



Non-stationary stochastic vessel roll dynamics with hereditary fractional-order hydrodynamic damping and added-inertia nonlinearity

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ABSTRACT

In this paper, an efficient semi-analytical methodology for the non-stationary stochastic analysis of nonlinear ship rolling in irregular seas is developed. To account for the underlying maritime complexity, the roll dynamics are described by a physically consistent fractional-order system that incorporates added inertia nonlinearity, cubic-quintic restoring characteristics, and nonlinear quadratic viscous damping. A defining feature of the model is the representation of hereditary hydrodynamic damping via a fractional derivative operator, which naturally captures the memory effects inherent in fluid-hull interactions. The non-stationary sea-wave excitation is modeled through a shaping-filter approach, specifically by means of a calibrated fourth-order filter combined with deterministic amplitude modulation, yielding a compact finite-dimensional Markovian representation. The proposed method leverages a refined blend of statistical linearization (SL), harmonic balancing (HB), and a time-varying covariance Lyapunov equation. Its effectiveness rests on a parsimonious yet physically consistent Markovian augmentation of the ship-roll dynamics. Local-in-time nonlinearities are treated via SL, while the fractional-memory contribution is embedded into an evolving equivalent-linear closure through time-dependent HB components. A key methodological refinement involves updating the representative carrier frequency used in the HB step solely from the statistically linearized backbone to avoid implicit coupling. The framework efficiently delivers estimations of time-dependent response variances, evolutionary power spectra, and roll-amplitude probability-density function surfaces. Validation against benchmark Monte Carlo simulations demonstrates a high level of agreement across various sea states and fractional orders. Ultimately, the proposed methodology offers a reliable, physically interpretable, and computationally efficient tool for stochastic ship-roll response and stability assessment in uncertain and dynamically evolving open sea environments.

1. Introduction

Ensuring the dynamic stability of vessels in heavy seas remains a cornerstone of naval engineering, underscored by the ongoing refinement of international safety standards such as the second generation of intact stability criteria by the International Maritime Organization (IMO) [1]. While vessels are typically represented as six-degree-of-freedom systems (Fig. 1) subjected to irregular wave loads [2–4], traditional assessment has often relied on quasi-static criteria. However, such approaches may prove inadequate for capturing the nonlinear response characteristics that can arise under severe ocean conditions [5]. Critical phenomena, such as parametric roll and the dead ship condition with lost propulsion, demand high-fidelity dynamic modeling. While contemporary research has matured in its treatment of nonlinear restoring moments (GZ curve) and viscous damping, the added mass

moment of inertia is still frequently treated as a constant, overlooking its significant, state-dependent role in large-amplitude motion [5,6].

The predictive capability of any ship motion model is contingent upon the fidelity of its hydrodynamic coefficients. In large-amplitude rolling, the fluid-hull interaction is inherently state-dependent; the added inertia varies as a function of the roll angle and its derivatives. Furthermore, these interactions exhibit a hereditary nature, where the motion at any instant is influenced by the history of the fluid flow. Traditionally, these memory effects are captured through Cummins-type impulse response formulations [7], which involve computationally burdensome convolution integrals. As a more efficient alternative, fractional-order derivative operators have emerged as a physically consistent paradigm for representing such non-local-in-time interactions. Building on early deterministic studies [8,9], fractional-order models serve as compact surrogate representations that capture frequency-dependent damping and radiation-memory behavior with high fidelity

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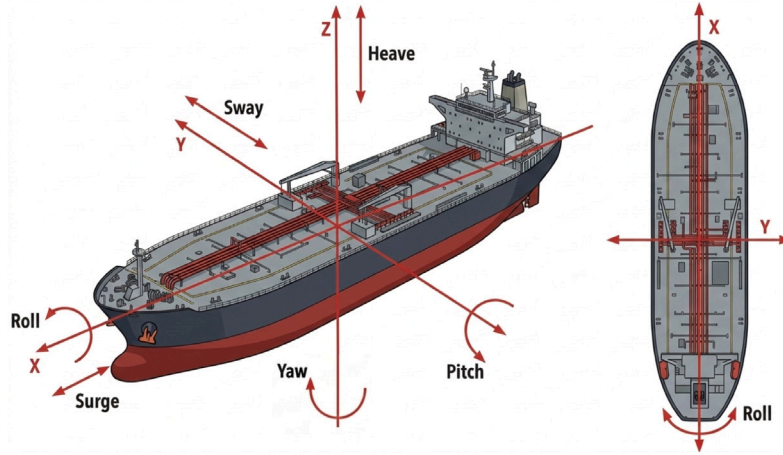


Fig. 1. Kinematic representation of the six degrees of freedom (6 DOF) for a ship model, illustrating translational motions (surge, sway, heave) and rotational motions (roll, pitch, yaw) relative to the body-fixed Cartesian coordinate system.

while maintaining a differential structure [10]. This mathematical compactness is particularly advantageous for stochastic analysis, as it facilitates the formulation of a finite-dimensional state-space, bridging the gap between history-dependent hydrodynamics and Markovian computational efficiency.

Despite these modeling advancements, the presence of multi-component nonlinearities precludes exact analytical solutions for the response probability density function (PDF), necessitating a reliance on approximate methodologies [2,11]. Since the introduction of stochastic process theory to ship motions by St. Denis and Pierson [12], progress has been measured, focusing on diverse frameworks to handle non-linearly driven dynamics. Pertinent efforts include statistical linearization [13–15] and Gaussian closure [16] for efficient moment estimation, as well as stochastic averaging for reduced-order Markovian representations [11,17]. More computationally intensive approaches include direct numerical solutions of the Fokker–Planck (FPK) equation via finite difference [18,19] or finite element methods [20], and versatile path integral (PI) and numerical path integral techniques [21–25]. Most recently, the literature has shifted toward non-stationary extensions, such as the exponential-polynomial-closure (EPC) method [26] and refined blends of stochastic averaging and statistical linearization within a Fokker–Planck representation for non-linear ship rolling dynamics [10]. Within a reliability-oriented context, approximate methods have also been developed to estimate the capsizing risk and seakeeping performance of ships undergoing nonlinear rolling motions [4], explicitly accounting for the critical behavior of both bounded and unbounded roll responses, including those associated with negative stiffness regions, through a suitably tailored non-stationary response-amplitude PDF.

The present work situates itself within the category of advanced moment-based treatments, evolving the traditional statistical linearization (SL) framework to address the curse of dimensionality and the limitations of stationary analysis. Unlike classical FPK-based models that are often restricted to steady-state sea conditions, the proposed methodology leverages an augmented state-space approach to accommodate non-stationary sea-wave excitations. By integrating Harmonic Balancing with a time-varying covariance Lyapunov equation, this framework reconciles fractional-order physics (e.g., [10,27–29]) with Markovian efficiency. It offers a high-fidelity alternative to the prohibitive cost of Monte Carlo simulations (MCS) [30,31] which remain the validation benchmark while providing deeper insights into the critical system dynamics in evolving maritime environments.

The remainder of the paper is organized as follows. Sections 2.1 to 2.6 present the mathematical foundations of the developed fractional-order stochastic framework. Section 2.7 discusses the practical implications and characteristics of the proposed methodology. Section 3 illustrates the framework's application through naval engineering case studies, where results are validated against MCS data obtained from non-linear response-history analysis (RHA). Finally, Section 4 summarizes the main findings of the study.

2. Mathematical formulation

This section develops the mathematical framework underpinning the proposed efficient fractional-order stochastic roll-dynamics approach. The emphasis is on the assumptions and simplifications introduced to render the formulation computationally efficient, while maintaining full consistency with the nonlinear governing equations of ship rolling systems endowed with fractional-order elements.

2.1. Nonlinear ship rolling fractional-order dynamics modeling

The ship rolling motion under stochastic sea-wave excitation is modeled through a nonlinear fractional-order dynamical system that retains four distinct physical mechanisms: state-dependent nonlinear added inertia, finite-amplitude nonlinear hydrostatic restoring, local nonlinear viscous dissipation, and nonlocal hereditary hydrodynamic damping. The proposed governing dynamics equation is expressed as

$$(I_{xx} + \delta I_{xx} \phi^2(t)) \ddot{\phi}(t) + \delta I_{xx} \phi(t) \dot{\phi}^2(t) + B_L D_{0,t}^\alpha \phi(t) + B_N \dot{\phi}(t) |\dot{\phi}(t)| + \Delta (C_1 \phi(t) + C_3 \phi^3(t) + C_5 \phi^5(t)) = W(t) \quad (1)$$

where $\phi(t)$ denotes the roll angle, I_{xx} is the ship roll moment of inertia, and δI_{xx} is an inertia coefficient controlling the amplitude-dependent added inertia contribution. The wave roll excitation moment $W(t)$ is taken as a zero-mean, non-stationary Gaussian process. The parameter Δ denotes the vessel displacement weight. Notably, the amplitude-dependent inertia contribution is represented through the effective roll inertia $I(\phi) = I_{xx} + \delta I_{xx} \phi^2$. A Lagrangian treatment of this angle-dependent inertia yields both the acceleration term $(I_{xx} + \delta I_{xx} \phi^2) \ddot{\phi}$ and the associated coupling term $\delta I_{xx} \phi \dot{\phi}^2$, so the latter follows consistently from the same state-dependent inertia representation. The damping terms distinguish between local-in-time nonlinear viscous dissipation, represented by $B_N \dot{\phi} |\dot{\phi}|$, and nonlocal-in-time hereditary hydrodynamic damping, represented by $B_L D_{0,t}^\alpha \phi$. The coefficients B_L and B_N correspond to the linear (memory-type) and nonlinear (quadratic) damping

contributions, respectively. The fractional operator $D_{0,t}^\alpha(\cdot)$ is defined in the Caputo sense as

$$D_{0,t}^\alpha \phi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{\phi}(\tau)}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1 \quad (2)$$

The restoring-moment coefficients C_1 , C_3 , and C_5 represent the linear, cubic, and quintic contributions extracted from the ship GZ curve, and are expressed as

$$C_1 = \frac{d(GZ)}{d\phi} = GM \quad (3)$$

$$C_3 = \frac{4}{\phi_v^4} (3A_{\phi_v} - GM \phi_v^2) \quad (4)$$

$$C_5 = -\frac{3}{\phi_v^6} (4A_{\phi_v} - GM \phi_v^2) \quad (5)$$

In these relations, GM is the metacentric height, ϕ_v denotes the vanishing stability angle, and A_{ϕ_v} represents the area enclosed by the GZ curve up to ϕ_v . The roll excitation moment $W(t)$ can be described by the wave excitation moment spectrum $S_{WW}(\omega)$. The latter is related to the wave energy spectrum $S_{\zeta\zeta}(\omega)$ following the relation [23,32]

$$S_{WW}(\omega) = |F_{\text{roll}}(\omega)|^2 S_{\zeta\zeta}(\omega) \quad (6)$$

where $F_{\text{roll}}(\omega)$ denotes the roll-excitation transfer function and $S_{\zeta\zeta}(\omega)$ is the stationary wave-elevation spectrum. In the present work, a modified Pierson-Moskowitz description is adopted, namely

$$S_{\zeta\zeta}(\omega; H_s, T_p) = \frac{5.058}{T_p^4} \frac{g^2 H_s^2}{\omega^5} \exp\left(-20 \frac{\pi^4}{T_p^4} \frac{1}{\omega^4}\right) \quad (7)$$

which provides a narrow-banded wave-energy distribution around the dominant wave frequency. In contrast to the original wind-speed-based Pierson-Moskowitz spectrum for fully developed wind seas, this engineering parameterization specifies the spectrum directly through the significant wave height H_s and peak period T_p . To facilitate the subsequent stochastic response analysis and to explicitly expose the influence of the nonlinear contributions, Eq. (1) is normalized by I_{xx} and rewritten by introducing distinct weighting factors for the fractional-memory hydrodynamic damping, local-in-time nonlinear viscous dissipation, restoring nonlinearities, and inertia nonlinearities. This yields the following normalized formulation

$$\ddot{\phi}(t) + \varepsilon_1 \mu \phi^2(t) \ddot{\phi}(t) + \varepsilon_1 \mu \phi(t) \ddot{\phi}^2(t) + \varepsilon_2 b_a D_{0,t}^\alpha \phi(t) + \varepsilon_3 b_N \phi(t) |\dot{\phi}(t)| + \omega_\phi^2 \phi(t) + \varepsilon_4 m_3 \phi^3(t) + \varepsilon_5 m_5 \phi^5(t) = w(t) \quad (8)$$

where the corresponding normalized coefficients are defined as

$$\omega_\phi^2 = \frac{AGM}{I_{xx}} \quad (9)$$

$$m_3 = \frac{4\omega_\phi^2}{\phi_v^2} \left(\frac{3A_{\phi_v}}{GM \phi_v^2} - 1 \right) \quad (10)$$

$$m_5 = -\frac{3\omega_\phi^2}{\phi_v^4} \left(\frac{4A_{\phi_v}}{GM \phi_v^2} - 1 \right) \quad (11)$$

$$b_a = \frac{B_L}{I_{xx}}, \quad b_N = \frac{B_N}{I_{xx}}, \quad \mu = \frac{\delta I_{xx}}{I_{xx}} \quad (12)$$

The weighting factors $\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$ enable controlled tuning of the fractional-memory and nonlinear local-in-time damping as well as the cubic/quintic restoring contributions, while ε_1 provide analogous flexibility for the inertia nonlinearities. In practical applications, these factors may be varied to represent ship-specific hydrodynamic and restoring characteristics within a compact yet physically meaningful modeling framework.

