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Uniqueness and stability of inverse space-dependent source problems in hyperbolic bio-heat transfer

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Abstract. This paper considers the linear but ill-posed inverse problems concerned with the determination of both the temperature and the source in the thermal-wave hyperbolic model of heat transfer in biological tissues. The heterogeneous source includes an unknown space-dependent component that has to be retrieved from extra temperature measurements at the final time or from time-average temperature integral observations. In contrast to the previous research on the parabolic bio-heat transfer equation, the present study takes into account for the realistic final speed of heat propagation in biological tissues. The approach to prove the uniqueness and stability is based on integral identities.

Keywords: bio-heat transfer; inverse source problems; thermal biology.

MSC 2020: 35A02, 35L20, 35R25, 35R30, 80A23.

1 Introduction

Heat transfer in biological tissues constitutes a fundamental area of research with significant implications for medical applications, including hyperthermia treatment, thermal ablation, cryosurgery and laser therapy [3, 4]. Accurate mathematical modeling of bio-heat transfer is essential for understanding temperature distributions in living tissues and for planning effective thermal treatments while minimizing damage to healthy tissue.

The classical approach to modeling heat transfer in perfused biological tissues is the Pennes bio-heat transfer equation [29]. However, being based on the classical Fourier law relating the heat flux to the temperature gradient, this parabolic equation assumes infinite speed of thermal signal propagation. This physical inconsistency becomes particularly problematic when modeling heat transfer in biological tissues in which this speed is finite. To address this limitation, generalized Fourier heat models accounting for a finite speed of heat propagation have been proposed. The Maxwell-Cattaneo-Vernotte (MCV) law, which introduces a thermal relaxation time τ into the constitutive relation between the heat flux and the temperature gradient, provides a physically realistic alternative. This modification transforms the parabolic heat-type equation into a hyperbolic wave-type equation, allowing thermal disturbances to propagate as waves with finite speed. Experimental studies have demonstrated that biological tissues exhibit significant relaxation times, ranging from 15 to 30 seconds for some tissues, and up to 100 seconds in certain cases [8, 35]. In this scenario, the MCV model provides a more accurate description than the classical Fourier model [26, 28, 39].

The incorporation of the MCV law into the Pennes equation yields the hyperbolic or thermal-wave model of bio-heat transfer, which demonstrates substantial differences in temperature predictions compared to parabolic models, especially during initial transients [19, 25, 28, 34]. This model is particularly relevant for applications involving high-intensity heating over short duration, such as laser therapy, radio-frequency ablation and high-intensity focused ultrasound.

Inverse bio-heat transfer problems commonly involve estimating thermophysical properties (thermal conductivity, heat capacity, blood perfusion rate), boundary conditions or internal heat sources from temperature measurements [4, 27, 32, 33]. The determination of heat sources is particularly important for optimizing thermal therapies. Several researchers have investigated inverse source problems in parabolic bio-heat transfer [4, 18, 38], while considerably less attention has been devoted to inverse problems in hyperbolic bio-heat transfer. References [23, 24] developed inverse hyperbolic analysis methods for estimating time-dependent surface heat fluxes, while [1, 2] investigated inverse problems in thermal-wave models using iterative methods.

An important class of inverse problems involves determining space-dependent source terms, which represent spatially varying heat generation in tissue arising from metabolic activity, electromagnetic heating or other therapeutic modalities. The mathematical analysis of these inverse problems requires establishing both uniqueness and stability results [13]. For general evolution equations, uniqueness theorems have been developed using Carleman estimates, unique continuation properties and convolution theorems [16, 20, 22, 37]. Stability estimates, typically of logarithmic or Hölder type, are essential for understanding how errors in measured data propagate to reconstructed source terms [6, 17, 21]. For inverse source problems governed by parabolic equations, Hasanov and coauthors [11, 12] developed variational methods and derived explicit formulas for Fréchet gradients using adjoint problems. Moreover, the use of integral identities has proven particularly effective for establishing both uniqueness and conditional stability estimates for inverse source problems in various evolution equations [31]. Finally, we mention that recently, Boulakia *et al.* [5] investigated inverse source problems for the wave equation (see also [10, 14, 36]), establishing connections between Carleman estimates and observer design.

This paper advances the theory of inverse problems in hyperbolic bio-heat transfer by considering the determination of space-dependent source components in the thermal-wave model from either final-time temperature measurements or time-average temperature observations. Unlike previous work on parabolic bio-heat transfer models or studies restricted to time-dependent sources, this research addresses the more challenging case of spatially varying sources in hyperbolic equations. The source term is assumed to have a separated form $F(x)G(x,t)$, where $F(x)$ represents the unknown spatial distribution to be recovered, and $G(x,t)$ is a known function. Using the framework of [31], the theoretical results communicated in this paper provide a foundation for developing numerical reconstruction algorithms and are essential for understanding the feasibility and reliability of recovering unknown heat sources in biomedical applications.

The plan for the paper is as follows. The mathematical formulation of the initial boundary value problem of hyperbolic bio-heat transfer in both dimensional and non-dimensional forms is presented in Section 2. An analysis of the direct problem is included in Section 3, whilst Sections 4 and 5 provide uniqueness and stability theorems with proofs and examples for the inverse source problem under final time or time-average integral observations. Final conclusions and directions for possible future work are highlighted in Section 6.

2 Mathematical formulation

Taking into account the fact that for biological tissues there is a non-negligible time lag τ (between 15-30 sec) necessary for the heat to propagate, this implies that the classical Fourier law of heat conduction needs to be generalized into the Maxwell-Cattaneo-Vernotte (MVC) law:

$$\underline{q} + \tau \frac{\partial \underline{q}}{\partial \bar{t}} = -\kappa \nabla \theta, \tag{1}$$

where $\bar{t}$ is the time $[s]$, θ is the temperature [°C], κ is the thermal conductivity [W/(m°C)] and $\underline{q}$ is the thermal heat flux vector [W/(m^2)]. At the same time, when analysing the heat transfer in biological tissues, one needs to consider taking into account physical phenomena such as conduction, convection, and metabolism. Therefore, the appropriate partial differential equation (PDE) governing such a model of bio-heat transfer in a biological medium with constant tissue and blood thermal properties in dimensioned form is given by [1, 4]

$$\begin{aligned} &C_{\text{tissue}} \tau \frac{\partial^2 \theta}{\partial \bar{t}^2} + (C_{\text{tissue}} + \tau C_{\text{blood}} w_{\text{blood}}) \frac{\partial \theta}{\partial \bar{t}} = \kappa \nabla^2 \theta + C_{\text{blood}} w_{\text{blood}} (\theta_{\text{blood}} - \theta) \\ &+ \overline{Q} + \tau \frac{\partial \overline{Q}}{\partial \bar{t}} \quad \text{in } \Omega \times (0, t_f], \end{aligned} \tag{2}$$

where C_{tissue} and C_{blood} are the heat capacities [J/m^3 °C] of the tissue and blood, respectively; w_{blood} is the blood perfusion rate $[s^{-1}]$, T_{blood} is the arterial blood temperature [°C] and $\overline{Q} = Q_e + Q_m$, where Q_e and Q_m are the heat generations [W/m^3] due to external heating and metabolism, respectively. Equation (2) holds in $\Omega \times (0, t_f]$, where $t_f > 0$ is a final time of interest $[s]$ and Ω is a bounded domain occupied by the tissue.

Assuming, for simplicity, that the initial temperature is uniform and equal to a constant $\theta_0 \neq 0$, and that the walls $\partial\Omega$ of the tissue Ω are kept at the same temperature θ_0 over duration $[0, t_f]$ of the experiment, the initial and boundary conditions are given by

$$\theta|_{\bar{t}=0} = \theta_0, \quad \left.\frac{\partial \theta}{\partial \bar{t}}\right|_{\bar{t}=0} = 0 \quad \text{in } \overline{\Omega} = \Omega \cup \partial\Omega, \tag{3}$$

$$\theta|_{\partial\Omega \times [0, t_f]} = \theta_0 \quad \text{on } \partial\Omega \times [0, t_f]. \tag{4}$$

Zero-flux Neumann adiabatic or Robin convective boundary conditions may also be considered in place of the Dirichlet boundary condition (4). In equation (2), we assume that

$$\overline{Q}(\overline{\mathbf{x}}, \bar{t}) = \overline{F}(\overline{\mathbf{x}}) g(\overline{\mathbf{x}}, \bar{t}), \tag{5}$$

where g is a known source [W/m^3] such that

$$\overline{Q}(\overline{\mathbf{x}}, \bar{t}) + \tau \frac{\partial \overline{Q}}{\partial \bar{t}} = \overline{F}(\overline{\mathbf{x}}) \left(g(\overline{\mathbf{x}}, \bar{t}) + \tau \frac{\partial g}{\partial \bar{t}}(\overline{\mathbf{x}}, \bar{t}) \right). \tag{6}$$

To determine the space component $F(\overline{\mathbf{x}})$ of the source in (5), we consider a space-dependent measurement of the temperature at the final time $\bar{t} = t_f$, namely,

$$\theta(\overline{\mathbf{x}}, t_f) = \theta_{t_f}(\overline{\mathbf{x}}), \quad \overline{\mathbf{x}} \in \Omega, \tag{7}$$

or the time-average temperature observation

$$\int_0^{t_f} \theta(\overline{\mathbf{x}}, \overline{t})dt = \Theta(\overline{\mathbf{x}}), \quad \overline{\mathbf{x}} \in \Omega, \tag{8}$$

In a recent paper [1], we have considered numerically the same inverse source problem, but for the particular case when $g(\overline{\mathbf{x}}, \overline{t}) = 1\,\mathrm{W/m^2}$.

