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Exploring how understanding of the equals sign relates to additive and multiplicative equivalence performance beyond arithmetic fluency

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ABSTRACT

A strong foundation in arithmetic relies on conceptualizing the equals sign as a symbol of equivalence. However, it remains unclear whether equals sign understanding predicts equivalence performance beyond arithmetic fluency. In this preregistered study, we examined these relations in a sample of 277 pupils ($M_{age} = 11.4$, $SD = 0.3$; 48% girls) in their final year of primary school. Despite almost all pupils recognizing the equals sign, most interpreted it as a signal to perform a calculation or provide an answer. Some demonstrated an emerging understanding of the symbol as an equivalence relation, defining it as a link between two expressions of the same value or, less frequently, recognizing that equal expressions can be interchanged. This equal value conceptualization significantly predicted both additive and multiplicative equivalence fluency, even after controlling for arithmetic fluency and gender. Notably, this understanding explained more variance in multiplicative than additive tasks, suggesting that multiplicative equivalence may be more demanding for learners. These findings suggest that while arithmetic proficiency is important for success on equivalence problems, knowledge of the relational meaning of the equals sign also plays a role, particularly in more complex equivalence contexts. Overall, the results highlight the need for instructional approaches that extend beyond practice in solving equations by explicitly supporting pupils' reasoning about equivalence.

A solid foundation in arithmetic is essential for children's broader mathematical development and later reasoning skills (McNeil et al., 2025). A key part of this foundation is understanding of the equals sign which provides a foundation for later algebraic thinking (Byrd et al., 2015; Fyfe et al., 2018; Hornburg et al., 2022; Matthews & Fuchs, 2020; Simsek et al., 2019). Rather than functioning simply as an operational prompt to compute, the equals sign denotes a relation of equivalence between two quantities or expressions that have the same value (Gould et al., 2024). Pupils who interpret the equals sign relationally are better able to recognize equivalence across different equation formats and are more likely to select and apply appropriate strategies when solving equivalence problems (Alibali et al., 2007; Knuth et al., 2008). While the link between equals sign interpretation and equivalence performance is well-documented in early learners (see a review by McNeil et al., 2017), it remains unclear whether understanding of the equals sign

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predicts equivalence performance after accounting for arithmetic fluency in more experienced pupils. Thus, the present study focused on pupils in their final year of primary school in Northern Ireland, a critical curricular transition from consolidated arithmetic towards foundational algebra and the use of symbols for unknown quantities (Council for the Curriculum, Examinations and Assessment, CCEA, 2019).

Examining understanding of the equals sign at this juncture is vital; a persistent operational bias may hinder success in secondary school, where relational understanding becomes essential for algebra (Alibali et al., 2007; Byrd et al., 2015; Knuth et al., 2006; McNeil et al., 2006; Simsek et al., 2019). Even among accurate problem-solvers, equivalence performance variability may reflect qualitative differences in underlying knowledge, with some pupils using relational reasoning to balance equations and others relying heavily on computation (cf. Rittle-Johnson et al., 2011). If relational interpretations explain unique variance in equivalence performance beyond arithmetic fluency, then this would reinforce the need for instructional approaches that move beyond equation-solving practice to explicitly support reasoning about equivalence.

Interpretations of the equals sign

Pupils' knowledge of equivalence is closely tied to their interpretation of the equals sign. Although most pupils can recognize the symbol once formally introduced, their interpretations vary considerably, reflecting key differences in their underlying understanding of equivalence (Sumpter & Löwenhielm, 2024). One way to assess these interpretations is through self-authored definitions, which generally align with two broad conceptual categories: operational and relational. An operational interpretation treats the symbol as a prompt to perform a calculation and produce an answer (Baroody & Ginsburg, 1983; Knuth et al., 2006; McAuliffe et al., 2020; Wardat et al., 2021). In contrast, a relational interpretation reflects an understanding that the equals sign expresses an equivalence relation between two expressions (Byrd et al., 2015; Knuth et al., 2006; Rittle-Johnson & Alibali, 1999). Research shows that most primary pupils hold operational views (e.g., '=' means to do something or answer comes next) rather than relational views (e.g., '=' reflects equivalence or sameness; Alibali et al., 2007; McNeil & Alibali, 2005a; Stephens et al., 2013).

Rather than an abrupt shift, frameworks suggest that understanding of the equals sign develops gradually toward increasingly flexible forms of relational reasoning (Lee & Pang, 2021, 2023; Rittle-Johnson et al., 2011; Simsek et al., 2019). Within these frameworks, relational understanding is differentiated into two distinct forms: *sameness-relational* (i.e., recognizing that two expressions represent the same value) and *substitutive-relational* (i.e., recognizing that if two expressions are equal, they can be interchanged or substituted to restructure or simplify a problem; Jones, 2008). The latter is grounded in the mathematical properties of equality, including reflexivity (i.e., any value equals itself; $7 = 7$), symmetry (e.g., if $3 + 4 = 7$, then $7 = 3 + 4$), and transitivity (e.g., if $5 + 2 = 7$ and $7 = 3 + 4$, then $5 + 2 = 3 + 4$), which together form the foundation for algebraic reasoning (Sfard & Linchevski, 1994; Stewart & Tall, 1977). Developmentally, sameness-relational conceptions may emerge earlier and support the later development of substitutive-relational reasoning (Donovan et al., 2022; Simsek et al., 2019), as many older pupils can recognize sameness yet still struggle to use substitution flexibly when solving equations (Filloy et al., 2010; Nogueira de Lima & Tall, 2008).

To better characterize these interpretations, Jones et al. (2012) developed a *Conception of Equals Sign* (CES) measure which evaluates how pupils rate different definitions of the equals sign. Their work confirmed three distinct interpretive categories: operational, sameness-relational, and substitutive-relational. In comparing 11- to 12-year-olds, English pupils favoured operational interpretations, whereas Chinese pupils favoured substitutive-relational interpretations, despite both groups providing similar cleverness ratings for sameness-relational definitions (Jones et al., 2012). Their findings indicate that by the end of primary school in the UK, many pupils still hold operational views, with relatively few demonstrating advanced substitutive understanding. This measure has since been used to examine links between interpretations of the equals sign and secondary school algebra performance (Simsek et al., 2019) and to inform interventions aimed at improving equivalence knowledge (Donovan et al., 2022).

