claim 4.3 and the property of being 1-close to stability assumed in Equation (12). By Remark 4.15, we can apply Claim 4.11 at time step t to the path Q between w_i to w_{i+2} and get a contradiction with $\mathcal{E}_1(P)$ due to (10).
- Finally, assume that u_{i+1} does not change its opinion in any even time step from $[0, t_4]$, and t_4 is the first even time step from $[0, t_4]$ when w_{i+2} changes its opinion. By Claim 4.3 and due to (12), w_{i+2} is weakly (t_4-2) -stable. So, by Claim 4.2, $\xi_{t_4-1}(u_{i+2}) = \xi$. Since $\xi_{t_4-1}(v_{i+1}) = \xi$, we get a contradiction with $\xi_{t_4}(v_{i+2}) \neq \xi$.

3. In case $v = w_{i+2}$, we can repeat the reasoning from the previous case, with the following amendments. First, everywhere in the proof we make the following replacements: $w_i \leftrightarrow w_{i+2}$, $u_i \leftrightarrow u_{i+2}$, $v_i \leftrightarrow v_{i+2}$. As in the previous case, we first get $\xi_{t_1}(w_{i+2}) = \xi_{t_1}(v_{i+2})$. Nevertheless, $\mathcal{E}_1(P)$ implies that $\xi_i(w_{i+2}) \neq \xi_i(v_{i+2})$. Recalling that $t_1 \leq i-2$, at least one of $\{v_{i+2}, w_{i+2}\}$ should change its opinion at time step not exceeding i . In the remaining part of the proof, it remains to make the following replacements: $i-2 \rightarrow i$ and $i-4 \rightarrow i-2$. Since now $t_4 \leq i-2$, in the same way as in the case “ $v = w_i$ ”, we can apply (10) to get a contradiction with $\mathcal{E}_1(P)$. This completes the proof of the claim. $\square$

In what follows, assuming that all vertices of U_i are 1-close to stability, we will get a contradiction with $\mathcal{E}_1(P)$ in some of the considered cases due to the observation (10).

4.3.2.2 | Consider the Case $\xi_0(v_i) = \xi_0(v_{i+2}) = 1$. First, notice that if $\mathcal{E}_2(P)$ holds and all three vertices from U_i are weakly 0-stable then (12) holds. Then, due to Claim 4.17, $\xi_{i-2}(w_i) = -1$, which contradicts $\mathcal{E}_1(P)$.

Now, let $\mathcal{E}_2(P)$ hold and there be a unique vertex $u \in U_i$ which is not weakly 0-stable. Then, let us prove that, to satisfy $\mathcal{E}_1(P)$, the following properties must hold:

- A1 the two vertices from $U_i \setminus \{u\}$ have initial opinion -1 ;
- A2 the two vertices from $U_i \setminus \{u\}$ are not strongly 0-stable;
- A3 for every vertex from $U_i \setminus \{u\}$, at least one of its grandchildren has initial opinion -1 ;
- A4 $\xi_t(u) = 1$ for some deterministic (that does not depend on $\xi_0 \in \mathcal{E}_1(P)$) time step $t := t(u, i) \in \{i-2, i\}$.

A1 follows from Equation (10) and the fact that Claims 4.11 and 4.12 are applicable to the path between the two vertices from $U_i \setminus \{u\}$ and time moment $t = 0$. Indeed, if both initial opinions are 1, then the considered path clearly satisfies the requirements of Claim 4.11 by Remark 4.15. If the initial opinions are different, then the only non-trivial case that may interfere with the requirements of Claim 4.12 is when $u = u_{i+1}$ and the path between the remaining two vertices from U_i has seven vertices (see Remark 4.15). However, since we assumed that $\xi_0(v_i) = \xi_0(v_{i+2})$, the initial opinions of $w_i, v_i, v_{i+2}, w_{i+2}$ along the path change only once, as required. The event $\mathcal{E}_1(P)$ along with A1 immediately imply A2. Assuming that some $u' \in U_i \setminus \{u\}$ has all four grandchildren with initial opinion 1 and recalling that $\xi_0(u') = -1$ due to A1, we get that u' is not weakly 0-stable. This proves A3. A4 is an instant corollary of $\mathcal{E}_1(P)$ —actually, t is either $i-2$ or i depending on the choice of $u \in U_i$. Due to Claim 4.14, the probability that u is not weakly 0-stable and A4 holds can be bounded from above by the probability of an event that only depends on $\xi_0|_{V(T_u)}$ which is, in turn, at most 0.038. Hence, by Remark 4.4 the latter event is independent of the events A1, A2, and A3. Recalling that, for any $u' \in U_i \setminus \{u\}$, the event that u' is strongly 0-stable implies A3, we get

$$\begin{aligned}
 q_3 &\leq 3 \cdot \mathbb{P}^*(\{u \text{ is not weakly 0-stable}\} \wedge A1 \wedge A2 \wedge A3 \wedge A4) \\
 &= 3 \cdot \mathbb{P}(\{u \text{ is not weakly 0-stable}\} \wedge A1 \wedge A2 \wedge A3 \wedge A4) \\
 &\leq 3 \cdot 0.038 \cdot \mathbb{P}(A1 \wedge A2 \wedge A3) \\
 &\leq 3 \cdot 0.038 \cdot \left(\frac{1}{2} \left(1 - \frac{1}{16} - p_s(0)\right)\right)^2 \stackrel{\text{Claim 4.6}}{<} 0.0055.
 \end{aligned} \tag{14}$$

4.3.2.3 | Consider the Case $\xi_0(v_i) = \xi_0(v_{i+2}) = -1$. We need to bound $q_3 = q_3^{(1)} + q_3^{(2)}$, where

$$q_3^{(1)} := \mathbb{P}^*(\mathcal{E}_2(P) \wedge \{\text{all vertices of } U_i \text{ are weakly 0-stable}\}),$$

$$q_3^{(2)} := \mathbb{P}^*(\mathcal{E}_2(P) \wedge \{\text{exactly one vertex of } U_i \text{ is not weakly 0-stable}\}).$$

In what follows, we prove bounds on $q_3^{(1)}$ and $q_3^{(2)}$ separately.

