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TRANSITIVITY OF MUTATION OF τ-EXCEPTIONAL SEQUENCES IN THE τ-TILTING FINITE CASE

ASLAK B. BUAN, ERIC J. HANSON, AND BETHANY R. MARSH

In memory of Idun Reiten

Abstract. We prove that mutation of complete τ-exceptional sequences is transitive for τ-tilting finite algebras.

Introduction

Inspired by τ-tilting theory [1] and, in particular, τ-tilting reduction [15], τ-exceptional sequences were introduced in [10]. These are certain finite sequences of indecomposable objects in the category of finite-dimensional modules over an arbitrary fixed finite-dimensional algebra. The sequences have properties similar to classical exceptional sequences over hereditary algebras [11, 18]. In particular, the rank of the algebra (that is, the number of isomorphism classes of simple modules) gives an upper bound of the length of a τ-exceptional sequence.

Exceptional sequences were first studied in a triangulated setting, where they come with a mutation operation that induces a braid group action on the set of such sequences; see [4, 14]. This specializes to an operation on exceptional sequences in the module category of a hereditary (finite-dimensional) algebra. Crawley-Boevey [11] and Ringel [18] proved that this mutation operation is transitive in the hereditary case.

In [8], a mutation operation on τ-exceptional sequences was introduced and proved to generalize the hereditary case. But, for arbitrary algebras, the operation does not respect the braid relations. Furthermore, it is not, in general, transitive on complete τ-exceptional sequences. However, it follows from [8, Thm. 0.6] that the mutation operation is transitive for algebras that are τ-tilting finite and have rank two. (Recall from [12] that an algebra is τ-tilting finite if there are only finitely many isomorphism classes of basic support τ-rigid modules.) Furthermore, in [6, Thm. 1.2] the authors obtained the same transitivity result for Nakayama algebras.

In this paper, we prove that mutation of complete τ-exceptional sequences is *always transitive* in the τ-tilting finite case. We prove our main result in Section 2, after giving the necessary notation and background in Section 1.

1. Background

We consider algebras which are basic and finite-dimensional over a field $\Bbbk$. We denote the category of finite-dimensional (left) Λ-modules over an algebra Λ by $\mathsf{mod}\,\Lambda$. We only consider basic modules and let $\delta(M)$ denote the number of indecomposable direct summands of a module M. We let n denote the rank of Λ. We consider Λ-modules up to isomorphism, and all subcategories considered are full and closed under isomorphisms.

Let τ denote the Auslander-Reiten translate in $\mathsf{mod}\,\Lambda$. Following [1], a module M is said to be *τ-rigid* if $\mathrm{Hom}(M, \tau M) = 0$. If, in addition, $\delta(M) = n$, the module M is said to be *τ-tilting*. We assume that Λ is τ-tilting finite throughout.

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We will consider pairs of objects (X, Y) in $\mathsf{mod}\,\Lambda$. In particular, a *support τ-rigid object* [1, Defn. 0.3] is a pair of modules (M, P), where M is τ-rigid and P is projective with $\mathrm{Hom}(P, M) = 0$. When convenient, we consider instead the corresponding object $X \oplus Y[1]$ in the subcategory $\mathsf{mod}\,\Lambda \amalg \mathsf{mod}\,\Lambda[1]$ of the bounded derived category of $\mathsf{mod}\,\Lambda$.

Given a Λ-module M, we have the left and right perpendicular subcategories

$$M^{\perp} = \{X \in \mathsf{mod}\,\Lambda \mid \mathrm{Hom}(M, X) = 0\} \text{ and } {}^{\perp}M = \{Y \in \mathsf{mod}\,\Lambda \mid \mathrm{Hom}(Y, M) = 0\}.$$

1.1. **Torsion classes and wide subcategories.** For a subcategory $\mathcal{X}$ of $\mathsf{mod}\,\Lambda$, we let $\mathsf{Gen}\,\mathcal{X}$ denote the closure of $\mathcal{X}$ under factor objects and we let $\mathsf{Filt}\,\mathcal{X}$ denote the closure under filtrations. We denote by $\mathrm{ind}\,\mathcal{X}$ the set of indecomposable objects in $\mathcal{X}$ (up to isomorphism). A module Q in $\mathcal{X}$ is called Ext-*projective* in $\mathcal{X}$ if $\mathrm{Ext}^1(Q, \mathcal{X}) = 0$. An indecomposable Ext-projective Q which is not in $\mathsf{Gen}(\mathcal{X} \setminus \mathsf{add}\,Q)$, is called *split projective*, and otherwise it is *non-split projective*. The set of Ext-projectives (respectively split projectives, non-split projectives) in $\mathcal{X}$ is denoted by $\mathcal{P}(\mathcal{X})$ (respectively $\mathcal{P}_{\mathrm{s}}(\mathcal{X}), \mathcal{P}_{\mathrm{ns}}(\mathcal{X})$). If $\mathcal{X}$ has a finite number of indecomposable Ext-projectives, we denote by $P(\mathcal{X})$ (respectively $P_{\mathrm{s}}(\mathcal{X}), P_{\mathrm{ns}}(\mathcal{X})$), the direct sum of one copy of each indecomposable Ext-projective (respectively, split projective, non-split projective) module. We let $\mathcal{P}(\Lambda)$ denote the subcategory of projective Λ-modules.

For a module M, we let $\mathsf{Gen}\,M = \mathsf{Gen}(\mathsf{add}\,M)$, where $\mathsf{add}\,M$ denotes the additive closure of M. A module M is *gen-minimal* if, for any proper direct summand M' of M, we have $\mathsf{Gen}\,M' \subsetneq \mathsf{Gen}\,M$.

Torsion classes are subcategories of $\mathsf{mod}\,\Lambda$ closed under factor modules and extensions, and for any subcategory $\mathcal{S}$ of $\mathsf{mod}\,\Lambda$, we have that $\mathsf{T}(\mathcal{S}) = \mathsf{Filt}(\mathsf{Gen}\,\mathcal{S})$ is the smallest torsion class containing $\mathcal{S}$ (see [16] or [8].) Dually, torsion-free classes are subcategories of $\mathsf{mod}\,\Lambda$ closed under submodules and extensions. For a τ-rigid module M, the subcategory $\mathsf{Gen}\,M$ is a torsion class by [3, Thm. 5.10]. A *torsion pair* in $\mathsf{mod}\,\Lambda$ is a pair of subcategories $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{T}^{\perp} = \mathcal{F}$. If $(\mathcal{T}, \mathcal{F})$ is a torsion pair, then $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion-free class. Moreover, if $\mathcal{T}$ is any torsion class in $\mathsf{mod}\,\Lambda$, then $(\mathcal{T}, \mathcal{T}^{\perp})$ is a torsion pair, and all torsion pairs are of this form.

Let $(\mathcal{T}, \mathcal{T}^{\perp})$ be a torsion pair and let M be a module. There is then a unique exact sequence $0 \to t_{\mathcal{T}}M \to M \to f_{\mathcal{T}}M \to 0$ with $t_{\mathcal{T}}M \in \mathcal{T}$ and $f_{\mathcal{T}}M \in \mathcal{T}^{\perp}$, and this induces functors $t_{\mathcal{T}} : \mathsf{mod}\,\Lambda \to \mathcal{T}$ and $f_{\mathcal{T}} : \mathsf{mod}\,\Lambda \to \mathcal{T}^{\perp}$ (see, for example, [2, VI.1]). If the torsion class $\mathcal{T}$ is equal to $\mathsf{Gen}\,M$ for some module M, we denote the functor $f_{\mathcal{T}}$ simply by f_M.

A subcategory $\mathcal{X}$ of $\mathsf{mod}\,\Lambda$ is called *wide* if it closed under kernels, cokernels and extensions. For a support τ-rigid object (M, P) we consider the τ-*perpendicular* category $\mathsf{J}(M, P) = M^{\perp} \cap {}^{\perp}\tau M \cap P^{\perp}$, which is wide by [13, Thm. 3.28] (see also [5, Cor. 3.22]). We let $\mathsf{J}(M) := \mathsf{J}(M, 0)$ for a τ-rigid module M. Note that *Serre subcategories* are exactly τ-perpendicular subcategories of the form $\mathsf{J}(P, 0) (= \mathsf{J}(0, P))$ for P a projective Λ-module.

There are precise interrelations between the sets of wide subcategories, torsion classes and gen-minimal τ-rigid modules. We will, in particular, be dealing with the τ-tilting finite case, where we have the following strong results.