Employing the non-dimensionalisation

$$\mathbf{x} = \frac{\overline{\mathbf{x}}}{L}, \quad t = \frac{\overline{t}}{t_f}, \quad u = \frac{\theta - \theta_0}{\theta_0}, \quad Q = \frac{\overline{Q} t_f^2}{\tau \theta_0 C_{\mathrm{tissue}}}, \tag{9}$$

where $L > 0$ is a characteristic length dimension of the tissue Ω, the non-dimensional version of equations (2)-(4) is

$$\frac{\partial^2 u}{\partial t^2} + a_1 \frac{\partial u}{\partial t} = a_2 \nabla^2 u - a_3 u + Q(\mathbf{x}, t) + \frac{\tau}{t_f} \frac{\partial Q}{\partial t}(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times (0, 1], \tag{10}$$

$$u(\mathbf{x}, 0) = \frac{\partial u}{\partial t}(\mathbf{x}, 0), \quad \mathbf{x} \in \overline{\Omega}, \tag{11}$$

$$u(\mathbf{x}, t) = 0, \quad (x, t) \in \partial\Omega \times (0, 1], \tag{12}$$

where we have also assumed, for simplicity, that $\theta_{\mathrm{blood}} = \theta_0$, and

$$a_1 = t_f \left(\frac{1}{\tau} + \frac{w_{\mathrm{blood}} C_{\mathrm{blood}}}{C_{\mathrm{tissue}}} \right), \quad a_2 = \frac{\kappa t_f^2}{\tau L^2 C_{\mathrm{tissue}}}, \quad a_3 = \frac{w_{\mathrm{blood}} C_{\mathrm{blood}} t_f^2}{\tau C_{\mathrm{tissue}}}. \tag{13}$$

We have also kept, for simplicity, the notation for the non-dimensional tissue domain unchanged as Ω. Also, denoting

$$U_1 = \frac{\theta_{t_f} - \theta_0}{\theta_0}, \quad E_1 = \frac{\Theta}{t_f \theta_0} - 1,$$

equations (7) and (8) in non-dimensional form, respectively, become

$$u(\mathbf{x}, 1) = U_1(\mathbf{x}), \quad \mathbf{x} \in \Omega \tag{14}$$

and

$$\int_0^1 u(\mathbf{x}, t)dt = E_1(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{15}$$

respectively. Finally, using (6) and (9), remark that

$$\begin{aligned} Q(\mathbf{x}, t) + \frac{\tau}{t_f} \frac{\partial Q(\mathbf{x}, t)}{\partial t} &= \frac{t_f^2}{\tau \theta_0 C_{\mathrm{tissue}}} \left(\overline{Q}(\overline{\mathbf{x}}, t) + \tau \frac{\partial \overline{Q}}{\partial \overline{t}}(\overline{\mathbf{x}}, \overline{t}) \right) \\ &= \frac{t_f^2 \overline{F}(\overline{\mathbf{x}})}{\tau \theta_0 C_{\mathrm{tissue}}} \left(g(\overline{\mathbf{x}}, \overline{t}) + \tau \frac{\partial g}{\partial \overline{t}}(\overline{\mathbf{x}}, \overline{t}) \right). \end{aligned}$$

Then, equation (10) can be rewritten as

$$u_{tt} + a_1 u_t = a_2 \nabla^2 u - a_3 u + F(\mathbf{x}) G(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times (0, 1], \tag{16}$$

where

$$F(\mathbf{x}) = \overline{F}(\overline{\mathbf{x}}), \quad G(\mathbf{x}, t) = \frac{t_f^2}{\tau \theta_0 C_{\mathrm{tissue}}} \left(g(\overline{\mathbf{x}}, \overline{t}) + \tau \frac{\partial g}{\partial \overline{t}}(\overline{\mathbf{x}}, \overline{t}) \right). \tag{17}$$

3 Analysis

For the mathematical analysis we consider a more general setup given by the hyperbolic PDE

$$\rho(x)u_{tt} + \mu(x)u_t - \nabla \cdot (r(x)\nabla u) + s(x)u = F(x)G(x,t) =: P(x,t), \\ (x,t) \in \Omega_T := \Omega \times (0,T], \tag{18}$$

subject to the initial conditions

$$u(x,0) = u_t(x,0) = 0, \quad x \in \overline{\Omega}, \tag{19}$$

and the homogeneous Dirichlet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{20}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary, and, for simplicity, we have replaced $\mathbf{x}$ by x. Obviously, equation (10) is a particular case of the more general equation (18) for $T = 1$, $\rho = 1$, $\mu = a_1$, $r = a_2$ and $s = a_3$.

Let us assume that

$$\begin{cases} r \in H^k(\Omega) \text{ with } k > n/2 + 1, \\ \rho, \mu, s \in H^2(\Omega), \\ 0 < \rho_0 \le \rho(x) \le \rho_1, \quad 0 \le \mu(x) \le \mu_1, \quad 0 < r_0 \le r(x) \le r_1, \quad 0 \le s(x) \le s_1 \text{ for } x \in \overline{\Omega}. \end{cases} \tag{21}$$

Then, under the assumptions (21) and that $\partial^l P/\partial t^l \in L^2(0,T;H^{2-l}(\Omega))$ for $l = 0,1,2$, along with the compatibility condition $P(\cdot,0) \in H_0^1(\Omega)$, there exists a unique weak solution $u \in L^\infty(0,T;H^3(\Omega))$ of the direct problem (18)-(20) with improved regularity

$$u_t \in L^\infty(0,T;H^2(\Omega)), \quad u_{tt} \in L^\infty(0,T;H^1(\Omega)), \quad u_{ttt} \in L^\infty(0,T;L^2(\Omega)), \tag{22}$$

by [9, Theorem 6 with $m = 2$, Section 7.2.3]. Conditions (21) may be relaxed to assume only that ρ, μ and s are smooth functions in $\overline{\Omega}$ and also the conductivity r may also be generalized to be a symmetric and positive-definite tensor modelling the anisotropy of the medium Ω, see [9, pages 378 and 388].

Consider now the measured quantities (14) and (15) in their more general form given by

$$u(x,T) = U_T(x), \quad x \in \Omega \tag{23}$$

or

$$\int_0^T u(x,t)dt = E_T(x), \quad x \in \Omega. \tag{24}$$

4 The inverse source problem (18)-(20) and (23)

Consider first the inverse source problem which requires finding the pair $(u(x,t), F(x))$ satisfying the equations (18)-(20) and (23). Then, we have the following theorem, which generalizes Theorem 4.1 of [31], in the sense that equation (18) includes the extra term $s(x)u$ and we also do not assume that $G(x,0) \equiv 1$.

Theorem 4.1. *Assume that in addition to the basic conditions (21), the source function $G(x,t)$ in (18) satisfies the following regularity and positivity conditions:*

$$\begin{cases} G \in H^2(\Omega_T), \quad G(x,t) > 0, \quad G_t(x,t) > 0, \quad (x,t) \in \Omega_T, \\ \rho(x) G_t^2(x,t) - r(x)|\nabla G(x,t)|^2 > 0, \quad (x,t) \in \Omega_T. \end{cases} \tag{25}$$

Suppose also that the measured output (23) satisfies the regularity condition $U_T \in H^2(\Omega) \cap H_0^1(\Omega)$. Then, the inverse source problem (18)-(20) and (23) has at most one solution $u \in L^\infty(0,T;H^3(\Omega))$ and $F \in H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, under the additional conditions

$$\begin{cases} G \in H^3(\Omega_T), \quad G_{tt}(x,t) \geq 0, \quad (x,t) \in \Omega_T, \\ \rho(x) G_{tt}^2(x,t) - r(x)|\nabla G_t(x,t)|^2 \geq 0, \quad (x,t) \in \Omega_T, \\ 0 < G_T = \inf_{x\in\Omega} G(x,T), \quad 0 < G_T^* = \sup_{x\in\Omega} G(x,T), \\ 0 < G_{01} = \inf_{x\in\Omega} G_t(x,0), \quad 0 < G_0^* = \sup_{x\in\Omega} G(x,0), \\ 0 < G_{T1}^* = \sup_{x\in\Omega} G_t(x,T), \end{cases} \tag{26}$$

and the condition

$$\sigma := \frac{4(G_0^*)^2[3\rho_1^2 G_{01} G_{T1}^* + \mu_1^2 (G_0^*)^2]}{\rho_0^2 G_T^2 G_{01}^2} < 1, \tag{27}$$

the following stability estimate holds:

$$\begin{aligned} \|F\|^2_{L^2(\Omega)} \leq & \frac{4\left[G_{01}^2 (G_T^*)^2 \left(\frac{\rho_1^2 G_{01} G_{T1}^*}{(G_0^*)^6} + \frac{\mu_1^2}{(G_0^*)^4}\right) + s_1^2\right]}{G_T^2(1-\sigma)} \|U_T\|^2_{L^2(\Omega)} \\ & + \frac{4(r_1^2 + \|\nabla r\|^2_{(L^\infty(\Omega))^n})}{G_T^2(1-\sigma)} \|U_T\|^2_{H^2(\Omega)}. \end{aligned} \tag{28}$$

Proof. The proof follows a similar path to that of [31, Theorem 4.1].