From interpretations of the equals sign to performance on equivalence problems

While equals sign interpretations reflect how pupils understand its meaning, it is also important to consider how these interpretations support equivalence performance. Such performance is commonly assessed using nonstandard formats (e.g., $a + \_ = b$; $a = b + \_$; $a + b = \_ + c$), rather than the standard arithmetic format where the equals sign always follows the operation (e.g., $a + b = \_$). Research consistently shows that pupils with relational views outperform those with operational views (Knuth et al., 2006). Instructional studies further suggest a directional link: instruction that incorporates relational understanding improves both equals sign knowledge and equivalence performance, whereas procedural instruction improves performance without strengthening understanding (Rittle-Johnson & Alibali, 1999). This pattern is consistent with the broader theoretical perspectives emphasizing that procedural skills are most effective when grounded in conceptual understanding (Fyfe et al., 2014).

Pupils with primarily operational views often find equivalence problems challenging. Evidence from studies with younger pupils (7-11 years) shows that operational interpretations are associated with systematic difficulties when solving nonstandard equations (McNeil et al., 2017). While such interpretations may be sufficient for standard arithmetic formats, they become problematic when equations require pupils to evaluate relations across both sides of the equals sign. This difficulty may be reinforced by instructional practices: curricula and textbooks provide limited support for teaching the relational meaning of the equals sign, offering minimal exposure to nonstandard equations or opportunities to engage with equivalence relationally, although the extent to which this issue is evident may vary across educational systems (Capraro et al., 2007; Essien, 2009; Li et al., 2008; McNeil et al., 2006; Pang & Lee, 2025; Powell, 2012). Furthermore, a persistent emphasis on practising arithmetic in standard formats can reinforce a rigid operational interpretation that is difficult to change (McNeil, 2014; Powell, 2015).

Without a relational understanding of the equals sign, many pupils default to left-to-right calculation routines and attempt to interpret equations as standard arithmetic tasks (McNeil et al., 2017). This operational approach makes it difficult for pupils to identify the underlying equation structure, leading to systematic errors (Falkner et al., 1999; Perry et al., 1988; Stephens et al., 2013; Sumpter et al., 2025). For example, when solving an equivalence problem such as $5 + 12 = \_ + 3$, some pupils may add only the numbers appearing before the equals sign (e.g., $5 + 12 = \underline{17} + 3$), while others may add all numbers in the equation regardless of structure (e.g., $5 + 12 = \underline{20} + 3$) or add backwards to the equals sign (e.g., $5 + 12 = \underline{9} + 3$). These patterns of computing errors reflect the constraints of operational interpretations and highlight the importance of relational understanding in solving equivalence problems.

However, equivalence performance requires both relational understanding and the ability to carry out the necessary calculations (McNeil et al., 2017). Pupils with strong arithmetic fluency retrieve number facts efficiently, reducing working memory load and freeing cognitive resources for problem solving (Tenison & Anderson, 2016). Conversely, pupils with weaker arithmetic fluency may expend excessive cognitive effort on computation, leaving fewer resources to apply relational understanding. Empirical evidence supports this view: Pupils who organize arithmetic facts around meaningful number relations demonstrate greater flexibility and accuracy when solving equivalence problems (Chesney et al., 2014), and interventions that build fluency through mental computation strengthen relational thinking (Kindrat & Osana, 2018). Longitudinal evidence further shows that arithmetic fluency more strongly supports growth in equivalence fluency than the reverse, underscoring its foundational role (Xu et al., 2024). Overall, while relational understanding of the equals sign is essential, arithmetic fluency plays a vital complementary role in successful equivalence performance. Notably, prior work has often controlled for general mathematics ability using standardized test scores, rather than directly measuring arithmetic fluency, when examining performance on equation-solving tasks (Knuth et al., 2006).

While there is substantial evidence that many late primary pupils still hold operational or partial relational views (McAuliffe et al., 2020; Rittle-Johnson et al., 2011; Stephens et al., 2013) and that arithmetic fluency supports equivalence performance (Xu et al., 2024), little is known about the factors that explain ongoing variability once pupils can solve these problems accurately. Specifically, it remains unknown whether equals sign interpretations continue to account for differences in equivalence fluency after arithmetic fluency is taken into consideration.

Moreover, arithmetic research suggests that multiplicative reasoning builds on and extends beyond additive reasoning, representing a more advanced form of mathematical thinking (Nunes et al., 2016; Steffe, 1992; Steffe & Olive, 2010). Although studied less extensively than additive reasoning, multiplicative reasoning is notably more complex and conceptually demanding for learners (Butterworth, 2005; Siemon, 2013; Vergnaud, 1994). For example, pupils are less likely to apply conceptually-based shortcuts, such as inversion or associativity, when solving multiplication and division problems compared to addition and subtraction (Robinson et al., 2006). However, whether these differences extend to equivalence contexts remains unclear: Do pupils approach additive and multiplicative equivalence tasks similarly, or do multiplicative formats introduce additional difficulty? If the latter is true, pupils may find multiplicative equivalence problems significantly more demanding than their additive counterparts.

The present study

In this preregistered study (https://osf.io/x5qmj/), Northern Irish pupils in their final year of primary school completed tasks assessing equals sign understanding, equivalence fluency, and arithmetic fluency. Equals sign understanding was operationalized in two ways: (1) pupils provided their own definitions of the equals sign, and (2) they rated the "cleverness" of alternative definitions. Including both self-generated and evaluative measures allowed us to examine whether pupils' explicit understanding aligns with their recognition-based evaluations, building on prior work suggesting these measures capture different facets of understanding (Donovan et al., 2022; Matthews et al., 2012; Sumpter & Löwenhielm, 2024). Furthermore, alignment between these measures may provide insight into the coherence of a pupil's knowledge representations, whereas a lack of alignment may indicate partial or unstable understanding that requires more explicit instruction to consolidate.

