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Geometric relationship between the generalized Komar energy and the Arnowitt-Deser-Misner mass in dynamical spacetimes

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The standard Komar mass provides an elegant, quasilocal measure of total gravitating energy for stationary spacetimes, but it conventionally fails in dynamical scenarios that lack an exact global timelike Killing vector field. In this paper, we explore a generalized Komar energy integral bounded by a spacelike two-surface, obtained by replacing the Killing vector with the purely kinematic normal evolution vector field $\xi^\mu = N\hat{T}^\mu$ associated with a $3+1$ spacelike foliation. Through a step-by-step mathematical derivation, we show that this generalized integral reduces to the spatial boundary flux of the Eulerian four-acceleration. Furthermore, by evaluating the asymptotic vacuum constraints in generic dynamical spacetimes without restricting the metric to an isotropic or transverse-traceless spatial gauge, we establish, at leading asymptotic order, a relationship mapping this flux to the global Arnowitt-Deser-Misner (ADM) mass. By constructing a superpotential for the linearized spatial Einstein tensor, we prove that the dynamically generalized Komar energy and the ADM mass are equivalent up to a boundary flux of the temporal evolution of the conjugate momentum. We verify this relationship through nontrivial analytical test cases involving dynamical gauge foliations and physical gravitational wave radiation, revealing that the geometric momentum flux isolates longitudinal time-slicing artifacts while naturally decoupling from transverse radiative degrees of freedom.

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I. INTRODUCTION

In general relativity, formulating a meaningful, coordinate-invariant definition of total mass energy in a dynamically evolving spacetime is a highly nontrivial endeavor. The difficulty fundamentally arises from the equivalence principle, which precludes a uniquely localizable energy density for the gravitational field [1,2]. For strictly stationary spacetimes, the standard Komar mass [3] circumvents this issue by offering a robust geometric definition of total mass energy. It exploits the continuous symmetries associated with an exact global timelike Killing vector field to construct a conserved boundary current. The foundational equivalence between the Komar mass and the canonical Arnowitt-Deser-Misner (ADM) mass [4] for strictly stationary spacetimes was rigorously established in seminal works by Beig [5] and Ashtekar and Magnon-Ashtekar [6]. In particular, Beig introduced the $W_i = x_j^{(3)} G_{ij}$ construction to prove this

stationary equivalence. Acknowledging these works properly frames the historical context and highlights why generalizing this equality to the dynamical regime ($\dot{\pi}_{ij} \neq 0$) represents a valuable extension of Beig's original framework. Indeed, in generic, physically realistic spacetimes—such as asymmetric gravitational collapses or coalescing binary systems—the geometry dynamically evolves, emits gravitational radiation, and inherently ceases to admit a global timelike Killing vector field.

The question of quasilocal energy in the absence of exact symmetries has been approached from several complementary perspectives. In the covariant phase space formalism of Iyer and Wald [7,8], the Hamiltonian conjugate to a diffeomorphism generator ξ^μ is expressed as a boundary integral of the Noether charge 2-form $\mathbf{Q}[\xi]$ plus a correction from the symplectic potential θ . For general relativity, the Noether charge integrand is proportional to $\nabla^{[\mu} \xi^{\nu]}$, which is precisely the Komar integrand. In the stationary case, where ξ is a Killing vector, the symplectic potential correction drops out, recovering the Komar integral (up to normalization). For a general vector field ξ , the full Hamiltonian differs from the pure Noether charge by the symplectic potential boundary terms [7,9]. Our result below shows that, for the specific choice $\xi^\mu = N\hat{T}^\mu$, the difference between the Noether charge (i.e., the generalized Komar energy) and the ADM Hamiltonian is controlled by the

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boundary flux of $\dot{\pi}_{ij}$. Within the Iyer-Wald framework, this flux can be understood as the nonvanishing symplectic potential contribution that arises because $\xi^\mu = N\hat{T}^\mu$ is not a Killing vector in a dynamical spacetime. The condition $\dot{\pi}_{ij} \rightarrow 0$ at spatial infinity—which restores the equality $E(\xi) = M_{\text{ADM}}$ —can thus be interpreted as the condition under which the symplectic potential correction to the Noether charge vanishes asymptotically, leaving the Noether charge as the sole contributor to the Hamiltonian. Other quasilocal energy constructions, such as the Brown-York energy [10] and the various proposals reviewed in Ref. [11], approach the problem from different angles; connecting the present result to these frameworks is an interesting direction for future work.

To formulate a dynamically robust quasilocal energy in the absence of exact continuous symmetries, one can generalize the Komar integral by coupling it directly to the kinematics of a $3 + 1$ spacelike foliation Σ_t [12,13]. We define the generalized Komar energy, enclosed by a two-dimensional spatial boundary $\partial\Sigma$, by evaluating the standard integral along the kinematic normal evolution vector field $\xi^\mu \equiv N\hat{T}^\mu$,

$$E(\xi) = \frac{1}{4\pi} \int_{\partial\Sigma} \xi^{[\nu;\mu]} \hat{N}_\nu \hat{T}_\mu \sqrt{\gamma^{(\partial\Sigma)}} d^2y, \quad (1)$$

where $\hat{T}^\mu$ is the hypersurface-orthogonal, future-directed timelike unit normal to the foliation, $\hat{N}^\mu$ is the outward-pointing spacelike unit normal to the boundary $\partial\Sigma$, N is the scalar lapse function, and $\gamma^{(\partial\Sigma)}$ is the determinant of the induced boundary metric.

