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Adequate Losses via Quantitative Linear Logic

MATTEO CAPUCCI*, University of Strathclyde and Independent Researcher, IT

ROBERT ATKEY*, University of Strathclyde, UK

CHARLES GRELOIS, Department of Computer Science, University of Sheffield, UK

EKATERINA KOMENDANTSKAYA*, Heriot-Watt and Southampton Universities, UK

MATTHEW DAGGITT, University of Western Australia, AU

As “neural” components are increasingly embedded in existing “symbolic” software—including safety-critical systems—the question arises of how to specify and enforce the safety of the newly introduced neural parts. Unlike traditional logical specifications, these must be amenable not only to the standard Boolean interpretation, but also to training and optimisation. The latter calls for a quantitative interpretation of the logical syntax, subject to further requirements such as smoothness and differentiability. Moreover, the qualitative and quantitative sides of the logic must share a unifying proof-theoretic and categorical semantics. Finally, the new logic should link cleanly to the substructural and program logics that underpin the verification of existing symbolic programs.

In this paper, we present a logic that ticks all of these boxes. We introduce a family of calculi, $p\text{QLL}$, indexed by a hardness degree p , prove a cut-elimination theorem for them, and establish completeness with respect to enriched residuated ‘soft’ lattices. At $p = \infty$, $p\text{QLL}$ reduces to multiplicative additive linear logic (MALL), and provability in $p\text{QLL}$ converges to provability in MALL as $p \rightarrow \infty$. We express optimisation objectives in the syntax of this logic and prove the quantitative adequacy of neuro-symbolic loss functions—a result that has eluded the neuro-symbolic machine learning community for nearly a decade.

CCS Concepts: • **Theory of computation** → **Linear logic; Proof theory; Semantics and reasoning; Logic and verification**; • **Computing methodologies** → **Machine learning approaches**.

Additional Key Words and Phrases: Linear Logic, Differentiable Logic, Fuzzy Logic, Sequent Calculus, Neuro-Symbolic AI, Neural Network Verification, Property-Driven Training, Enriched Category Theory

*Author supported by ARIA grant MSAI-PR01-P05.

Authors’ Contact Information: Matteo Capucci, University of Strathclyde and Independent Researcher, Modena, IT, matteo.capucci@gmail.com; Robert Atkey, University of Strathclyde, Glasgow, UK, robert.atkey@strath.ac.uk; Charles Grellois, Department of Computer Science, University of Sheffield, Sheffield, UK, c.grellois@sheffield.ac.uk; Ekaterina Komendantskaya, Heriot-Watt and Southampton Universities, Edinburgh and Southampton, UK, ek1u23@soton.ac.uk; Matthew Daggitt, University of Western Australia, Perth, AU, matthew.daggitt@uwa.edu.au.

1 Introduction

Neural networks are rapidly becoming ordinary software components: they are embedded in controllers, pipelines, and safety-critical systems, and so inherit the specification and verification obligations that programming language research has long articulated for conventional code. Unfortunately, Neural Network Verification (NNV) is hard [23]: it can be difficult both to formulate safety specifications and to ensure that networks satisfy them.

To address the former, the field has produced numerous domain-specific languages (DSLs) for encoding the interaction between logical reasoning and neural networks. These DSLs span a broad design space. Systems such as Scallop [43] and DeepProbLog [44] treat the outputs of neural networks as probabilistic inputs to a symbolic reasoning engine, so that logical inference can be integrated into the learning pipeline without directly constraining the learned function. In contrast, systems such as Logic Tensor Networks [9], DL2 [30], and **Vehicle** [25, 26] compile logical specifications into architectures or loss functions that push the network itself to satisfy the specified properties. This paper is motivated by the latter class of systems, and studies **property-driven training**, a set of methods that train neural networks directly on a spec¹.