Equivalence fluency was operationalized through timed additive and multiplicative tasks in which the position of the equals sign varied from the standard format. This timed format captured not only whether pupils could solve nonstandard equations, but also how efficiently they recognized and used the equivalence relation across both sides. This approach aligns with the view that stronger performance depends on relational recognition, rather than a reliance on slower, step-by-step computation (e.g., Carpenter et al., 2003; Kirkland et al., 2024). To account for foundational arithmetic competence, pupils also completed timed standard arithmetic fluency tasks, covering addition, subtraction, multiplication, and division.

Research question 1: How do pupils define the equals sign?

Based on previous research showing that many pupils can correctly identify the equals sign but often hold limited understanding of its meaning, we first predicted that most pupils would successfully name the equals sign (Hypothesis 1a) but that their definitions would differ, with operational definitions occurring more frequently than relational definitions (Hypothesis 1b). More specifically, we expected operational definitions to be the most frequent, followed by sameness-relational definitions, and finally substitutive-relational definitions, which were expected to be infrequent.

Research question 2: How do pupils' ratings of others' definitions of the equals sign relate to their own definitions?

Based on Jones et al. (2012), we predicted that the CES items would load onto three factors representing operational, sameness-relational and substitutive-relational conceptions, with operational and substitutive-relational factors negatively correlating and sameness-relational and substitutive-relational factors positively correlating (Hypothesis 2a). Moreover, consistent with the findings from the English sample in Jones et al. (2012), we predicted that Northern Irish pupils would rate operational definitions as "cleverer" than sameness-relational and substitutive-relational definitions (Hypothesis 2b). Lastly, extending prior work, we predicted that pupils' cleverness ratings would align with their own interpretations; for example, pupils offering an operational interpretation were expected to rate operational items as the cleverest, and the same alignment was expected for relational interpretations (Hypothesis 2c).

Research question 3: How do pupils' interpretations of the equals sign predict their equivalence fluency?

Previous work suggests that pupils' ability to solve equivalence problems involving multiple operations is negatively associated with operational interpretations and positively associated with relational interpretations of the equals sign (Alibali et al., 2007; Sumpter & Löwenhielm, 2024). Thus, we hypothesized that, after accounting for relevant arithmetic fluency, relational interpretations (both sameness-relational and substitutive-relational) would uniquely predict pupils' performance on both additive and multiplicative equivalence problems, whereas operational interpretations would negatively predict performance (Hypothesis 3).

Method

Participants

Data were collected from pupils in their final year of primary school (Primary 7) in Northern Ireland ($N = 277$; $M_{\text{age}} = 11.4$ years, $SD = 0.3$; 48% girls). These pupils were recruited from 11 schools in Belfast and surrounding regions, representing a range of socio-economic backgrounds, with free school meal eligibility ranging from 4% to 74%. Ethical approval for data collection was granted by the Faculty Ethics Committee at Queen's University Belfast.

Procedure

All data were collected in school classrooms through group testing, conducted by trained researchers under the supervision of classroom teachers. All tasks were administered using paper and pencil. Researchers explained the tasks, answered questions, and ensured pupils worked quietly.

Measures

Equals sign understanding

Define the equals sign. To assess pupils' ability to identify and define the equals sign, pupils were asked two open-ended questions: (i) "*What is the name of the symbol '=' in a mathematical context?*", and (ii) "*What does the symbol '=' mean in a mathematical context?*" All responses were independently coded by two of the authors using a predefined, preregistered coding scheme. For the naming question, pupils received a score of 1 for correctly identifying the equals sign (e.g., equal, equals), regardless of spelling errors, and a score of 0 for other responses. For the definition question, responses were coded into five categories: (i) Operational – equals sign indicates the answer or total; (ii) Sameness – indicates equality or balance between both sides; (iii) Substitutive – one side can be replaced by the other; (iv) Unspecified equals – responses stating "equals" or "equals to" without a clear interpretation; and (v) Unclassifiable – responses not fitting the above categories. Any discrepancies between coders were discussed amongst members of the research team until a consensus was reached.

Conceptions of the Equals Sign (CES). The CES (Jones et al., 2012) consists of 12 definitions from fictitious pupils presented in a pre-randomized fixed order, with three items each representing operational, sameness-relational, substitutive-relational and distractor definitions. Pupils rated each definition as *not so clever* (0), *sort of clever* (1) or *very clever* (2) by ticking a corresponding box. Internal consistency for the three conceptual categories was high, with Cronbach's α of .90 for operational, .89 for sameness-relational and .86 for substitutive-relational definitions.

Arithmetic fluency

Using the modified Arithmetical Ability subscale from the *Heidelberg Rechentest* (HRT; Haffner et al., 2005, adapted by Wu & Li, 2006) pupils completed an arithmetic task for each of the four operations (addition, subtraction, multiplication and division). Each task consisted of 40 items of increasing difficulty presented in two 20-item columns. Pupils were asked to complete as many items as they could within 1 min. Scoring consisted of the total correct for each operation, with error rates calculated as the proportion of incorrect responses relative to the total number of attempted items. Internal consistency for each operation was high, with Cronbach's α of .93 for addition, .93 for subtraction, .94 for multiplication and .96 for division.

Equivalence fluency

Pupils completed fluency tasks for both additive and multiplicative equivalence, each consisting of 40 items of increasing difficulty presented in two 20-item columns, with equations involving operations on only one side appearing in earlier items, and more challenging items involving operations on both sides mostly appearing in later items. The additive equivalence task was taken from the Arithmetical Ability subscale from the HRT (Haffner et al., 2005). The multiplicative task was developed for the present study, parallel to the additive format and difficulty. For both tasks, items consisted of a mixture of equations with a missing operand to the right of the equals sign (e.g., 5 = 8 –_; 5 = 15 ÷_), equations with a missing operand to the left of the equals sign (e.g., 6 +_= 7; 3 ×_= 6) and equations with operations on both sides of the equals sign (e.g., 10 + 1 = 9 +_; 5 × 6 = 10 ×_). Pupils were asked to complete as many items as they could within 1 min. Scoring consisted of the total correct for each equivalence task, with error rates calculated as the proportion of incorrect responses relative to the total number of attempted items. Internal consistency for each task was high, with Cronbach's α of .91 for additive and .92 for multiplicative equivalence fluency.