The objective of this paper is twofold. First, we aim to show that Eq. (1) algebraically reduces to the boundary flux of an effective gravitating force, specifically the spatial Eulerian four-acceleration. Second, we establish the precise relationship, at leading asymptotic order, between this purely kinematic quasilocal integral and the canonical ADM mass [4]. Completely bypassing restrictive spatial gauge choices, we analytically reveal how the choice of temporal gauge dictates the discrepancy between the two masses, and we verify this result analytically using non-trivial spacetimes featuring dynamical slicing artifacts and active gravitational wave radiation.

II. GEOMETRIC FRAMEWORK AND INTEGRAND EVALUATION

We adopt the standard $3 + 1$ ADM geometric decomposition framework [4]. The spacetime is foliated by a family of spacelike hypersurfaces Σ_t parametrized by a global coordinate time t . The requisite geometric definitions are as follows:

- (1) The unit normal to the spacelike hypersurface Σ_t , denoted as $\hat{T}^\mu$, is future directed and timelike, satisfying the normalization condition

$$\hat{T}_\mu \hat{T}^\mu = -1. \quad (2)$$

Taking the covariant derivative yields the orthogonality property $\hat{T}_\mu \nabla_\nu \hat{T}^\mu = 0$.

- (2) The hypersurface-tangent outward-pointing space-like unit normal to the boundary $\partial\Sigma$, denoted as $\hat{N}^\mu$, satisfies $\hat{N}_\mu \hat{N}^\mu = 1$. Because it is strictly tangent to the spacelike slice Σ_t , it is orthogonal to the timelike normal,

$$\hat{N}_\mu \hat{T}^\mu = 0. \quad (3)$$

- (3) The proper area element on the boundary is defined as

$$dA = \sqrt{\gamma^{(\partial\Sigma)}} d^2y. \quad (4)$$

A. Expansion of the covariant derivatives

We first focus on evaluating the antisymmetrized contraction inside the integral of Eq. (1), denoted by $\mathcal{I} = \xi^{[\nu;\mu]} \hat{N}_\nu \hat{T}_\mu$. Expanding the antisymmetrized covariant derivative $\xi^{[\nu;\mu]} = \frac{1}{2}(\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu)$, substituting $\xi^\mu = N\hat{T}^\mu$, and applying the Leibniz product rule, we split the integrand into two distinct parts,

$$\mathcal{I} = \frac{1}{2} \left[\nabla^\mu (N\hat{T}^\nu) \hat{N}_\nu \hat{T}_\mu - \nabla^\nu (N\hat{T}^\mu) \hat{N}_\mu \hat{T}_\nu \right]. \quad (5)$$

The first term in Eq. (5) expands to

$$\begin{aligned} \mathcal{I}_1 &= \frac{1}{2} \left[(\nabla^\mu N) \hat{T}^\nu + N \nabla^\mu \hat{T}^\nu \right] \hat{N}_\nu \hat{T}_\mu \\ &= \frac{1}{2} (\nabla^\mu N) \hat{T}_\mu (\hat{T}^\nu \hat{N}_\nu) + \frac{1}{2} N (\hat{T}_\mu \nabla^\mu \hat{T}^\nu) \hat{N}_\nu. \end{aligned} \quad (6)$$

Because of the boundary orthogonality condition ($\hat{T}^\nu \hat{N}_\nu = 0$), the first piece vanishes identically. In the second piece, we identify the Eulerian four-acceleration of the observers moving with the four-velocity $\hat{T}^\mu$, defined kinematically as $a^\nu \equiv \hat{T}^\mu \nabla_\mu \hat{T}^\nu = \hat{T}_\mu \nabla^\mu \hat{T}^\nu$. Thus, $\mathcal{I}_1$ simplifies to

$$\mathcal{I}_1 = \frac{1}{2} N a^\nu \hat{N}_\nu. \quad (7)$$

Similarly, expanding the second term yields

$$\begin{aligned} \mathcal{I}_2 &= -\frac{1}{2} \left[(\nabla^\nu N) \hat{T}^\mu + N \nabla^\nu \hat{T}^\mu \right] \hat{N}_\nu \hat{T}_\mu \\ &= -\frac{1}{2} (\nabla^\nu N) \hat{N}_\nu (\hat{T}^\mu \hat{T}_\mu) - \frac{1}{2} N \hat{N}_\nu (\hat{T}_\mu \nabla^\nu \hat{T}^\mu). \end{aligned} \quad (8)$$

From the normalization constraint $\hat{T}^\mu \hat{T}_\mu = -1$, its covariant derivative ensures that $\hat{T}_\mu \nabla^\mu \hat{T}^\mu = 0$. This forces the second piece to vanish. The remaining term evaluates to

$$\mathcal{I}_2 = -\frac{1}{2}(\nabla^\nu N)\hat{N}_\nu(-1) = \frac{1}{2}\hat{N}^\nu \nabla_\nu N. \quad (9)$$