Theorem 1.1. [1, 7, 8, 12, 13, 15, 16] *Let Λ be a τ-tilting finite algebra.*

(a) The indicated maps define inverse bijections

$$\begin{tikzcd}
\{\textit{Gen-minimal } \tau\textit{-rigid modules}\} \arrow[rr, shift right, "M\mapsto \mathsf{Gen}\,M"'] \arrow[dr, shift right, "M\mapsto \mathsf{J}(P_{\mathrm{ns}}(\mathsf{Gen}\,M), P_{\mathsf{Gen}\,M})"'] & & \{\textit{Torsion classes}\} \arrow[ll, shift right, "\mathcal{T}\mapsto P_{\mathrm{s}}(\mathcal{T})"'] \arrow[dl, shift left, "\mathcal{T}\mapsto \mathsf{J}(P_{\mathrm{ns}}(\mathcal{T}), P_{\mathcal{T}})"'] \\
& \{\textit{Wide subcategories}\} \arrow[ul, shift right, "\mathcal{W}\mapsto P_{\mathrm{s}}(\mathsf{T}(\mathcal{W}))"'] \arrow[ur, shift left, "\mathcal{W}\mapsto \mathsf{T}(\mathcal{W}) = \mathsf{Filt}(\mathsf{Gen}\,\mathcal{W})"'] &
\end{tikzcd}$$

where for a torsion-class $\mathcal{T}$, we let $P_{\mathcal{T}}$ denote the direct sum of one copy of each indecomposable projective Q with $\mathrm{Hom}(Q,\mathcal{T})=0$.

(b) *All wide subcategories are τ-perpendicular categories, and hence equivalent to module categories of τ-tilting finite algebras. Furthermore, the rank of a τ-perpendicular category $\mathsf{J}(M,P)$ (with (M,P) a support τ-rigid object) is given by $n-\delta(M)-\delta(P)$.*

Proof. We briefly explain how to extract these results from the literature.

Since Λ is τ-tilting finite, there are finitely many torsion classes, all of which are functorially finite, by [12, Thm. 1.2 and Cor. 2.9]. One can then obtain the horizontal bijections in (a) from [1, Sect. 2.3]; see also [8, Thm. 1.6]. A bijection α from torsion classes to wide subcategories was given by Marks-Šťovíček in [16, Cor. 3.11]. In [7, Lem. 4.3], it was proved that $\alpha(\mathcal{T})=\mathsf{J}(P_{\mathrm{ns}}(\mathcal{T}),P_{\mathcal{T}})$ (recalling the notation $\mathcal{W}_L$ for α introduced in [7, Def. 3.5]). Finally, the maps on the left hand side of the diagram are compositions of these bijections, or see [13, Thm. 4.18].

The fact that all wide subcategories are τ-perpendicular categories was proved in [13, Thm. 4.16], and is also contained in (a). The rest of statement (b) is from [13, Thm. 4.12] and [15, Thm. 1.4 and Thm. 1.5]. □

Let M be a τ-rigid module. Then the subcategory ${}^{\perp}\tau M$ is a functorially finite torsion class, and the module $P({}^{\perp}\tau M)$ is a τ-tilting module, called the *Bongartz completion* of M (see [1, Thm. 2.10]). Similarly, $\mathsf{Gen}\, M$ is a functorially finite torsion class and $P(\mathsf{Gen}\, M)$ is called the *co-Bongartz completion* of M [1, Sect. 2.3] (see also [10, Thm. 1.6]). It is a τ-tilting module in the Serre subcategory $\mathsf{mod}\,\Lambda/\langle e\rangle$, where e is the idempotent such that $\Lambda e = P_{\mathsf{Gen}\, M}$. The *Bongartz complement* of M is the basic module B_M such that $B_M\oplus M$ is the Bongartz completion of M, and the *co-Bongartz complement* M is the basic module C_M, such that $C_M\oplus M$ is the co-Bongartz completion of M.

We conclude this subsection with the following.

Lemma 1.2. *Let M and N be τ-rigid modules. Then any two of the following implies the third.*

(1) $\mathsf{Gen}\, M=\mathsf{Gen}\, N$.
(2) ${}^{\perp}\tau M={}^{\perp}\tau N$.
(3) $\mathsf{J}(M)=\mathsf{J}(N)$.

Moreover, these three conditions hold if and only if $M=N$.

Proof. It is clear that (1) and (2) together imply (3). Let $(\mathsf{Gen}\, M)*\mathsf{J}(M)$ denote all modules that appear as midterms of short exact sequences with the left term in $\mathsf{Gen}\, M$ and the right term in $\mathsf{J}(M)$. By [15, Thm. 3.12], we have

$$ {}^{\perp}\tau M=(\mathsf{Gen}\, M)*\mathsf{J}(M), $$

and likewise for N. This shows that (1) and (3) together imply (2). Finally, intersecting both sides of the equation above with ${}^{\perp}\mathsf{J}(M)$ yields

$$ {}^{\perp}\tau M\cap {}^{\perp}\mathsf{J}(M)=\mathsf{Gen}\, M, $$

and likewise for N. This shows that (2) and (3) together imply (1).

For the moreover part, suppose that (1) and (2) hold. Now (1) implies that $N\in\mathcal{P}(\mathsf{Gen}\, M)$, and so N is a direct summand of the co-Bongartz completion $M\oplus C_M$ of M. Similarly (2) implies that $N\in\mathcal{P}({}^{\perp}\tau M)$, and so N is a direct summand of the Bongartz completion $M\oplus B_M$ of M. Now the modules B_M and C_M have no direct summands in common by [8, Lem. 1.7], so together these facts imply that N is a direct summand of M. Reversing the roles of M and N, we see that also M is a direct summand of N. We conclude that $M=N$. □

1.2. **τ-exceptional sequences.** Following [10], a sequence $(X_k,X_{k+1},\dots,X_n)$ of indecomposable Λ-modules is called a *τ-exceptional sequence* in $\mathsf{mod}\,\Lambda$ if

- X_n is τ-rigid in $\mathsf{mod}\,\Lambda$, and

- if $k < n$, then $(X_k, \dots, X_{n-1})$ is a τ-exceptional sequence in $\mathsf{J}(X_n)$.

Remark 1.3. Note that, by Theorem 1.1(b), the wide subcategory $\mathsf{J}(X_n)$ is equivalent to a module category, so there is an Auslander-Reiten translate $\tau_{\mathsf{J}(X_n)}$, which in general is *different from* τ in $\mathsf{mod}\,\Lambda$. Saying that a module $M \in \mathsf{J}(X_n)$ is τ-rigid in $\mathsf{J}(X_n)$ thus means that M is $\tau_{\mathsf{J}(X_n)}$-rigid; that is, that $\mathrm{Hom}(M, \tau_{\mathsf{J}(X_n)} M) = 0$. Similarly, saying that a pair (M, P) (identified with the object $M \oplus P[1]$) is support τ-rigid in $\mathsf{J}(X_n)$ means that M is $\tau_{\mathsf{J}(X_n)}$-rigid, $P \in \mathcal{P}(\mathsf{J}(X_n))$, and $\mathrm{Hom}(P, M) = 0$.

Let M be a Λ-module with $\delta(M) = t$. A sequence $(M_1, M_2, \dots, M_t)$ of indecomposable modules is an *ordering* of M if $\oplus_{i=1}^t M_i = M$, and a *TF-ordering* [17, Defn. 3.1] if $M_i \notin \mathsf{Gen}(\oplus_{j>i} M_j)$ for all i. By a TF-ordered τ-rigid module, we will mean a τ-rigid module with a fixed TF-ordering. So, for example, the module ${}_\Lambda\Lambda$ gives rise to $n!$ different TF-ordered τ-rigid modules. Since τ-rigid modules have rank at most n, we typically denote a TF-ordered τ-rigid module by $\mathcal{Y} = (Y_k, \dots, Y_n)$. In this case, we write $\bigoplus \mathcal{Y} = Y_k \oplus \cdots \oplus Y_n$.

TF-ordered τ-rigid modules and τ-exceptional sequences are related by the following.

Theorem 1.4. [17, Thm. 5.1] *Let $\mathcal{Y} = (Y_k, Y_{k+1}, \dots, Y_n)$ be a TF-ordered τ-rigid module, and for all $k \leq i \leq n$, let $X_i = f_{\oplus Y_{j>i}} Y_i$. Then $\mathcal{X} = (X_k, X_{k+1}, \dots, X_n)$ is a τ-exceptional sequence. Moreover, the map $\omega : \mathcal{Y} \mapsto \mathcal{X}$ is a bijection from the set of TF-ordered τ-rigid modules with $\delta(\bigoplus \mathcal{Y}) = n - k + 1$ to the set of τ-exceptional sequences of length $n - k + 1$.*

The following map, extending Jasso's reduction map [15, Thm. 3.16], is crucial for our results.

Proposition 1.5. [10, Sect. 5] *For any support τ-rigid object $T = (M, P)$, there is a bijection*

$$\{\textit{indecomposable support } \tau\textit{-rigid objects } X \textit{ in } \mathsf{mod}\,\Lambda \textit{ with } T \oplus X \textit{ support } \tau\textit{-rigid}\}$$

$$\downarrow \mathsf{E}_T$$

$$\{\textit{indecomposable support } \tau\textit{-rigid objects } Y \textit{ in } \mathsf{J}(T)\}.$$

If $T = (M, 0)$ we let $\mathsf{E}_M := \mathsf{E}_{(M,0)}$, and if X is a module such that $(X, 0)$ is in the domain of E_T, we let $\mathsf{E}_T(X) := \mathsf{E}_T(X, 0)$.