The relative wave excitation moment $w(t)$ in Eq. (8) is described by the corresponding spectrum $S_{ww}(\omega)$. This is obtained from the roll-moment excitation spectrum $S_{WW}(\omega)$ of Eq. (6) according to

$$S_{ww}(\omega) = \frac{|F_{\text{roll}}(\omega)|^2 S_{\zeta\zeta}(\omega)}{I_{xx}^2} \quad (13)$$

In the herein study, the roll excitation is modeled as a zero-mean, Gaussian, non-stationary stochastic process described through an evolutionary power spectrum (EPS) written as

$$S_{ww}(\omega, t) = |g(t)|^2 S_{ww}(\omega), \quad (14)$$

where $g(t)$ is a deterministic time modulation function and $S_{ww}(\omega)$ is the stationary normalized moment excitation spectrum. The non-stationarity of the excitation is introduced through the deterministic envelope

$$g(t) = \left\{ g_0 + g_1 \left[\frac{t}{a_t} \exp\left(1 - \frac{t}{a_t}\right) \right]^{b_t} \right\}^{1/2} \quad (15)$$

where a_t and b_t control the temporal build-up and decay of the excitation, while g_0 and g_1 prescribe its baseline and modulation amplitude. In the stationary benchmark case, one simply sets $g(t) \equiv 1$. Hence, the complete excitation model is defined through Eqs. (7), and (13) to (15), with H_s and T_p controlling the sea state and $F_{\text{roll}}(\omega)$ controlling the mapping from wave energy to roll-moment excitation depending on ship geometry and frequency. In the present research study, $g(t)$ is used as a prescribed benchmark envelope that introduces controlled non-stationarity into the wave-induced roll excitation. The adopted parameters in the numerical section and the analysis window are not intended as a site-specific storm calibration, but define a finite-duration build-up, peak, and decay scenario for reproducible comparison with Monte Carlo simulations. In vessel-specific applications, $g(t)$ may be calibrated from measured records, hindcast/reanalysis data, evolving sea-state parameters, or time-varying spectral estimates.

2.2. Single-sided shaping filter technique of the fourth-order

To embed the prescribed normalized excitation into a finite-dimensional Markov framework suitable for time-domain moment propagation, the stationary target spectrum $S_{ww}(\omega)$ is approximated by the output of a single fourth-order linear shaping filter driven by Gaussian white noise. The present formulation employs a single fourth-order filter, consistent with shaping-filter practices [23,33,34] for ship rolling applications.

For the calibration stage, the modulation is neglected by setting $g(t) = 1$, so that the target spectrum reduces to the stationary normalized excitation spectrum $S_{ww}(\omega)$. The fourth-order shaping filter is written in state-space form as

$$\dot{\mathbf{x}}_f(t) = \mathbf{A}_4 \mathbf{x}_f(t) + \mathbf{B}_4 \xi(t), \quad m(t) = \mathbf{C}_4 \mathbf{x}_f(t) \quad (16)$$

where $\mathbf{x}_f(t) = [x_3(t) \ x_4(t) \ x_5(t) \ x_6(t)]^T$, $m(t)$ denotes the stationary colored process generated by the filter, and $\xi(t)$ is a zero-mean unit-intensity Gaussian white-noise process with two-sided spectral density $S_\xi = \frac{1}{2\pi}$. The non-stationary excitation is then obtained through deterministic modulation,

$$w(t) = g(t) m(t) \quad (17)$$

In the adopted parameterization, the filter matrices are expressed through the parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1)$, with

$$\begin{aligned} \lambda_1 &= \alpha_1 + \alpha_2, & \lambda_2 &= \beta_1 + \beta_2 + \alpha_1 \alpha_2 \\ \lambda_3 &= \alpha_1 \beta_2 + \alpha_2 \beta_1, & \lambda_4 &= \beta_1 \beta_2 \end{aligned} \quad (18)$$

and

$$\mathbf{A}_4 = \begin{bmatrix} -\lambda_1 & 1 & 0 & 0 \\ -\lambda_2 & 0 & 1 & 0 \\ -\lambda_3 & 0 & 0 & 1 \\ -\lambda_4 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_4 = \begin{bmatrix} 0 \\ \gamma_1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C}_4 = [1 \ 0 \ 0 \ 0] \quad (19)$$

The corresponding stationary spectrum generated by the proposed filter is

$$S_{4th}(\omega) = \frac{1}{2\pi} \frac{\gamma_1^2 \omega^4}{[(\beta_1 - \omega)^2 + (\alpha_1 \omega)^2][(\beta_2 - \omega)^2 + (\alpha_2 \omega)^2]} \quad (20)$$

It is important to note that $S_{4th}(\omega)$ is a double-sided spectrum, whereas the physical normalized target spectrum $S_{uw}(\omega)$ is single-sided. Accordingly, the filter calibration is performed against the single-sided quantity, i.e. $2S_{4th}(\omega) \approx S_{uw}(\omega)$. The filter parameters are identified numerically over a prescribed fitting window by minimizing the mismatch between $2S_{4th}(\omega)$ and the target $S_{uw}(\omega)$, while enforcing positivity of the filter parameters and asymptotic stability of the filter dynamics. In this way, the fourth-order shaping filter reproduces the stationary target moment excitation with sufficient fidelity while preserving a compact Markovian representation. Once coupled with the two-dimensional roll subsystem, the resulting augmented model is six-dimensional, which constitutes the basis for the time-varying covariance Lyapunov formulation provided in Section 2.5.

2.3. Statistical linearization approach

The normalized governing Eq. (8) can be written in the following compact form

$$\ddot{\phi}(t) + \varepsilon_2 b_\alpha D_{0,t}^\alpha \phi(t) + f(t, \phi, \dot{\phi}, \ddot{\phi}) = w(t) \quad (21)$$

with

$$f(t, \phi, \dot{\phi}, \ddot{\phi}) = \varepsilon_1 \mu \phi^2 \ddot{\phi} + \varepsilon_1 \mu \phi \dot{\phi}^2 + \varepsilon_3 b_N \phi |\dot{\phi}| + \omega_\phi^2 \phi + \varepsilon_4 m_3 \phi^3 + \varepsilon_5 m_5 \phi^5 \quad (22)$$

Then the equivalent statistically linearized model is taken as

$$I_{eq,SL} \ddot{\phi}(t) + \varepsilon_2 b_\alpha D_{0,t}^\alpha \phi(t) + c_{eq,SL} \dot{\phi}(t) + k_{eq,SL} \phi(t) = w(t) \quad (23)$$

Within the statistical linearization (SL) setting [13,15,35–37], the equivalent inertia, damping, and stiffness terms, namely $I_{eq,SL}$, $c_{eq,SL}$, and $k_{eq,SL}$, are determined by minimizing the expected value of the error $E[\varepsilon^2(t)]$

$$E[\varepsilon^2(t)] = \left\langle [f(t, \phi, \dot{\phi}, \ddot{\phi}) - (I_{eq,SL} - 1)\ddot{\phi} - c_{eq,SL}\dot{\phi} - k_{eq,SL}\phi]^2 \right\rangle \quad (24)$$

The minimization is then carried out with respect to $I_{eq,SL}$, $c_{eq,SL}$, and $k_{eq,SL}$ by imposing the conditions

$$\frac{\partial E[\varepsilon^2(t)]}{\partial I_{eq,SL}} = 0, \quad \frac{\partial E[\varepsilon^2(t)]}{\partial c_{eq,SL}} = 0, \quad \frac{\partial E[\varepsilon^2(t)]}{\partial k_{eq,SL}} = 0 \quad (25)$$

Assuming that $\phi(t)$ and $\dot{\phi}(t)$ can be approximated as Gaussian processes, these expressions simplify by invoking the moment properties of Gaussian random variables. In particular,

$$I_{eq,SL} = 1 + \left\langle \frac{\partial f(t, \phi, \dot{\phi}, \ddot{\phi})}{\partial \ddot{\phi}} \right\rangle \quad (26)$$

$$c_{eq,SL} = \left\langle \frac{\partial f(t, \phi, \dot{\phi}, \ddot{\phi})}{\partial \dot{\phi}} \right\rangle, \quad k_{eq,SL} = \left\langle \frac{\partial f(t, \phi, \dot{\phi}, \ddot{\phi})}{\partial \phi} \right\rangle \quad (27)$$

The equivalent elements in Eqs. (26) and (27) account for the linearization of the nonlinear component $f(t, \phi, \dot{\phi}, \ddot{\phi})$, while the hereditary fractional damping term is retained in its original form. In the non-stationary setting, these quantities are, in general, time-dependent. By considering Eqs. (22), (26), and (27), the process yields

$$I_{eq,SL}(t) = 1 + \langle \varepsilon_1 \mu \phi^2 \rangle = 1 + \varepsilon_1 \mu \langle \phi^2 \rangle = 1 + \varepsilon_1 \mu \sigma_\phi^2(t) \quad (28)$$

$$c_{eq,SL}(t) = \langle 2\varepsilon_1 \mu \phi \dot{\phi} + 2\varepsilon_3 b_N |\dot{\phi}| \rangle = 2\varepsilon_1 \mu \langle \phi \dot{\phi} \rangle + 2\varepsilon_3 b_N \langle |\dot{\phi}| \rangle \quad (29)$$

$$= 2\varepsilon_1 \mu \sigma_{\phi\dot{\phi}}(t) + 2\varepsilon_3 b_N \sqrt{\frac{2}{\pi}} \sigma_{\dot{\phi}}(t)$$

$$k_{eq,SL}(t) = \langle 2\varepsilon_1 \mu \phi \ddot{\phi} + \varepsilon_1 \mu \dot{\phi}^2 + \omega_\phi^2 \phi + 3\varepsilon_4 m_3 \phi^2 + 5\varepsilon_5 m_5 \phi^4 \rangle \quad (30)$$

$$= 2\varepsilon_1 \mu \langle \phi \ddot{\phi} \rangle + \varepsilon_1 \mu \langle \dot{\phi}^2 \rangle + \omega_\phi^2 + 3\varepsilon_4 m_3 \langle \phi^2 \rangle + 5\varepsilon_5 m_5 \langle \phi^4 \rangle$$

$$= 2\varepsilon_1 \mu (\sigma_{\phi\ddot{\phi}} - \sigma_\phi^2) + \varepsilon_1 \mu \sigma_{\dot{\phi}}^2 + \omega_\phi^2 + 3\varepsilon_4 m_3 \sigma_\phi^2 + 15\varepsilon_5 m_5 \sigma_\phi^4$$

$$= 2\varepsilon_1 \mu \dot{\sigma}_{\phi\ddot{\phi}}(t) - \varepsilon_1 \mu \sigma_\phi^2(t) + \omega_\phi^2 + 3\varepsilon_4 m_3 \sigma_\phi^2(t) + 15\varepsilon_5 m_5 \sigma_\phi^4(t)$$

In order to obtain closed-form expressions for the higher-order moments appearing in Eqs. (28) to (30), an instantaneous (time-local) Gaussian closure is adopted, i.e., it is assumed that for each time t

the response variables are approximately zero-mean Gaussian with the same second moments as the true process. Under this non-stationary closure, the relations $\langle |\dot{\phi}(t)| \rangle = \sqrt{2/\pi} \sigma_{\dot{\phi}}(t)$ and $\langle \phi^4(t) \rangle = 3\sigma_\phi^4(t)$ hold pointwise in time, yielding a tractable moment-based formulation. Note that, in doing this, it is assumed that the input process is Gaussian, hence the response of the equivalent linear system is also Gaussian.