Combining these evaluations, the total projected integrand becomes

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 = \frac{1}{2}\left(Na^\nu \hat{N}_\nu + \hat{N}^\nu \nabla_\nu N\right). \quad (10)$$

B. Relating acceleration to the lapse gradient

To simplify Eq. (10), we establish the fundamental geometric relationship between the Eulerian four-acceleration a^μ and the lapse function N . Because the foliation is strictly hypersurface orthogonal, the normal covector is proportional to the gradient of the global time scalar t ,

$$\hat{T}_\mu = -N\nabla_\mu t. \quad (11)$$

Taking the covariant derivative yields

$$\nabla_\nu \hat{T}_\mu = -(\nabla_\nu N)\nabla_\mu t - N\nabla_\nu \nabla_\mu t. \quad (12)$$

Since the Levi-Civita connection is torsion-free, the symmetric second covariant derivatives on scalar fields commute ($\nabla_\nu \nabla_\mu t = 0$). Antisymmetrizing the indices thus naturally isolates the gradient of the lapse,

$$\nabla_\nu \hat{T}_\mu - \nabla_\mu \hat{T}_\nu = \frac{1}{N}(\nabla_\nu N)\hat{T}_\mu - \frac{1}{N}(\nabla_\mu N)\hat{T}_\nu. \quad (13)$$

We algebraically isolate $a_\mu = \hat{T}^\nu \nabla_\nu \hat{T}_\mu$ by contracting Eq. (13) with $\hat{T}^\nu$,

$$a_\mu - \hat{T}^\nu \nabla_\mu \hat{T}_\nu = \frac{1}{N}(\hat{T}^\nu \nabla_\nu N)\hat{T}_\mu - \frac{1}{N}(\nabla_\mu N)(\hat{T}^\nu \hat{T}_\nu). \quad (14)$$

Enforcing the geometric identities $\hat{T}^\nu \nabla_\mu \hat{T}_\nu = 0$ and $\hat{T}^\nu \hat{T}_\nu = -1$, and subsequently multiplying both sides by N , yields

$$Na_\mu = (\hat{T}^\nu \nabla_\nu N)\hat{T}_\mu + \nabla_\mu N. \quad (15)$$

Projecting this equation purely onto the spacelike boundary normal $\hat{N}^\mu$ forces the temporal piece to vanish ($\hat{T}_\mu \hat{N}^\mu = 0$), establishing an exact geometric scalar identity,

$$Na_\mu \hat{N}^\mu = \hat{N}^\mu \nabla_\mu N. \quad (16)$$

Because $\hat{N}^\mu$ is a purely spatial vector strictly confined to the spacelike hypersurface Σ_t , contracting it with the

four-dimensional spacetime gradient $\nabla_\mu N$ is algebraically identical to contracting it with the three-dimensional intrinsic spatial covariant derivative $D_l N$. Defining the intrinsic spatial acceleration vector $a_l \equiv D_l N/N$, substituting Eq. (16) back into Eq. (10) collapses the terms,

$$\xi^{[\nu;\mu]} \hat{N}_\nu \hat{T}_\mu = Na_l \hat{N}^l. \quad (17)$$

Integrating this over the proper area dA , the generalized Komar energy evaluates to the flux of the intrinsic spatial acceleration, taking the form of a Newtonian-like Gauss law for gravity,

$$E(\xi) = \frac{1}{4\pi} \int_{\partial\Sigma} Na_l \hat{N}^l dA. \quad (18)$$

We note that this identity, $E(\xi) = \frac{1}{4\pi} \int Na_l \hat{N}^l dA$, is exact—it holds at any finite radius and in any dynamical spacetime, not only asymptotically. It relies only on the torsion-free, metric-compatible structure of the Levi-Civita connection and would therefore fail in theories with torsion or nonmetricity.

III. ASYMPTOTIC CONSTRAINTS IN DYNAMICAL SPACETIMES

To connect the boundary integral (18) to the global mass of the spacetime, we evaluate it asymptotically at spatial infinity (S_∞). The results in this section and the next hold at leading order in the far-field $1/r$ expansion under standard Weinberg-type asymptotic flatness conditions [14], which we now specify.

Asymptotic flatness dictates that $N \rightarrow 1$ as $r \rightarrow \infty$. We define the asymptotic perturbation $N = 1 + n$, where $n \sim \mathcal{O}(r^{-1})$. We assume throughout that the spatial metric admits an asymptotic expansion $\gamma_{ij} = \delta_{ij} + h_{ij}$ with $h_{ij} \sim \mathcal{O}(r^{-1})$ and $\partial_k h_{ij} \sim \mathcal{O}(r^{-2})$, that the extrinsic curvature satisfies $K_{ij} \sim \mathcal{O}(r^{-2})$, and that the shift vector falls off as $\beta^i \sim \mathcal{O}(r^{-1})$. All equalities in this section and the next are understood to hold at leading $\mathcal{O}(r^{-3})$ order in the asymptotic expansion, with corrections of $\mathcal{O}(r^{-4})$ or higher neglected. In particular, at linearized order, indices are raised and lowered with the flat metric δ_{ij} and partial derivatives replace covariant derivatives.