We start from $q_3^{(1)}$. Claim 4.17 gives us that, for every $u \in U_i$ and every even $t \leq i-2$, $\xi_t(u) = 1$. In particular, $\xi_0(u) = \xi_2(u)$. The strategy of the remaining proof is as follows: We assume that the event $\xi_0(w_i) = \xi_0(u_{i+1}) = \xi_0(w_{i+2}) = 1$ holds deterministically. Then, for every $u \in U_i$, we introduce an event that implies $\xi_0(u) \neq \xi_2(u)$ and such that all these three events (corresponding to w_i, u_{i+1}, w_{i+2}) are independent of each other and of initial opinions of all vertices from $U_i \cup \{v_i, v_{i+2}\}$. For $u \in \{w_i, w_{i+2}\}$, this is the event that all four grandchildren of u have initial opinion -1 . The probability of this event equals $1/16$. For $u = u_{i+1}$, the desired event is that u has a child with both children having initial opinion -1 (this event has probability $7/16$). Indeed, since $\xi_0(v_i) = \xi_0(v_{i+2}) = -1$, we have that $\xi_1(v_{i+1}) = -1$, and thus, for $u = u_{i+1}$, to change its opinion to -1 at time 2, it is sufficient to have one child that has opinion -1 at time 1. Due to Remark 4.4, the introduced “lower bound events” are independent, and thus

$$\begin{aligned}
 q_3^1 &\leq \mathbb{P}^* \left(\bigwedge_{u \in U_i} \{\xi_0(u) = 1\} \wedge \bigwedge_{u \in U_i} \{\xi_0(u) \neq \xi_2(u)\} \right) \\
 &\leq \mathbb{P} \left(\bigwedge_{u \in U_i} \{\xi_0(u) = 1\} \right) \left(1 - \frac{1}{16}\right)^2 \left(1 - \frac{7}{16}\right) \\
 &= \frac{1}{8} \cdot \left(\frac{15}{16}\right)^2 \cdot \frac{9}{16} < 0.0618.
 \end{aligned} \tag{15}$$

Next, let us give an upper bound on $q_3^{(2)}$. Let $u \in U_i$ be the unique vertex that is not weakly 0-stable. Let us prove that, to satisfy $\mathcal{E}_1(P)$, the following properties must hold:

B1 the two vertices from $U_i \setminus \{u\}$ have initial opinion 1;

B2 $\xi_t(u) = 1$ for some deterministic (that does not depend on $\xi_0 \in \mathcal{E}_1(P)$) time step $t := t(u, i) \in \{i - 2, i\}$.

B1 follows from Equation (10) and the fact that Claims 4.11 and 4.12 are applicable to the path between the two vertices from $U_i \setminus \{u\}$ and time moment $t = 0$. Indeed, if both initial opinions are -1 , then the considered path clearly satisfies the requirements of Claim 4.11 by Remark 4.15. If the initial opinions are different, then, by Remark 4.15, the only non-trivial case that may interfere with the requirements of Claim 4.12 is when $u = u_{i+1}$ and the path between the remaining two vertices from U_i has seven vertices. However, since we assumed that $\xi_0(v_i) = \xi_0(v_{i+2})$, the initial opinions of $w_i, v_i, v_{i+2}, w_{i+2}$ along the path change only once, as required by Remark 4.15. B2 is an instant corollary of $\mathcal{E}_1(P)$ —actually, t is either $i - 2$ or i depending on the choice of $u \in U_i$. Due to Claim 4.14, the probability that u is not weakly 0-stable and B2 holds can be bounded from above by the probability of an event that only depends on $\xi_0|_{V(T_u)}$ which is, in turn, at most 0.038. Hence, by Remark 4.4, the latter event is independent of the event B1. So,

$$q_3^{(2)} \leq 3 \cdot \mathbb{P}(\{u \text{ is not weakly 0-stable}\} \wedge \text{B1} \wedge \text{B2}) < 3 \cdot 0.038 \cdot \frac{1}{4} = 0.0285. \tag{16}$$

Finally, due to (15) and (16),

$$q_3 = q_3^{(1)} + q_3^{(2)} < 0.0618 + 0.0285 = 0.0903. \tag{17}$$

4.3.2.4 | Consider the Last Case $\xi_0(v_i) \neq \xi_0(v_{i+2})$. Here, we actually have two ways of giving initial opinions to v_i and v_{i+2} , but we plan to prove the same upper bound on q_3 in those cases.

Let us prove that if every $u \in U_i$ is weakly 0-stable, then $\mathcal{E}_1(P)$ does not hold. Indeed, since $\xi_0(v_i) \neq \xi_0(v_{i+2})$, there is an index $j \in \{i, i + 2\}$ such that $\xi_0(u_{i+1}) = \xi_0(v_j)$. Then, by Remark 4.15, we can apply either Claim 4.11 or Claim 4.12 to the path between u_{i+1} and w_j at time step 0, getting a contradiction with (10). Indeed, both u_{i+1} and w_j are weakly 0-stable. So, by Remark 4.15, since $\xi_0(u_{i+1}) = \xi_0(v_j)$, if $\xi_0(u_{i+1}) = \xi_0(w_j)$ then Claim 4.11 applies and if $\xi_0(u_{i+1}) \neq \xi_0(w_j)$ then Claim 4.12 applies.

Therefore, if $\mathcal{E}_2(P)$ holds, then there is a single vertex from U_i which is not weakly 0-stable; denote this vertex by u . Due to $\mathcal{E}_1(P)$, there exists a deterministic (that does not depend on $\xi_0 \in \mathcal{E}_1(P)$) time step $t := t(u, i) \in \{i - 2, i\}$ such that $\xi_t(u) = 1$. Due to Claim 4.14, the probability of the latter event jointly with the event that u is not weakly 0-stable can be bounded from above by the probability of an event that only depends on $\xi_0|_{V(T_u)}$, which is, in turn, at most 0.038. We further consider separately three cases, depending on the particular choice of u .

4.3.2.5 | 1. $u = w_{i+2}$. We have $\xi_0(w_i) = \xi_0(u_{i+1}) \neq \xi_0(v_i)$ due to (10) and Remark 4.15 by applying Claim 4.11 or Claim 4.12 at time step 0 to the path between w_i and u_{i+1} . Indeed, both u_{i+1} and w_i are weakly 0-stable. So, if $\xi_0(u_{i+1}) = \xi_0(w_i) = \xi_0(v_i)$ then, by Remark 4.15, Claim 4.11 applies and if $\xi_0(u_{i+1}) \neq \xi_0(w_i)$ then, by Remark 4.15, Claim 4.12 applies. So, $q_3 \leq 0.038 \cdot \frac{1}{4}$. Moreover, if $\xi_0(v_i) = 1$, we can improve this bound: Since $\xi_0(w_i) = \xi_0(u_{i+1}) = -1$, the event $\mathcal{E}_1(P)$ implies that both w_i and u_{i+1} are not strongly 0-stable. So, in this case, $\mathbb{P}^*(\mathcal{E}_2(P) \wedge \{u = w_{i+2}\}) \leq 0.038 \cdot \frac{1}{4} (1 - p_s(0))^2$. We get

$$\mathbb{P}^*(\mathcal{E}_2(P) \wedge \{u = w_{i+2}\}) \leq 0.038 \cdot \frac{1}{4} (\mathbb{1}_{\xi_0(v_i)=-1} + (1 - p_s(0))^2 \cdot \mathbb{1}_{\xi_0(v_i)=1}). \tag{18}$$

4.3.2.6 | 2. $u = w_i$. Since w_{i+2} and u_{i+1} are weakly 0-stable, in the same way as in the case (1), we get $\xi_0(w_{i+2}) = \xi_0(u_{i+1}) \neq \xi_0(v_{i+2})$. So, here we get that $\mathbb{P}^*(\mathcal{E}_2(P) \wedge \{u = w_{i+2}\}) \leq \frac{1}{4}$ as well. In a similar way, by distinguishing between $\xi_0(v_{i+2}) = -1$ and $\xi_0(v_{i+2}) = 1$, we get

$$\mathbb{P}^*(\mathcal{E}_2(P) \wedge \{u = w_i\}) \leq 0.038 \cdot \frac{1}{4} \left(\mathbb{1}_{\xi_0(v_{i+2})=-1} + (1 - p_s(0))^2 \cdot \mathbb{1}_{\xi_0(v_{i+2})=1} \right). \quad (19)$$