Transparency and openness

We adhered to the *American Psychological Association's Journal Article Reporting Standards* (Kazak, 2018). We report where and how all data were collected, the measures used, and any data exclusions, with justifications. All manipulations are fully disclosed. The present study was preregistered, and the data and coding scheme, along with the multiplicative equivalence fluency measure we developed, are available on the Open Science Framework (OSF; https://osf.io/4z2kw/). The pre-registration information is available at: https://osf.io/x5qmj/. The data were collected as part of a larger longitudinal project on the development of children's whole number and fraction knowledge. The present study addressed a unique set of theoretical questions that have not been published elsewhere.

Analysis plan

Data were analyzed using SPSS Version 30. All hypotheses were preregistered. For Research Question 1, percentages were used to assess correct identification of the equals sign (H1a), and a χ^2goodness-of-fit test examined differences in frequencies of definition types (H1b). For Research Question 2, principal component analysis (PCA) with oblimin rotation was conducted on nine CES items (excluding three distractors; Jones et al., 2012) after confirming suitability with Kaiser-Meyer-Olkin and Bartlett's tests (H2a). A one-way analysis of variance (ANOVA) was conducted to examine differences in pupils' rated cleverness of the three types of equals sign definitions (H2b). Following Field's (2013) recommendations, Greenhouse-Geisser corrections were applied to adjust for any degree of violation of the assumption of sphericity. Pupils' highest-rated factor scores were used to classify their "most clever" choice, and percentages were compared with their open-ended definitions to assess the consistency between them (H2c). For Research Question 3, at the time of preregistration we planned to average addition and subtraction scores, and multiplication and division scores, to create composite measures of additive and multiplicative fluency. However, to more precisely assess the operation-specific contributions to equivalence fluency beyond variance shared across operations, we retained the four operations separately. This decision to separate the operations did not change the overall pattern of results. We then conducted hierarchical linear regressions predicting equivalence performance. In Block 1, arithmetic fluency (addition and subtraction for the additive model; multiplication and division for the multiplicative model) was entered to account for general arithmetic skills. In Block 2, the cleverness ratings for the three equals-sign conceptions were added to examine their additional contribution (H3).

Data for a small number of pupils were missing across the measures (up to 2.2%). In accordance with best practices for maintaining unbiased results and statistical power (Woods et al., 2023), a multiple imputation approach with five imputations was used to estimate the missing values. Sensitivity analyses showed the pattern of results were consistent before and after imputation. Accordingly, pooled results from the imputations are reported for the PCA, correlation, and regression analyses.

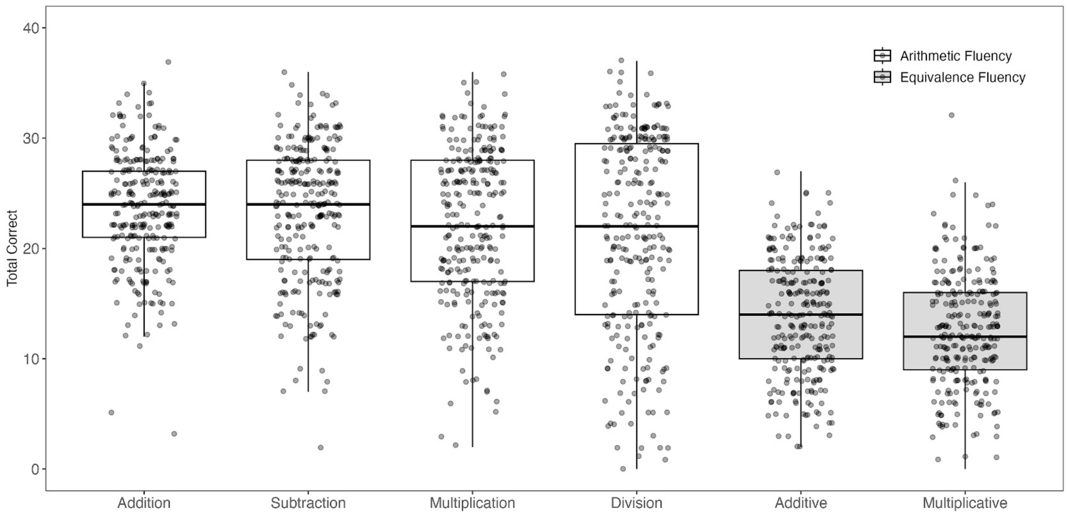

Fig. 1. Boxplots and Scatterplots of Arithmetic and Equivalence Fluency. Note. The horizontal black bar within the box and vertical black bar show the median and 1.5x the interquartile range, respectively.

Results

Preliminary analyses

Fig. 1 shows boxplots of arithmetic and equivalence fluency. Performance was strong across all tasks. For arithmetic fluency, error rates based on attempted items were low (2.1–5.9%). For equivalence fluency, we investigated error rates for both additive and multiplicative tasks. For additive fluency, pupils provided incorrect responses 5.2% of the time. When separating errors based on equation format, more errors were made when operations appeared on both sides (8.1%) than on one side (3.2%) of the equals sign. When investigating the types of errors pupils made, 62.3% of errors were computing errors (i.e., adding forward to the equals sign, adding backward to the equals sign, adding all numbers in the equation). A higher proportion of errors reflected computing errors when operations appeared on both sides (66.2%) compared to one side (55.8%) of the equals sign. For multiplicative fluency, pupils provided incorrect responses 8.1% of the time. When separating errors based on equation format, more errors were made when the operations appeared on both sides (12.2%) than on one side (5.6%) of the equals sign. When investigating the types of errors pupils made, 38.3% of errors were computing errors. A higher proportion of errors reflected computing errors when operations appeared on both sides (63.3%) compared to one side (6.9%) of the equals sign. Overall, the pattern of errors was broadly similar across additive and multiplicative equivalence fluency. Although error rates were generally low, pupils found the items more challenging when operations appeared on both sides. Computing errors were especially prominent in additive equivalence overall, whereas in multiplicative equivalence they were more strongly dependent on equation format.