2.4. Harmonic-balancing approximation of the hereditary fractional term

Under the assumption of a narrow-band (pseudo-harmonic) response with a slowly-evolving natural frequency, the roll angle $\phi(t)$ and roll rate or angular roll velocity $\dot{\phi}(t)$ are expressed in terms of a slowly varying amplitude $A(t)$ and phase $\theta(t)$ as

$$\phi(t) = A(t) \cos(\psi) \quad (31)$$

$$\dot{\phi}(t) = -\omega(A) A(t) \sin(\psi) \quad (32)$$

where ψ is defined as

$$\psi = \omega(A)t + \theta(t) \quad (33)$$

Here, $A(t)$ and $\theta(t)$ are treated as slowly varying functions of time, and thus may be considered approximately constant over a single oscillation cycle (e.g., [13,35,36,38]). The time-varying roll angle amplitude $A(t)$ and phase $\theta(t)$ are expressed as

$$A^2(t) = \phi^2(t) + \left(\frac{\dot{\phi}(t)}{\omega(A)} \right)^2 \quad (34)$$

$$\theta(t) = -\omega(A)t - \arctan\left(\frac{\dot{\phi}(t)}{\omega(A)\phi(t)} \right) \quad (35)$$

By resorting to the harmonic balancing (HB) method [13], the fractional-order term is approximated over a single oscillation cycle by an equivalent amplitude-dependent contribution to the effective damping and stiffness of the roll system, denoted by $c_{HB,\alpha}(A)$ and $k_{HB,\alpha}(A)$, respectively. These quantities are obtained by minimizing the mean-square error between the original fractional-order model and its linearized approximation over a single period. In this procedure, the amplitude A and phase θ are assumed to remain approximately constant within the cycle, consistent with the narrow-band (pseudo-harmonic) response assumption (e.g., [10,39]).

$$c_{HB,\alpha}(A) = -\frac{1}{\pi A \omega(A)} \left[\varepsilon_2 b_\alpha \int_0^{2\pi} D_{0,t}^\alpha (A \cos \psi) \sin \psi d\psi \right] \quad (36)$$

and

$$k_{HB,\alpha}(A) = \frac{1}{\pi A} \left[\varepsilon_2 b_\alpha \int_0^{2\pi} D_{0,t}^\alpha (A \cos \psi) \cos \psi d\psi \right] \quad (37)$$

The following compact quantities are defined (e.g., [10,38,40])

$$S_\alpha(A) = \int_0^{2\pi} D_{0,t}^\alpha (A \cos \psi) \sin \psi d\psi \quad (38)$$

and

$$F_\alpha(A) = \int_0^{2\pi} D_{0,t}^\alpha (A \cos \psi) \cos \psi d\psi \quad (39)$$

Approximating the fractional derivatives as in [41], in a manner consistent with the Caputo formulation [10,27–29], yields

$$c_{HB,\alpha}(A) = -\varepsilon_2 b_\alpha \frac{S_\alpha(A)}{\pi A \omega(A)} = \varepsilon_2 b_\alpha \omega^{\alpha-1}(A) \sin\left(\frac{\alpha\pi}{2}\right) \quad (40)$$

and

$$k_{HB,\alpha}(A) = \varepsilon_2 b_\alpha \frac{F_\alpha(A)}{\pi A} = \varepsilon_2 b_\alpha \omega^\alpha(A) \cos\left(\frac{\alpha\pi}{2}\right) \quad (41)$$

It follows from Eqs. (40) and (41) that the fractional-order contribution manifests itself through additional amplitude-dependent, and therefore time-dependent, damping and stiffness terms. Within this approximation, the fading-memory character of the operator is effectively

embedded in the response amplitude, which in turn reflects the prior oscillatory history of the system. Moreover, under stochastic roll excitation $w(t)$, the amplitude-dependent quantities $c_{HB,\alpha}(A)$ and $k_{HB,\alpha}(A)$ are replaced by their corresponding expected values, namely the associated time-dependent equivalent coefficients $c_{eq,HB}(t)$ and $k_{eq,HB}(t)$, respectively (e.g., [22,35]). Thus,

$$c_{eq,HB}(t) = \langle c_{HB,\alpha}(A) \rangle = \int_0^\infty c_{HB,\alpha}(A) p(A, t) dA \quad (42)$$

and

$$k_{eq,HB}(t) = \langle k_{HB,\alpha}(A) \rangle = \int_0^\infty k_{HB,\alpha}(A) p(A, t) dA \quad (43)$$

The response-amplitude probability density function $p(A, t)$ is approximated by a Rayleigh distribution with time-dependent parameters governed by the instantaneous variance of the ship roll angle. Specifically,

$$p(A, t) = \frac{A}{\sigma_\phi^2(t)} \exp \left[-\frac{A^2}{2\sigma_\phi^2(t)} \right], \quad A \geq 0 \quad (44)$$

It is noted that Eq. (44) is adopted as an instantaneous, or evolutionary, Rayleigh approximation, in which the scale parameter is updated through the time-dependent roll-angle variance $\sigma_\phi^2(t)$ obtained from the time-dependent covariance Lyapunov equation provided in Section 2.5. This treatment is consistent with previous non-stationary stochastic dynamics formulations in which time-dependent Rayleigh-type amplitude PDFs are used for response-envelope determination (e.g., [42]). The approximation is appropriate when the response remains locally narrow-band and dominated by a single slowly varying roll-frequency component, so that higher harmonics and strongly non-Gaussian effects remain secondary. The normalized ship-rolling governing Eq. (8) can be written in the following equivalent form, where the fractional terms ($0 < \alpha < 1$) are absorbed into the equivalent damping and stiffness coefficients $c_{eq}(t)$ and $k_{eq}(t)$, respectively

$$I_{eq}(t) \ddot{\phi}(t) + c_{eq}(t) \dot{\phi}(t) + k_{eq}(t) \phi(t) = w(t) \quad (45)$$

with equivalent inertia

$$I_{eq}(t) = I_{eq,SL}(t) \quad (46)$$

and equivalent damping and stiffness as sum of the SL and fractional HB parts

$$c_{eq}(t) = c_{eq,SL}(t) + c_{eq,HB}(t) \quad (47)$$

$$k_{eq}(t) = k_{eq,SL}(t) + k_{eq,HB}(t) \quad (48)$$

Eq. (45) defines a time-varying equivalent linear ship-roll system whose instantaneous properties evolve in accordance with the non-stationary response statistics. This remains fully consistent with the narrow-band interpretation adopted herein. For the equivalent system of Eq. (45), the pseudo-harmonic response representation of Eq. (31) is recast in the form

$$\phi(t) = A(t) \cos \left(\int_0^t \omega_{eq}(\tau) d\tau + \theta(t) \right) \quad (49)$$

where $A(t)$ and $\theta(t)$ are slowly varying in time. Under a slowly varying carrier approximation, Eq. (49) may be written in the shorthand form

$$\phi(t) \approx A(t) \cos(\omega_{eq}(t)t + \theta(t)) \quad (50)$$

The corresponding total equivalent frequency of the fully assembled equivalent ship-roll system is defined as

$$\omega_{eq}^2(t) = \frac{k_{eq}(t)}{I_{eq}(t)} = \frac{k_{eq,SL}(t) + k_{eq,HB}(t)}{I_{eq,SL}(t)} \quad (51)$$

Eq. (51) is introduced as a descriptive property of the total equivalent system, namely as the instantaneous dynamic natural frequency associated with the equivalent linearized system of Eq. (45). It is therefore an a posteriori characterization of the combined SL and HB closure and should not be confused with the auxiliary carrier frequency to evaluate the HB contribution in the proposed numerical coupling scheme.

2.5. Covariance propagation via the time-varying Lyapunov equation

To compute the non-stationary response statistics under the evolutionary sea-wave excitation of Eqs. (7), and (13) to (15), the equivalent ship-roll model of Eq. (45) is cast into an augmented state-space form by incorporating the calibrated fourth-order shaping filter of Section 2.2. In the present formulation, the stationary colored Gaussian excitation process is generated by the fourth-order filter state $\mathbf{x}_f(t) = [x_3(t) \ x_4(t) \ x_5(t) \ x_6(t)]^T$, while the prescribed non-stationarity is introduced through the deterministic modulation function $g(t)$ according to Eqs. (16) and (17) as

$$w(t) = g(t) \mathbf{C}_4 \mathbf{x}_f(t) \quad (52)$$

This construction yields a finite-dimensional Markov representation [11] and enables propagation of the time-dependent covariances through a Lyapunov differential equation (e.g., [13,43,44]). The augmented 6-state vector is defined as

$$\mathbf{x}(t) = [\phi(t) \ \dot{\phi}(t) \ x_3(t) \ x_4(t) \ x_5(t) \ x_6(t)]^T \quad (53)$$

so that the coupled roll-filter dynamics can be written in compact form as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}\xi(t), \quad (54)$$

The time-varying drift matrix is

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & \mathbf{0}_{1 \times 4} \\ -\frac{k_{eq}(t)}{I_{eq}(t)} & -\frac{c_{eq}(t)}{I_{eq}(t)} & \frac{g(t)}{I_{eq}(t)} \mathbf{C}_4 \\ \mathbf{0}_{4 \times 1} & \mathbf{0}_{4 \times 1} & \mathbf{A}_4 \end{bmatrix} \quad (55)$$

while the diffusion vector is

$$\mathbf{B} = [0 \ 0 \ \mathbf{B}_4]^T \quad (56)$$

with $\mathbf{A}_4$, $\mathbf{B}_4$, and $\mathbf{C}_4$ defined in Eq. (19). Hence, time variation enters $\mathbf{A}(t)$ explicitly through the modulation function $g(t)$ and implicitly through the evolving equivalent coefficients $I_{eq}(t)$, $c_{eq}(t)$ and $k_{eq}(t)$. Let

$$\mathbf{V}(t) = \langle \mathbf{x}(t)\mathbf{x}^T(t) \rangle \quad (57)$$

denote the time-varying covariance matrix of the augmented state. Then, $\mathbf{V}(t)$ satisfies the time-evolving Lyapunov differential equation

$$\dot{\mathbf{V}}(t) = \mathbf{A}(t)\mathbf{V}(t) + \mathbf{V}(t)\mathbf{A}^T(t) + \mathbf{Q} \quad (58)$$

with diffusion matrix

$$\mathbf{Q} = 2\pi S_\xi \mathbf{B}\mathbf{B}^T = \mathbf{B}\mathbf{B}^T \quad (59)$$

In the present implementation, the white-noise intensity is normalized consistently with Section 2.2, so that the excitation level is absorbed into the fitted filter gain parameter γ_1 of Eq. (20). The non-stationary ship rolling response statistics follow directly as

$$\sigma_\phi^2(t) = V_{11}(t), \quad \sigma_{\phi\dot{\phi}}(t) = V_{12}(t), \quad \sigma_{\dot{\phi}}^2(t) = V_{22}(t) \quad (60)$$

whereas $\dot{\sigma}_{\phi\phi}(t)$ is obtained from the (1,2) entry of $\dot{\mathbf{V}}(t)$ computed through Eq. (58). Therefore, all time-dependent second-order moments required for evaluating the equivalent coefficients $I_{eq,SL}(t)$, $c_{eq,SL}(t)$ and $k_{eq,SL}(t)$ in Eqs. (28) to (30) are obtained directly from the solution of the 6-dimensional Lyapunov equation.