4.3.2.7 | 3. $u = u_{i+1}$. Then w_i and w_{i+2} are weakly 0-stable. If $\xi_0(w_i) = \xi_0(v_i)$ and $\xi_0(w_{i+2}) = \xi_0(v_{i+2})$, then $\mathcal{E}_1(P)$ does not hold due to (10) combined with Remark 4.15 and Claim 4.12 applied to the path between w_i and w_{i+2} at time moment 0. Indeed, the initial opinion changes just once in the sequence $w_i, v_i, v_{i+2}, w_{i+2}$, and w_i and w_{i+2} are weakly 0-stable. Therefore, $\xi_0|_{\{w_i, w_{i+2}\}}$ has three possible initial values, excluding $(\xi_0(v_i), \xi_0(v_{i+2}))$; the latter vector has a single -1 . Note that $\mathcal{E}_1(P)$ implies the following: If $w \in \{w_i, w_{i+2}\}$ has initial opinion -1 then it is not strongly 0-stable. Summing up, we get

$$\mathbb{P}^*(\mathcal{E}_2(P) \wedge \{u = u_{i+1}\}) \leq 0.038 \cdot \frac{1}{4} (1 + (1 - p_s(0)) + (1 - p_s(0))^2). \quad (20)$$

The bounds (18), (19), and (20) imply

$$q_3 \leq 0.038 \cdot \left(\frac{1}{4} + \frac{1}{4}(1 - p_s(0))^2 + \frac{1}{4}(1 + (1 - p_s(0)) + (1 - p_s(0))^2) \right) \stackrel{\text{Claim 4.6}}{<} 0.0285. \quad (21)$$

Finally, bounds (14), (17), and (21) conclude the proof Lemma 4.8:

- if $\xi_0(v_i) = 1$ and $\xi_0(v_{i+2}) = 1$, then $k_+ = 2, k_- = 0$, and

$$\mathbb{P}^*(\mathcal{E}_1(P)) < 0.0117 + q_3 \leq 0.0117 + 0.0055 < 0.02 = 2p_+ = q(i)$$

due to (14);

- if $\xi_0(v_i) = -1$ and $\xi_0(v_{i+2}) = -1$, then $k_+ = 0, k_- = 2$, and

$$\mathbb{P}^*(\mathcal{E}_1(P)) < 0.0117 + q_3 \leq 0.0117 + 0.0903 < 0.1026 = 2p_- = q(i)$$

due to (17);

- if $\xi_0(v_i) \neq \xi_0(v_{i+2})$, then $k_+ = k_- = 1$, and

$$\mathbb{P}^*(\mathcal{E}_1(P)) < 0.0117 + q_3 \leq 0.0117 + 0.0285 = 0.0402 < 2\sqrt{p_-p_+} = q(i)$$

due to (21).

4.4 | Proof of Claim 4.6

Let us start with strong t -stability inequalities. We shall prove that there exists an $\varepsilon > 0$ such that, for every $h \in \mathbb{Z}_{>0}$ and for every non-leaf non-root $v \in V(T = T^{(h)})$,

$$p_s(3, h, v) > 0.4453 + \varepsilon, \quad p_s(2, h, v) > 0.5617 + \varepsilon, \quad p_s(1, h, v) \geq \frac{1}{2}p_w(0)^2, \quad p_s(0, h, w) > 0.5 + \varepsilon. \quad (22)$$

If $h_T(v) = 1$, then v is 0-stationary, hence $p_s(i, h, v) = 1$ for every $i \in \{0, 1, 2, 3\}$. This proves all four inequalities. If $h_T(v) = 2$, then v is 0-stable (and hence strongly 2-stable) if two non-sibling grandchildren of v share its initial opinion. The latter holds with probability $9/16 > 0.5617$ —this proves the second and the fourth inequalities. Also, v is strongly 1-stable (and hence strongly 3-stable) if both of its children share an initial opinion. This event holds with probability $1/2$, implying the first and the third inequalities. Now, we can assume that $h_T(v) \geq 3$.

Let us prove all four inequalities from Equation (22) one by one from the leftmost to the rightmost. In the proof of the j th bound, we will assume that all bounds to the right of the j th bound from the statement of Claim 4.6 hold true. First, let us bound $p_s(3, h, v)$ from below by $p_s(1, h, v) + \mathbb{P}(\mathcal{A})$ where $\mathcal{A}$ is an event, defined below, that implies that v is strongly 3-stable but not strongly 1-stable. This is indeed possible since strong 1-stability implies strong 3-stability. Let u_1, u_2 be the children of v . We define the event $\mathcal{A}$ as follows:

- u_1 is strongly 2-stable;
- initial opinion of u_2 is opposite to $\xi := \xi_2(u_1)$;
- all 4 grandchildren of u_2 have initial opinion ξ ;
- at least one of the 4 grandchildren of u_2 is weakly 0-stable.

Let us first show that $\mathcal{A}$ implies that v is not strongly 1-stable. Indeed, notice that the vertex u_2 has opinion opposite to ξ at time 0, while $\xi_2(u_2) = \xi$ since all grandchildren of u_2 have initial opinion ξ . Therefore, for the extension $\tilde{\xi}_0$ of $\xi_0|_{V(T_v)}$ to the whole tree such that all the vertices outside of the subtree T_v share the same initial opinion opposite to ξ , we get $\tilde{\xi}_1(v) \neq \xi$, while, for the extension $\tilde{\xi}'_0$ such that all the vertices outside T_v share the initial opinion ξ , we get $\tilde{\xi}'_3(v) = \xi$.

It remains to show that $\mathcal{A}$ implies strong 3-stability of v . Let us prove that, for any extension $\tilde{\xi}_0$ of $\xi|_{V(T_v)}$ to the whole tree, v is 3-stable and $\tilde{\xi}_3(v) = \xi$. Indeed, $\tilde{\xi}_2(u_2) = \xi$ since all grandchildren of u_2 have initial opinion ξ . Applying Claim 4.10 twice—to the parent of a weakly 0-stable grandchild of u_2 and then to u_2 itself, we get that u_2 is weakly 2-stable. Also, u_1 is weakly 2-stable as it is strongly 2-stable due to the definition of $\mathcal{A}$. By Claim 4.11, both u_1 and u_2 are 2-stable, and hence v is 3-stable. Since we conclude this only from the event $\mathcal{A}$ that is defined solely by the initial opinions on vertices of T_v , we get that v is strongly 3-stable, as needed.