Across the measures, a few outliers (defined as $|z| \geq 3.29$; Field, 2013) were identified for addition ($n = 2$), subtraction ($n = 1$), and multiplicative equivalence fluency ($n = 1$). Sensitivity analyses with and without these outliers yielded the same pattern of results; thus, the outliers were retained for subsequent analyses.

No gender differences were observed in equals sign understanding (i.e., definitions of equals sign and the CES scale). However, small but significant gender differences were found on the arithmetic and equivalence fluency measures, with boys outperforming girls ($ps < .05$; Cohen's d ranging from 0.31 to 0.40). Thus, we included gender as a control variable in the two regressions involving these fluency measures.

Research question 1: How do pupils define the equals sign?

Pupils' open-ended responses naming and describing the equals sign were analyzed. Supporting Hypothesis 1a, most pupils (96.4%) correctly identified the equals sign, with only 2.2% giving incorrect responses and 1.4% leaving the question blank. In their definitions, nearly half of the pupils (49.1%) provided an operational definition, 22.0% provided a sameness-relational definition, and none provided a substitutive-relational definition. The remaining responses were unspecified (20.2%; i.e., "equals" or "equals to" without elaboration), unclassifiable (5.8%), or missing (2.9%). Supporting Hypothesis 1b, excluding unspecified, unclassifiable, and missing responses, a χ^2 goodness-of-fit test comparing the proportions of operational and sameness-relational definitions showed a significant difference, $\chi^2(1, N = 197) = 28.55, p < .001$, indicating that pupils were more likely to give an operational than a relational definition. Overall, while most pupils correctly named the equals sign, their understanding of its meaning remained largely operational.

Research question 2: How do pupils' ratings of others' definitions of the equals sign relate to their own definitions?

We conducted a PCA with oblimin rotation on the nine items designed to tap into the three types of equals sign conceptions (operational, sameness-relational, and substitutive-relational), excluding the three distractor items (see Table S1 in Supplementary Material for descriptive statistics of all items). Confirming the suitability of the data for analysis, the Kaiser-Meyer-Olkin value was .77 and Bartlett's test of sphericity was significant ($p < .001$). The PCA revealed three factors explaining on average 34.9%, 14.7% and 11.4% of the variance across imputations. Consistent with Jones et al. (2012), items loaded strongly onto their respective factors (see Table 1), supporting Hypothesis 2a. Moreover, the operational factor significantly negatively correlated with both the sameness-relational factor, $r(273) = -.13$, $p = .039$, and substitutive-relational factor, $r(273) = -.28$, $p < .001$. The two relational factors

Table 1
Factor loadings from principal components analysis with Oblimin rotation

Definition '=' Means...	Operational	Sameness	Substitutive
...the total	**.59**	.15	–.32
... the answer to the problem	**.71**	.10	–.16
...work out the result	**.75**	–.20	.26
...the two amounts are the same	.01	**.70**	.21
...that something is equal to another thing	.05	**.77**	–.10
...that both sides have the same value	–.47	**.69**	.27
...the right side can be swapped for the left side	–.00	–.01	**.79**
...that one side can replace the other	–.02	.09	**.80**
...the two sides can be exchanged	–.02	.17	**.71**

Note. Factor loadings above .50 are bolded.

were significantly positively correlated, $r(273) = .22$, $p < .001$.

Next, we conducted a one-way within-subjects ANOVA on the summed scores representing the three conceptions of the equals sign to examine differences among them. In support of Hypothesis 2b, there were significant differences among the three types of definitions, $F(1.72, 470.43) = 173.44$, $\eta_p^2 = .49$, $p < .001$, with operational definitions rated the cleverest ($M = 4.1$, $SD = 1.5$), followed by sameness-relational ($M = 3.4$, $SD = 1.7$), and substitutive-relational definitions ($M = 1.6$, $SD = 1.6$). All pairwise comparisons were significant, $ps < .001$.

To examine the correspondence between pupils' explicit definitions of the equals sign and their judgments of item cleverness, we identified each pupil's "cleverest response" based on their highest summed score across the three conceptions. We found that 53.7% rated operational items as the cleverest, 25.5% rated sameness items as cleverest and 2.5% rated substitutive items as cleverest. Notably, some pupils rated multiple conceptions as equally clever: 13% rated operational and sameness items equally, 0.6% rated operational and substitutive items equally, and 4.0% rated sameness and substitutive items equally. Furthermore, Table 2 shows that most pupils who described the equals sign operationally also chose the operational definitions as the cleverest, while over half of the pupils who gave a sameness-relational definition selected sameness definitions as the cleverest. Although no pupils provided substitutive-relational definitions, seven pupils rated substitutive items as the cleverest, six of whom had given sameness-relational definitions. Overall, these patterns support Hypothesis 2c, showing that pupils' explicit definitions of the equals sign generally aligned with their judgments of item cleverness.

Research question 3: How do pupils' interpretations of the equals sign predict their equivalence fluency?

Hierarchical linear regression analyses were conducted to examine the extent to which pupils' understanding of the equals sign predicted their additive and multiplicative equivalence fluency beyond standard arithmetic fluency, controlling for gender. Given the strong correlations among the arithmetic and equivalence measures (see Table 3), multicollinearity was assessed. Following guidance from Field (2013), a variance inflation factor (VIF) ≥ 10 or a tolerance of ≤ 0.2 are problematic. Across all models and imputations, no such problems were observed (VIFs < 4.9; tolerance > 0.2), indicating that multicollinearity was not a concern.

The results of the hierarchical regression analyses are presented in Table 4. Gender, addition fluency, and subtraction fluency accounted for a substantial proportion of the variance in additive equivalence fluency (71.6%), however, gender was not a significant predictor. Adding the equals-sign conceptions explained an additional 1.1% of the variance, with the sameness-relational factor emerging as the only significant predictor. Similarly, gender, multiplication fluency, and division fluency accounted for a substantial proportion of the variance in multiplicative equivalence fluency (72.4%), however, gender was not a significant predictor. Adding the equals-sign conceptions explained an additional 2.2% of the variance, with the sameness-relational factor emerging as the only significant predictor. These findings partially support Hypothesis 3, indicating that relational understanding, specifically, sameness rather than substitutive, predicted equivalence fluency beyond arithmetic fluency. The pattern of results was consistent across both the additive and multiplicative models.