With this notation and with X a module in the domain of E_M, we have

(a) $\mathsf{E}_M(X) = f_M(X)$ for $X \notin \mathsf{Gen}\, M$, and

(b) $\mathsf{E}_M(X) = Q[1]$ for some module Q which is projective in $\mathsf{J}(M)$, otherwise.

Furthermore, extending E_T additively, we have that $\mathsf{E}_T(X)$ is support τ-rigid in $\mathsf{J}(T)$, for any X such that $T \oplus X$ is support τ-rigid in $\mathsf{mod}\,\Lambda$.

Remark 1.6. Note that, by [15, Thm. 3.8], τ-perpendicular categories of $\mathsf{mod}\,\Lambda$ are again equivalent to module categories. Hence, for some of the notions and constructions defined above, we can also use relative versions with respect to a τ-perpendicular, and hence wide, subcategory $\mathcal{W}$. In particular, by Theorem 1.1(b), τ-perpendicular subcategories of wide subcategories of $\mathsf{mod}\,\Lambda$ are also equivalent to module categories. Also note that in the τ-tilting finite case, we have, from Theorem 1.1(b), that all wide subcategories of $\mathsf{mod}\,\Lambda$ are again equivalent to module categories of τ-tilting finite algebras.

When using the relative versions, the wide subcategory $\mathcal{W}$ will appear in the sub- or superscript. For example, we will consider:

- $\mathsf{J}_{\mathcal{W}}(-)$
- $\mathsf{Gen}_{\mathcal{W}}(-)$
- $f_T^{\mathcal{W}}(-)$
- $\mathsf{E}_{(-)}^{\mathcal{W}}(-)$
- $\omega_{\mathcal{W}}(-)$
- $f_M^{\mathcal{W}}(-)$

Note that $\mathsf{Gen}_{\mathcal{W}}\, M = \mathsf{Gen}\, M \cap \mathcal{W}$.

Definition 1.7. (1) For a τ-exceptional sequence $(X_k, X_{k+1}, \dots, X_n)$, we define recursively
$$\mathsf{J}(X_k, X_{k+1}, \dots, X_n) = \mathsf{J}_{\mathsf{J}(X_{k+1},\dots,X_n)}(X_k),$$
for $k < n$.

(2) For a τ-perpendicular subcategory $\mathcal{W}$, we denote the set of τ-exceptional sequences (resp. TF-ordered τ-rigid modules) $\mathcal{X} = (X_k, \dots, X_n)$ such that $\mathsf{J}(\mathcal{X}) = \mathcal{W}$ (resp. $\mathsf{J}(\bigoplus \mathcal{X}) = \mathcal{W}$) by $\tau\text{-es}(\mathcal{W})$ (resp. $\mathrm{tf}(\mathcal{W})$)

We conclude this subsection by recalling and establishing the properties of the maps J, E_M, ω, and f_M that we need.

The following important property of the E-map was proved in the τ-tilting finite case in [9, Thm. 5.9], and then generalized in [7].

Lemma 1.8. [7, Thm. 6.12] *Let $X \oplus Y \oplus Z$ be a support τ-rigid object. Then*
$$\mathsf{E}_{Y\oplus Z}(X) = \mathsf{E}^{\mathsf{J}(Z)}_{\mathsf{E}_Z(Y)}(\mathsf{E}_Z(X)).$$

We will also need the following observation.

Lemma 1.9. *Let $X \oplus Y \oplus Z$ be a τ-rigid module with X and Y indecomposable. Suppose that $Y \notin \mathsf{Gen}\, Z$, that $X \notin \mathsf{Gen}\, Z$, and that $X \in \mathsf{Gen}(Y \oplus Z)$. Then $\mathsf{E}_Z(X) = f_Z X \in \mathsf{Gen}(f_Z Y) = \mathsf{Gen}(\mathsf{E}_Z(Y))$.*

Proof. We have $f_Z X = \mathsf{E}_Z(X)$ and $f_Z Y = \mathsf{E}_Z(Y)$ by Lemma 1.5. Since f_Z is right exact and $f_Z Z = 0$, the result follows from applying f_Z to an exact sequence of the form $(Y \oplus Z)^r \to X \to 0$. □

We next recall some key results that we will need when proving the main result in Section 2. Firstly, for a τ-rigid module M, there is a bijection between the indecomposable summands of the Bongartz complement of M and the indecomposable projectives in the corresponding τ-perpendicular category.

Lemma 1.10. [10, Lem. 4.9] *Let M be a τ-rigid module. Then f_M induces a bijection*
$$\mathrm{ind}(\mathcal{P}({}^{\perp}\tau M)) \setminus \mathrm{add}(M) \to \mathrm{ind}(\mathcal{P}(\mathsf{J}(M))).$$

Recall [9, Prop. 10.7], [7, Prop. 6.15] that, for indecomposable τ-rigid modules M and N, $\mathsf{J}(M) = \mathsf{J}(N)$ if and only if $M = N$. In general, a corank 2 wide subcategory does not have a unique description in the form $\mathsf{J}(M, N)$ for a τ-exceptional pair (M, N). However, we do have the following uniqueness result.

Proposition 1.11. [8, Cor. 3.6] *Let (X, Y) and (X', Y') be τ-exceptional sequences with $\mathsf{J}(X, Y) = \mathsf{J}(X', Y')$. Then $X = X'$ if and only if $Y = Y'$.*

We also recall the following result, which gives a description of an iterated τ-perpendicular subcategory of $\mathsf{mod}\,\Lambda$ as a τ-perpendicular subcategory of $\mathsf{mod}\,\Lambda$.

Lemma 1.12. [7, Thm. 6.4] *Let $X \oplus Y$ be a support τ-rigid object. Then $\mathsf{J}(X \oplus Y) = \mathsf{J}_{\mathsf{J}(Y)}(\mathsf{E}_Y(X))$ ($= \mathsf{J}(\mathsf{E}_Y(X), Y)$).*

We then have the following direct consequence of Lemmas 1.12 and 1.8.

Proposition 1.13. *Let $\mathcal{X}$ be a τ-exceptional sequence. Then $\mathsf{J}(\mathcal{X}) = \mathsf{J}(\bigoplus \omega^{-1}(\mathcal{X}))$. In particular, for $\mathcal{W}$ a τ-perpendicular subcategory, ω restricts to a bijection $\mathrm{tf}(\mathcal{W}) \to \tau\text{-es}(\mathcal{W})$.*

We now obtain the following description of the bijection in Theorem 1.4. This will be used in later proofs. See also [17, Thm. 5.1].

Proposition 1.14. *Let $(Y_k, \dots, Y_n)$ be a TF-ordered τ-rigid module and let $\omega(Y_k, \dots, Y_n) = (X_k, \dots, X_n)$. For $k \le i \le n+1$, denote $M_i = \bigoplus_{j=i+1}^n Y_j$. Then and, for $k \le i \le n$, we have*

$$X_i = \mathsf{E}_{M_i}(Y_i) = \mathsf{E}_{X_{i+1}}^{\mathsf{J}(M_{i+1})}(\mathsf{E}_{M_{i+1}}(Y_i)).$$

In particular, we have $X_n = Y_n$.

Note that here, and in the sequel, we use the convention that $M_n = M_{n+1} = X_{n+1} = 0$.

Proof. We prove the statement by reverse induction on i. The result for $i = n$ holds by the definition of ω (see Theorem 1.4).

Assume the result holds for $i+1$ where $1 \le i < n$. Then, since $(Y_k, \dots, Y_n)$ is TF-ordered, we have that $\mathsf{E}_{M_i}(Y_i) = f_{M_i}(Y_i)$ by Theorem 1.5(a). Applying Theorem 1.4, we have $X_i = \mathsf{E}_{M_i}(Y_i)$.

Furthermore, we have $M_i = Y_{i+1} \oplus M_{i+1}$, and so $\mathsf{E}_{M_i}(Y_i) = \mathsf{E}_{Y_{i+1} \oplus M_{i+1}}(Y_i) = \mathsf{E}_{X_{i+1}}^{\mathsf{J}(M_{i+1})}(\mathsf{E}_{M_{i+1}}(Y_i))$ where the second equation follows from Theorem 1.8 and the induction hypothesis. □

Corollary 1.15. *Let (X, Y) and (Z, Y) be TF-ordered τ-rigid modules with $\mathsf{J}(X \oplus Y) = \mathsf{J}(Z \oplus Y)$. Then $X = Z$.*

Proof. By Proposition 1.14, we get τ-exceptional sequences $\omega(X, Y) = (\mathsf{E}_Y(X), Y)$ and $\omega(Z, Y) = (\mathsf{E}_Y(Z), Y)$. Proposition 1.11 then implies that $\mathsf{E}_Y(X) = \mathsf{E}_Y(Z)$. Since E_Y is a bijection, this shows that $X = Z$. □

The following description of the torsion class generated by a τ-exceptional sequence will be useful.