Finally,

$$\begin{aligned} p_s(3, h, v) &\geq p_s(1, h, v) + \mathbb{P}(\mathcal{A}) \\ &\geq p_s(1) + p_s(2) \cdot \frac{1}{2} \cdot \frac{1}{16} \cdot (1 - (1 - p_w(0))^4) \\ &> 0.5 \cdot 0.925^2 + 0.5617 \cdot \frac{1}{32} \cdot (1 - 0.075^4) > 0.4453. \end{aligned}$$

The first bound in Equation (22) follows.

Now, let us bound $p_s(2, h, v)$. Similarly, we bound it from below by $p_s(0, h, v) + \mathbb{P}(\mathcal{B})$ for some $\mathcal{B}$ implying that v is strongly 2-stable but not strongly 0-stable. This event is defined as follows:

- all 4 grandchildren of v share the opposite to v initial opinion;
- 2 non-siblings among the grandchildren of v are weakly 0-stable.

Let us show that $\mathcal{B}$ is the desired event. Parents u_1 and u_2 of w_1 and w_2 (which are the weakly 0-stable non-siblings), that are precisely the two children of v , have opinion $\xi := \xi_0(w_1)$ at time Step 1. So, $\xi_2(v) = \xi \neq \xi_0(v)$ —thus, v is not strongly 0-stable. It remains to show that v is strongly 2-stable. By Claim 4.2, $\xi_2(w_1) = \xi_2(w_2) = \xi$. Hence, w_1 and w_2 also remain weakly 2-stable, due to Claim 4.3. Due to Claim 4.11, v is 2-stable. Since we conclude this only from the event $\mathcal{B}$ that is defined solely by the initial opinions on vertices of T_v , we get that v is strongly 2-stable, as needed.

Finally,

$$p_s(2, h, v) \geq p_s(0) + \frac{1}{16}(1 - (1 - p_w(0))^2)^2 > 0.5 + \frac{1}{16}(1 - 0.075^2)^2 > 0.5617.$$

The second bound in Equation (22) follows.

Let us prove the third inequality from Equation (22). Let u_1, u_2 be the children of v . By Claim 4.11, if u_1 and u_2 are both weakly 0-stable and share the initial opinion, then they are 0-stable, and hence v is strongly 1-stable. Due to Remarks 4.4 and 4.5, we have that

$$p_s(1, h, v) \geq \frac{1}{2}p_w(0, h, u_1)p_w(0, h, u_2).$$

So, $p_s(1, h, v) \geq \frac{1}{2}p_w(0)^2$ follows from taking the infimum of the right-hand side of the last inequality.

Let us finally prove the last inequality in Equation (22): $p_s(0, h, v) > 0.5 + \epsilon$. Let $\xi := \xi_0(v)$. Let u_1, u_2 be children of v . If u_1 has child w_1 and u_2 has child w_2 such that $\xi_0(w_1) = \xi_0(w_2) = \xi$, and w_1 and w_2 are weakly 0-stable, then v is 0-stable,

due to Claim 4.11. So, v is strongly 0-stable with probability at least $\left(1 - \left(1 - \frac{p_w(0)}{2}\right)^2\right)^2$. This probability is more than $1/2 + \varepsilon$ if and only if $p_w(0) > 2 - \sqrt{2(2 - \sqrt{2})}$. It remains to prove that $p_w(0) > 0.925$, as $0.925 > 2 - \sqrt{2(2 - \sqrt{2})}$.

Let us prove that $p_w(0) > 0.925$. Note that this inequality would complete the proof of Claim 4.6. First, if v is a leaf, then $p_w(0, h, v) = 1$ due to the definition. So, further, we assume that v is not a leaf. Let us denote by $\mathcal{N}$ the subset of all numbers from $\{0, \dots, h_T(v)\}$ with the same parity as $h_T(v)$, assuming that 0 is even. For every $t \in \mathcal{N}$, all vertices $u \in V(T_v)$ at distance t from leaves have the same $p_w(0, h, u)$, due to the definition of weak stability. For brevity, let us denote $1 - p_w(0, h, u)$ by q_t . Let $q > 0$ be the smallest positive real solution of $x = P(x)$, where

$$P(x) := P_1(x) + P_2(x)P_3(x)$$

and P_1, P_2, P_3 are defined by

$$P_1(x) := \left(\frac{1}{4} + \frac{x^2}{4}\right)^2, \quad (23)$$

$$P_2(x) := \left(1 - \frac{1 - x^2}{4}\right)^2, \quad (24)$$

$$P_3(x) := \left(\frac{1}{2} + \frac{x}{2}\right)^4 - \left(\frac{1}{4} + \frac{x^2}{4}\right)^2. \quad (25)$$

Since $P(0) = 1/16$, $P(3/40) < 3/40$, and P has only non-negative coefficients (implying that P increases on $(0, \infty)$), we get that $q \in (1/16, 3/40)$.

It is sufficient to prove by induction, that $q_t \leq q$ for every $t \in \mathcal{N}$, as $p_w(0, h, v) = 1 - q_{h_T(v)}$ and $q < 3/40 = 0.075$. For the base of induction, we check $t \in \{0, 1, 2, 3\}$. Clearly, $q_t = 0$ for $t \in \{0, 1\} \cap \mathcal{N}$. Moreover, $q_t = \frac{1}{16} < q$ for $t \in \{2, 3\} \cap \mathcal{N}$ since in these two cases u is weakly 0-stable if and only if it has a grandchild with the same initial opinion as u has.

For the induction step, let us fix $t \in \mathcal{N}$ ($t \geq 4$) and a vertex u at distance t from leaves. Let $\hat{\xi} := \xi_0(u)$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be three disjoint events defined by $\xi_0|_{T_u}$ in the following way:

1. $\mathcal{A}$ holds when
 - every grandchild of u with initial opinion $\hat{\xi}$ is not weakly 0-stable;
 - for every child of u , its children share the initial opinion (i.e., initial opinions of siblings that are grandchildren of u coincide).
2. $\mathcal{B}$ holds when at least one grandchild of u is weakly 0-stable with initial opinion $\hat{\xi}$;
3. $\mathcal{C} = \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$.

If $\mathcal{B}$ holds then u is weakly 0-stable due to Claim 4.9. Therefore,

$$\mathbb{P}(u \text{ is not weakly 0-stable} \mid \mathcal{B}) = 0$$

implying that

$$q_t = \sum_{\mathcal{Q} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}} \mathbb{P}(u \text{ is not weakly 0-stable} \mid \mathcal{Q}) \mathbb{P}(\mathcal{Q}) \leq \mathbb{P}(\mathcal{A}) + \mathbb{P}(u \text{ is not weakly 0-stable} \mid \mathcal{C}) \mathbb{P}(\mathcal{C}).$$

From the definition of events $\mathcal{A}, \mathcal{B}, \mathcal{C}$, it immediately follows that

$$\mathbb{P}(\mathcal{A}) = \left(\frac{1}{4} + \frac{q_{t-2}^2}{4}\right)^2 = P_1(q_{t-2}),$$

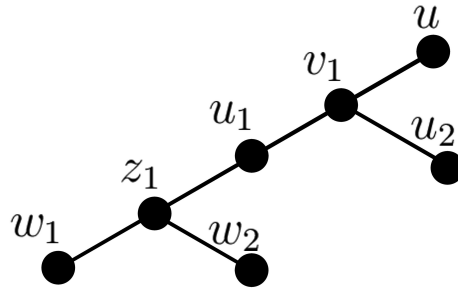


FIGURE 3 | Descendants of u .

where P_1 is defined in Equation (23), and

$$\mathbb{P}(C) = \mathbb{P}(\bar{B}) - \mathbb{P}(A) = \left(\frac{1}{2} + \frac{q_{t-2}}{2}\right)^4 - \left(\frac{1}{4} + \frac{q_{t-2}^2}{4}\right)^2 = P_3(q_{t-2}),$$

where P_3 is defined in Equation (25).