These methods share a common computational abstraction: given an architecture $f_\theta : \mathbb{R}^m \rightarrow \mathbb{R}^n$ parameterised by weights $\theta \in \mathbb{R}^v$, and a Boolean predicate $\varphi : (\mathbb{R}^m \rightarrow \mathbb{R}^n) \rightarrow \mathbf{Prop}$ over functions expressed in some domain-specific DSL, it is then translated into a differentiable real-valued function over the instances of the architecture $\llbracket \varphi \rrbracket : \mathbb{R}^v \rightarrow \mathbb{R}$. This translation is performed using what practitioners commonly call a *differentiable logic*: an interpretation of logical connectives and quantifiers as differentiable operations over real-valued truth degrees that allows the compiler to perform a compositional translation of the Boolean specification. Subsequently, the gradient $\partial_\theta \llbracket \varphi \rrbracket (f_\theta)$ can be used during training to guide the optimisation process towards a θ such that $\varphi(f_\theta)$ holds.

At first glance, differentiable logics appear to be little more than a modern incarnation of many-valued logics. Logics with truth values in some subset of the real line have been studied for at least a century, counting from the works of Łukasiewicz [60] and Gödel [48], and moving on to fuzzy logic [19, 59], continuous model theory [56], and recent work on quantitative algebra [6–8, 45].

Consequently, many differentiable logics and associated tools have been proposed and studied [9, 10, 30, 53, 54]. The central claim common to all of these works is that performing gradient descent to minimise the output of the compiled specification drives the network to satisfy the

¹Note that training is but one form of model repair/synthesis, see e.g. [35, 58] for other forms.

specification’s Boolean interpretation. From a logical perspective, this amounts to claiming *adequacy* of the differentiable logic (and hence of the real-valued semantics of the DSL) as a proxy for satisfaction: the logic should be *sound*, in the sense that a low loss guarantees the model satisfies the specification, and *complete*, in the sense that a satisfying model is reflected by a correspondingly low loss.

1.1 The problems

Yet a century of real-valued logic has not settled the matter [1, 2, 32, 53], because the mismatch runs deeper than the truth values of formulae. Using loss functions as logical semantics demands three things (Figure 1) that no existing logic can meet at once, which we now examine in turn.

The first aspect concerns the match between *satisfaction* of the specification and the result of *optimisation* of the loss: we call this *adequacy* (Figure 1). Consider, for example, the result in [30] that proves that the proposed differentiable logic (DL2) is adequate in the sense that $\llbracket \varphi \rrbracket = 0$ if and only if the network satisfies φ , as verified by a second Boolean semantics. This kind of *extremal* adequacy is of little use since, in practice, the loss is never *exactly* 0! Even worse, such a notion does not explain why *optimising* the logical loss should lead to better *satisfaction* of the spec—which, empirically, it does.

The hard-to-swallow pill here is that a **qualitative, ‘yes or no’ notion of satisfaction has little hope of ever being adequate**. Indeed, the logical loss is a real-valued quantity that ranges over a broad spectrum of values: if we want these values to mean something logical, then the logic must be able to accommodate this variation and the nuance that comes with it at a syntactic/proof-theoretic level. Secondly, the properties we are checking are seldom qualitative, clear-cut ones: they amount to measuring errors, distances, ratios, and the like.

For instance, a common specification enforced in neural classifiers is *robustness* [16], which is the property of predictions being robust under small perturbations of the input. One can formulate such a property as

$$\forall x \in X, |x - \hat{x}| < \varepsilon \implies |f_{\theta}(x) - f_{\theta}(\hat{x})| < \delta \quad (\text{robustness})$$

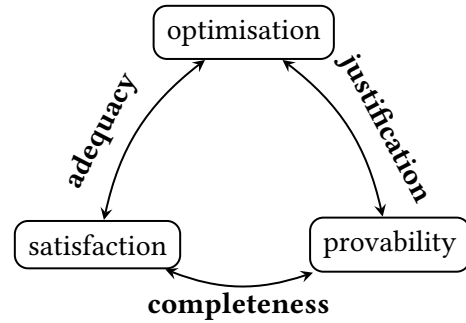


Fig. 1. The three facets a differentiable logic must reconcile.