Let $(u_i, F_i) \in L^\infty(0,T;H^3(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega))$ for $i = 1,2$ be two solutions of the linear inverse problem (18)–(20) and (23). Then, the pair of functions $w(x,t) := u_1(x,t) - u_2(x,t)$ and $\tilde{F}(x) := F_1(x) - F_2(x)$ solves the problem

$$\rho(x) w_{tt} + \mu(x) w_t - \nabla \cdot (r(x)\nabla w) + s(x) w = \tilde{F}(x) G(x,t), \quad (x,t) \in \Omega_T, \tag{29}$$

$$w(x,0) = w_t(x,0) = 0, \quad x \in \overline{\Omega}, \tag{30}$$

$$w(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{31}$$

$$w(x,T) = 0, \quad x \in \Omega. \tag{32}$$

We multiply both sides of equation (29) by $2w_t/G$ and integrate over Ω_T to obtain the integral identity

$$\begin{aligned} & \int_{\Omega_T} \frac{2}{G(x,t)} \left[\rho(x) w_{tt} w_t + \mu(x) w_t^2 - w_t \nabla \cdot (r(x)\nabla w) + s(x) w w_t\right] dx\, dt \\ & = 2 \int_\Omega \left(\int_0^T w_t(x,t)\, dt\right) \tilde{F}(x)\, dx = 0, \end{aligned} \tag{33}$$

which is well-defined for the regular weak solution $w \in L^\infty(0,T;H^3(\Omega))$ with improved regularity (22). We have the identities

$$
\begin{aligned}
&-\int_{\Omega_T} \frac{2w_t}{G(x,t)} \nabla\cdot(r(x)\nabla w)\,dx\,dt = -\int_{\Omega_T} \nabla\cdot\left(\frac{2}{G(x,t)} r(x) w_t \nabla w\right) dx\,dt \\
&+\int_{\Omega_T} \left[\frac{G_t(x,t)}{G^2(x,t)} r(x)|\nabla w|^2 + \left(\frac{1}{G(x,t)} r(x)|\nabla w|^2\right)_t - \frac{2}{G^2(x,t)} r(x) w_t(\nabla G(x,t)\cdot\nabla w)\right] dx\,dt,
\end{aligned} \tag{34}
$$

$$
\int_{\Omega_T} \frac{2}{G(x,t)} \rho(x) w_{tt} w_t\,dx\,dt = \int_{\Omega_T} \left[\left(\frac{1}{G(x,t)}\rho(x) w_t^2\right)_t + \frac{G_t(x,t)}{G^2(x,t)}\rho(x) w_t^2\right] dx\,dt, \tag{35}
$$

and, using that $w(x,T) = w(x,0) = 0$,

$$
\begin{aligned}
&\int_{\Omega_T} \frac{2s(x) w w_t}{G(x,t)}\,dx\,dt = \int_{\Omega_T} \frac{s(x)(w^2)_t}{G(x,t)}\,dx\,dt = \int_\Omega \left.\frac{s(x) w^2(x,t)}{G(x,t)}\right|_{t=0}^{t=T} dx \\
&+\int_{\Omega_T} \frac{w^2 s(x) G_t(x,t)}{G^2(x,t)}\,dx\,dt = \int_{\Omega_T} \frac{w^2 s(x) G_t(x,t)}{G^2(x,t)}\,dx\,dt.
\end{aligned} \tag{36}
$$

Introducing these identities into (33) we obtain

$$
\begin{aligned}
&\int_{\Omega_T} \left\{\left[\frac{1}{G(x,t)}(\rho(x) w_t^2 + r(x)|\nabla w|^2)\right]_t - \nabla\cdot\left(\frac{2}{G(x,t)} r(x) w_t \nabla w\right)\right\} dx\,dt \\
&\quad+\int_{\Omega_T} \left\{\frac{G_t(x,t)}{G^2(x,t)}[\rho(x) w_t^2 + r(x)|\nabla w|^2 + s(x) w^2] + \frac{2}{G(x,t)}\mu(x) w_t^2\right. \\
&\left.-\frac{2}{G^2(x,t)} r(x) w_t(\nabla G(x,t)\cdot\nabla w)\right\} dx\,dt = 0.
\end{aligned} \tag{37}
$$

In view of the homogeneous conditions (30)–(32) and the divergence theorem, equation (37) yields

$$
\begin{aligned}
&\int_\Omega \frac{1}{G(x,T)}\rho(x) w_t^2(x,T)\,dx + \int_{\Omega_T} \left\{\frac{G_t(x,t)}{G^2(x,t)}[\rho(x) w_t^2 + r(x)|\nabla w|^2 + s(x) w^2]\right. \\
&\left.+\frac{2}{G(x,t)}\mu(x) w_t^2 - \frac{2}{G^2(x,t)} r(x) w_t(\nabla G(x,t)\cdot\nabla w)\right\} dx\,dt = 0.
\end{aligned} \tag{38}
$$

Using (21) and that $G_t > 0$, we have

$$
\begin{aligned}
&G_t(x,t)[\rho(x) w_t^2 + r(x)|\nabla w|^2] - 2r(x) w_t(\nabla G(x,t)\cdot\nabla w) \\
&\geq \rho(x) G_t(x,t) w_t^2 + r(x) G_t(x,t)|\nabla w|^2 - 2r(x)|\nabla G(x,t)||\nabla w||w_t|.
\end{aligned} \tag{39}
$$

The right-hand side of (39) can be written in the quadratic form $\mathbb{W}^{\mathrm{T}} B_1 \mathbb{W}$ with respect to $\mathbb{W} = (|w_t|, |\nabla w|)$ and

$$
B_1 = \begin{pmatrix} \rho(x) G_t(x,t) & -r(x)|\nabla G(x,t)| \\ -r(x)|\nabla G(x,t)| & r(x) G_t(x,t) \end{pmatrix}. \tag{40}
$$

The leading principal minors of the real and symmetric matrix B_1 are positive due to conditions (21) and (25). Then, by Sylvester's criterion, B_1 is positive, which means that the second left-hand-side integral in (38) is also positive. As a consequence, $|\nabla w(x,t)| = 0$ and $w_t(x,t) = 0$. Hence, $w(x,t) = 0$ for all $(x,t) \in \Omega_T$, by the homogeneous initial and boundary conditions (30) and (31). This, in turn, from (29) implies that $F_1(x) = F_2(x)$ for all $(x,t) \in \Omega_T$. This concludes the proof of uniqueness of the solution.

To prove the stability estimate (28), we multiply (18) by $2u_t/G$ and integrate over Ω_T to obtain the integral identity

$$\int_0^T \int_\Omega \left[\frac{2}{G(x,t)}\rho(x)u_{tt}u_t + \frac{2}{G(x,t)}\mu(x)u_t^2 - \frac{2u_t}{G(x,t)}\nabla\cdot(r(x)\nabla u) + \frac{2}{G(x,t)}s(x)uu_t\right] dx\,dt$$
$$= 2\int_\Omega \left(\int_0^T u_t\,dt\right) F(x)\,dx, \tag{41}$$

As in the previous derivations, we use identities (34) and (35) with w replaced by u and

$$\int_{\Omega_T} \frac{2s(x)uu_t}{G(x,t)}\,dx\,dt = \int_{\Omega_T}\frac{s(x)(u^2)_t}{G(x,t)}\,dx\,dt = \int_\Omega \frac{s(x)u^2(x,t)}{G(x,t)}\bigg|_{t=0}^{t=T} dx$$
$$+ \int_{\Omega_T}\frac{u^2 s(x)G_t(x,t)}{G^2(x,t)}\,dx\,dt = \int_{\Omega_T}\frac{u^2 s(x)G_t(x,t)}{G^2(x,t)}\,dx\,dt + \int_\Omega \frac{s(x)U_T^2(x)}{G(x,T)}\,dx. \tag{42}$$

Then (41) yields

$$\int_{\Omega_T}\left\{\left[\frac{1}{G(x,t)}\left(\rho(x)u_t^2 + r(x)|\nabla u|^2 + s(x)u^2\right)\right]_t - \nabla\cdot\left(\frac{2}{G(x,t)}r(x)u_t\nabla u\right)\right\} dx\,dt$$
$$+\int_{\Omega_T}\left\{\frac{G_t(x,t)}{G^2(x,t)}\left[\rho(x)u_t^2 + r(x)|\nabla u|^2 + s(x)u^2\right] + \frac{2}{G(x,t)}\mu(x)u_t^2\right.$$
$$\left. -\frac{2}{G^2(x,t)}r(x)u_t(\nabla G(x,t)\cdot\nabla u)\right\} dx\,dt = 2\int_\Omega F(x)U_T(x)\,dx. \tag{43}$$

Using (19), (20) and (42), equation (43) recasts as

$$\int_\Omega \frac{1}{G(x,T)}\left(\rho(x)u_t^2(x,T) + r(x)|\nabla U_T(x)|^2 + s(x)U_T^2(x)\right)\,dx$$
$$+\int_{\Omega_T}\frac{1}{G^2(x,t)}\left\{G_t(x,t)\left[\rho(x)u_t^2 + r(x)|\nabla u|^2 + s(x)u^2\right] + 2G(x,t)\mu(x)u_t^2\right.$$
$$\left. -2r(x)u_t(\nabla G(x,t)\cdot\nabla u)\right\}\,dx\,dt = 2\int_\Omega F(x)U_T(x)\,dx. \tag{44}$$