In this far-field limit, the acceleration dominates as $Na_l \approx \partial_l n$. Orienting the surface element $dS_i = \hat{N}_i dA$, the generalized integral yields

$$E(\xi) \approx \frac{1}{4\pi} \int_{S_\infty} \partial_i n dS_i. \quad (19)$$

To tie n to the spatial geometry, we evaluate the full four-dimensional vacuum Einstein equations $R_{\mu\nu}^{(4)} = 0$ in the far-field initial-value regime. The standard Gauss-Codazzi relation structurally decomposes the spacetime curvature as

$$R_{ij}^{(4)} = {}^{(3)}R_{ij} + KK_{ij} - 2K_{im}K_j^m - \frac{1}{N}\mathcal{L}_{N\hat{T}}K_{ij} - \frac{1}{N}D_iD_jN = 0, \quad (20)$$

where K_{ij} is the extrinsic curvature. Under the asymptotic conditions specified above, terms quadratic in the extrinsic curvature fall off as $\mathcal{O}(r^{-4})$ and drop out of the leading-order field equations [15]. We do not restrict the temporal gauge to an asymptotically stationary slicing. The Lie derivative along the shift vector $\mathcal{L}_\beta K_{ij}$ scales as $\mathcal{O}(r^{-4})$ under the assumed falloff of β^i and drops out at leading order, so that the time evolution of the extrinsic curvature reduces to $\mathcal{L}_{N\hat{T}}K_{ij} \approx \partial_t K_{ij} \equiv \dot{K}_{ij}$. We require only that $\dot{K}_{ij}$ falls off at $\mathcal{O}(r^{-3})$ or faster. Truncating at leading $\mathcal{O}(r^{-3})$ order, the vacuum equation dictates

$${}^{(3)}R_{ij} \approx \partial_i\partial_j n + \dot{K}_{ij}. \quad (21)$$

Taking the spatial trace imposes that the Ricci scalar is ${}^{(3)}R \approx \partial^2 n + \dot{K}$. Concurrently, the leading-order vacuum Hamiltonian constraint (${}^{(3)}R + K^2 - K_{ij}K^{ij} = 0$) forces ${}^{(3)}R \sim \mathcal{O}(r^{-4}) \approx 0$ at this order, which requires $\partial^2 n \approx -\dot{K}$.

Constructing the asymptotic three-dimensional Einstein tensor ${}^{(3)}G_{ij} = {}^{(3)}R_{ij} - \frac{1}{2}\delta_{ij}{}^{(3)}R$ and noting that the Hamiltonian constraint enforces ${}^{(3)}R \approx 0$, it follows that ${}^{(3)}G_{ij} \approx {}^{(3)}R_{ij}$. Defining the conjugate gravitational momentum tensor as $\pi_{ij} \equiv K_{ij} - K\delta_{ij}$, we express the constraint asymptotically as

$$\begin{aligned} {}^{(3)}G_{ij} &\approx \partial_i\partial_j n + \dot{K}_{ij} - \delta_{ij}(\partial^2 n + \dot{K}) \\ &= \partial_i\partial_j n - \delta_{ij}\partial^2 n + (\dot{K}_{ij} - \delta_{ij}\dot{K}) \\ &= \partial_i\partial_j n - \delta_{ij}\partial^2 n + \dot{\pi}_{ij}. \end{aligned} \quad (22)$$

This establishes a purely geometric initial-value constraint linking the lapse gradient to both the spatial curvature strains and the temporal evolution of the conjugate momentum.

IV. RELATIONSHIP TO THE ADM MASS

To finalize the mathematical relationship, we define an auxiliary spatial vector field $W_i \equiv x_j {}^{(3)}G_{ij}$ and calculate its flux over the infinite sphere S_∞ using two distinct methods. We employ the general asymptotically flat metric expansion $\gamma_{ij} = \delta_{ij} + h_{ij}$ without forcing any specialized canonical spatial coordinate gauge (such as isotropic or transverse traceless).

A. Method A: Evaluation via the dynamical lapse constraint

Substituting the dynamical vacuum constraint from Eq. (22) into W_i ,

$$\begin{aligned} W_i &= x_j(\partial_i\partial_j n - \delta_{ij}\partial^2 n + \dot{\pi}_{ij}) \\ &= x_j\partial_i\partial_j n - x_i\partial^2 n + x_j\dot{\pi}_{ij}. \end{aligned} \quad (23)$$

We introduce the antisymmetric tensor $A_{ij} \equiv x_j\partial_i n - x_i\partial_j n$. Taking its Euclidean spatial divergence, and noting that in three spatial dimensions $\partial_j x_j = 3$ and $\partial_j x_i = \delta_{ij}$,

$$\begin{aligned} \partial_j A_{ij} &= \partial_j(x_j\partial_i n) - \partial_j(x_i\partial_j n) \\ &= (\delta_{ij}\partial_j n + x_j\partial_i\partial_j n) - (\delta_{ij}\partial_j n + x_i\partial^2 n) \\ &= 3\partial_i n + x_j\partial_i\partial_j n - \partial_i n - x_i\partial^2 n \\ &= 2\partial_i n + (x_j\partial_i\partial_j n - x_i\partial^2 n). \end{aligned} \quad (24)$$