2.6. Iterative SL-HB mechanism and representative-carrier frequency update

Although the formal HB development of Section 2.4 motivates amplitude-dependent equivalent damping and stiffness contributions for the fractional term, the present numerical implementation adopts a representative-carrier closure in order to obtain a robust and computationally efficient non-stationary coupling with the SL and Lyapunov

stages. Specifically, the carrier frequency used in the HB update is taken from the SL backbone only, rather than from the fully assembled equivalent system. This avoids introducing an additional implicit coupling between the HB correction and the carrier employed to evaluate it. According to the proposed numerical stabilization strategy, the SL carrier frequency is defined as

$$\omega_c(t) \equiv \omega_{eq,SL}(t) = \sqrt{\frac{k_{eq,SL}(t)}{I_{eq,SL}(t)}} \quad (61)$$

At outer iteration $j+1$, once the updated SL quantities $I_{eq,SL}^{(j+1)}(t)$, $c_{eq,SL}^{(j+1)}(t)$, and $k_{eq,SL}^{(j+1)}(t)$ have been obtained from the propagated moments, the carrier is updated according to

$$\omega_c^{(j+1)}(t) = \sqrt{\frac{k_{eq,SL}^{(j+1)}(t)}{I_{eq,SL}^{(j+1)}(t)}} \quad (62)$$

The corresponding HB fractional contributions at iteration j are then evaluated directly at the representative carrier frequency $\omega_c^{(j)}(t)$. In this sense, the resulting formulation may be viewed as a practical bridge between the local HB characterization of the fractional contribution and the ensemble-based, time-varying equivalent linear system employed for covariance propagation

$$c_{eq,HB}^{(j)}(t) = \int_0^\infty \varepsilon_2 b_\alpha (\omega_c^{(j)}(A))^{\alpha-1} \sin\left(\frac{\alpha\pi}{2}\right) p(A, t) dA \quad (63)$$

$$\approx \varepsilon_2 b_\alpha (\omega_c^{(j)}(t))^{\alpha-1} \sin\left(\frac{\alpha\pi}{2}\right)$$

$$k_{eq,HB}^{(j)}(t) = \int_0^\infty \varepsilon_2 b_\alpha (\omega_c^{(j)}(A))^\alpha \cos\left(\frac{\alpha\pi}{2}\right) p(A, t) dA \quad (64)$$

$$\approx \varepsilon_2 b_\alpha (\omega_c^{(j)}(t))^\alpha \cos\left(\frac{\alpha\pi}{2}\right)$$

Eqs. (63) and (64) distinguish between the formal HB ensemble-average definitions and the representative-carrier closure adopted in the numerical implementation. In this sense, the resulting coefficients should be interpreted as moment-closed, ensemble-informed approximations of the corresponding amplitude-averaged HB contributions; see also [38]. For a given outer iteration j , the total equivalent coefficients are assembled as

$$I_{eq}^{(j)}(t) = I_{eq,SL}^{(j)}(t), \quad c_{eq}^{(j)}(t) = c_{eq,SL}^{(j)}(t) + c_{eq,HB}^{(j)}(t)$$

$$k_{eq}^{(j)}(t) = k_{eq,SL}^{(j)}(t) + k_{eq,HB}^{(j)}(t) \quad (65)$$

Here, the superscript (j) denotes the outer fixed-point iteration of the coupled SL-HB procedure over the full time interval, whereas the variable t denotes physical time. In the proposed non-stationary framework, the time dependence and the physical effects are captured by the evolving equivalent coefficients $I_{eq}(t)$, $c_{eq}(t)$ and $k_{eq}(t)$, obtained from the coupled statistical linearization and harmonic balancing procedure using Lyapunov-propagated second-order moments. The iterative mechanism proceeds over the full time grid as follows

- **Initialization:** Set the initial statistically linearized equivalent elements as

$$I_{eq,SL}^{(0)}(t) = 1, \quad c_{eq,SL}^{(0)}(t) = 0, \quad k_{eq,SL}^{(0)}(t) = \omega_\phi^2, \quad \forall t$$

and initialize the representative carrier frequency according to

$$\omega_c^{(0)}(t) = \omega_\phi$$

- **Initial HB evaluation:** Using the initial carrier frequency $\omega_c^{(0)}(t)$, evaluate the fractional HB equivalent contributions $c_{eq,HB}^{(0)}(t)$ and $k_{eq,HB}^{(0)}(t)$ from Eqs. (63) and (64).
- **Formation of total equivalent coefficients:** For iteration $j = 0, 1, \dots$, assemble the total equivalent elements $I_{eq}^{(j)}(t)$, $c_{eq}^{(j)}(t)$, and $k_{eq}^{(j)}(t)$ according to Eq. (65).

- **Lyapunov propagation (LTVE):** Integrate the time-varying covariance Lyapunov equation associated with the current equivalent system in order to obtain the non-stationary second-order moments, such as $\sigma_\phi^{2,(j)}(t)$, $\sigma_\phi^{(j)}(t)$, and $\sigma_\phi^{2,(j)}(t)$.
- **Update of SL equivalent elements:** Update the statistically linearized equivalent quantities $I_{eq,SL}^{(j+1)}(t)$, $c_{eq,SL}^{(j+1)}(t)$, and $k_{eq,SL}^{(j+1)}(t)$ from the propagated Lyapunov moments through Eqs. (28)–(30).
- **Update of carrier frequency:** Update the carrier frequency $\omega_c^{(j+1)}(t)$ according to Eq. (62).
- **Update of HB fractional contributions:** Using the updated carrier frequency $\omega_c^{(j+1)}(t)$, evaluate $c_{eq,HB}^{(j+1)}(t)$ and $k_{eq,HB}^{(j+1)}(t)$ from Eqs. (63) and (64).
- **Convergence check:** Repeat the above steps until convergence is achieved, namely until the equivalent coefficients, or equivalently the propagated response moments, exhibit negligible variation between two successive outer iterations.

In this way, the non-stationary response statistics propagated by the Lyapunov equation define the evolving SL backbone, and this backbone, in turn, determines the representative carrier used to parameterize the fractional-memory contribution. As a result, the HB contribution associated with the fractional term remains time-dependent through the evolving response statistics, without requiring repeated evaluation of the amplitude-PDF integrals of the formal HB theory at each outer iteration. This provides an efficient and practically robust closure that remains fully consistent with the implemented algorithm.

The quantity $\omega_{eq}(t)$ defined in Eq. (51) is retained as the effective natural frequency of the converged total equivalent oscillator of Eq. (45). It is thus used for interpretation of the final equivalent system, whereas $\omega_c(t)$ is the auxiliary carrier used in the HB update itself. In the present methodology, the representative carrier frequency is updated solely from the statistically linearized backbone, thereby avoiding the additional implicit coupling that would occur if the HB correction contributed directly to the definition of the carrier used for its own evaluation.

2.7. Discussion on attributes of the proposed nonlinear fractional-order stochastic roll dynamics method

The proposed methodology integrates statistical linearization (SL), harmonic balancing (HB), and Lyapunov-based moment propagation into a unified non-stationary framework for ship rolling. Its primary merit lies in its ability to retain, within a computationally efficient equivalent-linear setting, the principal physical mechanisms governing realistic roll response: state-dependent nonlinear inertia, cubic-quintic restoring effects, nonlinear viscous dissipation, and hereditary fractional-order hydrodynamic damping under evolutionary sea-wave excitation.

A defining attribute of the method is that non-stationarity is introduced at two coupled levels. First, the sea-wave excitation is rendered non-stationary via the deterministic modulation function $g(t)$ modulating the filtered stochastic process. Second, the equivalent coefficients $I_{eq}(t)$, $c_{eq}(t)$, and $k_{eq}(t)$ evolve concurrently with the response moments derived from the time-varying Lyapunov equation. Consequently, the framework does not merely describe a transient approach to a steady state, but rather a genuinely time-varying response where the nonlinear equivalent physics of the system adapt to the evolving energy of the sea.

The present formulation may be viewed as complementary to the earlier fractional ship-roll dynamics framework of [10], while addressing a higher-complexity stochastic modeling setting. Specifically, [10] belongs to the stochastic-averaging/Fokker–Planck amplitude-envelope class of methods, whereas the present contribution belongs to the augmented-state, moment-based SL-HB-Lyapunov class. The present work follows an efficient augmented-state moment-propagation route suited to a richer dynamical model: the sea wave-induced roll excitation is embedded through a calibrated fourth-order shaping filter,

and the resulting filter states are combined with the roll variables into a six-dimensional Markovian system. In addition, the local-in-time nonlinearities associated with added inertia, cubic-quintic restoring, and quadratic viscous damping are treated through an extended statistical-linearization treatment, yielding time-dependent equivalent inertia, damping, and stiffness coefficients, whereas the nonlocal-in-time fractional hereditary damping is incorporated through harmonic-balancing-based equivalent damping and stiffness contributions. The response statistics are then propagated directly through a time-varying covariance Lyapunov equation. Overall, the proposed SL-HB-Lyapunov formulation extends the range of tractable stochastic roll models by combining extended nonlinear roll effects, colored excitation embedding, hereditary fractional damping, evolutionary response spectra, and roll-amplitude PDF estimates within a unified time-varying augmented state-space setting.

The nonlinear structure of the adopted ship-roll model also distinguishes the present formulation from standard reduced-order idealizations. While many stochastic roll models rely on Duffing-type descriptions in which the restoring nonlinearity is represented mainly through a cubic term (e.g., [2,11,22,23,45]), the present formulation incorporates both amplitude-dependent inertia and a cubic-quintic restoring representation. The latter is consistent with established nonlinear roll-modeling practice, where cubic and quintic polynomial approximations of the righting-arm curve are commonly employed, while even higher-order odd-polynomial descriptions may be introduced when more detailed vessel-specific GZ fitting is required [46]. In this work, the fifth-order representation is adopted as a compact and physically meaningful benchmark model, since it incorporates the initial metacentric stiffness, the vanishing stability angle, and the area under the righting-arm curve, while retaining analytical and computational tractability. Thus, the model can capture richer softening-hardening restoring trends together with state-dependent added-inertia effects that become relevant in large-amplitude rolling. For vessel-specific applications, the same SL-HB-Lyapunov framework can be extended to higher-order restoring expansions, with the additional terms entering the equivalent stiffness through the corresponding Gaussian moments.

Furthermore, the treatment of the fractional-order term provides a computationally efficient bridge between frequency-domain and time-domain representations. The HB step provides a localized representation of the fractional operator in terms of equivalent damping and stiffness, while the SL-Lyapunov stage supplies the ensemble-based statistics required for non-stationary closure. By employing a representative carrier frequency derived from the SL-backbone, the method circumvents the need for repeated evaluation of the amplitude-PDF integrals at every outer iteration, ensuring a robust coupling between the harmonic-balance description and the time-varying equivalent linear system. In other words, the adopted scheme provides a computationally efficient bridge between cycle-based and moment-based representations of fractional-memory effects.

The proposed excitation embedding is a critical element of the framework, enabling the transition from a stationary sea-state description to a non-stationary one while maintaining Markovian efficiency. The stationary target roll-excitation spectrum is represented through a calibrated fourth-order shaping filter driven by white noise, with the non-stationarity introduced via deterministic amplitude modulation. This construction yields a finite-dimensional Markov representation of the excitation and leads, after coupling with the roll subsystem, to a six-dimensional augmented state-space model. This representation reproduces the dominant part of the target narrow-band excitation while retaining a compact state-space form suitable for time-domain covariance propagation through the time-varying Lyapunov formulation (e.g., [43,47]).

A key element of the excitation modeling choice lies in the assumption of separable (uniformly modulated) non-stationarity. While ocean waves can theoretically exhibit non-separable evolutionary power spectra where the spectral peak shifts in frequency over time, the situation

in the maritime environment during extreme storm events is typically dominated by the evolution of wave intensity $g(t)$ rather than rapid shifts in the dominant wave period. Because the timescale of a ship's resonant roll response is significantly shorter than the duration required for a sea state's spectral shape to transform, the uniformly modulated model serves as an adequate and physically justifiable surrogate. This separable treatment allows the framework to leverage the Markovian efficiency. Furthermore, the adoption of a stationary shaping filter avoids the significant modeling uncertainty inherent in linear time-varying filters. Implementing a non-separable model would require the identification of time-varying filter coefficients $A_4(t)$ and $B_4(t)$ to match an evolving spectral kernel, a complex inverse problem that is often poorly constrained by available maritime data. By focusing the non-stationarity on the amplitude modulation, the proposed framework reconciles the need for evolutionary energy intensity with the computational robustness of a constant coefficient excitation filter, effectively shielding the analysis from the curse of dimensionality and the parametric uncertainties associated with non-separable evolutionary kernels.

The methodology also relies on a narrow-band response interpretation. As the excitation is modeled through a narrow-band evolutionary spectrum and the roll response is expected to be dominated by a single, slowly-varying equivalent frequency, the response envelope is approximated via a Rayleigh amplitude PDF. This approximation remains consistent with the equivalent oscillator interpretation and is expected to maintain fidelity as long as the dominant roll component remains the primary energy carrier of the system. Finally, the fractional operator introduces history dependence and frequency-dependent phase lag in a low-parameter, economical manner. It serves as a reduced-order alternative to both high-order integer approximations and traditional Cummins-type impulse response functions, where radiation effects require computationally expensive convolution integrals. Within the range $0 < \alpha < 1$, the fractional order α acts as a memory tuner: smaller values of α intensify the hereditary character of the fluid-hull interaction, while values approaching unity recover local-in-time behavior.