By the conditions (25) of the theorem and employing the same arguments as in (39), using the ε-inequality $2ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon}$, we conclude that

$$\frac{\rho_0}{G_T^*}\int_\Omega u_t^2(x,T)\,dx \le \varepsilon\|F\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|U_T\|^2_{L^2(\Omega)}, \quad \varepsilon > 0. \tag{45}$$

Now, differentiate equation (18) with respect to t, multiply both sides by $2u_{tt}/G_t$ and integrate over Ω_T to obtain

$$\int_0^T\int_\Omega \frac{2}{G_t(x,t)}\left[\rho(x)u_{ttt}u_{tt} + \mu(x)u_{tt}^2 - u_{tt}\nabla\cdot(r(x)\nabla u_t) + s(x)u_tu_{tt}\right]\,dx\,dt$$
$$= 2\int_\Omega F(x)u_t(x,T)\,dx, \tag{46}$$

corresponding to the weak solution with improved regularity of the direct problem (18)-(20). In the same way as in the derivation of the integral identity (43), we obtain

$$\begin{aligned}
&\int_{\Omega_T} \left\{ \left[\frac{1}{G_t(x,t)} \left(\rho(x) u_{tt}^2 + r(x)|\nabla u_t|^2 \right) + s(x) u_t^2 \right]_t - \nabla \cdot \left(\frac{2r(x)}{G_t(x,t)} u_{tt} \nabla u_t \right) \right\} dx\, dt \\
&\quad + \int_{\Omega_T} \frac{1}{G_t^2(x,t)} \Big\{ G_{tt}(x,t) \left[\rho(x) u_{tt}^2 + r(x)|\nabla u_t|^2 + s(x) u_t^2 \right] + 2G_t(x,t)\mu(x) u_{tt}^2 \\
&- 2r(x) u_{tt} (\nabla G_t(x,t) \cdot \nabla u_t) \Big\} dx\, dt = 2 \int_\Omega F(x) u_t(x,T)\, dx. \qquad (47)
\end{aligned}$$

Evaluating the first left-hand-side integral in (47), using (20) and taking into account the limit equation obtained by letting $t \searrow 0$ in (18), namely,

$$\rho(x) u_{tt}(x, 0^+) = F(x) G(x, 0), \qquad (48)$$

we arrive at the following identity:

$$\begin{aligned}
&\int_\Omega \frac{1}{G_t(x,T)} \left[\rho(x) u_{tt}^2(x,T) + r(x)|\nabla u_t(x,T)|^2 + s(x) u_t^2(x,T) \right] dx \\
&\quad + \int_{\Omega_T} \frac{1}{G_t^2(x,t)} \Big\{ G_{tt}(x,t) \left[\rho(x) u_{tt}^2 + r(x)|\nabla u_t|^2 + s(x) u_t^2 \right] + 2G_t(x,t)\mu(x) u_{tt}^2 \\
&- 2r(x) u_{tt} (\nabla G_t(x,t) \cdot \nabla u_t) \Big\} dx\, dt = \int_\Omega \frac{G^2(x,0)}{\rho(x) G_t(x,0)} F^2(x)\, dx + 2 \int_\Omega F(x) u_t(x,T)\, dx. \qquad (49)
\end{aligned}$$

By the conditions (25) and (26) and employing the same arguments as in (39) we obtain

$$\frac{\rho_0}{G_{T1}^*} \int_\Omega u_{tt}^2(x,T)\, dx \le \frac{(G_0^*)^2}{\rho_0 G_{01}} \int_\Omega F^2(x)\, dx + 2 \int_\Omega F(x) u_t(x,T)\, dx. \qquad (50)$$

Now, the ε-inequality applied with $\varepsilon = \frac{(G_0^*)^2}{\rho_0 G_{01}}$ to the right-hand-side of (50) yields the following estimate for the norm $\|u_{tt}(\cdot, T)\|^2_{L^2(\Omega)}$:

$$\int_\Omega u_{tt}^2(x,T)\, dx \le \frac{2(G_0^*)^2 G_{T1}^*}{\rho_0^2 G_{01}} \|F\|^2_{L^2(\Omega)} + \frac{G_{01} G_{T1}^*}{(G_0^*)^2} \int_\Omega u_t^2(x,T)\, dx. \qquad (51)$$

Choose now $\varepsilon > 0$ in (45), as follows:

$$\varepsilon = \frac{(G_0^*)^4}{\rho_0 G_T^* G_{01}^2}. \qquad (52)$$

Then

$$\int_\Omega u_t^2(x,T)\, dx \le \frac{(G_0^*)^4}{\rho_0^2 G_{01}^2} \|F\|^2_{L^2(\Omega)} + \frac{(G_T^*)^2 G_{01}^2}{(G_0^*)^4} \|U_T\|^2_{L^2(\Omega)}. \qquad (53)$$

Substituting this into (51), we conclude that

$$\int_\Omega u_{tt}^2(x,T)\, dx \le \frac{3(G_0^*)^2 G_{T1}^*}{\rho_0^2 G_{01}} \|F\|^2_{L^2(\Omega)} + \frac{(G_T^*)^2 G_{01}^3 G_{T1}^*}{(G_0^*)^6} \|U_T\|^2_{L^2(\Omega)}. \qquad (54)$$

Finally, let $t = T$ in (18) to obtain

$$\rho(x)u_{tt}(x,T) + \mu(x)u_t(x,T) - \nabla\cdot(r(x)\nabla u(x,T)) + s(x)u(x,T) = F(x)G(x,T). \tag{55}$$

This implies

$$\|F\|^2_{L^2(\Omega)} \le \frac{4}{G_T^2}\int_\Omega \left\{\rho_1^2 u_{tt}^2(x,T) + \mu_1^2 u_t^2(x,T) + [\nabla\cdot(r(x)\nabla U_T(x))]^2 + s^2(x)U_T^2(x)\right\} dx. \tag{56}$$

Using the inequalities (53) and (54) into (56) yield

$$\begin{aligned}\|F\|^2_{L^2(\Omega)} &\le \frac{4(G_0^*)^2[3\rho_1^2 G_{01}G_{T1}^* + \mu_1^2(G_0^*)^2]}{\rho_0^2 G_T^2 G_{01}^2}\|F\|^2_{L^2(\Omega)}\\ &+ \frac{4}{G_T^2}\left\{s_1^2 + \frac{G_{01}^2(G_T^*)^2}{(G_0^*)^4}\left[\mu_1^2 + \frac{\rho_1^2 G_{01}G_{T1}^*}{(G_0^*)^2}\right]\right\}\|U_T\|^2_{L^2(\Omega)} + \frac{4(r_1^2 + \|\nabla r\|^2_{(L^\infty(\Omega))^n})}{G_T^2}\|U_T\|^2_{H^2(\Omega)}.\end{aligned} \tag{57}$$

The required stability estimate (28) is a consequence of this inequality and condition (27). $\square$

Remark 4.2. In case $G_0^* = 1$ and $s_1 = 0$, equations (27) and (28) may be simplified to take the form of equations (4.4) and (4.5) of [31]. We also note that the last term in the equation (4.5) of [31] (and in (57)) holds only if $n = 1$ under the assumption $r \in H^2(\Omega)$. Higher regularity is required for $n > 1$, such as $r \in H^k(\Omega)$ with $k > n/2 + 1$, as stated in equation (21) for r, to ensure, by Sobolev embedding, that $\nabla r \in (L^\infty(\Omega))^n$.

Remark 4.3. In case $\mu(x) = 0$ in (18), the analysis for the resulting wave equation remains the same, but for the simplified versions of (27) and (28) obtained when taking $\mu_1 = 0$ in their expressions.

Example 1. We provide an explicit example of a source function $G(x,t)$ that satisfies the assumptions (25)-(27) of Theorem 4.1. Let $\Omega = (0,1)$, $\rho(x) = 1+x$, $\mu(x) = 1+x$, $r(x) = 1+x$ and

$$G(x,t) = e^t(x^2+1), \tag{58}$$

(or in higher dimensions $G(\mathbf{x},t) = e^t(|\mathbf{x}|^2+1)$). Then, $G_T = e^T$, $G_T^* = 2e^T$, $G_{01} = 1$, $G_{T1}^* = 2e^T$, $G_0^* = 2$ and one can check that the conditions (25) and (26) are satisfied. Furthermore, with $\rho_0 = 1$ and $\rho_1 = \mu_1 = 2$, if $e^T > 96 + 16\sqrt{37}$, e.g. if $T \ge 5.3$, then condition (27) is also satisfied.