Rearranging, we express the auxiliary vector field as $W_i = \partial_j A_{ij} - 2\partial_i n + x_j\dot{\pi}_{ij}$. Integrating over the boundary surface S_∞ yields

$$\int_{S_\infty} W_i dS_i = \int_{S_\infty} \partial_j A_{ij} dS_i - 2 \int_{S_\infty} \partial_i n dS_i + \int_{S_\infty} x_j \dot{\pi}_{ij} dS_i. \quad (25)$$

To evaluate the first term, we employ a purely intrinsic 2D surface geometric argument that does not require the existence of a regular interior volume (and therefore remains valid in the presence of apparent horizons or singularities). Let $\hat{n}_i = x_i/r$ be the outward unit normal to the two-sphere boundary, so that the surface element is $dS_i = \hat{n}_i dA$. The flux integrand is $\hat{n}_i \partial_j A_{ij}$. By the product rule, $\partial_j(\hat{n}_i A_{ij}) = (\partial_j \hat{n}_i) A_{ij} + \hat{n}_i \partial_j A_{ij}$. Because $\hat{n}_i$ is purely radial, its spatial derivative $\partial_j \hat{n}_i = \frac{1}{r}(\delta_{ij} - \hat{n}_i \hat{n}_j)$ is symmetric in i and j . Because A_{ij} is antisymmetric, their contraction vanishes identically, leaving $\hat{n}_i \partial_j A_{ij} = \partial_j(\hat{n}_i A_{ij})$. Defining $v_j \equiv \hat{n}_i A_{ij}$, we note that its radial projection evaluates to $v_j \hat{n}_j = \hat{n}_i \hat{n}_j A_{ij} = 0$ by antisymmetry, so v_j is a vector field entirely tangent to the two-sphere boundary. Since $\hat{n}_i = x_i/r$ is the radial unit vector, $v_r = 0$ everywhere, and the 3D spatial divergence reduces to the 2D surface divergence (up to factors of r). By the divergence theorem on a closed two-manifold, the total integrated flux of a tangent vector field vanishes identically ($\oint_{S^2} \nabla_{S^2} \cdot \mathbf{v}_{\text{tangent}} dA = 0$).

Recalling from Eq. (19) that $E(\xi) = \frac{1}{4\pi} \int_{S_\infty} \partial_i n dS_i$, we obtain

$$\int_{S_\infty} W_i dS_i = -8\pi E(\xi) + \int_{S_\infty} x_j \dot{\pi}_{ij} dS_i. \quad (26)$$

B. Method B: Evaluation via the generic linearized metric expansion

The canonical ADM mass is defined for any generic asymptotically flat spatial metric $\gamma_{ij} = \delta_{ij} + h_{ij}$ as [4,16]

$$M_{\text{ADM}} = \frac{1}{16\pi} \int_{S_\infty} V_i dS_i, \quad (27)$$

where the ADM integrand vector is $V_i \equiv \partial_j h_{ij} - \partial_i h$, and $h = \delta^{ij} h_{ij}$ is the trace. The linearized three-dimensional Einstein tensor $^{(3)}G_{ij}$ for a general metric perturbation (without spatial gauge fixing) is

$$^{(3)}G_{ij} = \frac{1}{2} \left(\partial_k \partial_i h_{jk} + \partial_k \partial_j h_{ik} - \partial^2 h_{ij} - \partial_i \partial_j h - \delta_{ij} \partial_a \partial_b h_{ab} + \delta_{ij} \partial^2 h \right). \quad (28)$$

To analytically evaluate the integral of $W_i = x_j {}^{(3)}G_{ij}$, we construct a rank-three tensor H_{kij} whose divergence reproduces the linearized Einstein tensor. This object is structurally equivalent to the standard flat-background superpotential for linearized gravity, related to the Abbott-Deser-Tekin construction [17,18],

$$H_{kij} \equiv \frac{1}{2} \left(\partial_i h_{kj} - \partial_k h_{ij} + \delta_{kj} \partial_m h_{im} - \delta_{ij} \partial_m h_{km} + \delta_{ij} \partial_k h - \delta_{kj} \partial_i h \right). \quad (29)$$

By inspection, swapping the indices k and i negates every term, confirming that H_{kij} is antisymmetric in its first two indices ($H_{kij} = -H_{ikj}$). Taking its spatial divergence $\partial_k H_{kij}$ regenerates all six components of $^{(3)}G_{ij}$, as may be verified by direct computation.