Lemma 1.16. *Let $\mathcal{X}$ be a τ-exceptional sequence. Then $\mathsf{Gen}(\omega^{-1}(\mathcal{X})) = \mathsf{T}(\mathcal{X})$.*

Proof. Write $\omega^{-1}(\mathcal{X}) = (Y_k, \dots, Y_n)$, $\mathcal{X} = (X_k, \dots, X_n)$, and for $k \le i \le n+1$ denote $M_i = \bigoplus_{j=i+1}^n Y_j$. For $k \le i \le n$, Proposition 1.14 and Theorem 1.5 say that there is a short exact sequence

$$0 \to t_{M_i} Y_i \to Y_i \to X_i \to 0$$

with $X_i = f_{M_i} Y_i \in M_i^{\perp}$. We see immediately that $X_i \in \mathsf{Gen}\, Y_i$, and so $\mathsf{T}(\mathcal{X}) \subseteq \mathsf{Gen}(\omega^{-1}(\mathcal{X}))$. For the reverse inclusion we use reverse induction on k. For $k = n$, we have $Y_n = X_n$. For $k < n$, we have that $t_{M_k} Y_k \in \mathsf{T}(\mathcal{X})$ by the induction hypothesis, so the short exact sequence implies that also $Y_k \in \mathsf{T}(\mathcal{X})$. □

1.3. **Mutation of τ-exceptional sequences.** A τ-exceptional sequence of length two will be referred to as a *τ-exceptional pair*. Following [8], a τ-exceptional pair (B, C) is *left regular* if C is projective or $C \notin \mathcal{P}({}^{\perp}\tau \mathsf{E}_C^{-1}(B))$. Otherwise, it is called *left irregular*. A τ-exceptional pair (X, Y) is *right regular* if $\mathsf{E}_Y^{-1}(X) \in \mathcal{P}({}^{\perp}\tau Y)$ or $Y \notin \mathsf{Gen}\, \mathsf{E}_Y^{-1}(X)$. Otherwise it is *right irregular*.

For U indecomposable in $\mathsf{mod}\,\Lambda \amalg \mathsf{mod}\,\Lambda[1]$, we denote

$$|U| = \begin{cases} U, & \text{if } U \in \mathsf{mod}\,\Lambda; \\ U[-1], & \text{if } U \in \mathsf{mod}\,\Lambda[1]. \end{cases}$$

For a τ-exceptional pair (M, N), we let

$$N_+ := \begin{cases} N[1], & \text{if } N \text{ is projective;} \\ N, & \text{otherwise;} \end{cases} \qquad M^+ := \begin{cases} M[1], & \text{if } M \text{ is projective in } \mathsf{J}(N); \\ M, & \text{otherwise.} \end{cases}$$

Furthermore, we let $M_{N\uparrow} := \mathsf{E}_{N_+}^{-1}(M)$ and $M_{N\uparrow}^+ := \mathsf{E}_N^{-1}(M^+)$ and note that we have $|N_+| = N$ and $|M^+| = M$.

In [8, Sect. 4], we introduced partially-defined mutually inverse "mutation operators" φ (left mutation) and ψ (right mutation) on the set of τ-exceptional pairs. For τ-tilting finite algebras, all τ-exceptional pairs are (left and right) "mutable"; that is, every τ-exceptional pair is in the domain of both φ and ψ.

For a left regular τ-exceptional pair (B, C), we have

$$\varphi(B, C) = (|\mathsf{E}_{B_{C\uparrow}}(C_+)|, B_{C\uparrow}). \tag{1}$$

For a right regular τ-exceptional pair (X, Y), we have

$$\psi(X, Y) = (\mathsf{E}_{X^+_{Y\uparrow}}(Y), |X^+_{Y\uparrow}|). \tag{2}$$

Remark 1.17. We give a more explicit description of left and right mutation in the regular case.

(a) For left regular mutation, suppose that (B, C) is a left regular τ-exceptional pair. Let $B' = \mathsf{E}^{-1}_C(B)$, so that $B' \oplus C$ is τ-rigid, and let $\mathsf{Bg}(B')$ and $\mathsf{CoBg}(B')$ be the set of indecomposable direct summands in the Bongartz and co-Bongartz complements of B', respectively. The condition that (B, C) is left regular means that $C \notin \mathsf{Bg}(B')$ or $C \in \mathcal{P}(\Lambda)$. Note that if $C \in \mathcal{P}(\Lambda)$, then, since $C \in {}^\perp\tau B$, we have $C \in \mathsf{Bg}(B')$, so it is not possible for both conditions to hold.

Consider first the case where $C \notin \mathsf{Bg}(B')$, and $C \notin \mathsf{CoBg}(B')$. Since $C \notin \mathsf{Bg}(B')$, then $C \notin \mathcal{P}(\Lambda)$, and so $C_+ = C$ and $B_{C\uparrow} = \mathsf{E}^{-1}_{C_+}(B) = \mathsf{E}^{-1}_C(B) = B'$. Moreover, since $C \notin \mathsf{CoBg}(B')$, we have that $C \notin \mathsf{Gen}\, B'$, and so $\mathsf{E}_{B'}(C) = f_{B'}(C)$ by Theorem 1.5(a). Hence, we have $|\mathsf{E}_{B_{C\uparrow}}(C_+)| = |\mathsf{E}_{B'}(C)| = \mathsf{E}_{B'}(C)$.

Next, assume that $C \notin \mathsf{Bg}(B')$, and $C \in \mathsf{CoBg}(B')$. As in the previous case, we have that C is not projective (in $\mathsf{mod}\,\Lambda$), so $C_+ = C$ and $B_{C\uparrow} = \mathsf{E}^{-1}_{C_+}(B) = \mathsf{E}^{-1}_C(B) = B'$. Since $C \in \mathsf{CoBg}(B')$, we now have that $C \in \mathsf{Gen}\, B'$, and so $\mathsf{E}_{B'}(C) = Q[1]$, for some module Q which is projective in $\mathsf{J}(B')$, by Theorem 1.5(b). Hence we have $|\mathsf{E}_{B_{C\uparrow}}(C_+)| = |\mathsf{E}_{B'}(C)| = \mathsf{E}_{B'}(C)[-1]$ in this case.

Finally, assume $C \in \mathcal{P}(\Lambda)$. Then $C_+ = C[1]$, and $B_{C\uparrow} = \mathsf{E}^{-1}_{C_+}(B) = \mathsf{E}^{-1}_{C[1]}(B) = B$. See [8, Thm. 2.8(a)] for the last equation. Also from [8, Thm. 2.8(a)], we have that $E_B(C[1])$ is in $\mathcal{P}(\mathsf{J}_B)[1]$, and so $|\mathsf{E}_{B_{C\uparrow}}(C_+)| = |\mathsf{E}_B(C[1])| = \mathsf{E}_B(C[1])[-1]$ in this case.

Noting that $C \in \mathsf{CoBg}(B')$ implies, by [8, Lemma 1.7], that $C \notin \mathsf{Bg}(B')$, we have:

$$\varphi(B, C) = \begin{cases} (\mathsf{E}_{B'}(C), B') & C \notin \mathsf{Bg}(B') \cup \mathsf{CoBg}(B'); \quad \text{(L1)} \\ (\mathsf{E}_{B'}(C)[-1], B'), & C \in \mathsf{CoBg}(B'); \quad \text{(L2)} \\ (\mathsf{E}_B(C[1])[-1], B), & C \in \mathcal{P}(\Lambda). \quad \text{(L3)} \end{cases}$$

(b) A similar description can also be given for right regular mutation, ψ. Let (X, Y) be a right regular τ-exceptional pair. Then we have:

$$\psi(X, Y) = \begin{cases} (\mathsf{E}_{\mathsf{E}^{-1}_Y(X[1])}(Y), \mathsf{E}^{-1}_Y(X[1])), & \mathsf{E}^{-1}_Y(X) \in \mathsf{Bg}(Y), \mathsf{E}^{-1}_Y(X[1]) \in \mathsf{CoBg}(Y); \quad \text{(R1)} \\ (\mathsf{E}_{\mathsf{E}^{-1}_Y(X[1])}(Y), \mathsf{E}^{-1}_Y(X[1])[-1]), & \mathsf{E}^{-1}_Y(X) \in \mathsf{Bg}(Y), \mathsf{E}^{-1}_Y(X[1]) \in (\mathcal{P}(\Lambda) \cap {}^\perp Y)[1]; \quad \text{(R2)} \\ (\mathsf{E}_{\mathsf{E}^{-1}_Y(X)}(Y), \mathsf{E}^{-1}_Y(X)), & \mathsf{E}^{-1}_Y(X) \notin \mathsf{Bg}(Y), Y \notin \mathsf{CoBg}(\mathsf{E}^{-1}_Y(X)). \quad \text{(R3)} \end{cases}$$