Example 2. In case $G(x,t) \equiv 1$, the strict monotonicity in t in equation (25) is violated and so the uniqueness of solution cannot be established using Theorem 4.1. However, we can establish the uniqueness of solution by utilizing the method of separation of variables, as follows. Let us consider equation (16) in one dimension with $\Omega = (0,1)$ and $G \equiv 1$, given by

$$u_{tt} + a_1 u_t = a_2 u_{xx} - a_3 u + F(x), \quad (x,t) \in (0,1)\times(0,1], \tag{59}$$

where the non-dimensional coefficients a_1, a_2 and a_3 are given by (13), subject to the initial and boundary conditions

$$u(x,0) = u_t(x,0) = 0, \quad x \in [0,1], \tag{60}$$

$$u(0,t) = u(1,t) = 0, \quad t \in (0,1], \tag{61}$$

and the over-determination condition (23) given by

$$u(x, 1) = U_1(x), \quad x \in (0, 1). \tag{62}$$

Then the solution of the above problem is unique.

A similar inverse source problem for the wave equation $u_{tt} = u_{xx} + F(x)$ was considered in [7, 15].

Proof. Let $(u_i(x,t), F_i(x))$ for $i = 1, 2$ be two solutions of the inverse source problem (59)–(62). Denote $w := u_1 - u_2$ and $\tilde{F}(x) := F_1(x) - F_2(x)$ satisfying

$$w_{tt} + a_1 w_t = a_2 w_{xx} - a_3 w + \tilde{F}(x), \quad (x,t) \in (0,1) \times (0,1], \tag{63}$$

$$w(x,0) = w_t(x,0) = 0, \quad x \in [0,1], \tag{64}$$

$$w(0,t) = w(1,t) = 0, \quad t \in (0,1], \tag{65}$$

$$w(x,1) = 0, \quad x \in (0,1). \tag{66}$$

Differentiate equation (63) with respect to t to eliminate the unknown source $\tilde{F}(x)$ and obtain

$$w_{ttt} + a_1 w_{tt} = a_2 w_{xxt} - a_3 w_t, \quad (x,t) \in (0,1) \times (0,1]. \tag{67}$$

On solving this third-order linear PDE using the method of separating variables we obtain

$$w(x,t) = X(x)Z(t), \tag{68}$$

where

$$\frac{Z''' + a_1 Z'' + a_3 Z'}{Z'} = \frac{a_2 X''}{X} = -\lambda^2, \tag{69}$$

for some constant λ. The homogeneous Dirichlet boundary conditions (65) yield the eigenvalues and eigenfunctions

$$\lambda_n = (n\pi)^2 a_2 \quad \text{and} \quad X_n(x) = \sin(n\pi x) \quad \text{for } n \in \mathbb{N}^*. \tag{70}$$

On denoting $R = Z'$, we have

$$R'' + a_1 R' + (a_3 + n^2\pi^2 a_2)R = 0, \quad t \in (0,1]. \tag{71}$$

Using (13), the discriminant of the characteristic equation (71) can be expressed as

$$\Delta_n = a_1^2 - 4(a_3 + n^2\pi^2 a_2) = \frac{t_f^2}{\tau C_{\text{tissue}}} \left\{ \frac{(C_{\text{tissue}} - W_{\text{blood}} C_{\text{blood}})^2}{\tau C_{\text{tissue}}} - \frac{4\kappa n^2 \pi^2}{L^2} \right\}. \tag{72}$$

Let

$$N = \left\lceil \frac{|C_{\text{tissue}} - W_{\text{blood}} C_{\text{blood}}|}{2\pi\sqrt{\tau C_{\text{tissue}} \kappa}} \right\rceil \in \mathbb{N}, \tag{73}$$

where $\lceil \alpha \rceil$ denotes the integer part of α. If $\frac{|C_{\text{tissue}} - W_{\text{blood}} C_{\text{blood}}|}{2\pi\sqrt{\tau C_{\text{tissue}} \kappa}} = \tilde{N} \in \mathbb{N}$, then the general solution is:

$$w_t(x,t) = \sum_{n=1}^{\tilde{N}-1} \left(C_{1,n} e^{\lambda_{+,n} t} + C_{2,n} e^{\lambda_{-,n} t} \right) \sin(n\pi x) + \left(D_{1,\tilde{N}} t + D_{2,\tilde{N}} \right) e^{-a_1 t/2} \sin(\tilde{N}\pi x)$$
$$+ \sum_{n=\tilde{N}+1}^{\infty} \left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right) e^{-a_1 t/2} \sin(n\pi x), \tag{74}$$

otherwise

$$w_t(x,t) = \sum_{n=1}^{N} \left(C_{1,n}e^{\lambda_{+,n}t} + C_{2,n}e^{\lambda_{-,n}t}\right)\sin(n\pi x)$$
$$+ \sum_{n=N+1}^{\infty} \left(A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\right)e^{-a_1 t/2}\sin(n\pi x), \tag{75}$$

where

$$\lambda_{\pm,n} = \frac{-a_1 \pm \sqrt{\Delta_n}}{2} \quad \text{for } n = \overline{1, \tilde{N}-1} \text{ or } n = \overline{1,N} \tag{76}$$

and

$$\omega_n = \frac{\sqrt{-\Delta_n}}{2} \quad \text{for } n \geq \tilde{N}+1 \text{ or } n \geq N+1. \tag{77}$$

For simplicity, let us consider only the latter situation from (75). From the initial condition $w_t(x,0) = 0$ we obtain $A_n = 0$, $\forall n \geq N+1$, and $C_{2,n} = -C_{1,n}$, $\forall n = \overline{1,N}$. Then, (75) simplifies as

$$w_t(x,t) = \sum_{n=1}^{N} C_{1,n}\left(e^{\lambda_{+,n}t} - e^{\lambda_{-,n}t}\right)\sin(n\pi x) + \sum_{n=N+1}^{\infty} B_n\sin(\omega_n t)e^{-a_1 t/2}\sin(n\pi x), \tag{78}$$

Integrating this expression with respect to t and using that $w(x,0) = 0$ we get

$$w(x,t) = -2\sum_{n=N+1}^{\infty}\frac{B_n}{a_1^2 + 4\omega_n^2}\left[(2\omega_n\cos(\omega_n t) + a_1\sin(\omega_n t))e^{-a_1 t/2} - 2\omega_n\right]\sin(n\pi x)$$
$$+ \sum_{n=1}^{N} C_{1,n}\left(\frac{e^{\lambda_{+,n}t}}{\lambda_{+,n}} - \frac{e^{\lambda_{-,n}t}}{\lambda_{-,n}}\right)\sin(n\pi x). \tag{79}$$

Observing that (since $a_1 > 0$)

$$e^{a_1/2} > 1 + \frac{a_1}{2} \geq \cos(\omega_n) + \frac{a_1}{2}\frac{\sin(\omega_n)}{\omega_n} \quad \text{for } n \geq N+1, \tag{80}$$

that $\lambda_{-,n} < \lambda_{+,n} < 0$ for $n = \overline{1,N}$, and that the function e^z/z is strictly decreasing for $z < 0$, imposing $w(x,1) = 0$ immediately implies that $B_n = 0$ for $n \geq N+1$ and $C_{1,n} = 0$ for $n = \overline{1,N}$. Therefore, $w \equiv 0$ and, finally from (63), we also get that $\tilde{F} = 0$. □

4.1 Parabolic particularisation

If we take $\rho(x) = 0$, then the hyperbolic PDE (18) degenerates into the parabolic PDE

$$\mu(x)u_t - \nabla\cdot(r(x)\nabla u) + s(x)u = F(x)G(x,t) = P(x,t), \quad (x,t) \in \Omega_T, \tag{81}$$

and only the initial condition

$$u(x,0) = 0, \quad x \in \overline{\Omega} \tag{82}$$

needs to be imposed. Also, in place of (21) we assume

$$\begin{cases} r \in H^k(\Omega) \text{ with } k > n/2+1, \\ \mu, s \in H^2(\Omega), \\ 0 < \mu_0 \leq \mu(x) \leq \mu_1, \quad 0 < r_0 \leq r(x) \leq r_1, \quad 0 \leq s(x) \leq s_1 \text{ for } x \in \overline{\Omega}. \end{cases} \tag{83}$$

Then, under the assumptions (83) and that $\partial^l P/\partial t^l \in L^2(0,T;H^{2-l}(\Omega))$ for $l = 0, 1$, along with the compatibility condition $P(\cdot, 0) \in H_0^1(\Omega)$, there exists a unique weak solution $u \in L^2(0,T;H^4(\Omega))$ of the direct problem (81), (82) and (20) with improved regularity $u_t \in L^2(0,T;H^2(\Omega))$ and $u_{tt} \in L^2(0,T;L^2(\Omega))$, by [9, Theorem 6 with $m = 1$, Section 7.1.3].

For the inverse source problem given by equations (81), (82), (20) and (23), we cannot directly particularise Theorem 4.1 by taking $\rho(x) = 0$, for instance, we cannot divide with $\rho_0 = 0$ in (27), but we can follow a similar analysis, see also [31, Theorem 3.4], to establish the following uniqueness and stability theorem.