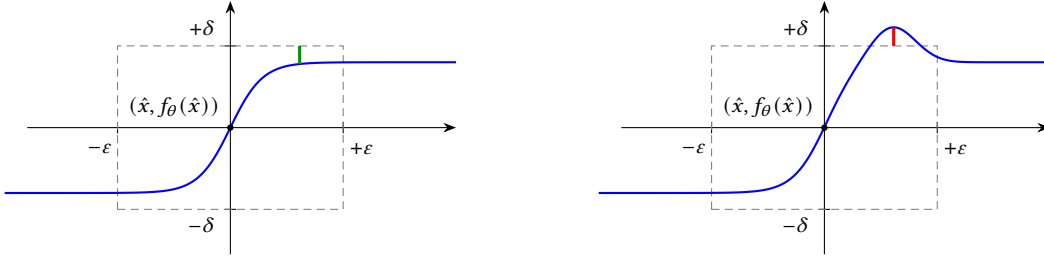


Fig. 2. Two near-identical behaviours judged very differently by a binary notion of satisfaction.

for chosen constants ε and δ and a fixed datapoint $(\hat{x}, \hat{y}) \in D$. Our intuition tells us that there are degrees to the violation of this spec, and it is somewhat unreasonable that two nearly identical functions, like those in Figure 2, should receive completely opposite verdicts. The main thesis of this paper is thus that

*the verification of machine learning models should be
founded on a **quantitative** reconception of logic.*

The second aspect concerns the seemingly irreconcilable tension plaguing differentiable logic: the trade-off between the algebraic/proof-theoretic and analytic properties of the connectives. In fact, to behave well during gradient-descent, the connectives of a differentiable logic should be entrywise strictly increasing differentiable operations, i.e. such that $\partial_i \odot > 0$ for both arguments $i = 1, 2$ of $\odot$. This allows both operands to receive a meaningful gradient signal, avoiding situations in which one *shadows* (in the words of [54]) the other since its truth already ‘suffices’ to guarantee the truth of the whole expression. The paradigmatic example is $\max(\vee)$: when $f(x) > g(x)$ then none of the gradient signal coming from $f(x) \vee g(x)$ will flow in the right branch, leading to fragile results (see [32, 53]).

On the other hand, if we entrust substructural logic as a proof-theoretical foundation, we are led to organize our logical connectives in either *additive* or *multiplicative* ones (using Girard’s terminology for linear logic [34]). Roughly speaking, additives are the connectives users expect to write their specs in, since their rules reflect the most intuitive usage, compared to multiplicatives. Then the established algebraic semantics based on residuated lattices² [20, 33] forces the additives to be interpreted as the lattice operations of join ($\vee$) and meet ($\wedge$).

²Specifically, residuated structures on sublattices of $([-\infty, +\infty], \leq)$. Recall that a residuated lattice is given by $(R, \wedge, \vee, \odot, 1, \multimap)$, where $(R, \wedge, \vee)$ is a lattice, $(R, \odot, 1)$ is a monoid with a neutral element 1, and the following *residuation property* holds: $a \odot b \leq c$ iff $a \leq b \multimap c$.

Here’s then the dilemma: the lattice operations on $\mathbb{R}$ do not enjoy the analytical properties mentioned above, and thus differentiable logics are forced to choose between empirical performance and logical elegance (the *justification* arrow in Figure 1). As one might expect, it is often the latter which takes the fall. For example, Varnai et al. prove a famous *no-go result* that no logic can have connectives that are simultaneously idempotent, associative and shadow-lifting, and propose to sacrifice associativity [54].

And yet, on the reals there are obvious candidate ‘additive’ and ‘multiplicative’ operations, *namely addition and multiplication themselves!* Most importantly, these are readily interpretable operations which are already used in quantitative mathematics (probability theory, statistics, machine learning, etc.), in the very losses where the semantics of a spec is put in property-driven training.

When this **naturality principle** is not respected, even sacrificing logic for analysis does not guarantee success: [31] shows that, on a representative set of verification problems, none of the most popular differentiable logics can successfully translate minimisation of the logical loss into good verification performance, as measured by special-purpose verifiers such as Marabou [37].