The co-Bongartz complement of Y in $C(\Lambda) = \mathsf{mod}\,\Lambda \amalg \mathsf{mod}\,\Lambda[1]$ is

$$\mathsf{CoBg}_{C(\Lambda)}(Y) = \mathsf{CoBg}(Y) \oplus Q[1],$$

where Q is an additive generator of $\mathcal{P}(\Lambda) \cap {}^\perp Y$ (see [10, §3]). In this notation, the formula for ψ can be replaced with

$$\psi(X, Y) = \begin{cases} (\mathsf{E}_{\mathsf{E}^{-1}_Y(X[1])}(Y), |\mathsf{E}^{-1}_Y(X[1])|), & \mathsf{E}^{-1}_Y(X) \in \mathsf{Bg}(Y), \mathsf{E}^{-1}_Y(X[1]) \in \mathsf{CoBg}_{C(\Lambda)}(Y); \quad \text{(R1-R2)} \\ (\mathsf{E}_{\mathsf{E}^{-1}_Y(X)}(Y), \mathsf{E}^{-1}_Y(X)), & \mathsf{E}^{-1}_Y(X) \notin \mathsf{Bg}(Y), Y \notin \mathsf{CoBg}(\mathsf{E}^{-1}_Y(X)). \quad \text{(R3)} \end{cases}$$

There are also explicit formulas for φ and ψ in the left/right irregular cases; see [8, Sec. 4]. These formulas were used to extend mutation to τ-exceptional sequences in [8, Defn. 5.2] as follows (noting [8, Cor. 5.4] that all τ-exceptional sequences are left and right mutable in the τ-tilting finite case):

Definition 1.18. [8, Defn. 5.2] Left mutation in a wide subcategory $\mathcal{W}$ is denoted by $\varphi^{\mathcal{W}}$, and for a τ-exceptional sequence $(X_k, X_{k+1}, \ldots, X_n)$, and $i \geq k$, we define the *left i-mutation* φ_i by replacing the pair (X_i, X_{i+1}) with $\varphi^{\mathsf{J}(X_{i+2},\ldots,X_n)}(X_i, X_{i+1})$. *Right i-mutation* ψ_i is defined similarly.

Mutation preserves τ-perpendicular categories, in the following sense. This holds both in the regular and irregular cases.

Proposition 1.19. [8, Cor. 5.3] *Let $\mathcal{X} = (X_k, \ldots, X_n)$ be a τ-exceptional sequence and consider $k \leq i < n$. Then $\mathsf{J}(\varphi_i(\mathcal{X})) = \mathsf{J}(\mathcal{X}) = \mathsf{J}(\psi_i(\mathcal{X}))$.*

1.4. **Gen-minimal τ-rigid modules.** For the proof of our main result, we will need some further results on Gen-minimal τ-rigid modules, so we recall additional results from [8], and point out some easy consequences.

Lemma 1.20. [8, Lem. 1.12] *Let $U \neq V$ be indecomposable τ-rigid modules and suppose that $V \in \mathsf{Gen}\, U$. Then $\mathsf{Gen}\, V \subsetneq \mathsf{Gen}\, U$.*

Recall that for any subcategory $\mathcal{S}$, we have that ${}^{\perp}\mathcal{S}$ is a torsion class.

Theorem 1.21. [8, Thm. 2.14] *A τ-rigid module M is gen-minimal if and only if $M = P_{\mathrm{s}}({}^{\perp}\mathsf{J}(M))$.*

The following consequence is an important ingredient for the proof of our main result.

Corollary 1.22. *Let $\mathcal{W}$ be a τ-perpendicular category and let $\mathcal{X}, \mathcal{X}' \in \tau\text{-es}(\mathcal{W})$. If $\bigoplus \omega^{-1}(\mathcal{X})$ and $\bigoplus \omega^{-1}(\mathcal{X}')$ are both gen-minimal, then $\bigoplus \omega^{-1}(\mathcal{X}) = \bigoplus \omega^{-1}(\mathcal{X}')$.*

Proof. Since $\mathcal{X}, \mathcal{X}' \in \tau\text{-es}\mathcal{W}$, we have $\mathsf{J}(\mathcal{X}) = \mathsf{J}(\mathcal{X}') = \mathcal{W}$. Hence $\mathsf{J}(\bigoplus \omega^{-1}\mathcal{X}) = \mathsf{J}(\bigoplus \omega^{-1}\mathcal{X}')$ by Proposition 1.13. This implies that $P_{\mathrm{s}}({}^{\perp}\mathsf{J}(\bigoplus \omega^{-1}\mathcal{X})) = P_{\mathrm{s}}({}^{\perp}\mathsf{J}(\bigoplus \omega^{-1}\mathcal{X}'))$. The result now follows from Theorem 1.21. $\square$

2. Transitivity in the τ-tilting finite case

In this section we prove our main result, transitivity of mutation of complete τ-exceptional sequences. Suppose throughout this section that Λ is τ-tilting finite. We freely use the facts that, in this case, every wide subcategory is τ-perpendicular (recalled in Theorem 1.1), and that every τ-exceptional sequence is left and right i-mutable for any i [8, Cor. 0.4].

We give a more general formulation, where we first fix a wide subcategory $\mathcal{W}$ and then consider the transitivity of mutation for all τ-exceptional sequences $\mathcal{X}$ with $\mathsf{J}(\mathcal{X}) = \mathcal{W}$. For $\mathcal{W} = 0$, this gives transitivity of mutation for all complete τ-exceptional sequences.

Lemma 2.1. *Let $\mathcal{W}$ be a τ-perpendicular subcategory of rank $n-2$. Then $\mathsf{J}(P_{\mathrm{s}}({}^{\perp}\mathcal{W})) = \mathcal{W}$ and there are indecomposable U and V, such that $P_{\mathrm{s}}({}^{\perp}\mathcal{W}) = U \oplus V$. Moreover $U \oplus V$ is gen-minimal.*

Proof. The fact that $\mathsf{J}(P_{\mathrm{s}}({}^{\perp}\mathcal{W})) = \mathcal{W}$ follows from [8, Lem. 2.21]. Note that the part of the proof of [8, Lem. 2.11] showing this does not need the sincerity assumption made in the statement. The decomposition then follows from Theorem 1.1(b) and the assumption that $\mathcal{W}$ has rank $n-2$. The last statement follows from the definition of split projectivity. $\square$

Setup 2.2. *We suppose that $\mathcal{W}$ is a wide subcategory of $\mathsf{mod}\,\Lambda$ of rank $n-2$. As in Lemma 2.1, we decompose $P_{\mathrm{s}}({}^{\perp}\mathcal{W}) = U \oplus V$ with U and V indecomposable.*

Remark 2.3. Note that in Setup 2.2, we have $\mathsf{J}(U \oplus V) = \mathcal{W}$, where $U \oplus V$ is gen-minimal by Lemma 2.1. Thus, both (U, V) and (V, U) are TF-ordered and hence $\omega(U, V), \omega(V, U) \in \tau\text{-es}(\mathcal{W})$, using Theorem 1.13.

Lemma 2.4. *Suppose that, in Setup 2.2, $\mathcal{W}$ is sincere. Then there exists $\ell \in \mathbb{Z}$ such that $\varphi^\ell(\omega(U,V)) = \omega(V,U)$.*

Proof. Since $\mathcal{W}$ is sincere, we must have $U, V \notin \mathcal{P}(\Lambda)$. By [8, Prop. 3.15], there is at most one left irregular τ-exceptional pair (X,Y) with $\mathsf{J}(X,Y) = \mathcal{W}$. Hence, we can, without loss of generality, assume that $\omega(V,U)$ is left regular (otherwise we just exchange the roles of U and V). Since $U \notin \mathcal{P}(\Lambda)$ by assumption and $U \notin \mathsf{Gen}\, V$ by gen-minimality, by Equation (1) we have that

$$\varphi(\omega(V,U)) = \varphi(\mathsf{E}_U(V), U) = (|\mathsf{E}_V(U)|, \mathsf{E}_U^{-1}\mathsf{E}_U(V)) = (\mathsf{E}_V(U), V) = \omega(U,V).$$

□

Lemma 2.5. *Suppose that, in Setup 2.2, $\mathcal{W}$ is not sincere and not a Serre subcategory of $\mathsf{mod}\,\Lambda$. Then there exists $\ell \in \mathbb{Z}$ such that $\varphi^\ell(\omega(U,V)) = \omega(V,U)$.*

Proof. Without loss of generality, we have that $V \in \mathcal{P}(\Lambda)$ and $U \notin \mathcal{P}(\Lambda)$. Since $\mathcal{W}$ is not sincere, $\omega(V,U)$ is left regular; see [8, Prop. 3.13]. As in Lemma 2.4, we have that $U \notin \mathsf{Gen}\, V$ by gen-minimality, and $U \notin \mathcal{P}(\Lambda)$ by assumption. Hence, $\varphi(\omega(V,U))$ is computed as in Lemma 2.4, so we have that $\varphi(\omega(V,U)) = \omega(U,V)$. □

Remark 2.6. Note that if $\mathcal{W}$ is a Serre subcategory in Setup 2.2, we have $U \oplus V \in \mathcal{P}(\Lambda)$. In this case, every τ-exceptional sequence in τ-es$(\mathcal{W})$ is right regular; see [8, Prop. 3.14]. In fact, every such sequence is also left regular by [8, Prop. 3.13].