Theorem 4.4. *Assume that in addition to the basic conditions (83), the source function $G(x,t)$ in (81) satisfies the following regularity and positivity conditions:*

$$\begin{cases} G \in H^2(\Omega_T), \quad G(x,t) > 0, \quad G_t(x,t) > 0, \quad (x,t) \in \Omega_T, \\ \mu(x)G(x,t)G_t(x,t) - r(x)|\nabla G(x,t)|^2 > 0, \quad (x,t) \in \Omega_T. \end{cases} \tag{84}$$

Suppose also that the measured output (23) satisfies the regularity condition $U_T \in H^2(\Omega) \cap H_0^1(\Omega)$. Then, the inverse source problem (81), (82), (20) and (23) has at most one solution $u \in L^2(0,T;H^4(\Omega))$ and $F \in H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, under the additional conditions

$$\begin{cases} G \in H^3(\Omega_T), \quad G_{tt}(x,t) \geq 0, \quad (x,t) \in \Omega_T, \\ \mu(x)G_t(x,t)G_{tt}(x,t) - r(x)|\nabla G_t(x,t)|^2 \geq 0, \quad (x,t) \in \Omega_T, \end{cases} \tag{85}$$

and the condition

$$\sigma_1 := \frac{6\mu_1^2 G_{T1}^* G_0^*}{\mu_0^2 G_T^2 G_{01}} < 1,$$

the following stability estimate holds:

$$\|F\|_{L^2(\Omega)}^2 \leq \frac{3}{G_T^2(1-\sigma_1)} \left[\left(\frac{\mu_1^2 G_{T1}^* G_{01}}{G_0^*} + s_1^2 \right) \|U_T\|_{L^2(\Omega)}^2 + (r_1^2 + \|\nabla r\|_{(L^\infty(\Omega))^n}^2) \|U_T\|_{H^2(\Omega)}^2 \right].$$

5 The inverse source problem (18)-(20) and (24)

First, as with the previous analysis of Section 3, we have that under the assumptions (21) and that $\partial^l P/\partial t^l \in L^2(0,T;H^{1-l}(\Omega))$ for $l = 0, 1$, there exists a unique weak solution $u \in L^\infty(0,T;H^2(\Omega))$ of the direct problem (18)-(20) with improved regularity $u_t \in L^\infty(0,T;H^1(\Omega))$ and $u_{tt} \in L^\infty(0,T;L^2(\Omega))$, by [9, Theorem 6 with $m = 1$, Section 7.2.3].

Now, consider the inverse problem (18)-(20) with the integral condition (24) instead of the final time condition (23). Let us denote

$$\tilde{G}(x,t) := \int_0^t G(x,\tau)d\tau, \quad (x,t) \in \Omega_T, \tag{86}$$

are remark that $\tilde{G}(x,0) = 0$ for $x \in \Omega$ and $\tilde{G}_t = G$.

Theorem 5.1. *Assume that in addition to the basic conditions (21), we have satisfied the following regularity and positivity conditions:*

$$\begin{cases} G \in H^1(\Omega_T), \quad G(x,t) > 0, \quad (x,t) \in \Omega_T, \\ \rho(x)G^2(x,t) - r(x)|\nabla \tilde{G}(x,t)|^2 > 0, \quad (x,t) \in \Omega_T. \end{cases} \tag{87}$$

Suppose also that the measured output (24) satisfies the regularity condition $E_T \in H^2(\Omega) \cap H_0^1(\Omega)$. Then the inverse source problem (18)-(20) and (24) has at most one solution $u \in L^\infty(0,T;H^2(\Omega))$ and $F \in H^2(\Omega)$. Moreover, under the additional conditions

$$\begin{cases} G \in H^2(\Omega_T),\ G_t(x,t) > 0, & (x,t) \in \Omega_T, \\ \rho(x)G_t^2(x,t) - r(x)|\nabla G(x,t)|^2 \geq 0, & (x,t) \in \Omega_T, \end{cases} \tag{88}$$

we have the stability estimate

$$\begin{aligned} (1-\tilde{\sigma})\|F\|^2_{L^2(\Omega)} \leq \frac{4}{\mathrm{G}^2T^2}\left\{s_1^2 + \rho_1^2\frac{8(\mathrm{G}^*)^3T^2}{\rho_0^2\alpha_+^3} + \mu_1^2\frac{4(\mathrm{G}^*)^2T^2}{\rho_0\alpha_+^2}\right\}\|E_T\|^2_{L^2(\Omega)} \\ + \frac{4(r_1^2 + \|\nabla r\|^2_{(L^\infty(\Omega))^n})}{G^2T^2}\|E_T\|^2_{H^2(\Omega)}, \end{aligned} \tag{89}$$

where $\mathrm{G} = \inf_{(x,t)\in\Omega_T} G(x,t)$, $\mathrm{G}^* = \sup_{(x,t)\in\Omega_T} G(x,t)$,

$$\tilde{\sigma} := \frac{4}{T^2\mathrm{G}^2}\left[\frac{\rho_1^2\mathrm{G}^*}{2\rho_0}\left(\alpha_+ + \frac{\alpha_+}{\rho_0}\right) + \mu_1^2\frac{\alpha_+^2}{4\rho_0}\right] < 1 \tag{90}$$

and

$$\alpha_\pm := \begin{cases} \frac{\rho_0}{2\mu_1^2}\left(-\frac{\rho_1^2\mathrm{G}^*}{\rho_0^2}(\rho_0+1) \pm \sqrt{\frac{\rho_1^4(\mathrm{G}^*)^2}{\rho_0^4}(\rho_0+1)^2 + \frac{\mu_1^2\mathrm{G}^2T^2}{\rho_0}}\right) & \text{if } \mu_1 > 0, \\ \frac{\mathrm{G}^2T^2\rho_0^2}{4\rho_1^2\mathrm{G}^*(\rho_0+1)} & \text{if } \mu_1 = 0. \end{cases} \tag{91}$$

Proof. Let us consider the problem (18) together with the initial and boundary conditions (19) and (20), and the integral over-determination condition (24). We denote

$$E(x,t) := \int_0^t u(x,\tau)d\tau, \quad (x,t) \in \Omega_T. \tag{92}$$

Next, we integrate (18)-(20) with respect to t to obtain:

$$\rho(x)E_{tt} + \mu(x)E_t - \nabla\cdot(r(x)\nabla E) + s(x)E = F(x)\tilde{G}(x,t) =: \tilde{P}(x,t), \quad (x,t) \in \Omega_T, \tag{93}$$

$$E(x,0) = 0, \quad E_t(x,0) = 0, \quad x \in \overline{\Omega}, \tag{94}$$

$$E(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{95}$$

$$E(x,T) = E_T(x), \quad x \in \Omega. \tag{96}$$

Using the equivalence between (25) and (87) via (86), by Theorem 4.1, the inverse source problem (93)-(96) has at most one solution $E(x,t) \in L^\infty(0,T;H^3(\Omega))$ and $F \in H^2(\Omega)$ (note that the compatibility condition $\tilde{P}(\cdot,0) \in H_0^1(\Omega)$ is satisfied because of the definition (86)).

Finally, from (92) we have that $u \in L^{\infty}(0,T;H^2(\Omega))$ along with $F \in H^2(\Omega)$ is the unique solution of the original inverse problem (18)-(20) and (24).

We now establish the stability of the solution to (18)-(20) and (24) by proving the stability of solution to the problem (93)–(96). Namely, following the steps from the proof of the Theorem 4.1, we will derive the estimate for the norm of F.

Firstly, we multiply the equation (93) by $2E_t/\tilde{G} = 2u/\tilde{G}$ and integrate over Ω_T to obtain the integral identity

$$\int_0^T \int_\Omega \left[\frac{2}{\tilde{G}(x,t)}\rho(x)E_{tt}E_t + \frac{2}{\tilde{G}(x,t)}\mu(x)E_t^2 - \frac{2E_t}{\tilde{G}(x,t)}\nabla\cdot(r(x)\nabla E) \right.$$
$$\left. + \frac{2}{\tilde{G}(x,t)}s(x)EE_t\right] dx\,dt = 2\int_\Omega \left(\int_0^T E_t\,dt\right) F(x)\,dx \tag{97}$$

or

$$\int_{\Omega_T}\left\{\left[\frac{1}{\tilde{G}(x,t)}\left(\rho(x)E_t^2 + r(x)|\nabla E|^2 + s(x)E^2\right)\right]_t - \nabla\cdot\left(\frac{2}{\tilde{G}(x,t)}r(x)E_t\nabla E\right)\right\} dx\,dt$$
$$+\int_{\Omega_T}\left\{\frac{\tilde{G}_t(x,t)}{\tilde{G}^2(x,t)}\left[\rho(x)E_t^2 + r(x)|\nabla E|^2 + s(x)E^2\right] + \frac{2}{\tilde{G}(x,t)}\mu(x)E_t^2\right.$$
$$\left. - \frac{2}{\tilde{G}^2(x,t)}r(x)E_t(\nabla\tilde{G}(x,t)\cdot\nabla E)\right\} dx\,dt = 2\int_\Omega F(x)E_T(x)\,dx. \tag{98}$$

In view of the homogeneous initial and boundary conditions (94) and (95), we get

$$\int_\Omega \frac{1}{\tilde{G}(x,T)}\left(\rho(x)E_t^2(x,T) + r(x)|\nabla E_T(x)|^2 + s(x)E_T^2(x)\right) dx$$
$$+\int_{\Omega_T}\frac{1}{\tilde{G}^2(x,t)}\left\{\tilde{G}_t(x,t)\left[\rho(x)E_t^2 + r(x)|\nabla E|^2 + s(x)E^2\right] + 2\tilde{G}(x,t)\mu(x)E_t^2\right.$$
$$\left. - 2r(x)E_t(\nabla\tilde{G}(x,t)\cdot\nabla E)\right\} dx\,dt = 2\int_\Omega F(x)E_T(x)\,dx. \tag{99}$$