Hence our second thesis is that

*a logic for NNV should have semantics in **natural** and **analytically adequate** operations,
without sacrificing the additives of the proof calculus.*

1.2 Our contribution: quantitative logic as a new paradigm

In this paper we propose a new logical paradigm to solve all these challenges at once, **quantitative logic**. Logics *about quantity* (such as fuzzy logics, continuous model theory [56], or the logics of the Lawvere quantale from [6, 7]), concern formulae with intended semantics in the reals, but their notion of logical validity remains a yes-or-no affair. Rather, following Martin-Löf [46] and Lawvere [41], we maintain that **a quantitative logic shall have quantitative proofs and quantitative judgments**, and thus also a quantitative notion of *satisfaction*. In particular, Lawvere argued convincingly that such *generalized logics* [41] arise by *enriching* the entailment relation in a suitably structured **alethic theory** (our terminology) $\mathcal{V}$ —for us, a structure of extended reals.

Our first contribution is to propose a proof-theoretic counterpart to Lawvere’s ideas in the quantitative realm. We introduce **quantitative sequent calculi**, which are calculi ‘enriched’ in one of the algebraic structures presented in Table 1 (and later in Section 2). In such an ‘enriched’ calculus, we can choose any operation on $\mathcal{V}$ to join the premises of a rule and any constant on

	polarity	additive	multiplicative
duality $a^* := 1/a$	positive	$\perp := 0$ $a \oplus^p b \xrightarrow{p \rightarrow \infty} a \oplus^\infty b := a \vee b$	$1 := 1$ $a \otimes b \quad \infty \otimes 0 := 0$
	negative	$\top := \infty$ $a \oplus^{-p} b \xrightarrow{p \rightarrow \infty} a \oplus^{-\infty} b := a \wedge b$	$1 := 1$ $a \otimes^* b \quad \infty \otimes^* 0 := \infty$

Table 1. The family of algebraic structures we collectively refer to as the **multiplicative reals** $[0, \infty]_{\oplus^p, \otimes}$. Each $0 < p \leq \infty$ gives a choice of additives. Red and blue backgrounds denote, respectively, disjunctive and conjunctive operations, based on the behaviour of their restriction to $\{0, +\infty\}$. Positive and negative polarities relate to each other via De Morgan duality.

$\mathcal{V}$ to be the premise of an axiom. The proof-theoretic properties of the calculus become then tightly related to the way this assignment is done.

Thus the choices of Table 1 and Figure 3 are essentially dictated by cut-elimination (Theorem 4.1), for which we need that the operations used for multiplicative rules distribute over the ones we choose for additive ones. Indeed, having decided to use multiplication for multiplicative rules, our additives must necessarily [3, 42] be p -sums³ which are the operations defined as:

$$0 < p < \infty, \quad a \oplus^p b = (a^p + b^p)^{1/p}, \quad a \oplus^{-p} b = (a^{-p} + b^{-p})^{-1/p},$$

with the convention $a + \infty = \infty$. Notably, by taking the limit $p \rightarrow \infty$ one recovers $\oplus^{-\infty} = \wedge$ and $\oplus^\infty = \vee$. We describe this structure in more detail in Section 2.

By decorating multiplicative rules with multiplication and additive rules with (harmonic) p -sums, we then obtain a family of quantitative analogues to (isomix) MALL, which we call p QLL (p -hard Quantitative Linear Logic)—see Figure 3. The annotations on the rules are then used to define a quantitative notion of *validity*, in the form of a function $|-|$ that evaluates proof trees by traversing them and aggregating the constants in the leaves using the operations specified by the rules. A sequent’s *provability* is then the supremum of the validity of all its proofs.

With this notion in hand, we can solve all the aforementioned problems. Here is an outline of the paper and of such a solution:

- (1) After elaborating on Table 1 in Section 2, in Section 3 we introduce quantitative calculi and p QLL specifically. In Section 4, we prove various proof-theoretic results: p QLL admits cut-elimination (Theorem 4.1 proven in Appendix B), with all its usual consequences; we show provability of a sequent in MALL is equivalent to that sequent having positive provability in

³Note these are Yager’s ‘ s -norms’ from [57] but without truncation to $[0, 1]$.