Let us notice, that all three polynomials P_1, P_2, P_3 defined in Equations (23) to (25) have only non-negative coefficients, hence they increase on $[0, q]$. Since $P(q) = q$, we get that, for any $0 \leq x_1, x_2, x_3 \leq q$,

$$P_1(x_1) + P_2(x_2)P_3(x_3) \leq P_1(q) + P_2(q)P_3(q) = P(q) = q.$$

So, it remains to prove the first inequality in

$$\mathbb{P}(u \text{ is not weakly 0-stable} \mid C) \leq \left(1 - \frac{1 - q_{t-4}^2}{4}\right)^2 = P_2(q_{t-4}).$$

Let us enumerate all 16 possible initial opinions of the four grandchildren of u arbitrarily. We decompose $C = \sqcup_{j=1}^{16} C_j$, where C_j says that C holds and the four grandchildren of u have the j -th 4-tuple of opinions. Fix $j \in [16]$ such that $\mathbb{P}(C_j) \neq 0$. By the definition, C (and therefore C_j) implies that there is a child v_1 of u and two children u_1, u_2 of v_1 such that $\xi_0(u_1) \neq \hat{\xi}$ and $\xi_0(u_2) = \hat{\xi}$. Let us notice that, since $\xi_0(u_1) \neq \hat{\xi}$, any event defined by $\xi_0|_{V(T_{u_1}) \setminus \{u_1\}}$ is independent from C_j . To conclude the proof, let us construct an event D defined by $\xi_0|_{V(T_{u_1}) \setminus \{u_1\}}$ with probability at least $1 - (1 - (1 - q_{t-4}^2)/4)^2$ such that $C_j \cap D$ implies the weak 0-stability of u .

The desired event D is defined in the following way. It says that there are two grandchildren w_1, w_2 of u_1 (see Figure 3) such that

- w_1, w_2 are siblings (i.e., they share a parent);
- $\xi_0(w_1) = \xi_0(w_2) = \hat{\xi}$;
- at least one of w_1, w_2 is weakly 0-stable.

The required condition that D is defined by $\xi_0|_{V(T_{u_1}) \setminus \{u_1\}}$ and the inequality

$$\mathbb{P}(D) \geq 1 - \left(1 - \frac{1 - q_{t-4}^2}{4}\right)^2$$

are immediate. Now, suppose that $C_j \cap D$ holds.

We have to derive the weak 0-stability of u . Let us recall that $\xi_0(u) = \xi_0(u_2) = \hat{\xi}$, so $\xi_1(v_1) = \hat{\xi}$ as u, u_2 are neighbors of v_1 . In addition, at time 1, the parent z_1 of w_1 and w_2 holds the opinion $\hat{\xi}$. Since u_1 is adjacent to z_1 and to v_1 , we get that $\xi_2(u_1) = \hat{\xi}$. Then, applying Claim 4.10 twice, we first get that z_1 is weakly 1-stable, and then that u_1 is weakly 2-stable.

Since $\xi_1(v_1) = \hat{\xi}$, the extension $\tilde{\xi}_0$ of $\xi_0|_{T_u}$ to the entire tree T satisfying $\tilde{\xi}_0(u^*) = \hat{\xi}$ for all $u^* \notin V(T_u)$ forces u to inherit the opinion $\hat{\xi}$ at time 2, that is, $\tilde{\xi}_2(u) = \hat{\xi}$. Note that u_1 is weakly 2-stable w.r.t. $\tilde{\xi}_0$ as well since the event $C_j \cap \mathcal{D}$ is defined by $\xi_0|_{T_u}$. Then, by Claim 4.9, u is weakly 2-stable w.r.t. $\tilde{\xi}_0$. It is also 2-stable (w.r.t. to $\tilde{\xi}_0$) by the second equivalent definition of weak 2-stability from Claim 4.1, due to the substitution in the definition $\tilde{\xi}_0^* := \tilde{\xi}_2$, and hence u is 0-stable. So finally we get that u is weakly 0-stable w.r.t. ξ_0 , due to the second equivalent definition of weak 0-stability, once again by substitution $\tilde{\xi}_0^* := \tilde{\xi}_0$, completing the proof.

5 | Perfect k -ary Trees: Proof of Theorem 1.3

In this section, we prove Theorem 1.3, which asserts both the lower and the upper bound for the typical stabilization time on a perfect k -ary tree. Our upper bound is an immediate corollary of Theorem 1.1. Indeed, $\mathcal{Q}$ only consists of paths with ends at distance at least 2 from the leaves. Hence, the maximum of $t(\mathcal{Q})$ over all paths in $\mathcal{Q}$ is achieved at the longest such path. So, the maximum equals $D - 3$.

In this section, we prove the lower bound. We first introduce the notion of being $(\leq t)$ -stable, then state the lower bound in a separate Lemma 5.2 and prove it, completing the proof of Theorem 1.3.

For every positive integer h , we fix a perfect k -ary tree $T^{(h)}$ of depth h rooted in a vertex R . When the height is clear from the context, we omit the superscript and write simply T . For a uniformly random ξ_0 on $V(T)$, we let $\tau := \tau(T; \xi_0)$.

Let us say that a vertex $v \in V(T)$ is $(\leq t)$ -stable, if $\tilde{\xi}_{t'}(v) = \tilde{\xi}_t(v)$ holds for every vector of initial opinions $\tilde{\xi}_0$ such that $\tilde{\xi}_0|_{V(T_v)} = \xi_0|_{V(T_v)}$ and for every time step $t' < t$ with the same parity as t . Let us note that being $(\leq t)$ -stable is the property of $\xi_0|_{V(T_v)}$. So, Remark 4.4 is applicable here too (i.e., events $\{v \text{ is } (\leq t)\text{-stable}\}$ are independent for vertices v which are not descendants of each other).

Claim 5.1. There exists $C > 0$ such that, for every even $k > 2$, every opinion $\xi \in \{-1, 1\}$, every non-root non-leaf vertex $v \in V(T)$ of a perfect k -ary tree T , and every time step $t \in \mathbb{Z}_{\geq 2}$, probability that v is $(\leq t)$ -stable and $\xi_t(v) = \xi$ is at least 2^{-Ctk} .

Proof. Without loss of generality, we may assume that $h_T(v) \geq 2$ since otherwise the desired event follows from the property that more than half of the leaves adjacent to v have opinion ξ ; and the latter event has probability at least $1/4$.