We rewrite the auxiliary vector field via the superpotential,

$$\begin{aligned} W_i &= x_j {}^{(3)}G_{ij} = x_j \partial_k H_{kij} \\ &= \partial_k (x_j H_{kij}) - (\partial_k x_j) H_{kij} \\ &= \partial_k (x_j H_{kij}) - \delta_{kj} H_{kij} \\ &= \partial_k (x_j H_{kij}) - H_{jij}. \end{aligned} \quad (30)$$

We evaluate the trace term H_{jij} (using $\delta_{jj} = 3$ and the summation $h_{jj} = h$),

$$\begin{aligned} H_{jij} &= \frac{1}{2} \left(\partial_i h_{jj} - \partial_j h_{ij} + \delta_{jj} \partial_m h_{im} - \delta_{ij} \partial_m h_{jm} + \delta_{ij} \partial_j h - \delta_{jj} \partial_i h \right) \\ &= \frac{1}{2} \left(\partial_i h - \partial_j h_{ij} + 3 \partial_m h_{im} - \partial_m h_{im} + \partial_i h - 3 \partial_i h \right) \\ &= \frac{1}{2} \left((-1 + 3 - 1) \partial_j h_{ij} + (1 + 1 - 3) \partial_i h \right) \\ &= \frac{1}{2} \left(\partial_j h_{ij} - \partial_i h \right) \equiv \frac{1}{2} V_i. \end{aligned} \quad (31)$$

The trace H_{jij} thus evaluates to exactly half the ADM integrand vector V_i . Substituting back, the flux of W_i over spatial infinity is

$$\int_{S_\infty} W_i dS_i = \int_{S_\infty} \partial_k (x_j H_{kij}) dS_i - \frac{1}{2} \int_{S_\infty} V_i dS_i. \quad (32)$$

Defining $F_{ki} \equiv x_j H_{kij}$, the antisymmetry of H_{kij} in k and i ensures that F_{ki} is also antisymmetric ($F_{ki} = -F_{ik}$). Applying the same 2D surface divergence argument used in method A, the flux of $\partial_k F_{ki}$ over the closed sphere vanishes identically. We are left with

$$\int_{S_\infty} W_i dS_i = -\frac{1}{2} (16\pi M_{\text{ADM}}) = -8\pi M_{\text{ADM}}. \quad (33)$$

C. Final relationship

Equating the evaluations from method A [Eq. (26)] and method B [Eq. (33)] yields

$$-8\pi E(\xi) + \int_{S_\infty} x_j \dot{\pi}_{ij} dS_i = -8\pi M_{\text{ADM}}. \quad (34)$$

Dividing by -8π and rearranging,

$$E(\xi) = M_{\text{ADM}} + \frac{1}{8\pi} \int_{S_\infty} x_j \dot{\pi}_{ij} dS_i. \quad (35)$$

The generalized Komar energy tracks the canonical ADM mass, deviating by the boundary flux of the extrinsic curvature's temporal evolution. If one selects an asymptotically stationary temporal gauge [such that $\dot{\pi}_{ij} \sim \mathcal{O}(r^{-4})$], the flux integral vanishes and $E(\xi) = M_{\text{ADM}}$. If a dynamical temporal gauge is chosen (such as geodesic slicing where $N = 1$), the generalized Komar energy accounts for the deviation through this asymptotic momentum flux.

We note that both methods A and B operate at leading asymptotic order, using the linearized $^{(3)}G_{ij}$ and flat-space identities such as $\partial_i x_j = \delta_{ij}$. The relationship (35) is therefore established at leading order in the far-field expansion. However, the individual quantities $E(\xi)$ and M_{ADM} are each independently well defined and exact; the asymptotic approximation enters only in the identification of the $\dot{\pi}_{ij}$ correction term.

V. ANALYTICAL VERIFICATION

While Sec. IV establishes the relationship abstractly, evaluating Eq. (35) against explicit test cases provides concrete verification and physical intuition. We construct spacetimes that are intentionally subjected to highly dynamical time slicings and genuine radiative degrees of freedom (ensuring $\dot{\pi}_{ij} \neq 0$), demonstrating that the

momentum flux accurately isolates kinematic coordinate artifacts without suppressing true physical mass.

A. Pure gauge dynamics in flat spacetime

Consider Minkowski spacetime, which possesses zero mass. In standard inertial coordinates (t, x^i) , the metric is $ds^2 = -dt^2 + \delta_{ij}dx^i dx^j$. We impose a dynamical time slicing by transforming the global time t to a new coordinate time $\bar{t}$ via a generic, time-dependent, spatially decaying gauge transformation,

$$t = \bar{t} + \frac{C(\bar{t})}{r}, \quad (36)$$

where $r = \sqrt{\delta_{ij}x^i x^j}$. The coordinate differential yields $dt = (1 + \frac{\dot{C}}{r})d\bar{t} - \frac{\dot{C}}{r^2}x_i dx^i$, where $\dot{C} \equiv \partial_{\bar{t}}C(\bar{t})$. Substituting back into the Minkowski line element generates a dynamical 3 + 1 asymptotic foliation at leading $\mathcal{O}(r^{-1})$ order,

$$ds^2 \approx -\left(1 + \frac{2\dot{C}}{r}\right)d\bar{t}^2 + \frac{2C}{r^3}x_i dx^i d\bar{t} + \delta_{ij}dx^i dx^j. \quad (37)$$

We systematically evaluate the components of the theorem:

- (1) Canonical ADM mass: Because the spatial metric remains flat at $\mathcal{O}(r^{-1})$ ($\gamma_{ij} \approx \delta_{ij}$), the ADM integrand vanishes: $M_{\text{ADM}} = 0$.
- (2) Generalized Komar energy: The lapse function is $N \approx 1 + \frac{\dot{C}(\bar{t})}{r}$, yielding $n = \frac{\dot{C}}{r}$. Taking the spatial gradient gives $\partial_i n = -\frac{\dot{C}}{r^2}x_i$. Evaluating the boundary integral over S_∞ (where $dS_i = \frac{x_i}{r}r^2 d\Omega$),

$$\begin{aligned} E(\xi) &= \frac{1}{4\pi} \oint_{S_\infty} \left(-\frac{\dot{C}}{r^3}x_i\right) \left(\frac{x_i}{r}r^2 d\Omega\right) \\ &= -\frac{\dot{C}}{4\pi} \oint d\Omega = -\dot{C}(\bar{t}). \end{aligned} \quad (38)$$

The Komar evaluation registers a nonzero “fictitious” quasilocal energy arising from the observers’ noninertial kinematics.