By the conditions (87) of the theorem and employing the same arguments as in (39), and using the ε-inequality $2ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon}$, we conclude that (similar to (50))

$$\frac{\rho_0}{\sup_{x\in\Omega}\tilde{G}(x,T)}\int_\Omega E_t^2(x,T)\,dx \le \varepsilon\|F\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|E_T\|^2_{L^2(\Omega)}, \quad \varepsilon > 0. \tag{100}$$

Now, differentiate equation (93) with respect to t, multiply both sides by $2E_{tt}/\tilde{G}_t = 2u_t/G$ and integrate over Ω_T to obtain

$$\int_0^T\int_\Omega \frac{2}{\tilde{G}_t(x,t)}\left[\rho(x)E_{ttt}E_{tt} + \mu(x)E_{tt}^2 - E_{tt}\nabla\cdot(r(x)\nabla E_t) + s(x)E_tE_{tt}\right] dx\,dt$$
$$= 2\int_\Omega F(x)E_t(x,T)\,dx. \tag{101}$$

In the same way as in the derivation of the integral identity (98) (and similar to (47)), we obtain

$$\int_{\Omega_T} \left\{ \left[\frac{1}{\tilde{G}_t(x,t)} \left(\rho(x) E_{tt}^2 + r(x) |\nabla E_t|^2 \right) + s(x) E_t^2 \right]_t - \nabla \cdot \left(\frac{2r(x)}{\tilde{G}_t(x,t)} E_{tt} \nabla E_t \right) \right\} dx\, dt$$
$$+ \int_{\Omega_T} \frac{1}{\tilde{G}_t^2(x,t)} \Big\{ \tilde{G}_{tt}(x,t) \left[\rho(x) E_{tt}^2 + r(x) |\nabla E_t|^2 + s(x) E_t^2 \right] + 2\tilde{G}_t(x,t) \mu(x) E_{tt}^2$$
$$- 2r(x) E_{tt} (\nabla \tilde{G}_t(x,t) \cdot \nabla E_t) \Big\} dx\, dt = 2 \int_\Omega F(x) E_t(x,T)\, dx. \tag{102}$$

Evaluating the first left-hand-side integral in (102), using (20) and taking into account the limit equation obtained by letting $t \searrow 0$ in (93), namely,

$$\rho(x) E_{tt}(x, 0^+) = F(x) \tilde{G}(x, 0) = 0, \tag{103}$$

we arrive at the identity:

$$\int_\Omega \frac{1}{\tilde{G}_t(x,T)} \left[\rho(x) E_{tt}^2(x,T) + r(x) |\nabla E_t(x,T)|^2 + s(x) E_t^2(x,T) \right] dx$$
$$+ \int_{\Omega_T} \frac{1}{\tilde{G}_t^2(x,t)} \Big\{ \tilde{G}_{tt}(x,t) \left[\rho(x) E_{tt}^2 + r(x) |\nabla E_t|^2 + s(x) E_t^2 \right] + 2\tilde{G}_t(x,t) \mu(x) E_{tt}^2$$
$$- 2r(x) E_{tt} (\nabla \tilde{G}_t(x,t) \cdot \nabla E_t) \Big\} dx\, dt = 2 \int_\Omega F(x) E_t(x,T)\, dx. \tag{104}$$

By the conditions (87) and (88), and employing the same arguments as in (39), identity (104) implies

$$\frac{\rho_0}{\sup_{x\in\Omega} \tilde{G}_t(x,T)} \int_\Omega E_{tt}^2(x,T)\, dx \le 2 \int_\Omega F(x) E_t(x,T)\, dx. \tag{105}$$

With the ε-inequality applied to (105) in view of (100) we get

$$\int_\Omega E_{tt}^2(x,T)\, dx \le \frac{\sup_{x\in\Omega} \tilde{G}_t(x,T)}{\rho_0} \left[\varepsilon_1 \|F\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon_1} \frac{\sup_{x\in\Omega} \tilde{G}(x,T)}{\rho_0} \right.$$
$$\left. \times \left(\varepsilon \|F\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon} \|E_T\|_{L^2(\Omega)}^2 \right) \right]. \tag{106}$$

Now, we let $t = T$ in (93) to obtain

$$\rho(x) E_{tt}(x,T) + \mu(x) E_t(x,T) - \nabla \cdot (r(x) \nabla E(x,T)) + s(x) E(x,T) = F(x) \tilde{G}(x,T), \tag{107}$$

which implies

$$\|F\|_{L^2(\Omega)}^2 \le \frac{4}{(\inf_{x\in\Omega} \tilde{G}(x,T))^2} \int_\Omega \left\{ \rho_1^2 E_{tt}^2(x,T) + \mu_1^2 E_t^2(x,T) + [\nabla \cdot (r(x) \nabla E_T(x))]^2 \right.$$
$$\left. + s^2(x) E_T^2(x) \right\} dx. \tag{108}$$

Let us point out that $\sup_{x\in\Omega}\tilde{G}_t(x,T)\leq \mathrm{G}^*$, $\sup_{x\in\Omega}\tilde{G}(x,T)\leq \mathrm{G}^*T$ and $\inf_{x\in\Omega}\tilde{G}(x,T)\geq \mathrm{G}T$. Then from the inequalities (100) and (106) applied into (108) we obtain

$$\begin{aligned}\|F\|^2_{L^2(\Omega)} &\leq \frac{4}{\mathrm{G}^2T^2}\left[\frac{\rho_1^2\mathrm{G}^*}{\rho_0}\left(\varepsilon_1+\frac{\mathrm{G}^*T\varepsilon}{\rho_0\varepsilon_1}\right)+\mu_1^2\frac{\mathrm{G}^*T\varepsilon}{\rho_0}\right]\|F\|^2_{L^2(\Omega)}\\ &+\frac{4}{\mathrm{G}^2T^2}\left\{s_1^2+\rho_1^2\frac{(\mathrm{G}^*)^2T}{\rho_0^2\varepsilon\varepsilon_1}+\mu_1^2\frac{\mathrm{G}^*T}{\rho_0\varepsilon}\right\}\|E_T\|^2_{L^2(\Omega)}+\frac{4(r_1^2+\|\nabla r\|^2_{(L^\infty(\Omega))^n})}{\mathrm{G}^2T^2}\|E_T\|^2_{H^2(\Omega)}.\end{aligned} \tag{109}$$

In order to get the stability estimate from (109) we must ensure that

$$\tilde{\sigma}:=\frac{4}{T^2\mathrm{G}^2}\left[\frac{\rho_1^2\mathrm{G}^*}{\rho_0}\left(\varepsilon_1+\frac{\mathrm{G}^*T\varepsilon}{\rho_0\varepsilon_1}\right]+\mu_1^2\frac{\mathrm{G}^*T\varepsilon}{\rho_0}\right)<1. \tag{110}$$

This is equivalent to

$$\frac{\rho_1^2\mathrm{G}^*\varepsilon_1}{\rho_0}+\frac{\rho_1^2(\mathrm{G}^*)^2T\varepsilon}{\rho_0^2\varepsilon_1}+\mu_1^2\frac{\mathrm{G}^*T\varepsilon}{\rho_0}<\frac{\mathrm{G}^2T^2}{4}. \tag{111}$$

Let us take $\varepsilon=\frac{\varepsilon_1^2}{\mathrm{G}^*T}$ in (111). Then, we get

$$\frac{\mu_1^2}{\rho_0}\varepsilon_1^2+\frac{\rho_1^2\mathrm{G}^*}{\rho_0^2}(\rho_0+1)\varepsilon_1-\frac{\mathrm{G}^2T^2}{4}<0. \tag{112}$$

If $\mu_1>0$, this quadratic equation with respect to ε_1 has 2 distinct real roots; one positive and one negative given by (91). Given that ε_1 must be positive, we conclude that the inequality (112) holds on the interval $(0,\alpha_+)$. If $\mu_1=0$, we take

$$\alpha_+=\frac{\mathrm{G}^2T^2\rho_0^2}{4\rho_1^2\mathrm{G}^*(\rho_0+1)},$$

yielding $\tilde{\sigma}=\frac{1}{2}<1$ in (90). The stability estimate (89) with $\tilde{\sigma}$ given by (90) is finally obtained by plugging in the values $\varepsilon_1=\alpha_+/2$ and $\varepsilon=\frac{\alpha_+^2}{4\mathrm{G}^*T}$ in (109) and (110). $\square$

Remark 5.2. On comparing (48) with (103), we observe that in Theorem 5.1, $\tilde{\sigma}$ given by (90) is always less than 1, whilst in Theorem 4.1, σ given by (27) was conditioned to be less than 1.

Example 3. We investigate the analog of the problem in Example 2 with $G(x,t)\equiv 1$. Namely, let us consider equation (59) subject to the initial and boundary conditions (60) and (61); however, this time with the integral over-determination condition

$$\int_0^1 u(x,t)\,dt=E_1(x),\quad x\in(0,1). \tag{113}$$

Then, the solution of the resulting inverse source problem is unique.