Discussion

In this preregistered study, we examined pupils' equals sign understanding and explored the extent to which relational understanding predicts equivalence fluency beyond foundational arithmetic skills. Previous studies have found that pupils often hold a rigid, operational view of the equals sign, treating it as a command to calculate rather than a symbol of relational equivalence (Baroody & Ginsburg, 1983; Knuth et al., 2006; McAuliffe et al., 2020). Extending these findings, we found that while nearly all final year primary pupils could correctly identify the equals sign, their definitions were mostly operational, with very few demonstrating advanced substitutive-relational knowledge. Crucially, however, sameness-relational understanding emerged as a significant, unique predictor of performance on both additive and multiplicative equivalence tasks, even after controlling for arithmetic fluency and gender. Thus, while arithmetic proficiency is foundational for equivalence performance, specific relational understanding distinctly contributes to pupils' success in solving nonstandard equations.

Table 2
Percentage of pupils selecting each definition as the "cleverest" grouped by their own equals sign definition category.

Definition Category (*n*)	***Cleverest definition type (%)***			***Mixed cleverness definition type (%)***			**Missing (%)**
	Operational	Sameness	Substitutive	Operational & Sameness	Operational & Substitutive	Sameness & Substitutive	
Operational (*136*)	66.9	14.7	0.7	13.2	0.0	2.2	2.2
Sameness (*61*)	18.0	52.5	9.8	11.5	0.0	6.6	1.6
Unspecified (*56*)	51.8	19.6	0.0	17.9	1.8	7.1	1.8
Unclassified (*16*)	68.8	25.0	0.0	6.3	0.0	0.0	0.0
Missing (*8*)	62.5	25.0	0.0	0.0	0.0	0.0	12.5

Note. The mixed cleverness percentages represent pupils who rated two definition types as equally clever.

Table 3
Correlations among gender, arithmetic, equivalence, and equals sign rating factors.

Variable	1	2	3	4	5	6	7	8	9
1. Gender	–								
2. Addition[a]	–.20***	–							
3. Subtraction[a]	–.19***	.88***	–						
4. Multiplication[a]	–.17**	.81***	.78***	–					
5. Division[a]	–.17**	.78***	.79***	.89***	–				
6. Additive[b]	–.15**	.83***	.81***	.78***	.80***	–			
7. Multiplicative[b]	–.16**	.78***	.75***	.80***	.84***	.86***	–		
8. Operational[c]	.02	–.05	–.04	–.01	.09	–.09	–.11*	–	
9. Sameness[c]	.04	.15*	.11*	.11*	.18**	.21***	.28***	–.13*	–
10. Substitutive[c]	–.03	.06	.05	.02	–.08	.08	.13*	–.28***	.22***

Note. $*p < .05$, $**p \leq .01$, $***p \leq .001$; [a]arithmetic fluency measures, [b]equivalence fluency measures, [c]equals sign rating factors

Table 4
Hierarchical regressions predicting additive and multiplicative equivalence fluency

Variable	**Additive equivalence fluency**[a]						**Multiplicative equivalence fluency**[b]					
	B	***SE***	***β***	***t***	***p***	**Unique r^2**	***B***	***SE***	***β***	***t***	***p***	**Unique r^2**
Block 1												
Gender	0.19	0.35	.02	0.53	.594	< .001	− 0.02	0.34	−. 00	− 0.05	.961	< .001
Addition /Multiplication	0.50	0.07	.49	7.04	< .001	.053	0.17	0.05	.23	3.23	.001	.011
Subtraction /Division	0.34	0.06	.39	5.63	< .001	.034	0.38	0.04	.63	9.18	< .001	.089
R^2	71.6%						72.4%					
Block 2												
Gender	0.12	0.35	.01	0.34	.737	< .001	− 0.12	0.33	−. 01	− 0.37	.713	< .001
Addition /Multiplication	0.47	0.07	.46	6.74	< .001	.047	0.20	0.05	.27	3.99	< .001	.016
Subtraction /Division	0.34	0.06	.40	5.83	< .001	.035	0.33	0.04	.57	8.18	< .001	.066
Operational	− 0.22	0.18	−. 04	− 1.23	.218	.002	− 0.18	0.17	−. 03	− 1.02	.306	.001
Sameness-Relational	0.48	0.18	.09	2.68	.007	.008	0.69	0.17	.13	4.06	< .001	.016
Substitutive-Relational	0.00	0.18	.00	0.00	.951	< .001	0.17	0.18	.03	0.96	.340	.001
Δ R^2	1.1%						2.2%					

Note. Unique r^2 values are squared part correlations within each model. Coefficients are pooled across imputations.
[a] For the additive equivalence model, gender, addition fluency, and subtraction fluency were entered as Block 1 predictors.
[b] For the multiplicative equivalence model, gender, multiplication fluency, and division fluency were entered as Block 1 predictors.

Pupils' definitions of the equals sign

Our first research question examined how pupils name and define the equals sign. Consistent with recent literature, nearly all pupils were able to successfully identify the equals sign (Sumpter & Löwenhielm, 2024), yet their interpretations of its meaning varied significantly. Supporting our hypotheses, pupils were more likely to provide an operational than relational definition. Specifically, nearly half of the cohort viewed the equals sign operationally, while approximately 22% provided a sameness-relational definition. This high prevalence of operational definitions aligns with prior work suggesting that standard arithmetic instruction often reinforces a "do-something" interpretation of the equals sign (McAuliffe et al., 2020; Wardat et al., 2021). Specifically, our findings with Northern Irish pupils align with the findings from other Western countries in which inconsistency in definitions and rigid traditional arithmetic practice leads to rigid operational conceptions (McNeil, 2014; Powell, 2012). Within the Northern Ireland Primary Curriculum there is minimal guidance on how to introduce pupils to the equals sign (CCEA, 2019). Thus, a re-evaluation of the current guidance may be necessary to better facilitate the development of relational understanding.