For the case where $\mathcal{W}$ is a Serre subcategory, we need the following technical lemma, which forms the base case for an induction needed in the proof of the lemma following it (Lemma 2.8).

Lemma 2.7. *Suppose that, in Setup 2.2, $\mathcal{W}$ is a Serre subcategory of $\mathsf{mod}\,\Lambda$. Let $(V_1, U_1) := \omega^{-1}(\psi(\omega(V,U)))$. Then:*

(I1) $\mathsf{Gen}\, U_1 \subsetneq \mathsf{Gen}\, U$,
(I2) $V_1 = U$, *and*
(I3) $\mathsf{E}_{U_1}(V_1) \in \mathcal{P}(\mathsf{J}(U_1))$.

Proof. By Remark 2.6, we have $V \in \mathcal{P}(\Lambda)$, so $\mathsf{E}_U(V) \in \mathcal{P}(\mathsf{J}(U))$ by Theorem 1.5 and Lemma 1.10. Thus, by Remark 1.17(b) cases (R1) and (R2) we have

$$U_1 = |\mathsf{E}_U^{-1}((\mathsf{E}_U(V))[1])|$$

and either

(i) $U_1 \in \mathsf{Gen}\, U$, in which case $U_1 = \mathsf{E}_U^{-1}((\mathsf{E}_U(V))[1])$.
(ii) $U_1 \in \mathcal{P}(\Lambda)$, in which case $U \oplus U_1[1]$ is support τ-rigid, or

We first claim that Case (ii) cannot occur. We have $\mathcal{W} = \mathsf{J}(U_1 \oplus V_1)$ by Remark 2.3 and Theorem 1.19, so $\mathcal{W} \subseteq \mathsf{J}(U_1) = U_1^\perp$. Since $\mathcal{W} = (U \oplus V)^\perp$, with U, V, U_1 indecomposable projectives, this implies $U_1 \in \mathsf{add}(U \oplus V)$. Since, by assumption, $U_1 \neq V$, we must have $U_1 = U$. But then (ii) says that $U \oplus U[1]$ is support τ-rigid, which is a contradiction. We conclude that we are in Case (i). Then $U_1 \neq U$ since U is not in the image of E_U^{-1}. Since (i) says that $U_1 \in \mathsf{Gen}\, U$, it follows from Lemma 1.20 that (I1) holds, that is: $\mathsf{Gen}\, U_1 \subsetneq \mathsf{Gen}\, U$. Then, using Remark 1.17(b) case (R1) for ψ yields

$$(\mathsf{E}_{U_1}(V_1), U_1) = \psi(\mathsf{E}_U(V), U) = (\mathsf{E}_{U_1}(U), U_1),$$

and so $V_1 = U$ since E_{U_1} is a bijection. Thus (I2) holds. Finally, since $V_1 \in \mathcal{P}(\Lambda)$ and $U_1 \oplus V_1$ is τ-rigid, also $V_1 \in \mathcal{P}({}^\perp\tau U_1)$. Thus $\mathsf{E}_{U_1}(V_1) \in \mathcal{P}(\mathsf{J}(U_1))$ by Lemma 1.10, so also (I3) holds. □

Lemma 2.8. *Suppose that, in Setup 2.2, $\mathcal{W}$ is a Serre subcategory of $\mathsf{mod}\,\Lambda$. Then there exists $\ell \in \mathbb{Z}$ such that $\varphi^\ell(\omega(U,V)) = \omega(V,U)$.*

Proof. For $\ell \in \mathbb{N}$, let $(V_\ell, U_\ell) := \omega^{-1}(\psi^\ell(\omega(V,U)))$. Note that by Remark 2.6, (V_ℓ, U_ℓ) is right regular for all ℓ. We first prove that there is $\ell \in \mathbb{N}$ such that $U_\ell = V$. Suppose, for a contradiction, that $U_\ell \neq V$ for all $\ell \in \mathbb{N}$. We will show by induction on ℓ that:

(I1) $\mathsf{Gen}\, U_{\ell+1} \subsetneq \mathsf{Gen}\, U_\ell$,
(I2) $V_{\ell+1} = U_\ell$, and
(I3) $\mathsf{E}_{U_{\ell+1}}(V_{\ell+1}) \in \mathcal{P}(\mathsf{J}(U_{\ell+1}))$

for all $\ell \in \mathbb{N}$.

The base case $\ell = 0$ is Lemma 2.7. So, suppose that $\ell > 0$ and that the inductive claim holds for all $\ell' < \ell$. By the induction hypothesis, we have $\mathsf{E}_{U_\ell}(V_\ell) \in \mathcal{P}(\mathsf{J}(U_\ell))$. As in the base case (Lemma 2.7), the definition of ψ (Remark 1.17(b), cases (R1) and (R2)) then says that

$$U_{\ell+1} = |\mathsf{E}_{U_\ell}^{-1}((\mathsf{E}_{U_\ell}(V_\ell))[1])|$$

and that either

(i) $U_{\ell+1} \in \mathsf{Gen}\, U_\ell$ and $U_{\ell+1} = \mathsf{E}_{U_\ell}^{-1}((\mathsf{E}_{U_\ell}(V_\ell))[1])$.
(ii) $U_{\ell+1} \in \mathcal{P}(\Lambda)$ and $U_\ell \oplus U_{\ell+1}[1]$ is support τ-rigid, or

In Case (ii), we would have $\mathsf{J}(U_{\ell+1} \oplus V_{\ell+1}) = \mathcal{W}$, so $\mathsf{J}(U \oplus V) = \mathcal{W} \subseteq U_{\ell+1}^{\perp}$, with $U, V, U_{\ell+1}$ indecomposable projectives. Since, by assumption, $U_{\ell+1} \neq V$, we must have $U_{\ell+1} = U$, and hence $\mathrm{Hom}(U, U_\ell) = 0$, since $U_\ell \oplus U_{\ell+1}[1]$ is support τ-rigid. But the induction hypothesis says $U_\ell \in \mathsf{Gen}(U_{\ell-1}) \subseteq \mathsf{Gen}\, U$, so this is a contradiction.

Thus we are in Case (i), and moreover $U_{\ell+1} \neq U_\ell$ since U_ℓ is not in the image of $\mathsf{E}_{U_\ell}^{-1}$. Since (i) says that $U_{\ell+1} \in \mathsf{Gen}\, U_\ell$, it follows from Lemma 1.20 that $\mathsf{Gen}\, U_{\ell+1} \subsetneq \mathsf{Gen}\, U_\ell$, so (I1) holds. Then, as in the proof of Lemma 2.7, the definition of ψ (Remark 1.17(b), case (R1)) yields $V_{\ell+1} = U_\ell$, so (I2) holds.

It remains then to show that also (I3) holds, that is $\mathsf{E}_{U_{\ell+1}}(V_{\ell+1}) \in \mathcal{P}(\mathsf{J}(U_{\ell+1}))$. Suppose, for a contradiction, that $\mathsf{E}_{U_{\ell+1}}(V_{\ell+1}) \notin \mathcal{P}(\mathsf{J}(U_{\ell+1}))$. Then, by the definition of ψ (Remark 1.17(b) case (R3), noting Remark 2.6), we have

$$U_{\ell+2} = \mathsf{E}_{U_{\ell+1}}^{-1}(\mathsf{E}_{U_{\ell+1}}(V_{\ell+1})) = V_{\ell+1},$$

and so $U_{\ell+2} = V_{\ell+1} = U_\ell$. Moreover,

$$\mathsf{J}(V_{\ell+2} \oplus U_{\ell+2}) = \mathcal{W} = \mathsf{J}(V_\ell \oplus U_\ell)$$

by Propositions 1.19 and 1.13, so Corollary 1.15 implies that also $V_\ell = V_{\ell+2}$. Now since ψ and ω are bijections, the equality $(V_{\ell+2}, U_{\ell+2}) = (V_\ell, U_\ell)$ implies the equality $(V_{\ell+1}, U_{\ell+1}) = (V_{\ell-1}, U_{\ell-1})$. This gives us the contradiction $\mathsf{Gen}\, U_{\ell-1} = \mathsf{Gen}\, U_{\ell+1} \subsetneq \mathsf{Gen}\, U_\ell \subsetneq \mathsf{Gen}\, U_{\ell-1}$, where the last inclusion is by the induction hypothesis. So (I3) holds also for ℓ. The induction is complete, and we have shown that (I1), (I2) and (I3) hold for all $\ell \in \mathbb{N}$. It follows that there is an infinite chain $\cdots \subsetneq \mathsf{Gen}\, U_2 \subsetneq \mathsf{Gen}\, U_1 \subsetneq \mathsf{Gen}\, U_0$. Since Λ is τ-tilting finite, there are only finitely many torsion classes in $\mathsf{mod}\,\Lambda$, all of which are functorially finite, by [12, Thm. 1.2 and Cor. 2.9], so we have a contradiction. We can conclude that there is $\ell \in \mathbb{N}$ such that $U_\ell = V$. We have $\mathsf{J}(V_\ell \oplus U_\ell) = \mathcal{W}$ by Propositions 1.19 and 1.13. Since $U_\ell = V$, Corollary 1.15 then implies that $V_\ell = U$. We conclude that $\psi^\ell(\omega(V,U)) = \omega(V_\ell, U_\ell) = \omega(U,V)$, or equivalently that $\varphi^{-\ell}(\omega(V,U)) = \omega(U,V)$, as required. $\square$

Combining Lemmas 2.4, 2.5 and 2.8, we obtain:

Proposition 2.9. *Let $\mathcal{W}$ be a wide subcategory of $\mathsf{mod}\,\Lambda$ of rank $n-2$. As in Lemma 2.1, decompose $P_{\mathrm{s}}({}^{\perp}\mathcal{W}) = U \oplus V$ with U and V indecomposable. Then there exists $\ell \in \mathbb{Z}$ such that $\varphi^\ell(\omega(U,V)) = \omega(V,U)$.*

The following application of Proposition 2.9 generalizes [6, Lem. 5.1].