The accuracy of the proposed semi-analytical framework should be interpreted in light of its approximation assumptions. The statistical linearization step relies on a locally Gaussian equivalent response, the harmonic-balancing treatment of the fractional derivative assumes a locally narrow-band pseudo-harmonic motion, and the roll-amplitude PDF is reconstructed through an evolutionary Rayleigh envelope approximation. Therefore, the method is expected to be most reliable for non-capsizing responses that remain predominantly narrow-band and lightly to moderately nonlinear under smoothly varying non-stationary excitation. Qualitatively, the accuracy is expected to be most sensitive to the strength of the restoring, inertia, and damping nonlinearities, the overall dissipation level, and the rate and bandwidth of the non-stationary sea-wave excitation. Larger discrepancies may arise for strongly broadband or multi-modal responses, abrupt excitation modulation, extreme nonlinear regimes close to capsizing, or strongly non-Gaussian tail-dominated response statistics. In particular, the fractional order affects the phase lag and frequency dependence of the hereditary damping term and, consequently, the time-dependent equivalent damping and stiffness contributions entering the covariance propagation. These limitations are consistent with the use of time-local equivalent coefficients and with the pseudo-harmonic assumption underlying the SL-HB approximation.

From a broader stochastic dynamics perspective, the proposed methodology may be viewed as a hybrid treatment that reconciles the physical consistency of fractional-order hereditary dynamics with the computational efficiency of moment-based stochastic analysis. Unlike Fokker-Planck approaches, which aim to resolve the full probability density function often at considerable numerical cost, or Path Integral techniques, which propagate the response through discrete transition kernels, the present framework evolves only the response

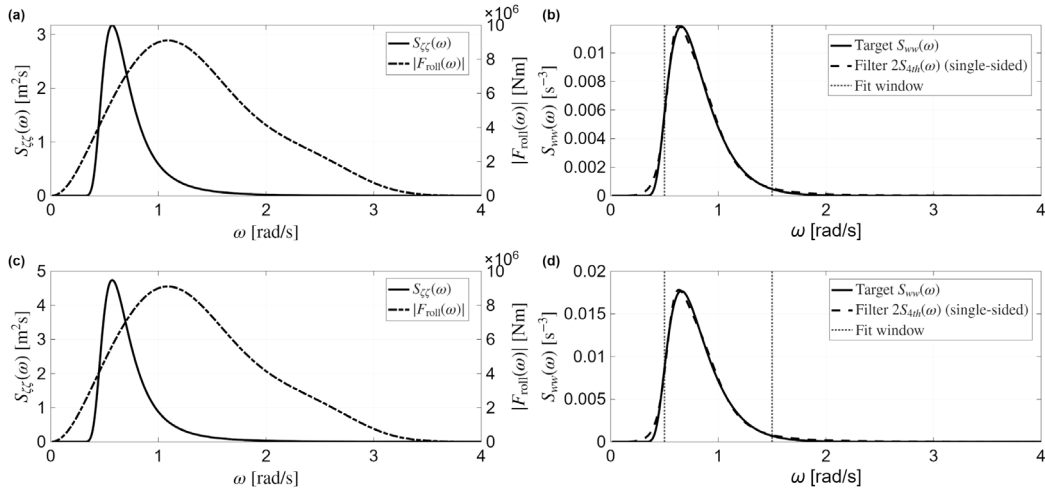


Fig. 2. Wave-energy spectrum $S_{\zeta\zeta}(\omega)$, roll-excitation moment per unit wave height $|F_{\text{roll}}(\omega)|$, and the single-sided fourth-order shaping-filter approximation $2S_{4\text{th}}(\omega)$ to the target normalized excitation spectrum $S_{ww}(\omega)$. Panels (a) and (b) correspond to a significant wave height of $H_s = 4.5$ m and peak period $T_p = 11.0$ s, while panels (c) and (d) correspond to $H_s = 5.5$ m and $T_p = 11.0$ s.

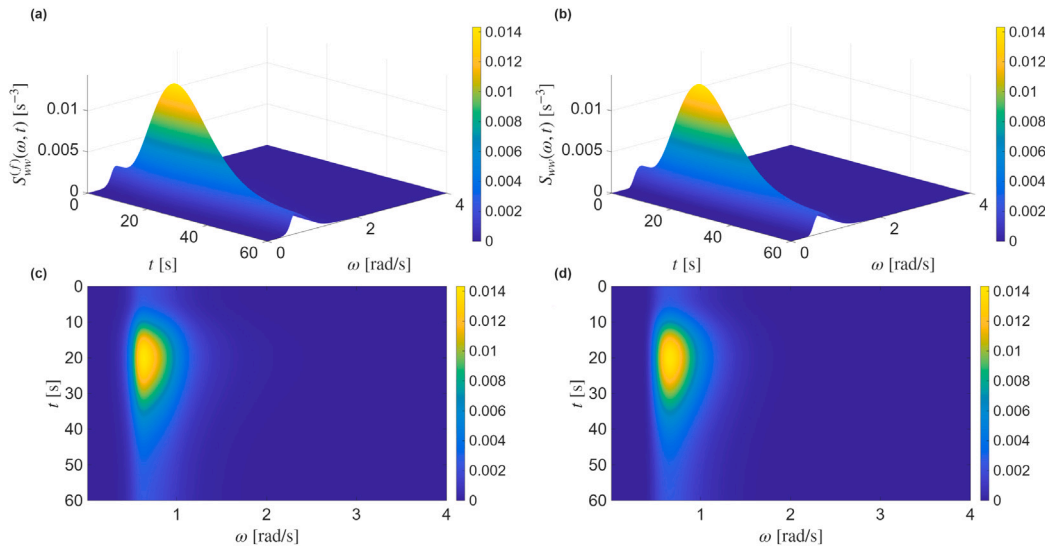


Fig. 3. Non-stationary excitation spectra for the sea state defined by $H_s = 4.5$ m and $T_p = 11.0$ s. Panels (a) and (c) show the fourth-order shaping-filter representation $S_{ww}^{(f)}(\omega, t) = |g(t)|^2 2S_{4\text{th}}(\omega)$, while panels (b) and (d) show the target normalized excitation spectrum $S_{ww}(\omega, t)$. Panels (a) and (b) are presented in three-dimensional view, whereas panels (c) and (d) give the corresponding top-view representations.

moments through the time-varying Lyapunov equation. Its methodological position is therefore distinct: the Markovian structure required for Lyapunov-based propagation is preserved through the use of shaping filters for colored excitation and harmonic-balancing localization for the fractional-memory contribution. This enables the treatment of non-stationary ship-roll response with a level of efficiency that is generally inaccessible to standard Monte Carlo simulations, while still retaining the essential effects of excitation spectral color and fluid-hull memory. Accordingly, the formulation provides a tractable basis for non-stationary ship-roll response prediction, seakeeping assessment, and reliability-oriented analyses in settings where repeated nonlinear Monte Carlo simulations may be computationally demanding.

3. Numerical application

This section provides numerical results illustrating the performance of the proposed fractional-order framework for nonlinear stochastic ship rolling, following the developments of Sections 2.1 to 2.6. Parametric variations in the significant wave height H_s , the peak period

T_p , and the fractional derivative order α are examined, while the weighting factors are chosen to reflect practically relevant nonlinear roll-dynamics characteristics. The derived results are validated against pertinent Monte Carlo simulation (MCS) data in order to assess the accuracy, robustness, and computational efficiency of the proposed methodology. The reference ship is characterized by vanishing stability angle $\phi_v = 0.96$ rad, area under the righting-arm curve up to this angle $A_{\phi_v} = 0.67$ m rad, metacentric height $GM = 3$ m, roll moment of inertia $I_{xx} = 1 \times 10^8$ kg m², added inertia $\delta I_{xx} = 0.2I_{xx}$, displacement $\Delta = 4 \times 10^7$ N, generalized fractional damping coefficient $B_L = 1.3 \times 10^7$ N m s ^{α} , and nonlinear quadratic damping constant $B_N = 1.3 \times 10^7$ N m s². The restoring arm is represented by a fifth-order polynomial in the roll angle, with coefficients chosen in line with realistic ship GZ characteristics. The excitation is specified through the significant wave height H_s and the peak period T_p of the underlying Pierson-Moskowitz spectrum, together with the envelope parameters $g_0 = 0.2$, $g_1 = 1.0$, $a_t = 20$, and $b_t = 5$, which control the temporal build-up and decay of sea severity; Eqs. (7), and (13) to (15). For all cases, the performance of the proposed stochastic dynamics fractional-order methodology is assessed

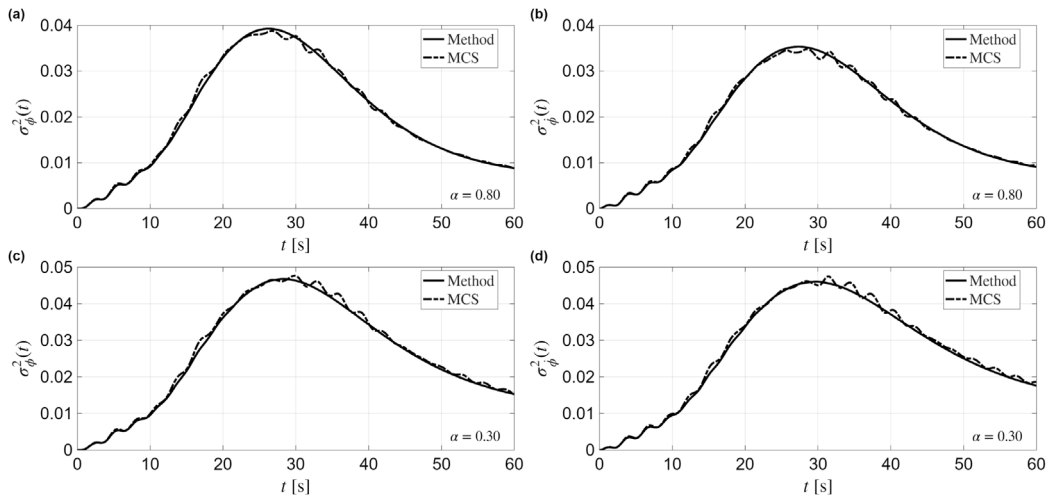


Fig. 4. Comparison of the time-dependent roll response variances obtained from the proposed method and Monte Carlo simulation for the sea state defined by $H_s = 4.5$ m and $T_p = 11.0$ s. Panels (a) and (b) correspond to the case $\alpha = 0.80$, showing the roll-angle variance $\sigma_\phi^2(t)$ and the roll-rate variance $\sigma_{\dot{\phi}}^2(t)$, respectively. Panels (c) and (d) correspond to the case $\alpha = 0.30$, showing the corresponding roll-angle and roll-rate variances.

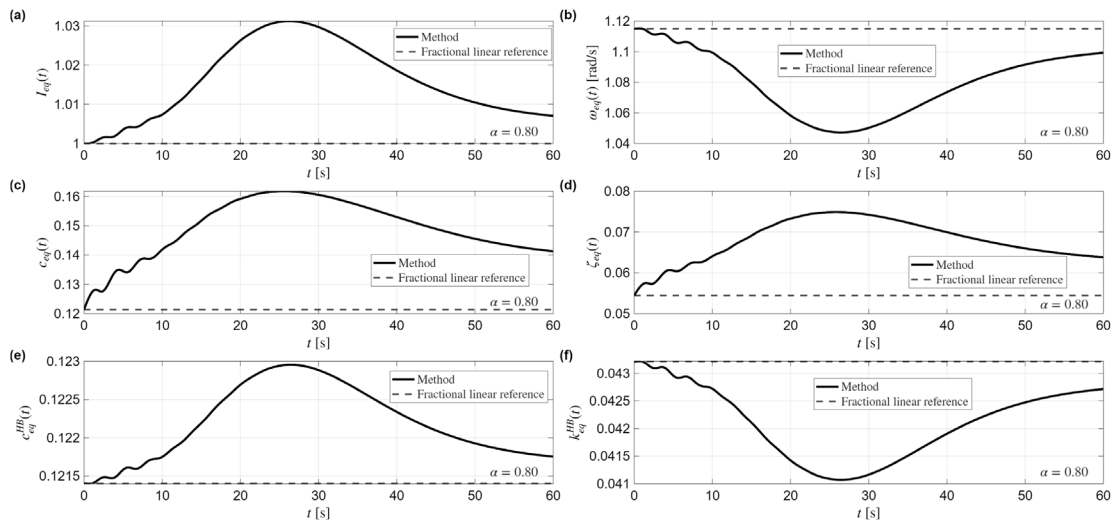


Fig. 5. Time evolution of the equivalent coefficients and fractional harmonic-balance contributions for the case $H_s = 4.5$ m, $T_p = 11.0$ s, and $\alpha = 0.80$. Panels (a) and (b) show the equivalent inertia $I_{eq}(t)$ and equivalent natural frequency $\omega_{eq}(t)$, respectively. Panels (c) and (d) present the equivalent damping coefficient $c_{eq}(t)$ and equivalent damping ratio $\zeta_{eq}(t)$, while panels (e) and (f) show the HB fractional contributions to the equivalent damping and stiffness, namely $c_{eq}^{HB}(t)$ and $k_{eq}^{HB}(t)$. In all panels, the method prediction is compared with the corresponding fractional linear reference.

by direct comparison with MCS results based on 5000 realizations (e.g., [4,48]). More precisely, the spectral representation method [49] is employed to generate the ensemble of excitation realizations compatible with the prescribed normalized excitation spectrum, while the corresponding nonlinear fractional-order roll response is obtained by numerical integration of the governing Eq. (8) by resorting to an L1-algorithm (e.g., [50]). In the present study, every ship configuration is subjected to two sea states, namely $H_s = 4.5$ m and $H_s = 5.5$ m, with $T_p = 11.0$ s. For completeness and reproducibility, the calibrated parameters of the fourth-order shaping filter introduced in Section 2.2 are reported in Table 1. These values were identified by fitting the single-sided filter spectrum to the target normalized excitation spectrum over the prescribed frequency window, while enforcing positivity of the filter parameters and asymptotic stability of the corresponding state-space realization. Since the peak period T_p and the roll-excitation transfer function $F_{roll}(\omega)$ remain fixed in the two considered sea states, whereas H_s primarily affects the excitation level, the identified shape parameters are expected to remain close across the two cases, with the gain parameter γ_1 accounting for most of the amplitude adjustment.