Notably, a substantial proportion of pupil responses were categorized as "unspecified", where pupils simply wrote 'equals' or 'equals to' with no elaboration. It is unclear whether these pupils are simply repeating the terms they have been exposed to during classroom instruction without true understanding, or if they are beginning to understand the relations but cannot explain them. The latter has been documented in prior research where pupils had difficulties providing a relational definition even if their equivalence performance reflected a relational understanding (Alibali et al., 2007; McNeil & Alibali, 2005a; Sumpter & Löwenhielm, 2024). These findings highlight the importance of instructional language; in the absence of explicit guidance on the relational meaning of symbols, pupils often adopt rigid operational definitions (McNeil, 2014; Powell, 2012). Thus, there is a need for instruction that explicitly features relational language to help pupils articulate and solidify their equals sign understanding (Chesney et al., 2018; Hattikudur & Alibali, 2010).

Comparing equals sign definitions and cleverness ratings

Our second research question explored the structure of pupils' equals-sign conceptions and the alignment between their own definitions and their evaluation of others'. Consistent with the findings of Jones et al. (2012), the PCA reproduced a three-factor structure, distinguishing operational, sameness-relational, and substitutive-relational conceptions. Mirroring the pupil-provided definitions and the findings with English pupils (Jones et al., 2012), operational definitions received the highest cleverness ratings. Thus, beyond articulation, most pupils believed that operational definitions most accurately captured the meaning of the equals sign, suggesting their own definitions can be indicative of their wider knowledge of the equals sign. This preference for operational definitions likely reflects common instructional experiences in which operational interpretations are most frequently modelled (Capraro et al., 2007; Essien, 2009; Li et al., 2008; McNeil et al., 2006; Powell, 2012). Conversely, evidence from China, where the equals sign is taught using a relational framework (Li et al., 2008) shows that pupils of the same age achieve a wider relational understanding (Sun et al., 2024). These findings reinforce the necessity of engaging with relational language during mathematics instruction to foster pupils' understanding.

The observed alignment between pupils' own definitions and their cleverness ratings suggests a level of coherence in pupils' knowledge representations. Pupils who defined the equals sign operationally typically rated operational items as cleverest and those who provided sameness-relational definitions tended to rate sameness-relational definitions as cleverest. Notably, although no pupils provided substitutive-relational definitions, a small minority rated substitutive-relational items as cleverest; nearly all of these pupils also provided sameness-relational definitions. This finding is consistent with proposals that substitutive understanding builds from sameness-relational understanding (Jones et al., 2012; Simsek et al., 2019). Supporting this view, Donovan et al. (2022) found that nearly all pupils who held a substitutive understanding also simultaneously held a sameness understanding, whereas the reverse pattern was not found. Thus, our pattern of findings hints at the emergence of differentiated interpretations in a small subset of pupils, even if explicit substitutive reasoning remains rare.

Together the findings from our first two research questions suggest that by the end of primary school relational understandings of the equals sign are beginning to emerge for some pupils, but operational conceptions remain dominant. These results align with prior research (Baroody & Ginsburg, 1983; Knuth et al., 2006; McAuliffe et al., 2020; Wardat et al., 2021), reinforcing that although pupils are capable of developing relational conceptions (Knuth et al., 2006), they may struggle to articulate this understanding (Alibali et al., 2007; Byrd et al., 2015; Knuth et al., 2006; McNeil & Alibali, 2005a; Sumpter & Löwenhielm, 2024). This discrepancy points to a need for more consistent engagement with instruction that explicitly supports pupils to understand and articulate the relational meaning of the equals sign as well as provide varied equation experience to support wider equivalence understanding (Powell, 2015).

Equals sign understanding and equivalence fluency

Our third research question explored whether pupils' equals-sign conceptions predicted equivalence fluency. Arithmetic fluency accounted for most of the variance in both additive and multiplicative equivalence performance, consistent with accounts emphasizing that efficient fact retrieval supports higher-level mathematical reasoning (Chesney et al., 2014; Tenison & Anderson, 2016; Xu et al., 2024). Compared to additive problems, pupils were less accurate when solving multiplicative problems, which aligns with the greater cognitive and conceptual demands of multiplicative reasoning (Nunes et al., 2016; Steffe, 1992; Steffe & Olive, 2010).

Gender differences were observed across all problem-solving tasks, a finding well documented by international testing like the Programme for International Student Assessment (PISA), where half the participating countries found that boys outperformed girls in the 2022 session (OECD, 2023). These performance gaps are likely not due to innate abilities but are instead influenced by external factors such as societal gender norms or stereotypes (Gonzalez et al., 2021). Importantly, however, our findings indicate that gender differences are not significant at a conceptual level, aligning with Hornburg et al.'s (2017) meta-analysis which suggests equals sign definitions do not significantly differ by gender. Moreover, we found that gender did not significantly predict equivalence fluency, further supporting the argument that observed differences stem from external influences, not innate mathematical ability (Gonzalez et al., 2021). Where gender differences are noted in equivalence tasks, they are often attributed to girls' tendencies to adhere too closely to instruction or rely more on incorrect strategies like 'add-all', leading to greater errors (Hornburg et al., 2017). Consistent with this, our results, which show performance differences despite a lack of differences in equals sign understanding, highlight the need for further research on instructional influences on task interpretation and strategy use in equivalence. Future longitudinal research could also examine whether associations between gender and equivalence fluency are mediated through arithmetic fluency, helping to clarify how such differences emerge over time.

Despite the strong influence of arithmetic skills, sameness-relational understanding predicted unique variance in both equivalence domains after controlling for standard arithmetic fluency and gender, suggesting that equivalence fluency is influenced not only by a pupil's procedural and arithmetic competence but also their relational conceptual understanding that the equals sign represents balance (McNeil et al., 2017). This finding is consistent with the view that recognizing the equals sign as expressing a relation between expressions supports flexible engagement with nonstandard equation structures (Alibali et al., 2007; Knuth et al., 2006; Rittle-Johnson et al., 2011).