Proof. Let $(u_i(x,t),F_i(x))$ for $i=1,2$ be two solutions of the inverse source problem (59)–(61) and (113). Denote $w:=u_1-u_2$ and $\tilde{F}(x):=F_1(x)-F_2(x)$ satisfying (63)–(65) and

$$\int_0^1 w(x,t)\,dt=0. \tag{114}$$

Again the method of separation of variables is sufficient to establish the uniqueness. We proceed exactly the same way as in Example 2 up to the point when we utilize the over-determination condition (114) instead of the time-time condition (66). Let us define the functions h_1, $h_2 : [0,1] \to \mathbb{R}$ by

$$h_1(t) := (2\omega_n \cos(\omega_n t) + a_1 \sin(\omega_n t))e^{-a_1 t/2} - 2\omega_n \quad \text{for } n \geq N+1$$

and

$$h_2(t) := \frac{e^{\lambda_{+,n} t}}{\lambda_{+,n}} - \frac{e^{\lambda_{-,n} t}}{\lambda_{-,n}} \quad \text{for } n = \overline{1,N},$$

which appear in the expression (79) for w. For $t = 0$, we have that $h_1(0) = h_2(0) = 0$, but for $t \in (0,1]$, using that

$$e^{a_1 t/2} > 1 + \frac{a_1 t}{2} \geq \cos(\omega_n t) + \frac{a_1 t}{2} \frac{\sin(\omega_n t)}{\omega_n t} \quad \text{for } n \geq N+1,$$

that $\lambda_{-,n} t < \lambda_{+,n} t < 0$ for $n = \overline{1,N}$, and that the function e^z/z is strictly decreasing for $z < 0$, we obtain that $h_1(t) < 0$ and $h_2(t) < 0$ for $t \in (0,1]$. Then, on integrating (79), for the over-determination condition (113) to hold both B_n and $C_{1,n}$ must be zero, implying $w = 0$. Finally, equation (63) implies that $\tilde{F} = 0$. $\square$

5.1 Parabolic particularisation

For the inverse source problem given by the parabolic PDE (81) subjected to (82), (20) and (24), we cannot directly particularise Theorem 5.1 by taking $\rho(x) = 0$, for instance, we cannot divide with $\rho_0 = 0$ in (89) or (90), but we can follow a similar analysis to establish the following uniqueness and stability theorem.

Theorem 5.3. *Assume that in addition to the basic conditions (83), we have satisfied the following regularity and positivity conditions:*

$$\begin{cases} G \in H^1(\Omega_T), \quad G(x,t) > 0, \quad (x,t) \in \Omega_T, \\ \mu(x) G(x,t) \tilde{G}(x,t) - r(x) |\nabla \tilde{G}(x,t)|^2 > 0, \quad (x,t) \in \Omega_T. \end{cases} \tag{115}$$

Suppose also that the measured output (24) satisfies the regularity condition $E_T \in H^2(\Omega) \cap H_0^1(\Omega)$. Then, the inverse source problem (81), (82), (20) and (24) has at most one solution $u \in L^2(0,T;H^2(\Omega))$ and $F \in H^2(\Omega)$. Moreover, under the additional conditions

$$\begin{cases} G \in H^2(\Omega_T), \quad G_t(x,t) \geq 0, \quad (x,t) \in \Omega_T, \\ \mu(x) G_t(x,t) G(x,t) - r(x) |\nabla G(x,t)|^2 \geq 0, \quad (x,t) \in \Omega_T, \end{cases} \tag{116}$$

we have the stability estimate

$$\|F\|^2_{L^2(\Omega)} \leq \frac{6}{\mathrm{G}^2 T^2} \left\{ \left(s_1^2 + \frac{6 G_{T1}^2 \mu_1^4}{\mu_0^2 \mathrm{G}^2 T^2} \right) \|E_T\|^2_{L^2(\Omega)} + (r_1^2 + \|\nabla r\|^2_{(L^\infty(\Omega))^n}) \|E_T\|^2_{H^2(\Omega)} \right\}. \tag{117}$$

Proof. Similarly to obtaining (93)–(96), integrating the equation (81) with respect to t gives:

$$\mu(x)E_t - \nabla\cdot(r(x)\nabla E) + s(x)E = F(x)\tilde{G}(x,t) =: \tilde{P}(x,t), \quad (x,t)\in\Omega_T, \tag{118}$$
$$E(x,0) = 0, \quad x\in\overline{\Omega}, \tag{119}$$
$$E(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T], \tag{120}$$
$$E(x,T) = E_T(x), \quad x\in\Omega. \tag{121}$$

Using the equivalence between (84) and (115) via (86), by Theorem 4.4, the inverse source problem (118)-(121) has at most one solution $E(x,t)\in L^2(0,T;H^4(\Omega))$ and $F\in H^2(\Omega)$ (note that the compatibility condition $\tilde{P}(\cdot,0)\in H_0^1(\Omega)$ is satisfied because of the definition (86)). Finally, from (92) and the statement underneath equation (83) we have that $u\in L^2(0,T;H^2(\Omega))$ along with $F\in H^2(\Omega)$ is the unique solution of the original inverse problem given by equations (81), (82), (20) and (24).

We now establish the stability estimate (117). Differentiating (118) with respect to t and multiplying by $2E_t/G$ yields

$$\frac{1}{G(x,t)}\left(2\mu(x)E_tE_{tt} - 2E_t\nabla\cdot(r(x)\nabla E_t) + 2s(x)E_t^2\right) = 2F(x)E_t. \tag{122}$$

Using the identities

$$\frac{2E_tE_{tt}}{G(x,t)} = \left(\frac{E_t^2}{G(x,t)}\right)_t + \frac{E_t^2G_t}{G^2(x,t)},$$
$$\nabla\cdot\left(\frac{E_tr(x)\nabla E_t}{G(x,t)}\right) = \frac{E_t\nabla\cdot(r(x)\nabla E_t)}{G(x,t)} + \frac{r(x)|\nabla E_t|^2}{G(x,t)} - \frac{r(x)E_t(\nabla G\cdot\nabla E_t)}{G^2(x,t)}$$

into (122), integrating over Ω_T and using the homogeneous initial and boundary conditions (119) and (120) result in

$$\begin{aligned}\epsilon\|F\|^2_{L^2(\Omega)} + \frac{1}{\epsilon}\|E_T\|^2_{L^2(\Omega)} \geq 2\int_\Omega F(x)E_T(x)dx \geq \int_\Omega \frac{\mu(x)E_t^2(x,T)}{G(x,T)}dx \\ + \int_{\Omega_T}\frac{\mu(x)E_t^2G_t}{G^2}dxdt + \int_{\Omega_T}\left(\frac{2r(x)|\nabla E_t|^2}{G} - \frac{2r(x)E_t(\nabla G\cdot\nabla E_t)}{G^2}\right)dxdt,\end{aligned} \tag{123}$$

for any $\epsilon > 0$. Similar arguments to those used to derive (100), but now based on the conditions (116) implies

$$\epsilon\|F\|^2_{L^2(\Omega)} + \frac{1}{\epsilon}\|E_T\|^2_{L^2(\Omega)} \geq \frac{\mu_0}{\sup_{x\in\Omega}G(x,T)}\int_\Omega E_t^2(x,T)dx. \tag{124}$$

Now, let $t = T$ in (118) and use (121) to obtain

$$\mu(x)E_t(x,T) - \nabla\cdot(r(x)\nabla E_T(x)) + s(x)E_T(x) = F(x)\tilde{G}(x,T). \tag{125}$$

This implies

$$\begin{aligned}\|F\|^2_{L^2(\Omega)} \leq \frac{3}{\mathrm{G}^2T^2}\int_\Omega\left[\mu_1^2E_t^2(x,T) + s_1^2E_T^2(x) + |\nabla\cdot(r(x)\nabla E_T(x))|^2\right]dx \\ \leq \frac{3}{\mathrm{G}^2T^2}\left[\left(\frac{G_{T1}\mu_1^2}{\epsilon\mu_0} + s_1^2\right)\|E_T\|^2_{L^2(\Omega)} + \frac{\mu_1^2G_{T1}\epsilon}{\mu_0}\|F\|^2_{L^2(\Omega)} + (r_1^2 + \|\nabla r\|^2_{(L^\infty(\Omega))^n})\|E_T\|^2_{H^2(\Omega)}\right],\end{aligned}$$

which, on choosing $\epsilon = \frac{\mu_0\mathrm{G}^2T^2}{6G_{T1}\mu_1^2}$, yields the desired stability estimate (117). □

6 Conclusions

The theoretical results established in this paper provide a rigorous mathematical foundation for developing numerical reconstruction algorithms in thermal therapy planning. The established uniqueness and stability estimates are valuable assets to modern medical treatments including hyperthermia, radio-frequency ablation, laser therapy and high-intensity focused ultrasound, where the finite speed of thermal wave propagation and significant thermal relaxation times make the hyperbolic model essential for accurate temperature prediction. The derived stability bounds enable the development of stable regularization methods for source identification that can handle measurement noise inherent in clinical applications.

As further work, it would be interesting to investigate whether the Fredholm property of the inverse source problem (with final or integral over-determination) for parabolic PDEs [30] can be extended to hyperbolic PDEs, such that we can also establish the existence of solution. Also, potential directions for the future research could be:

• numerical implementation of reconstruction algorithms and their validation with experimental and clinical data;

• simultaneous identification of multiple unknown parameters alongside source term;

• analysis of nonlinear models with temperature-dependent coefficients;

• application to coupled systems that incorporate tissue damage evolution and phase change phenomena relevant to ablation procedures for medical treatment.

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