Table 1

Parameter values of the calibrated fourth-order shaping filter for the considered sea states.

Sea states		Fourth-order filter								
H_s (m)	T_p (s)	α_1	α_2	β_1	β_2	γ_1	λ_1	λ_2	λ_3	λ_4
4.5	11.0	0.274	0.643	0.323	0.901	0.059	0.917	1.400	0.455	0.291
5.5	11.0	0.643	0.274	0.901	0.323	0.072	0.917	1.400	0.455	0.291

In the first set of results (Ship Configuration A), the weighting factors are chosen as $\varepsilon_1 = 4$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 1.0$, $\varepsilon_4 = 0.5$, and $\varepsilon_5 = 0.5$, while the fractional derivative order is taken as $\alpha = 0.80$. This choice yields a roll model in which the cubic softening contribution reasonably dominates the restoring behavior, whereas the quintic term only mildly modifies the restoring curve at larger angles. In the second set of results (Ship Configuration B), the weighting factors are retained at the same values, whereas the fractional derivative order is reduced to $\alpha = 0.30$. In this way, the role of memory can be investigated systematically

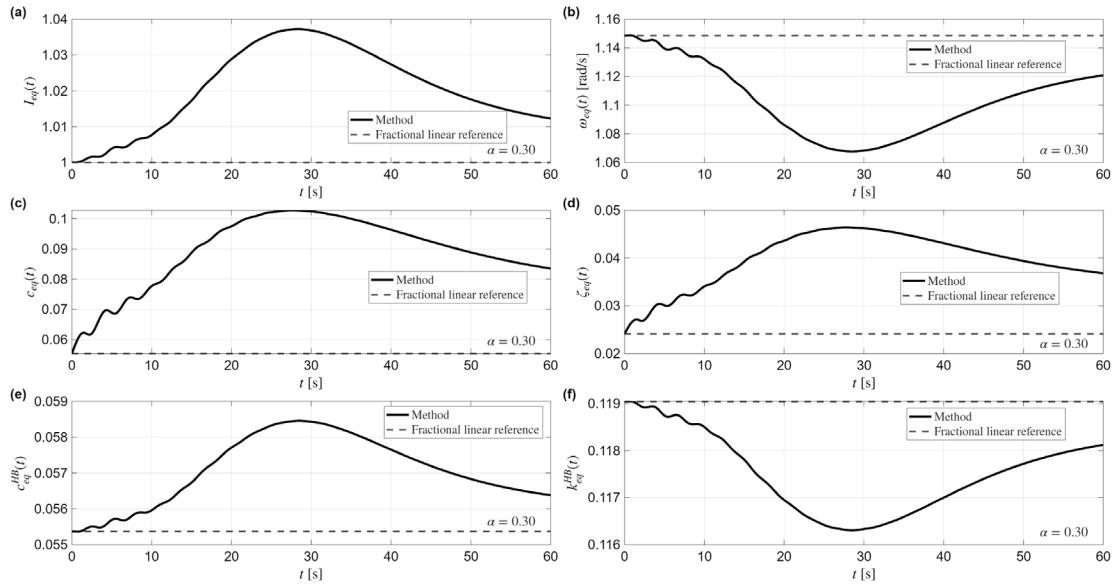


Fig. 6. Time evolution of the equivalent coefficients and fractional harmonic-balance contributions for the case $H_s = 4.5$ m, $T_p = 11.0$ s, and $\alpha = 0.30$. Panels (a) and (b) show the equivalent inertia $I_{eq}(t)$ and equivalent natural frequency $\omega_{eq}(t)$, respectively. Panels (c) and (d) present the equivalent damping coefficient $c_{eq}(t)$ and equivalent damping ratio $\zeta_{eq}(t)$, while panels (e) and (f) show the HB fractional contributions to the equivalent damping and stiffness, namely $c_{eq}^{HB}(t)$ and $k_{eq}^{HB}(t)$. In all panels, the method prediction is compared with the corresponding fractional linear reference.

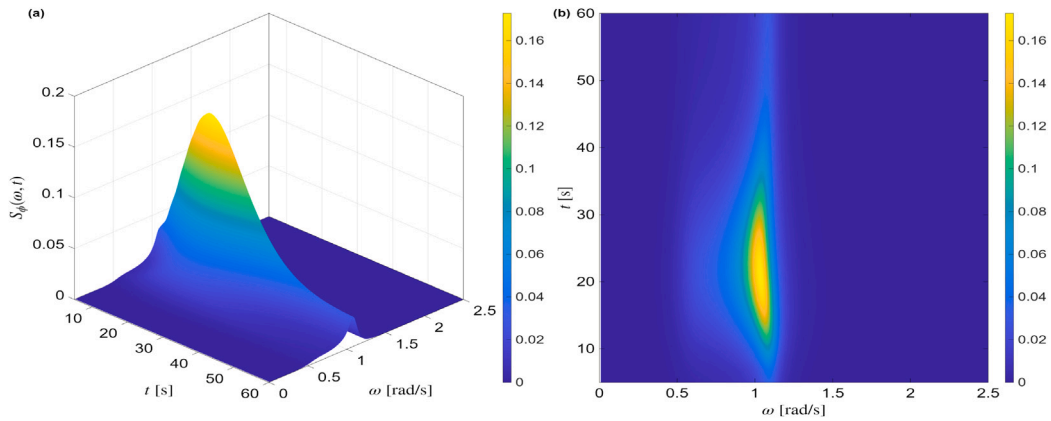


Fig. 7. Evolutionary power spectrum (EPS) of the roll response predicted by the proposed method. Panel (a) shows the three-dimensional representation of $S_\phi(\omega, t)$, while panel (b) presents the corresponding top-view representation; case $H_s = 4.5$ m, $T_p = 11.0$ s, and $\alpha = 0.80$.

through the variation of α , with smaller values corresponding to a more pronounced hereditary character of the fractional damping operator.

Regarding the quality of the excitation embedding, Fig. 2 presents, for both considered sea states, the wave-energy spectrum $S_{\zeta\zeta}(\omega)$, the roll-excitation moment per unit wave height $|F_{roll}(\omega)|$, and the corresponding fourth-order shaping-filter approximation of the target normalized excitation spectrum $S_{uw}(\omega)$. In both cases, the agreement between the target spectrum and its filter-based approximation seems to be very good over the dominant energetic range. It demonstrates that the reduced-order Markovian excitation embedding preserves with high fidelity the part of the spectrum that is most relevant for the roll response. In this sense, the calibrated fourth-order filter may be viewed as a compact spectral surrogate of the original excitation environment, sufficiently reliable for time-domain covariance propagation while remaining computationally tractable.

The corresponding non-stationary excitation representation for the case $H_s = 4.5$ m is shown in Fig. 3, where the filter-generated evolutionary spectrum is compared with the target non-stationary excitation spectrum of Eq. (14). The comparison confirms that the good quality

of the stationary fit is preserved under deterministic amplitude modulation. Both the three-dimensional and top-view representations indicate that the filtered excitation reproduces reliably the temporal build-up, the concentration of energy in the dominant frequency band, and the subsequent decay of the target non-stationary spectrum. Hence, the adopted modulation strategy does not compromise the effective spectral representation achieved at the stationary calibration stage.

The response level validation is presented in Fig. 4, which compares the time-dependent roll-angle and roll-rate variances obtained from the proposed method and from MCS for the lower sea state, $H_s = 4.5$ m. In both fractional-order cases, namely $\alpha = 0.80$ and $\alpha = 0.30$, the agreement is very good over the full time interval. The proposed method reproduces both the transient build-up and the subsequent decay of the response variances with high accuracy, including the timing and level of the dominant peaks. This provides strong evidence that the equivalent-linear stochastic closure captures the essential second-order response statistics with good fidelity, despite the simultaneous presence of nonlinear restoring terms, amplitude-dependent inertia, quadratic local damping, and hereditary fractional damping. The response is assumed to start from quiescent initial conditions at $t = 0$. Accordingly,

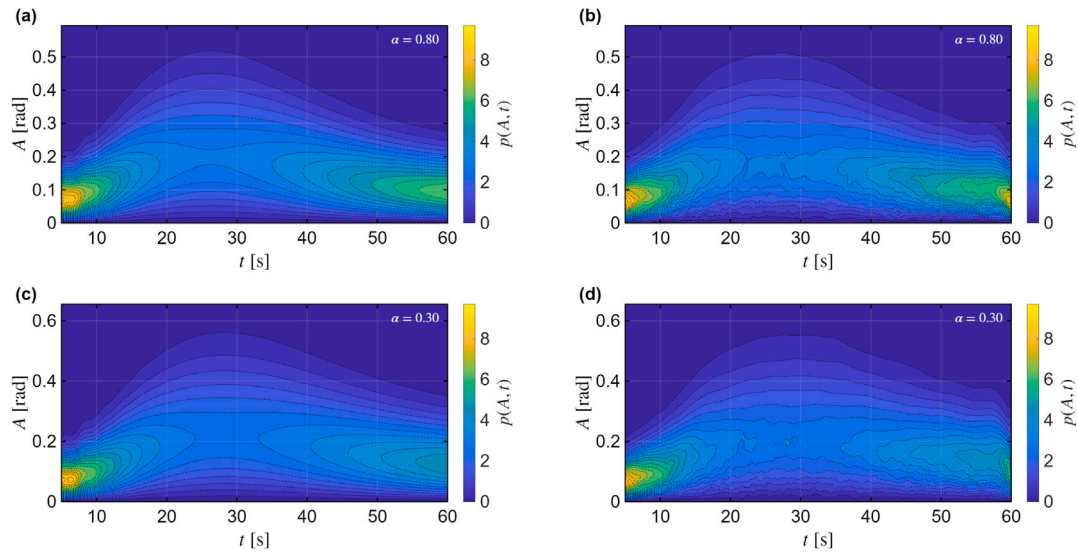


Fig. 8. Time-dependent response amplitude probability density functions $p(A, t)$ obtained using the proposed stochastic dynamics method and Monte Carlo simulation (MCS) based on 5000 realizations for the sea state defined by $H_s = 4.5$ m and $T_p = 11.0$ s. Panel (a): method, $\alpha = 0.80$; panel (b): MCS, $\alpha = 0.80$; panel (c): method, $\alpha = 0.30$; and panel (d): MCS, $\alpha = 0.30$.