Proposition 2.10. *Suppose that we have TF-orderings*

$$\mathcal{Y} = (Y_k, \dots, Y_{i-1}, Y_i, Y_{i+1}, \dots, Y_n) \text{ and } \mathcal{Y}' = (Y_k, \dots, Y_{i-1}, Y_{i+1}, Y_i, Y_{i+2}, \dots, Y_n)$$

of some τ-rigid module $\bigoplus \mathcal{Y}$. Then there exists $\ell \in \mathbb{Z}$ such that $\varphi_i^\ell(\omega(\mathcal{Y})) = \omega(\mathcal{Y}')$.

Proof. Write $\omega(\mathcal{Y}) = (X_k, \dots, X_n)$. Then, by Proposition 1.14, we have

$$\omega(\mathcal{Y}') = (X_k, \dots, X_{i-1}, Z_i, Z_{i+1}, X_{i+2}, \dots, X_n),$$

for some modules Z_i and Z_{i+1}. For $k \leq j \leq n$, let $M_j = Y_{j+1} \oplus \cdots \oplus Y_n$. To compute Z_i, Z_{i+1}, let $\mathcal{V} = \mathsf{J}(M_{i+1})$. By Propositions 1.8 and 1.14 we have

$$\mathsf{E}_{M_i} = \mathsf{E}_{Y_{i+1} \oplus M_{i+1}} = \mathsf{E}^{\mathsf{J}(M_{i+1})}_{\mathsf{E}_{M_{i+1}}(Y_{i+1})} \mathsf{E}_{M_{i+1}} = \mathsf{E}^{\mathcal{V}}_{X_{i+1}} \mathsf{E}_{M_{i+1}}.$$

It follows that

$$Z_{i+1} = \mathsf{E}_{M_{i+1}}(Y_i) = \mathsf{E}_{M_{i+1}}(\mathsf{E}_{M_i})^{-1}(X_i) = \mathsf{E}_{M_{i+1}} \mathsf{E}^{-1}_{M_{i+1}} (\mathsf{E}^{\mathcal{V}}_{X_{i+1}})^{-1}(X_i) = (\mathsf{E}^{\mathcal{V}}_{X_{i+1}})^{-1}(X_i).$$

Applying Propositions 1.8 and 1.14 again, we have

$$\begin{aligned} Z_i = \mathsf{E}_{Y_i \oplus M_{i+1}}(Y_{i+1}) = \mathsf{E}_{Y_i \oplus M_{i+1}} \mathsf{E}^{-1}_{M_{i+1}}(X_{i+1}) &= \mathsf{E}^{\mathsf{J}(M_{i+1})}_{\mathsf{E}_{M_{i+1}}(Y_i)} \mathsf{E}_{M_{i+1}} \mathsf{E}^{-1}_{M_{i+1}}(X_{i+1}) \\ &= \mathsf{E}^{\mathcal{V}}_{Z_{i+1}} \mathsf{E}_{M_{i+1}} \mathsf{E}^{-1}_{M_{i+1}}(X_{i+1}) = \mathsf{E}^{\mathcal{V}}_{Z_{i+1}}(X_{i+1}). \end{aligned}$$

Thus, we have $\omega_{\mathcal{V}}(X_{i+1}, Z_{i+1}) = (Z_i, Z_{i+1})$ and $\omega_{\mathcal{V}}(Z_{i+1}, X_{i+1}) = (X_i, X_{i+1})$. It follows that $X_{i+1} \oplus Z_{i+1}$ is a gen-minimal $\tau_{\mathcal{V}}$-rigid module, since both orderings are TF-orderings. Now Theorem 1.21 says that $X_{i+1} \oplus Z_{i+1} = P_{\mathrm{s}}(\mathcal{V} \cap {}^\perp \mathsf{J}_{\mathcal{V}}(X_{i+1} \oplus Z_{i+1}))$. We therefore obtain the result by applying Proposition 2.9 in the subcategory $\mathcal{V}$ with $\mathcal{W} = \mathsf{J}_{\mathcal{V}}(X_{i+1} \oplus Z_{i+1})$. ☐

As a consequence of Proposition 2.10 we obtain the following.

Corollary 2.11. *Let $\mathcal{W}$ be a wide subcategory. Let r be the rank of $\mathcal{W}$, and let $\mathcal{X}, \mathcal{X}' \in \tau\text{-es}(\mathcal{W})$. If $\bigoplus \omega^{-1}(\mathcal{X})$ and $\bigoplus \omega^{-1}(\mathcal{X}')$ are both gen-minimal, then $\mathcal{X}$ and $\mathcal{X}'$ are in the same orbit under $\varphi_{r+1}, \psi_{r+1}, \dots, \varphi_{n-1}, \psi_{n-1}$.*

Proof. By Corollary 1.22, we have that $\bigoplus \omega^{-1}(\mathcal{X}) = \bigoplus \omega^{-1}(\mathcal{X}')$. Now the fact that this module is gen-minimal means that every ordering of its direct summands is a TF-ordering. Thus the result follows immediately from Proposition 2.10. ☐

Our approach to transitivity is now to show that every mutation orbit of τ-es$(\mathcal{W})$ contains a τ-exceptional sequence for which the corresponding ordered TF-rigid module has a gen-minimal direct sum. This, combined with Corollary 2.11, will give the main result.

Lemma 2.12. *Let $\mathcal{Y} = (Y_k, \dots, Y_n)$ be a TF-ordered sequence, with the property that Y_{i+1} is in $\mathsf{Gen}(Y_i \oplus Y_{i+2} \oplus \cdots \oplus Y_n)$, for some $k \leq i < n$. Let $\mathcal{X} = \omega(\mathcal{Y})$. Then $\mathsf{T}(\varphi_i(\mathcal{X})) \supsetneq \mathsf{T}(\mathcal{X})$.*

Proof. Write $\mathcal{X} = (X_k, \dots, X_n)$ and $\mathcal{X}' = \varphi_i(\mathcal{X}) = (X'_k, \dots, X'_n)$, and let $M = M_{i+1} = Y_{i+2} \oplus \cdots \oplus Y_n$. By Theorem 1.8 and the definition of the map ω, we have

$$\begin{aligned} \omega^{-1}_{\mathsf{J}(M)}(X_i, X_{i+1}) &= \left(\left(\mathsf{E}^{\mathsf{J}(M)}_{X_{i+1}} \right)^{-1} (X_i), X_{i+1} \right) \\ &= \left(\left(\mathsf{E}^{\mathsf{J}(M)}_{X_{i+1}} \right)^{-1} \mathsf{E}_{M \oplus Y_{i+1}}(Y_i), \mathsf{E}_M(Y_{i+1}) \right) \\ &= \left(\left(\mathsf{E}^{\mathsf{J}(M)}_{X_{i+1}} \right)^{-1} \mathsf{E}^{\mathsf{J}(M)}_{X_{i+1}} \mathsf{E}_M(Y_i), \mathsf{E}_M(Y_{i+1}) \right) \\ &= (\mathsf{E}_M(Y_i), \mathsf{E}_M(Y_{i+1})). \end{aligned} \tag{3}$$

Since Y_{i+1} is in $\mathsf{Gen}(Y_i \oplus Y_{i+2} \oplus \cdots \oplus Y_n) = \mathsf{Gen}(Y_i \oplus M)$, Lemma 1.9 (with $X = Y_{i+1}$, $Y = Y_i$ and $Z = M$) implies that

$$X_{i+1} = \mathsf{E}_M(Y_{i+1}) \in \mathsf{Gen}\,\mathsf{E}_M(Y_i), \tag{4}$$

where the equality $X_{i+1} = \mathsf{E}_M(Y_{i+1})$ is from (3). This implies that $X_{i+1} \notin \mathcal{P}(\mathsf{J}(M))$ and, by [8, Lem. 3.12], that $\mathcal{X}$ is left i-regular, noting that the computation in Eq. (3) in particular gives that

$$\left(\mathsf{E}^{\mathsf{J}(M)}_{X_{i+1}}\right)^{-1}(X_i) = \mathsf{E}_M(Y_i). \tag{5}$$