We will show a sufficiently large class $\mathcal{F} = \mathcal{F}(k, \xi, v, t)$ of vectors of initial opinions $\xi_0^* : V(T_v) \rightarrow \{-1, 1\}$ such that, for any extension of ξ_0^* to the whole tree, the vertex v is $(\leq t)$ -stable and $\xi_t(v) = \xi$.

For a non-root vertex u , let the random vector $\hat{\xi}_0^{(u)} : V(T) \rightarrow \{-1, 1\}$ have uniformly random distribution on $V(T_u)$ and $\hat{\xi}_0|_{V(T) \setminus V(T_u)} \equiv \xi$. Let $u_1, \dots, u_k$ be children of v , $w_1, \dots, w_k$ be children of u_1 ; and $z_1, \dots, z_k$ be children of u_2 .

Let $U = \{u_3, \dots, u_k, w_1, \dots, w_k, z_1, \dots, z_k\}$. Let $t(u) : U \rightarrow \mathbb{Z}_{\geq 0}$ equal t if $u = u_i$ for some $i \in \{3, \dots, k\}$, and $t - 1$, otherwise. For convenience, for vertices $u, w \in U$ such that $t(u) = t(w)$ we introduce the event

$$E(u, w) := \left\{ \text{for all } 0 \leq i \leq t(u) - 1 \text{ with the same parity as } t(u) - 1, \hat{\xi}_i^{(u)}(u) = \xi \text{ or } \hat{\xi}_i^{(w)}(w) = \xi \right\}.$$

We then define $\mathcal{F}$ as the set of ξ_0^* such that:

1. for every odd $i \in \{3, \dots, k - 1\}$, if $\hat{\xi}_0^{(u_i)}|_{V(T_{u_i})} = \xi_0^*|_{V(T_{u_i})}$ and $\hat{\xi}_0^{(u_{i+1})}|_{V(T_{u_{i+1}})} = \xi_0^*|_{V(T_{u_{i+1}})}$, then $E(u_i, u_{i+1})$ holds;
2. for every odd $i \in [k - 1]$, if $\hat{\xi}_0^{(w_i)}|_{V(T_{w_i})} = \xi_0^*|_{V(T_{w_i})}$ and $\hat{\xi}_0^{(w_{i+1})}|_{V(T_{w_{i+1}})} = \xi_0^*|_{V(T_{w_{i+1}})}$, then $E(w_i, w_{i+1})$ holds;
3. for every odd $i \in [k - 1]$, if $\hat{\xi}_0^{(z_i)}|_{V(T_{z_i})} = \xi_0^*|_{V(T_{z_i})}$ and $\hat{\xi}_0^{(z_{i+1})}|_{V(T_{z_{i+1}})} = \xi_0^*|_{V(T_{z_{i+1}})}$, then $E(z_i, z_{i+1})$ holds;
4. $\xi_0^*(v) = \xi$ and $\xi_0^*(u_1) = \xi_0^*(u_2) = \xi$.

Let ξ_0 be an extension of ξ_0^* to the entire T . Let us first show that indeed v is $(\leq t)$ -stable w.r.t. ξ_0 and $\xi_t(v) = \xi$. It is sufficient for us to show the following stronger statement:

- 1 for every $u \in U$ and every $s \in \{0, \dots, t(u) - 1\}$ with the same parity as $t(u) - 1$, if $\hat{\xi}_0^{(u)}|_{V(T_u)} = \xi_0^*|_{V(T_u)}$, then $\xi_s(u) = \hat{\xi}_s^{(u)}(u)$;
- 2 $\xi_s(u_1) = \xi_s(u_2) = \xi$ for every $s \in \{0, \dots, t - 1\}$ with the same parity as $t - 1$;
- 3 $\xi_s(v) = \xi$, for every $s \in \{0, \dots, t - 1\}$ with the same parity as t .

Indeed, if that holds, recalling that $\xi_0|_{V(T_v)} \in \mathcal{F}$, we get that more than a half of children of v at time step $t - 1$ have opinion ξ , and so $\xi_t(v) = \xi$. Along with property (3), it implies $(\leq t)$ -stability of v as needed.

To prove the statement, suppose, towards contradiction, that $s < t$ is the earliest time step when at least one vertex from $U \sqcup \{v, u_1, u_2\}$ has an opinion other than the one we declared. Let us first assume that one of the vertices v, u_1, u_2 has such a different opinion at time s . By Condition 4 in the definition of $\mathcal{F}$, they have the desired opinion at time step 0, that is, $s \geq 1$. If s has the same parity as t , then at least $k/2 + 1$ of the children of v have opinion ξ at time $s - 1$ since (1) holds for time step $s - 1$ and due to Condition 1 in the definition of $\mathcal{F}$. So $\xi_s(v) = \xi$. In addition, if s has different parity than t , then at least $k/2$ of the children of u_1 have opinion ξ at time $s - 1$ since (1) holds at time step $s - 1$ and due to Condition 2 in the definition of $\mathcal{F}$. Since the parent of u_1 also has opinion ξ at time $s - 1$, we get that $\xi_s(u_1) = \xi$. Similarly, $\xi_s(u_2) = \xi$ — a contradiction. Now, assume that a vertex $u \in U$ has an opinion at time s which is not declared by (1) and the definition of $\mathcal{F}$. Due to the assumption that (1) to (3) hold for time moments less than s , the parent of u has opinion ξ at every time $s' < s$ with different parity; hence u inherits the desired opinion — a contradiction.

It remains to show that $\mathcal{F}$ is large. Let a pair of vertices (u, w) be one of the following:

- (u_i, u_{i+1}) , for some odd $i \in \{3, \dots, k - 1\}$;
- (w_i, w_{i+1}) , for some odd $i \in [k - 1]$;
- (z_i, z_{i+1}) , for some odd $i \in [k - 1]$.

Hence, $t(u) = t(w)$. To bound the probability of $\mathcal{F}$, first, let us bound the probability of the event $E(u, w)$. Let us note that vectors

$$\phi(\hat{\xi}_0^{(u)}) := \left(\hat{\xi}_{((t(u)-1) \bmod 2)}^{(u)}(u), \hat{\xi}_{((t(u)-1) \bmod 2)+2}^{(u)}(u) \dots, \hat{\xi}_{t(u)-1}^{(u)}(u) \right)$$

and

$$\phi(\hat{\xi}_0^{(w)}) := \left(\hat{\xi}_{((t(w)-1) \bmod 2)}^{(w)}(w), \hat{\xi}_{((t(w)-1) \bmod 2)+2}^{(w)}(w) \dots, \hat{\xi}_{t(w)-1}^{(w)}(w) \right)$$

are independent and identically distributed. Let Φ be the set of all atoms of this distribution.