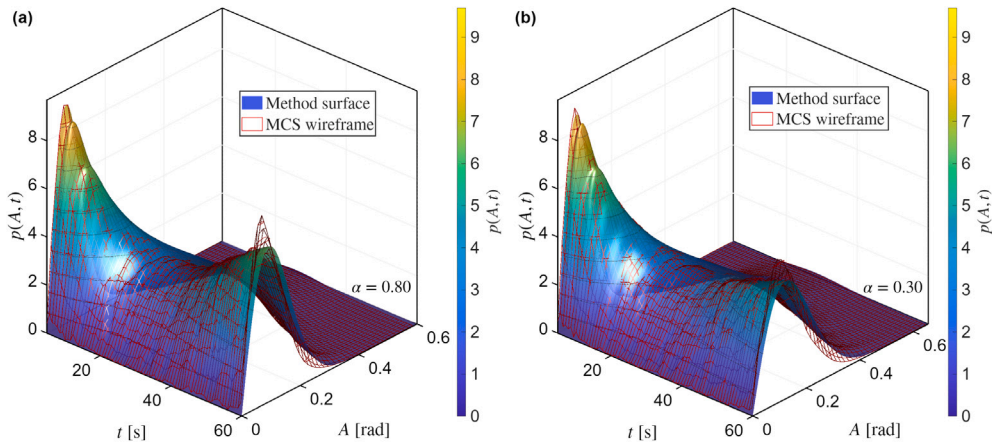


Fig. 9. Overlaid roll-amplitude PDF evolution obtained from the proposed stochastic dynamics method and benchmark Monte Carlo simulations. The colored surface denotes the proposed method, while the red wireframe denotes the MCS-based PDF estimates. Panels (a) and (b) correspond to fractional orders $\alpha = 0.80$ and $\alpha = 0.30$ for the sea state $H_s = 4.5$ m and $T_p = 11.0$ s.

although the excitation envelope possesses a nonzero baseline level, the response variances exhibit an initial transient build-up from zero as energy accumulates in the ship rolling system.

At the same time, Fig. 4 already reveals the effect of the fractional order on the response evolution. When the order is reduced to $\alpha = 0.30$, both the roll-angle variance and the roll-rate variance remain at higher levels during the later stages of the response. Moreover, the corresponding peaks are shifted slightly toward later times. This behavior is consistent with a stronger memory effect: the lower-order fractional operator produces a stronger hereditary fractional effect, leading to a more persistent and temporally delayed response evolution. By contrast, for $\alpha = 0.80$, the dissipation mechanism behaves in a more local-in-time manner, with weaker hereditary influence, lower late-time variance levels, and a somewhat earlier peak response.

Further insight into the internal mechanics of the proposed framework is provided by Figs. 5 and 6, which show the time evolution of the equivalent coefficients and of the fractional harmonic-balance contributions for the same sea state $H_s = 4.5$ m, for $\alpha = 0.80$ and $\alpha = 0.30$, respectively. These results provide considerably more information than a simple comparison of response histories, since they effectively reveal the evolving equivalent oscillator identified by the methodology.

In this sense, the proposed framework exhibits a system-identification character: beyond predicting the non-stationary response statistics, it also identifies the time-dependent inertia, damping, stiffness, damping ratio, and hereditary HB contributions that govern the instantaneous equivalent dynamics.

In Figs. 5 and 6, the dashed horizontal levels correspond to the associated fractional-linear reference values, namely the coefficients of the linear fractional oscillator obtained by suppressing the nonlinear contributions while retaining the same fractional order and baseline hydrodynamic parameters. These reference values should therefore be interpreted as fractional-linear baselines for the selected value of α , rather than as asymptotic targets that the time-evolving equivalent elements must recover. Since the excitation remains non-zero over the considered time window and the response level stays appreciable, the nonlinear contributions remain active, and the equivalent coefficients are naturally expected to remain offset from their corresponding reference levels.

For the case $\alpha = 0.80$ shown in Fig. 5, the equivalent quantities indicate a moderately nonlinear response regime. The equivalent inertia remains close to unity, with only a limited increase above its linear reference level. The equivalent natural frequency undergoes a moderate

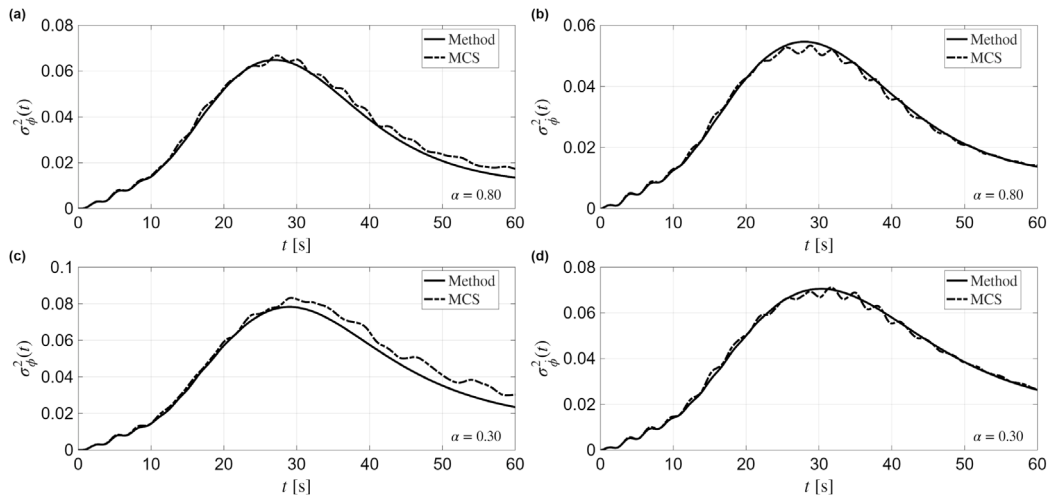


Fig. 10. Comparison of the time-dependent roll response variances obtained from the proposed method and Monte Carlo simulation for the sea state defined by $H_s = 5.5$ m and $T_p = 11.0$ s. Panels (a) and (b) correspond to the case $\alpha = 0.80$, showing the roll-angle variance $\sigma_\phi^2(t)$ and the roll-rate variance $\sigma_{\dot{\phi}}^2(t)$, respectively. Panels (c) and (d) correspond to the case $\alpha = 0.30$, showing the corresponding roll-angle and roll-rate variances.

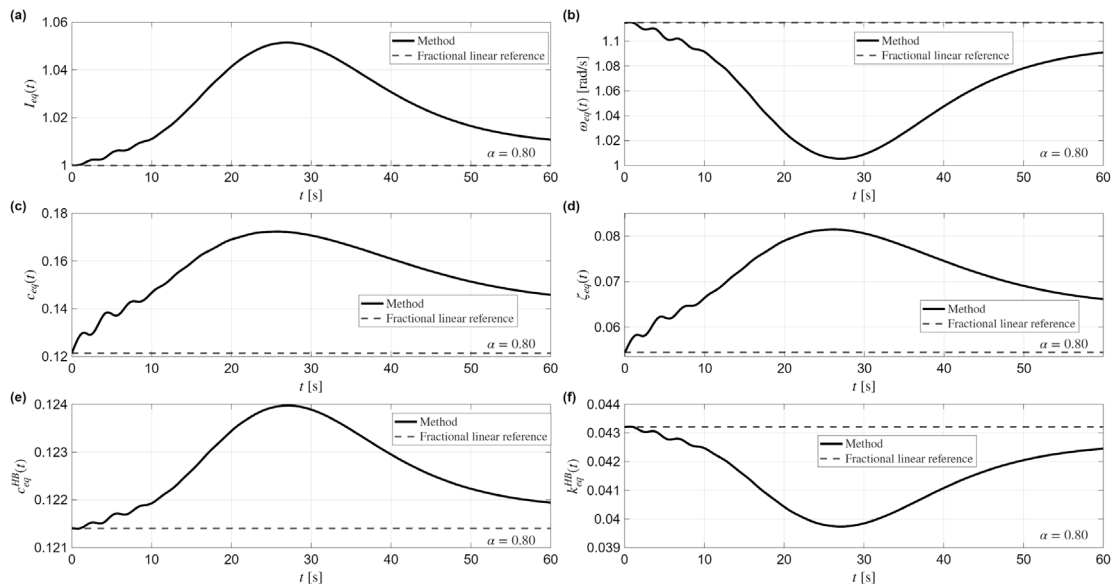


Fig. 11. Time evolution of the equivalent coefficients and fractional harmonic-balance contributions for the case $H_s = 5.5$ m, $T_p = 11.0$ s, and $\alpha = 0.80$. Panels (a) and (b) show the equivalent inertia $I_{eq}(t)$ and equivalent natural frequency $\omega_{eq}(t)$, respectively. Panels (c) and (d) present the equivalent damping coefficient $c_{eq}(t)$ and equivalent damping ratio $\zeta_{eq}(t)$, while panels (e) and (f) show the HB fractional contributions to the equivalent damping and stiffness, namely $c_{eq}^{HB}(t)$ and $k_{eq}^{HB}(t)$. In all panels, the method prediction is compared with the corresponding fractional linear reference.

transient reduction, while the equivalent damping coefficient increases above the corresponding fractional-linear baseline. These variations are smooth and moderate, showing that the nonlinear mechanisms are clearly activated but do not dominate the dynamics. This is fully consistent with the high agreement already observed between the proposed method and the MCS results in the variance and roll-rate variance predictions in Fig. 4. When the fractional order is reduced to $\alpha = 0.30$, as shown in Fig. 6, the same qualitative picture is preserved, but the relative balance between damping-like and stiffness-like hereditary effects changes. In particular, the equivalent damping becomes smaller, whereas the equivalent natural frequency becomes larger. This reflects the fact that a lower fractional order weakens the dissipative character of the fractional term while strengthening its stiffness-like contribution. The equivalent inertia also reaches a somewhat larger peak, which is consistent with the slightly stronger response attained in this case. Comparison of Figs. 5 and 6 further shows that the hereditary effect

is indeed visible, especially in the later parts of the histories, since for $\alpha = 0.30$ the equivalent coefficients remain more persistently shifted from their corresponding fractional-linear baselines, in line with the stronger memory effect already inferred from the variance histories.

The corresponding spectral content of the response is illustrated in Fig. 7, which shows the EPS predicted by the proposed method for the case $H_s = 4.5$ m, $T_p = 11.0$ s, and $\alpha = 0.80$. The response is seen to remain distinctly narrow-band, with its energy concentrated along a relatively compact time-evolving ridge. The motion of the dominant spectral ridge follows the evolution of the equivalent natural frequency $\omega_{eq}(t)$, thereby confirming that $\omega_{eq}(t)$ is not merely a formal closure quantity, but a meaningful tracker of the instantaneous dominant response band.

Figs. 8 and 9 report the time-dependent response roll amplitude probability-density functions $p(A, t)$ for the same sea state, comparing the method-based and MCS-based predictions for both $\alpha = 0.80$ and $\alpha = 0.30$.

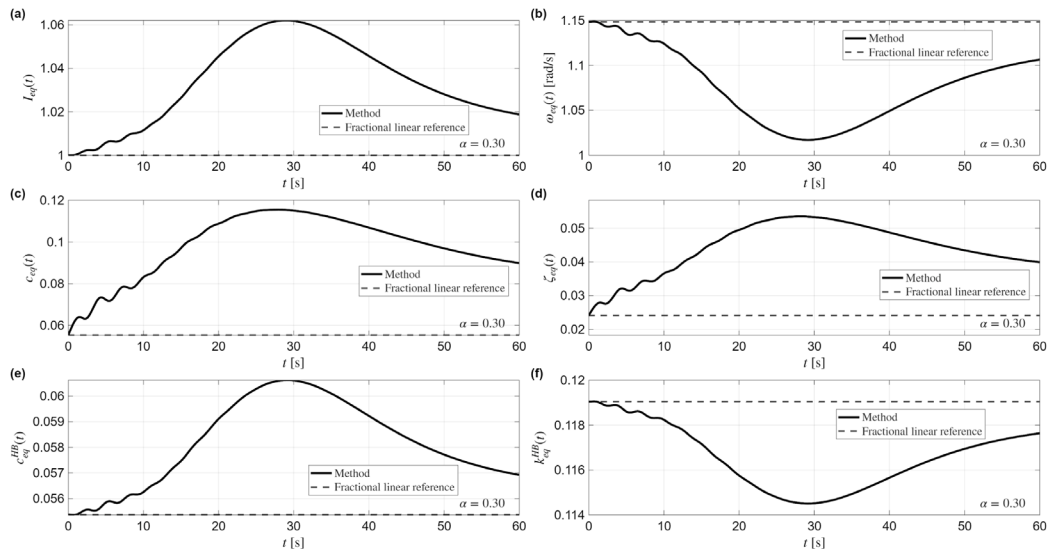


Fig. 12. Time evolution of the equivalent coefficients and fractional harmonic-balance contributions for the case $H_s = 5.5$ m, $T_p = 11.0$ s, and $\alpha = 0.30$. Panels (a) and (b) show the equivalent inertia $I_{eq}(t)$ and equivalent natural frequency $\omega_{eq}(t)$, respectively. Panels (c) and (d) present the equivalent damping coefficient $c_{eq}(t)$ and equivalent damping ratio $\zeta_{eq}(t)$, while panels (e) and (f) show the HB fractional contributions to the equivalent damping and stiffness, namely $c_{eq}^{HB}(t)$ and $k_{eq}^{HB}(t)$. In all panels, the method prediction is compared with the corresponding fractional linear reference.