Using the formula for φ_i in Theorem 1.18 (see also Equation (1)) yields

$$X'_{i+1} = \mathsf{E}_M(Y_i). \tag{6}$$

We are now prepared to show that $\mathsf{T}(\mathcal{X}') \supseteq \mathsf{T}(\mathcal{X})$. First note that $X'_j = X_j$ for $j \notin \{i, i+1\}$ (since φ_i can only change the terms in positions i and $i+1$). Thus it suffices to show that $X_i \oplus X_{i+1} \in \mathsf{Gen}\,X'_{i+1}$. The fact that $X_{i+1} \in \mathsf{Gen}\,X'_{i+1}$ follows from (4) and (6). The fact that $X_i \in \mathsf{Gen}\,X'_{i+1}$ follows from noting that

$$X_i = \mathsf{E}^{\mathsf{J}(M)}_{X_{i+1}}(X'_{i+1})$$

by (5) and (6)(and noting that X_i is a module, so $X_i = f^{\mathsf{J}(M)}_{X_{i+1}}(X'_{i+1})$ by Proposition 1.5).

We have shown that $\mathsf{T}(\mathcal{X}') \supseteq \mathsf{T}(\mathcal{X})$. Suppose now for a contradiction that $\mathsf{T}(\mathcal{X}') = \mathsf{T}(\mathcal{X})$. Then $\mathsf{Gen}(\omega^{-1}(\mathcal{X})) = \mathsf{T}(\mathcal{X}) = \mathsf{Gen}(\omega^{-1}(\mathcal{X}'))$ by Lemma 1.16 and $\mathsf{J}(\bigoplus \omega^{-1}(\mathcal{X})) = \mathsf{J}(\bigoplus \omega^{-1}(\mathcal{X}'))$ by Propositions 1.13 and 1.19. By Lemma 1.2, this means $\bigoplus \omega^{-1}(\mathcal{X}) = \bigoplus \omega^{-1}(\mathcal{X}')$. Denoting $\omega^{-1}(\mathcal{X}') = (W_k, \ldots, W_n)$, it follows that there exists some index j such that $W_j = Y_{i+1}$. On the other hand, note that $W_\ell = Y_\ell$ for $i+1 < \ell$ and that $W_{i+1} = Y_i$ (using $\mathsf{E}_M(Y_i) = X'_{i+1} = \mathsf{E}_M(W_{i+1})$ from Proposition 1.14 and Equation (6)). Thus $j \leq i$, but this contradicts the assumption that $(Y_{i+1}, Y_i, Y_{i+2}, \ldots, Y_n)$ is not TF-ordered. We conclude that $\mathsf{T}(\mathcal{X}') \supsetneq \mathsf{T}(\mathcal{X})$, which contradicts the maximality of $\mathcal{X}$. □

Lemma 2.13. *Let $\mathcal{W}$ be a wide subcategory. Let r be the rank of $\mathcal{W}$, and let $\mathcal{O}$ be an orbit of τ-es$(\mathcal{W})$ under $\varphi_{r+1}, \psi_{r+1}, \ldots, \varphi_{n-1}, \psi_{n-1}$. Then there exists $\mathcal{X} \in \mathcal{O}$ such that $\bigoplus \omega^{-1}(\mathcal{X})$ is gen-minimal.*

Proof. Choose $\mathcal{X} \in \mathcal{O}$ such that $\mathsf{T}(\mathcal{X})$ is maximal, in the sense that we do not have $\mathsf{T}(\mathcal{X}) \subsetneq \mathsf{T}(\mathcal{X}')$ for any $\mathcal{X}'$ in $\mathcal{O}$. Note that such an $\mathcal{X}$ exists because there are finitely many τ-exceptional sequences for Λ, since Λ is assumed to be τ-tilting finite. We claim that $\bigoplus \omega^{-1}(\mathcal{X})$ is gen-minimal. Assume for a contradiction that it is not.

Write $\omega^{-1}(\mathcal{X}) = \mathcal{Y} = (Y_k, \ldots, Y_n)$ and denote $\mathcal{T} := \mathsf{T}(\mathcal{X})$. By Lemma 1.16, we have

$$\mathsf{T}(\mathcal{X}) = \mathsf{Gen}(Y_k \oplus \cdots \oplus Y_n),$$

with each Y_i in $\mathcal{P}(\mathcal{T})$ (by [1, Prop. 2.9]). By the assumption that $\bigoplus \omega^{-1}(\mathcal{X})$ is not gen-minimal, there exists an integer r with $k \leq r \leq n$ such that $Y_r \in P_{\mathrm{ns}}(\mathcal{T})$. Let s be maximal such that $(Y_k, \ldots Y_{s-1}, Y_r, Y_s, \ldots, \hat{Y}_r, \ldots, Y_n)$ is not TF-ordered (where the hat indicates omission). Then we have:

$$Y_r \in \mathsf{Gen}(Y_s \oplus Y_{s+1} \oplus \cdots \oplus Y_n). \tag{7}$$

Note that, since $(Y_r, Y_k, Y_{k+1}, \ldots, \hat{Y}_r, \ldots, Y_n)$ is not TF-ordered, such an $s \geq k$ exists, and we have $s \leq r-1$ since $(Y_{r+1}, \ldots, Y_n)$ is TF-ordered. Note also that, by the definition of s, we have that

$$\widetilde{\mathcal{Y}} = (Y_k, \ldots, Y_{s-1}, Y_s, Y_r, Y_{s+1}, \ldots, \hat{Y}_r, \ldots, Y_n)$$

is TF-ordered.

By iterated applications of Proposition 2.10, we have that the τ-exceptional sequence

$$\widetilde{\mathcal{X}} = \omega(\widetilde{\mathcal{Y}}) = \omega(Y_k, \dots, Y_{s-1}, Y_s, Y_r, Y_{s+1}, \dots, \hat{Y}_r, \dots, Y_n)$$

is in the same orbit $\mathcal{O}$ as $\mathcal{X}$. Moreover, we have

$$\begin{aligned}\mathsf{T}(\mathcal{X}) &= \mathsf{Gen}(\bigoplus \mathcal{Y}) = \mathsf{Gen}(Y_k \oplus \ldots \oplus Y_n)\\ &= \mathsf{Gen}(Y_k \oplus \cdots \oplus Y_{s-1} \oplus Y_s \oplus Y_r \oplus Y_{s+1} \oplus \cdots \oplus \hat{Y}_r \oplus \cdots \oplus Y_n)\\ &= \mathsf{Gen}(\bigoplus \widetilde{\mathcal{Y}}) = \mathsf{T}(\widetilde{\mathcal{X}}).\end{aligned}$$

By Theorem 2.12 applied to $\widetilde{\mathcal{Y}}$ with $i = s$ (noting equation (7)), we have that $\mathsf{T}(\widetilde{\mathcal{X}}) \subsetneq \mathsf{T}(\phi_i(\widetilde{\mathcal{X}}))$. Hence we have a contradiction to the maximality of $\mathsf{T}(\widetilde{\mathcal{X}}) = \mathsf{T}(\mathcal{X})$. Therefore $\bigoplus \omega^{-1}(\mathcal{X})$ is gen-minimal. □

We are now prepared to prove our main result.

Theorem 2.14. *Let $\mathcal{W}$ be a wide subcategory and let r be the rank of $\mathcal{W}$. Then any two τ-exceptional sequences in τ-es$(\mathcal{W})$ are in the same orbit under $\varphi_{r+1}, \psi_{r+1}, \dots, \varphi_{n-1}, \psi_{n-1}$. In particular, letting $\mathcal{W} = 0$, any two complete τ-exceptional sequences are in the same orbit under the action of left and right mutations.*

Proof. Let $\mathcal{X}_1, \mathcal{X}_2 \in \tau$-es$(\mathcal{W})$. By Lemma 2.13, $\mathcal{X}_1$ can be mutated into some $\mathcal{X}_1' \in \tau$-es$(\mathcal{W})$ with $\bigoplus \omega^{-1}(\mathcal{X}_1')$ gen-minimal. Likewise $\mathcal{X}_2$ can be mutated into some $\mathcal{X}_2' \in \tau$-es$(\mathcal{W})$ with $\bigoplus \omega^{-1}(\mathcal{X}_2')$ gen-minimal. Finally, Corollary 2.11 implies that $\mathcal{X}_1'$ and $\mathcal{X}_2'$ are related by a sequence of mutations. □

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Department of Mathematical Sciences, Norwegian University of Science and Technology (NTNU), 7491 Trondheim, NORWAY
Email address: aslak.buan@ntnu.no

Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA
Email address: ehansonmath@gmail.com

School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K.
Email address: B.R.Marsh@leeds.ac.uk