Let us fix some isomorphism φ from T_w to T_u . Suppose that $\hat{\xi}_0^{(w)}|_{V(T_w)} = -\hat{\xi}_0^{(u)} \circ \varphi|_{V(T_w)}$. Let $\hat{\xi}_s^{(u)}(u) \neq \xi$, for some s . Then, due to symmetry and the fact that $\hat{\xi}_0^{(w)}|_{V(T) \setminus V(T_w)} \equiv \xi$, we get $\hat{\xi}_s^{(w)}(w) = \xi$ as well. Therefore, $E(u, w)$ holds. So, for every $\phi \in \Phi$,

$$\mathbb{P}(E(u, w) | \phi(\hat{\xi}_0^{(u)}) = \phi) \geq \mathbb{P}(\phi(\hat{\xi}_0^{(w)}) = \phi) = \mathbb{P}(\phi(\hat{\xi}_0^{(u)}) = \phi).$$

Thus,

$$\begin{aligned} \mathbb{P}(E(u, w)) &= \sum_{\phi \in \Phi} \mathbb{P}(E(u, w) | \phi(\hat{\xi}_0^{(u)}) = \phi) \mathbb{P}(\phi(\hat{\xi}_0^{(u)}) = \phi) \geq \sum_{\phi \in \Phi} \mathbb{P}(\phi(\hat{\xi}_s^{(u)}) = \phi)^2 \\ &\stackrel{\text{QM-AM}}{\geq} \frac{1}{|\Phi|} \left(\sum_{\phi \in \Phi} \mathbb{P}(\phi(\hat{\xi}_s^{(u)}) = \phi) \right)^2 = \frac{1}{|\Phi|} \geq 2^{-\lceil t(u)/2 \rceil}. \end{aligned}$$

Since the event $E(u, w)$ is a function of random variables associated with vertices of T_u and T_w , we get that these events are independent over the set of considered pairs (u, w) . So, applying the latter bound to every pair (u, w) , we conclude that

$$\mathbb{P}\left(\hat{\xi}_0^{(v)} |_{V(T_v)} \in \mathcal{F}\right) \geq 2^{-3} \left(\frac{1}{2^{\lceil t/2 \rceil}}\right)^{\frac{k-2}{2}} \left(\frac{1}{2^{\lceil (t-1)/2 \rceil}}\right)^k > 2^{-Ctk},$$

for some constant $C > 0$, completing the proof. $\square$

Lemma 5.2. *There exists $c_- > 0$ such that for any even $k > 2$, any R -rooted perfect k -ary tree T of height h , and uniformly random $\xi_0 \in \{-1, 1\}^{V(T)}$, whp, as $h \rightarrow \infty$, there is at least one not $\lfloor c_- \cdot \sqrt{h \ln k/k} \rfloor$ -stationary vertex.*

Proof. Let d be the smallest even number bigger than $\lfloor c_- \cdot \sqrt{h \ln k/k} \rfloor + 2$. We will choose the value of the constant c_- at the end of the proof.

Let $R^* \in V(T)$ be at distance d from a leaf, define $T^* := T_{R^*}$. First, we will define a class $\mathcal{F}$ of functions $\xi_0^* : V(T^*) \rightarrow \{-1, 1\}$ such that for each initial opinion $\xi_0^* \in \mathcal{F}$ and every its extension to the entire $V(T)$, there exists a not $(d-3)$ -stable vertex. Secondly, we will show that this class is so large that whp, for a uniformly random $\xi_0 \in \{-1, 1\}^{V(T)}$, there exists R^* as above so that $\xi_0|_{V(T^*)} \in \mathcal{F}$ leading to the fact that $\tau \geq d-2$ whp. The latter immediately implies the statement of Lemma 5.2 due to the definition of d . $\square$

5.1 | Definition of the Desired Set $\mathcal{F}$

The following requirements on ξ_0^* define the set $\mathcal{F}$. There exists a path $v_2 v_3 \cdots v_d$ with $v_d := R^*$ in T^* such that for any extension ξ_0 to the entire $V(T)$

- A1 for every even $t \in [d]$, $\xi_0(v_t) = 1$ holds;
- A2 for every $t \in \{3, \dots, d\}$, all children of v_t , other than v_{t-1} , are $(\leq t-1)$ -stable;
- A3 for every $t \in \{3, \dots, d-1\}$, the number of children u of v_t such that $\xi_{t-1}(u) = -1$ and $u \neq v_{t-1}$ equals $k/2$;
- A4 for every child u of v_d , other than v_{d-1} , $\xi_{d-1}(u) = 1$ holds;
- A5 all grandchildren of v_2 are leaves with initial opinion -1 .

Let $\xi_0^* \in \mathcal{F}$ and let ξ_0 be an extension of ξ_0^* to the entire $V(T)$. Let us show that the vertex v_{d-1} is not $(d-3)$ -stable w.r.t. ξ_0 . To prove that, it is sufficient to show that for every $s \in \{2, \dots, d-1\}$

- $\xi_s(v_s) = -1$;
- $\xi_s(v_i) = 1$, for every $i \in [d] \setminus [s]$ with the same parity as s .

Let us prove the latter statement by induction. First, by A2 and A4, v_d has at least $k-1$ children with opinion 1 at every odd time step less than d . So, $\xi_0(v_d) = \xi_2(v_d) = \cdots = \xi_d(v_d) = 1$. Next, let us note that all children of v_2 have opinion -1 at time 1, so $\xi_2(v_2) = -1$. We will further rely on the following fact: For every time step $t \in [d-1]$ and every $i \in \{\max\{t, 3\}, \dots, d-1\}$ with the same parity as t ,

$$\xi_t(v_i) = -1 \text{ if and only if either } \xi_{t-1}(v_{i-1}) = -1 \text{ or } \xi_{t-1}(v_{i+1}) = -1. \quad (26)$$

Indeed, first, by A2, for every neighbor u of v_i other than v_{i-1} and v_{i+1} , $\xi_{i-1}(u) = \xi_{t-1}(u)$, and, second, by A3, for exactly $k/2$ neighbors u of v_i other than v_{i-1} and v_{i+1} , $\xi_{i-1}(u) = -1$. So, combining these two statements, we conclude that, for exactly $k/2$ neighbors u of v_i other than v_{i-1} and v_{i+1} , $\xi_{t-1}(u) = -1$, implying (26).

Now, let us derive both the base of induction ($s = 2$) and the induction step from Equation (26). To prove the base, let us first apply (26) with $t = 1$ and every odd $i \in \{3, \dots, d-1\}$. Due to A1, $\xi_0(v_{i-1}) = \xi_0(v_{i+1}) = 1$, hence $\xi_1(v_i) = 1$. This allows us to apply (26) with $t = 2$ and every even $i \in \{3, \dots, d-1\}$ and get $\xi_2(v_i) = 1$, concluding the base of induction (we already showed $\xi_2(v_2) = -1$ and $\xi_2(v_d) = 1$). Finally, for the induction step, we assume that $s \in \{3, \dots, d-1\}$ and that for smaller time steps the statement is true. We apply (26) with $t = s$ and $i = s$ and get $\xi_s(v_s) = -1$, since $\xi_{s-1}(v_{s-1}) = -1$. Eventually, we observe that from the induction assumption it follows that $\xi_{s-1}(v_i) = 1$ for all $i \in \{s+1, \dots, d\}$ with the same parity as $s-1$. Thus we may apply (26) with $t = s$ and every $i \in \{t+2, \dots, d-1\}$ with the same parity as t and get $\xi_s(v_i) = 1$. Recalling $\xi_s(v_d) = 1$ whenever s is even, we complete the induction step.