- (3) Conjugate momentum flux: The leading-order shift vector is $N_i \approx \frac{C x_i}{r^3} = -C(\bar{t})\partial_i(1/r)$. The dominant extrinsic curvature is $K_{ij} \approx \partial_i N_j$, which evaluates to

$$K_{ij} \approx -C(\bar{t})\partial_i \partial_j \left(\frac{1}{r}\right) = C(\bar{t})\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right). \quad (39)$$

The trace K vanishes since $\nabla^2(1/r) = 0$ for $r > 0$. Therefore, the conjugate momentum is $\pi_{ij} = K_{ij}$. Taking the temporal derivative,

$$\dot{\pi}_{ij} = \dot{C}(\bar{t})\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right). \quad (40)$$

Constructing the flux density $x_j \dot{\pi}_{ij}$ and using $x_j x_j = r^2$,

$$x_j \dot{\pi}_{ij} = \dot{C}(\bar{t})\left(\frac{x_i}{r^3} - \frac{3x_i r^2}{r^5}\right) = -2\dot{C}(\bar{t})\frac{x_i}{r^3}. \quad (41)$$

Integrating over S_∞ ,

$$\frac{1}{8\pi} \oint_{S_\infty} x_j \dot{\pi}_{ij} dS_i = \frac{-2\dot{C}}{8\pi}(4\pi) = -\dot{C}(\bar{t}). \quad (42)$$

Substituting into Eq. (35),

$$\begin{aligned} E(\xi) &= M_{\text{ADM}} + \frac{1}{8\pi} \int_{S_\infty} x_j \dot{\pi}_{ij} dS_i \\ -\dot{C}(\bar{t}) &= 0 - \dot{C}(\bar{t}). \quad \checkmark \end{aligned} \quad (43)$$

The momentum flux correctly offsets the fictitious energy recorded by the nonstationary foliation.

B. Dynamically sliced Schwarzschild spacetime

To test the relationship in a spacetime with genuine intrinsic mass, we apply the same slicing transformation to the Schwarzschild solution. In isotropic coordinates at leading order:

$$ds^2 \approx -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)\delta_{ij}dx^i dx^j. \quad (44)$$

Applying the dynamical shift $t = \bar{t} + \frac{C(\bar{t})}{r}$:

- (1) ADM mass: The $\mathcal{O}(r^{-1})$ spatial metric is unaffected by the temporal transformation [$\gamma_{ij} \approx (1 + \frac{2M}{r})\delta_{ij}$], so $M_{\text{ADM}} = M$.
- (2) Generalized Komar energy: The temporal metric component becomes $g_{\bar{t}\bar{t}} \approx -(1 - \frac{2M}{r})(1 + \frac{\dot{C}}{r})^2 \approx -(1 - \frac{2(M-\dot{C})}{r})$. The lapse is $N \approx 1 - \frac{M-\dot{C}}{r}$, yielding $E(\xi) = M - \dot{C}(\bar{t})$.
- (3) Momentum flux: The mass metric deviations ($\sim M/r$) multiply the gauge variations ($\sim C/r^2$) to produce $\mathcal{O}(r^{-4})$ cross terms in the extrinsic curvature dynamics. The leading-order shift and momentum gradients remain identical to the flat-spacetime case. The integrated momentum flux evaluates to $-\dot{C}(\bar{t})$.

Evaluating the theorem,

$$\begin{aligned} E(\xi) &= M_{\text{ADM}} + \text{flux} \\ M - \dot{C}(\bar{t}) &= M - \dot{C}(\bar{t}). \quad \checkmark \end{aligned} \quad (45)$$

The momentum flux bridges the local kinematics of the nonstationary observers back to the ADM mass.

C. Decoupling of physical gravitational wave radiation

A robust result in dynamical spacetimes must handle the presence of propagating gravitational waves. We verify that the theorem structurally isolates kinematic slicing artifacts without corrupting physical radiative degrees of freedom.