Building on the literature which has predominantly focused on pupils' performance in additive equivalence (Donovan et al., 2022; Hornburg et al., 2022; Rittle-Johnson et al., 2011), in the present study we examined both additive and multiplicative equivalence fluency. We found that relational understanding of the equals sign predicted unique variance in both equivalence domains. Such understanding explained approximately twice the variance in multiplicative equivalence fluency compared to additive equivalence fluency. One potential reason multiplicative equivalence may be more demanding is that successful performance requires abstracting the relational structure of equivalence across problems that differ substantially in surface features and operations. Prior intervention work suggests that transfer from additive to multiplicative equivalence relies on pupils recognizing the abstract relational structure shared across equations, rather than applying operation-specific procedures (Alibali et al., 2009). Consistent with this account, Rickard et al. (1994) showed that tasks requiring abstraction beyond surface operations place greater cognitive demands on learners, which may help explain the increased conceptual demands of multiplicative equivalence. Together with theoretical accounts highlighting the greater complexity of multiplicative reasoning (Butterworth, 2005; Siemon, 2013; Vergnaud, 1994), these findings highlight the need to strengthen pupils' relational understanding of the equals sign to support performance on more conceptually demanding equivalence tasks.

Although the literature suggests that operational understanding may hinder equivalence performance when arithmetic problems deviate from the expected traditional format (Falkner et al., 1999; Perry et al., 1988; Stephens et al., 2013; Sumpter et al., 2025), in the present study, operational conceptions did not predict equivalence fluency. Possibly, by the end of primary school our pupils had developed a flexible operational understanding that allowed them to recognize that the equals sign can move while still holding operational definitions about its meaning (Jones et al., 2012; Lee & Pang, 2021, 2023; Rittle-Johnson et al., 2011).

Substitutive-relational conceptions also did not significantly predict equivalence performance, contrasting with prior findings highlighting its importance for equivalence fluency (Hornburg et al., 2022) and later algebra (Simsek et al., 2019). This non-significance likely reflects the low expression of substitutive understanding in our sample, which constrained variability and limited the power to detect significant predictive relations. Given that the substitutive conception is regarded as a higher-level understanding that develops from earlier sameness-relational conceptions (Donovan et al., 2022; Simsek et al., 2019), our findings may simply reflect that most pupils at this stage have not yet developed this explicit knowledge. Alternatively, the specific equivalence problems used may not have been sufficiently challenging to require such nuanced understanding for a correct solution; consequently, even if students engaged in these ways of thinking, the assessment may not have effectively elicited or demonstrated its use. In summary, within this sample and task context the findings highlight a closer relation between equivalence performance and sameness-relational understanding than a substitutive-relational one, which may not have been fully developed or strongly elicited by the problems used.

Limitations and future directions

The present study is not without limitations. First, the cross-sectional design prevents conclusions about developmental trajectories. Longitudinal work would help clarify how conceptual and procedural understanding of equivalence co-develop and how early relational understanding supports later algebra (Alibali et al., 2007; Hornburg et al., 2022; Matthews & Fuchs, 2020). A substantial proportion of work in algebra, and by extension equivalence, has been completed within the United States (Veith et al., 2023); therefore, conducting such work outside of this context would extend the existing evidence base and provide insights into global patterns and cultural differences.

Second, the measures of equals sign understanding may not have fully captured pupils' understanding. Pupils were asked to define the equals sign only once; prior research shows that when pupils are asked whether the symbol can mean anything else, they sometimes provide both operational and relational definitions (Alibali et al., 2007; Sumpter & Löwenhielm, 2024). Including follow-up prompts could better reveal partial or emerging understanding. Moreover, researchers have suggested that within operational definitions of the equals sign, definitions that use general words (e.g., answer, end, result) may be more helpful for future learning than arithmetic-specific definitions (e.g., add, sum; Byrd et al., 2015; McNeil & Alibali, 2005b). However, in the present study a nontrivial number of pupils provided responses that used a combination of general and arithmetic-specific words and phrases. As such, we could not reliably differentiate among these non-relational interpretations. Relatedly, in the present study, some pupils rated two types of conceptions as equally clever, which may indicate overlapping interpretations. Future work could refine the measure to make distinctions between conceptions more explicit, for example, by asking pupils to rank definitions from most to least clever to better capture subtle differences in their understanding.

Finally, although this study accounted for arithmetic fluency, it did not directly examine strategy use or error types. Prior research suggests that reliance on "add-all" or left-to-right routines is a primary driver of equivalence errors (Falkner et al., 1999; Stephens et al., 2013; Sumpter et al., 2025), and mapping these error patterns across different problem types remains an important next step. In our study, the speeded nature of our tasks, together with low attempt rates on later and more challenging items featuring equations on both sides, precluded a robust error analysis. Future research using computer-based assessments could capture both reaction times and specific response patterns to more precisely model the reasoning underlying equivalence performance.

Conclusion

This preregistered study contributes new evidence about pupils' understanding of the equals sign at the end of primary school. Operational understanding was widespread, and while some pupils demonstrated emerging relational understanding, substitutive-relational knowledge was seldom expressed. Sameness-relational understanding predicted unique variance in equivalence fluency, beyond standard arithmetic proficiency, reinforcing its importance for flexible engagement with nonstandard equation structures. The replication of Jones et al.'s (2012) three-factor structure of equals-sign conceptions further emphasizes the distinction between sameness and substitution as separate components of relational understanding. Overall, our findings suggest the need for consistent relational classroom instruction to support pupils' equals sign understanding and for longitudinal research to clarify how this understanding and equivalence performance develop and interact over time.

Data availability statement

In the interest of transparency and openness, anonymised data, code and materials analysed in the current study are available to

download from the Open Science Framework (OSF): https://osf.io/4z2kw/.

CRediT authorship contribution statement

Phoebe Mills: Writing – review & editing, Writing – original draft, Visualization, Resources, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Sabrina Di Lonardo Burr:** Writing – review & editing, Writing – original draft, Visualization, Methodology, Formal analysis, Conceptualization. **Lisa Jane Rodgers:** Writing – review & editing, Resources, Project administration, Investigation, Data curation. **Ella Denvir:** Writing – review & editing, Investigation. **Judith Wylie:** Writing – review & editing, Writing – original draft, Supervision, Resources, Project administration, Methodology, Investigation, Funding acquisition, Data curation, Conceptualization. **Chang Xu:** Writing – review & editing, Writing – original draft, Visualization, Supervision, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.jecp.2026.106652.

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