5.2 | The Existence of R^*

First, let us note that probabilities of the events A1 and A5 are $2^{-d/2}$ and 2^{-k^2} respectively. Next, let U be the set of all children of vertices from the path $v_3 \cdots v_d$, excluding the vertices of the path and v_2 . Next, let $I : U \rightarrow \{2, \dots, d-1\}$ be such that

$$I(u) = i - 1 \quad \text{if and only if} \quad v_i \text{ is the parent of } u.$$

Next, consider an arbitrary function $O : U \rightarrow \{-1, 1\}$ such that

- $O(u) = 1$ for every u with $I(u) = d-1$;
- for every $i \in \{2, \dots, d-2\}$, the number of vertices $u \in U$ with $I(u) = i$ and $O(u) = -1$ equals $k/2$.

Note that such a function exists, since, for every $i \in \{2, \dots, d-2\}$, the number of vertices $u \in U$ with $I(u) = i$ equals $k-1 > k/2$.

We note that A2, A3, and A4 follow immediately if every vertex $u \in U$ is $(\leq I(u))$ -stable and has $\xi_{I(u)}(u) = O(u)$. Next, let us note, that events $\{u \text{ is } (\leq I(u))\text{-stable and } \xi_{I(u)}(u) = O(u)\}$ are independent over $u \in U$ since these events depend only on vectors of initial opinions $\xi_0|_{V(T_u)}$. Therefore, the events A1, A5, and $\{u \text{ is } (\leq I(u))\text{-stable and } \xi_{I(u)}(u) = O(u)\}$, $u \in U$, are independent, as they are defined by initial opinions on disjoint sets of vertices. Therefore,

$$\mathbb{P}(\xi_0^* \in \mathcal{F}) \geq 2^{-d/2-k^2} \prod_{u \in U} \mathbb{P}(u \text{ is } (\leq I(u))\text{-stable and } \xi_{I(u)}(u) = O(u)).$$

Due to Claim 5.1,

$$\mathbb{P}(u \text{ is } (\leq I(u))\text{-stable and } \xi_{I(u)}(u) = O(u)) \geq 2^{-CkI(u)}$$

for every $u \in U$. Thus, we conclude

$$\mathbb{P}(\xi_0^* \in \mathcal{F}) \geq 2^{-d/2-k^2} \prod_{u \in U} 2^{-CkI(u)} = 2^{-d/2-k^2} \prod_{i \in \{2, \dots, d-1\}} (2^{-Cki})^{k-1} \geq 2^{-(C+2)d^2k^2}. \quad (27)$$

In addition, events $\{\xi_0|_{V(T_{R^*})} \in \mathcal{F} = \mathcal{F}(R^*)\}$ are independent over all vertices R^* at distance d from leaves. So, since the number of vertices at distance d from leaves is $(k+1)k^{h-d-1}$, it is sufficient for us to show

$$(k+1)k^{(h-d-1)} \cdot \mathbb{P}(\xi_0^* \in \mathcal{F}) = \omega(1). \quad (28)$$

Due to (27), we achieve (28) by choosing small enough constant c_- in the definition of d .

6 | Discussions

Two immediate questions remain unanswered. First, what is the exact asymptotics in Theorem 1.2? We suspect, for a perfect binary tree T and a uniformly random ξ_0 , there exists $c \in (1/4, 1/3)$ such that $\frac{\tau(T; \xi_0)}{n} \xrightarrow{P} c$. Determining this limit would be of great interest, within the scope of majority dynamics. Second, in Theorem 1.3, our methods do not capture the correct dependency on D , so we leave that as an open problem. Below, we propose several other directions for future investigation.

6.1 | Stabilization Time

In all the processes that we observed on finite graphs, there is always at least one vertex which is stable from the beginning. Is it always the case? More formally, is it true that for every finite graph G and for every vector of initial opinions $\xi_0 \in \{-1, 1\}^{V(G)}$, there exists a vertex $v \in V(G)$ such that $\xi_{t+2}(v) = \xi_t(v)$ for all $t \in \mathbb{Z}_{\geq 0}$?

Recall that $\tau(G) < |E(G)|$ for every graph G . Let $\delta(G)$ be the minimum degree of G . It seems plausible that there exists a universal constant $C > 0$ such that, for every graph G , $\tau(G) \leq C \frac{|E(G)|}{\delta(G)}$. This bound clearly holds for all trees due to Theorem 1.1. Note that it cannot be improved in such generality since, in particular, it is achieved by the wheel graph.

6.2 | Stable Partitions

Let us say that (U, W) is a *stable partition* of $V(G)$, if, letting $\xi_0(u) = 1$ for $u \in U$ and $\xi_0(w) = -1$ for $w \in W$, we get that all vertices of G are 0-stationary. Observe that any unfriendly partition (including trivial cases $U = \emptyset$ or $W = \emptyset$) is stable. In particular, these are the only stable partitions of a clique. Can one characterize graphs with only these stable partitions? What are the stable partitions of the random graph $G(n, 1/2)$?

Clearly, for every G and ξ_0 , letting $\tau := \tau(G; \xi_0)$, we get that ξ_τ induces a stable partition. More generally, given a graph G and a positive integer t , what is the set of partitions $\Pi_t(G)$ achievable by the process at time t ? How fast does $|\Pi_t(G)|$ decay? Note that there are graphs with exponentially many stable partitions. For instance, identify every vertex of an n -vertex cubic graph with a clique of size 5, such that all n cliques are vertex-disjoint. Then, every partition that has a trivial intersection with every 5-clique is stable.

6.3 | Average-Case Dynamics

Here, we assume that the initial configuration ξ_0 is uniformly random. Recall that, for (asymptotically) almost all graphs on n vertices Fountoulakis et al. (2020) and for perfect trees G (due to Theorem 1.3), whp $\tau(G; \xi_0) = O(\log n)$. It seems plausible that there exists a universal constant $C > 0$ such that $\mathbb{E}\tau(G; \xi_0) < C \log n$ for every graph G . To support this conjecture, note that our proofs, in part, rely on the fact that, in perfect trees, distant vertices have low influence on each other's dynamics. This leads naturally to the following question: For an arbitrary graph G , how does the distribution of the final stabilized opinion of v , conditioned on $\xi_0(u) = 1$, depend on the graph distance between the vertices v and u ?

In perfect binary trees, exponential decay of this dependency implies that, with high probability, the final stable partition of opinions is almost balanced. How does the final partition behave on other graph families? For instance, if G is planar, may one of the two stabilized opinion classes contain a large connected component?

Acknowledgments

The authors are grateful to Elad Tzalik for useful discussions. The authors thank the anonymous reviewers for their helpful comments and suggestions on an earlier version of this paper.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Endnotes

¹With probability approaching 1 as $n \rightarrow \infty$. In what follows, we will write simply 'whp' for brevity.

²Hereinafter, a vertex is a descendant of itself.

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