In the transverse-traceless (TT) gauge, the physical radiation is characterized by a spatial strain perturbation $h_{ij}^{TT} = \frac{f_{ij}(t-r)}{r}$, satisfying the traceless condition $h^{TT} = 0$ and the transversality condition $\partial_j h_{ij}^{TT} = 0$. In the far-field limit, the spatial divergence is dominated by the radial derivative: $\partial_j h_{ij}^{TT} \approx -\hat{n}_j \frac{\dot{f}_{ij}}{r} = 0$, which enforces the geometric orthogonality $\hat{n}_j f_{ij} = 0$ (where $\hat{n}_j = x_j/r$). We evaluate the interaction of the gravitational waves with each component of the theorem:

- (1) ADM mass: The ADM integrand is $V_i = \partial_j h_{ij} - \partial_i h$. The TT constraints enforce $V_i^{GW} = 0$: the wave is divergence-free and traceless. Thus, the linear TT wave carries zero ADM mass [consistent with the standard result that gravitational wave energy is a nonlinear $\mathcal{O}(h^2)$ effect]: $M_{\text{ADM}} = M$.
- (2) Generalized Komar energy: Purely spatial TT waves do not perturb the temporal metric component at linear order, so the lapse is governed by the background mass $N = 1 - M/r$. The Komar integral yields $E(\xi) = M$.
- (3) Conjugate momentum flux: The dynamical spatial strains produce an active extrinsic curvature. In the $N = 1, \beta_i = 0$ gauge, the extrinsic curvature for a purely spatial TT wave is $K_{ij} \approx -\frac{1}{2} \dot{h}_{ij}^{TT}$. Since the wave propagates as $f_{ij}(t-r)/r$, the temporal derivative is $\dot{h}_{ij}^{TT} = \frac{\dot{f}_{ij}}{r}$. Because the trace is zero ($K = 0$), the conjugate momentum is $\pi_{ij} = -\frac{1}{2} \frac{\dot{f}_{ij}}{r}$.

Constructing the boundary flux integrand by contracting the temporal derivative against the outgoing radial vector $x_j = r\hat{n}_j$,

$$x_j \dot{\pi}_{ij} = r \hat{n}_j \left(-\frac{\ddot{f}_{ij}}{2r} \right) = -\frac{1}{2} \hat{n}_j \ddot{f}_{ij}. \quad (46)$$

Because the wave's polarization is geometrically transverse to the propagation direction ($\hat{n}_j f_{ij} = 0$), and because in the far-field regime the wave equation $\ddot{f}_{ij} \approx \partial_r^2 f_{ij}$ preserves the transversality condition, we have $\hat{n}_j \ddot{f}_{ij} = 0$. The integrated momentum flux of a linear TT gravitational wave therefore vanishes.

Evaluating the theorem,

$$E(\xi) = M_{\text{ADM}} + \frac{1}{8\pi} \int_{S_\infty} x_j \dot{\pi}_{ij} dS_i$$

$$M = M + 0. \quad \checkmark \quad (47)$$

This demonstrates an elegant structural feature of Eq. (35): the geometric momentum flux acts as a filter that separates longitudinal from transverse degrees of freedom. Longitudinal coordinate gauge artifacts—such as the dynamical time slicing $\dot{C}$ evaluated in Secs. VA and VB—survive the radial projection $x_j \dot{\pi}_{ij}$ and correctly compensate the observer-dependent Komar energy. Physical gravitational waves, being transverse, are annihilated by contraction with the radial vector x_j . This decouples genuine transverse radiation from the longitudinal coordinate fluxes, ensuring that dynamical spatial radiation does not pollute the measurement of the quasilocal bound mass.

VI. CONCLUSION

Through step-by-step geometric derivations, we have shown that by replacing the timelike Killing vector field with the purely kinematic normal evolution vector field $\xi^\mu = N\hat{T}^\mu$, the generalized Komar energy integral reduces to the spatial boundary flux of the Eulerian four-acceleration a_l . This identity, Eq. (18), is exact and holds at any finite radius in any dynamical spacetime, revealing this quasilocal energy in general relativity can be expressed as a Newtonian-like Gauss law for the acceleration.

By constructing an antisymmetric superpotential H_{kij} for the linearized spatial Einstein tensor, we bypassed the historical necessity of forcing the metric parametrization into a specialized canonical spatial coordinate gauge. Furthermore, by evaluating the complete far-field vacuum constraints without restricting to an asymptotically stationary temporal gauge [$\mathcal{L}_{N\hat{T}} K_{ij} \sim \mathcal{O}(r^{-3})$], we established, at leading asymptotic order, a relationship between the generalized Komar energy and the ADM mass: the two are equal up to the integrated boundary flux of the temporal derivative of the conjugate momentum tensor, Eq. (35).

Within the covariant phase space formalism [7,8], this momentum flux can be understood as the nonvanishing symplectic potential contribution that distinguishes the Noether charge from the full Hamiltonian for the non-Killing vector field $\xi^\mu = N\hat{T}^\mu$. The condition $\dot{\pi}_{ij} \rightarrow 0$ at spatial infinity—which restores the equality $E(\xi) = M_{\text{ADM}}$ —corresponds to the condition under which the symplectic potential correction vanishes, so that the Noether charge alone yields the Hamiltonian.

The result has been analytically verified against explicit solutions for empty gauge dynamics, Schwarzschild black holes with dynamical slicing, and actively radiating gravitational wave spacetimes. These test cases confirm that the theorem correctly quantifies the role of the temporal gauge in the discrepancy between quasilocal kinematics and the global mass, and that transverse-traceless radiative fluxes naturally decouple from the global mass relationship, leaving the structural equivalence intact in the presence of dynamical radiative degrees of freedom.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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