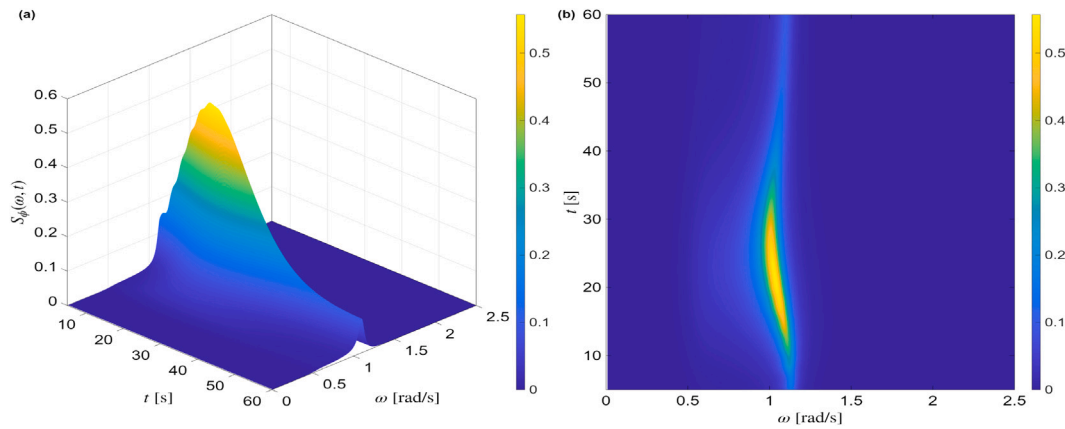


Fig. 13. Evolutionary power spectrum (EPS) of the roll response predicted by the proposed method. Panel (a) shows the three-dimensional representation of $S_\phi(\omega, t)$, while panel (b) presents the corresponding top-view representation; case $H_s = 5.5$ m, $T_p = 11.0$ s, and $\alpha = 0.30$.

The agreement is found to be very good, which is especially important because these quantities represent higher-level probabilistic descriptors derived from the second-order response statistics through the narrow-band envelope approximation. For $\alpha = 0.80$, the PDF surface remains more compact and decays more rapidly in time, whereas for $\alpha = 0.30$ it remains broader and more persistent toward the later stages of the response. Thus, the PDF results corroborate the same physical interpretation already suggested by the response variances and equivalent coefficients: decreasing the fractional order intensifies the hereditary character of the damping mechanism and leads to a more persistent response.

The second sea state corresponds to a stronger excitation level that of $H_s = 5.5$ m. Fig. 10 presents the time-dependent roll-angle and roll-rate variances for both $\alpha = 0.80$ and $\alpha = 0.30$. The agreement between the proposed method and the MCS remains satisfactory, even though the response levels are now higher than in Fig. 4. This demonstrates that the method preserves its predictive accuracy also under more energetic excitation conditions. As in the lower sea-state case, the smaller fractional order leads to higher late-time variances and a slightly delayed peak response, which again indicates the stronger hereditary effect associated with lower values of α .

Figs. 11 and 12 show the time evolution of the equivalent coefficients and the HB fractional contributions for $H_s = 5.5$ m and for the two fractional orders $\alpha = 0.80$ and $\alpha = 0.30$, respectively. The same qualitative trends already identified for the lower sea state are preserved, but the departures from the fractional-linear reference levels become more pronounced, reflecting the stronger excitation and the increased activation of the nonlinear mechanisms. For $\alpha = 0.80$, the equivalent coefficients still indicate a moderately nonlinear regime, with smooth and physically interpretable variations in time. For $\alpha = 0.30$, the equivalent damping is further reduced, the equivalent natural frequency is increased, and the equivalent inertia reaches a larger peak, consistently with the stronger stiffness-like and more weakly dissipative role of the fractional term at lower order. Overall, these results reinforce the interpretation that the examined cases correspond to moderately nonlinear fractional-response regimes for which the proposed methodology remains satisfactorily accurate.

Fig. 13 presents the EPS of the roll response for the higher sea state in the representative case $\alpha = 0.30$. As in Fig. 7, the response remains clearly narrow-band, with the dominant energy concentrated along a time-varying ridge. The stronger excitation increases the overall magnitude of the EPS, but the spectral organization remains fully

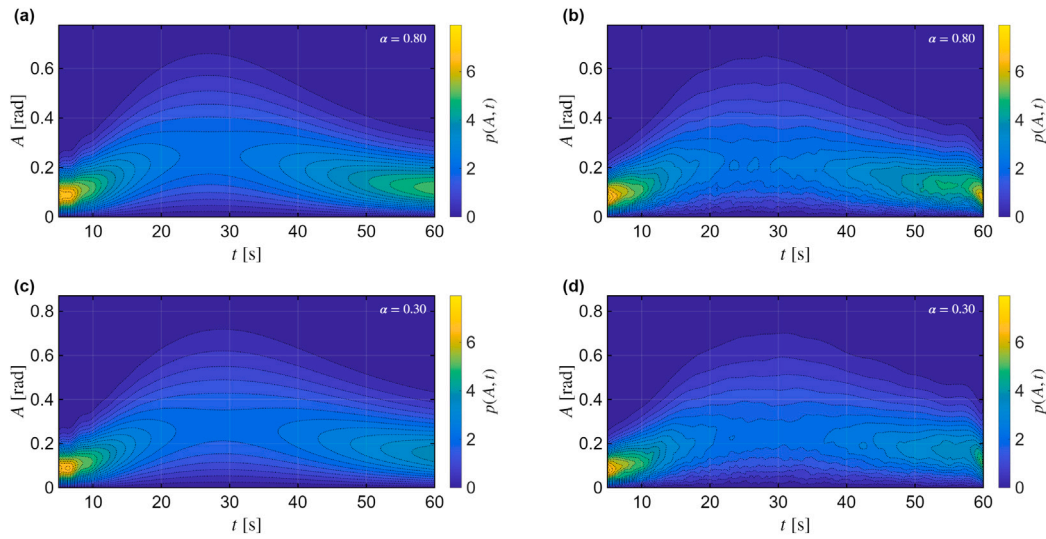


Fig. 14. Time-dependent response amplitude probability density functions $p(A, t)$ obtained using the proposed stochastic dynamics method and Monte Carlo simulation (MCS) based on 5000 realizations for the sea state defined by $H_s = 5.5$ m and $T_p = 11.0$ s. Panel (a): method, $\alpha = 0.80$; panel (b): MCS, $\alpha = 0.80$; panel (c): method, $\alpha = 0.30$; and panel (d): MCS, $\alpha = 0.30$.

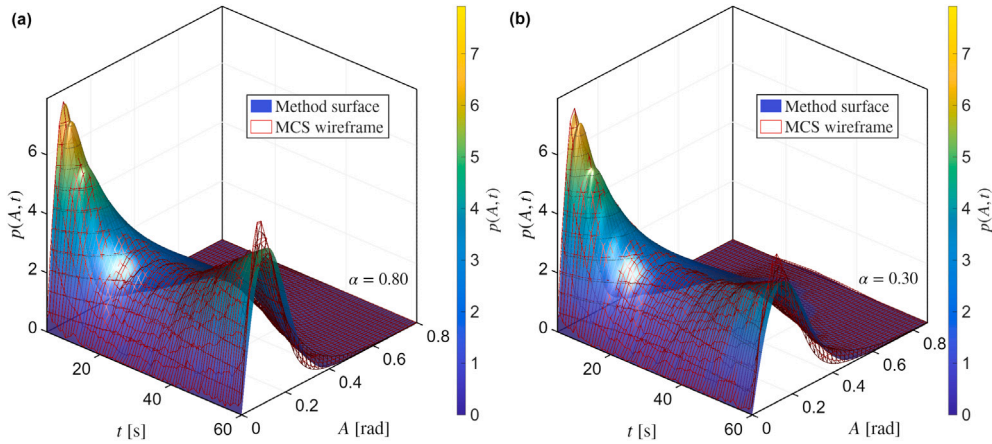


Fig. 15. Overlaid roll-amplitude PDF evolution obtained from the proposed stochastic dynamics method and benchmark Monte Carlo simulations. The colored surface denotes the proposed method, while the red wireframe denotes the MCS-based PDF estimates. Panels (a) and (b) correspond to fractional orders $\alpha = 0.80$ and $\alpha = 0.30$ for the sea state $H_s = 5.5$ m and $T_p = 11.0$ s.

compatible with the equivalent-oscillator interpretation adopted in the method. The time-varying ridge again follows the evolution of the corresponding equivalent natural frequency, which further supports the interpretation of $\omega_{eq}(t)$ as a reliable tracker of the evolving dominant response band.

Finally, Figs. 14 and 15 show the method-based and MCS-based amplitude PDF surfaces for the sea state $H_s = 5.5$ m. The agreement remains strong for both fractional orders, confirming that the proposed methodology provides reliable non-stationary amplitude statistics also under the more energetic wave environment. As in the lower sea state, the case $\alpha = 0.30$ produces broader and more persistent probability-density surfaces than the case $\alpha = 0.80$, once again reflecting the stronger hereditary character of the lower-order fractional damping operator. Figs. 10 to 15 show that the proposed framework remains robust as the excitation level increases, while preserving the same physically interpretable trends already identified for the lower sea state.

Overall, the numerical results show that the proposed methodology provides satisfactory accuracy while offering physical insight into the evolving roll dynamics. It reproduces the target excitation spectrum over the dominant frequency range, captures the non-stationary second-order response statistics and the corresponding amplitude

probability-density functions with good agreement relative to MCS, and at the same time provides information about the time-varying equivalent system through the evolution of $I_{eq}(t)$, $c_{eq}(t)$, $k_{eq}(t)$, $\zeta_{eq}(t)$, and the hereditary harmonic-balance contributions. The results therefore confirm both the efficiency and consistency of the proposed stochastic dynamics framework for nonlinear ship rolling under non-stationary sea-wave excitation.

4. Concluding remarks

In this paper, an efficient semi-analytical framework has been developed for the non-stationary stochastic dynamic analysis of nonlinear ship rolling under irregular sea-wave excitation. The proposed methodology combines a physically consistent fractional-order roll model with a compact Markovian excitation embedding and an equivalent-linear stochastic solution strategy based on statistical linearization, harmonic balancing, and a time-varying covariance Lyapunov equation. In this way, the framework incorporates within a unified non-stationary setting the principal mechanisms governing genuine roll response, namely amplitude-dependent added inertia nonlinearity, nonlinear restoring behavior, nonlinear quadratic viscous dissipation, nonlocal hereditary

hydrodynamic damping, and explicitly time-varying wave excitation. A key feature of the formulation is the statistical linearization and harmonic balancing coupling adopted for the fractional-memory contribution, together with the representative-carrier update from the statistically linearized backbone, which yields a numerically robust and computationally efficient iterative closure.

The numerical applications demonstrate that the proposed framework reproduces the target excitation spectrum with fidelity and provides reliable predictions of the non-stationary roll-angle and roll-rate variances, the response evolutionary power spectra, and the time-dependent roll-amplitude probability-density functions when compared with benchmark Monte Carlo simulations. The response remains distinctly narrow-band, and the evolving equivalent natural frequency acts as a meaningful tracker of the dominant response band. At the same time, the time histories of the equivalent inertia, damping, stiffness, damping ratio, and harmonic-balance fractional terms reveal a direct insight into the evolving ship roll dynamics.

The results also clarify the influence of the fractional order on the non-stationary roll response. Decreasing α intensifies the hereditary character of the damping mechanism, leading to higher late-time response levels, broader and more persistent amplitude probability-density surfaces, and slightly delayed peak behavior, whereas larger values of α recover a more local-in-time dissipation mechanism. Overall, the proposed methodology achieves a favorable balance between physical fidelity, interpretability, and computational tractability. It extends conventional stochastic ship-roll formulations by incorporating hereditary damping and amplitude-dependent added inertia non-linearity within a non-stationary equivalent-linear framework, while preserving the efficiency required for practical naval engineering use. It therefore constitutes a reliable and efficient tool for stochastic ship-roll response prediction, probabilistic seakeeping assessment, and stability-oriented analysis in dynamically evolving maritime environments.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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