(a) *Structural Rules*

$$\frac{1}{A \vdash A} \text{AX} \quad \frac{1}{\vdash} \text{EMP} \quad \frac{0}{\Gamma \vdash \Delta} \text{EFQ} \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT} \quad \frac{\Gamma \vdash \Delta \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{MIX}$$

(b) *Rules for Multiplicatives*

$$\frac{1}{1 \vdash} 1_L \quad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^\perp \vdash \Delta} (-)^\perp_L \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} (-)^\perp_R \quad \frac{1}{\vdash 1} 1_R$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \wp B \vdash \Delta, \Delta'} \wp_L \quad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \otimes_L \quad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \wp B, \Delta} \wp_R \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma', \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \otimes_R$$

(c) *Rules for Additives*

$$\frac{\infty}{\Gamma, \perp \vdash \Delta} \perp_L \quad \frac{\infty}{\Gamma \vdash \top, \Delta} \top_R$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee_L \quad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge_L \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee_R \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge_R$$

Fig. 3. Rules of p -hard Quantitative Linear Logic ($pQLL$).

$pQLL$ (Theorem 4.4) or provability above 1 in ∞QLL (Theorem 4.7); and finally that provability in $pQLL$ converges to provability in ∞QLL as $p \rightarrow +\infty$.

- (2) Then, in Section 5 we look at the model theory. We formulate a notion of algebraic model, **softales**, which generalizes residuated lattices and metric spaces by enriching the order structure in the multiplicative reals, and supports a ‘soft’ notion of lattice connectives: in this way, for $p < \infty$, **the proof calculi $pQLL$ provide the first instance in the literature of additive rules with a differentiable semantics**, specifically in the canonical model $\mathcal{V} = [0, \infty]_\otimes$. This also allows us to have a quantitative notion of satisfaction, for which we prove a kind of completeness result (Theorem 5.18)—this addresses the last of the arrows in Figure 1.
- (3) Finally, in Section 6 we show how our theory works as a foundation for NNV in the restricted case of propositional specifications. There we formulate the **first quantitative adequacy theorem for a differentiable logic** (Theorem 6.4), in the form $e^{-\llbracket \varphi \rrbracket} = |\vdash_{\mathbb{T}} \varphi|$ where $\mathbb{T}$ is a suitably defined $pQLL$ -theory pertaining to the specific verification problem at hand.

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2 The positive reals

Consider the extended positive reals $[0, \infty]$ with their usual order $\leq$. It supports a breadth of structure we now describe. We start from **(conjunctive) multiplication**, i.e. the following extension of the usual arithmetic multiplication:

$$\begin{array}{c|ccc}
 a \otimes b & 0 & a \in (0, \infty) & \infty \\
 \hline
 0 & 0 & 0 & 0 \\
 b \in (0, \infty) & 0 & ab & \infty \\
 \infty & 0 & \infty & \infty
 \end{array} \tag{1}$$

To see why we call it conjunctive, look at its action on $\{0, \infty\}$. It is commutative, associative, and unital, and we call the resulting ordered monoid the **multiplicative reals** and denote it by $[0, \infty]_{\otimes}$.

Crucially, **inversion** yields a duality $(-)^* : [0, \infty]^{\text{op}} \rightarrow [0, \infty]$

$$\begin{array}{c|ccc}
 a^* & 0 & a \in (0, \infty) & \infty \\
 \hline
 & \infty & 1/a & 0
 \end{array} \tag{2}$$

which is also uniquely characterized by the property

$$a \otimes b \leq c^* \iff a \leq (b \otimes c)^*. \tag{3}$$

This makes $([0, \infty], \leq, 1, \otimes, (-)^*)$ into a $*$ -autonomous monoidal poset [13], which is a kind of (totally ordered and) involutive commutative residuated lattice (see [19, Chapter IV]).

By De Morgan duality, we obtain a second ‘multiplicative’ operation, $\otimes^*$ (**disjunctive multiplication**), which differs from the first only in that $0 \otimes^* \infty = \infty$. In particular, $\otimes$ *linearly distributes* over $\otimes^*$ in the sense of [22]:

$$(a \otimes^* b) \otimes c \leq a \otimes^* (b \otimes c). \tag{4}$$

Moreover, since $1^* = 1$ —and thus $a \otimes b \leq a \otimes^* b$ —we say the operations are *isomix*, in the terminology of [21]. The residuation is explicitly given by $a \multimap b = a^* \otimes^* b$. This means $a \leq$

$$b \multimap c \iff a \otimes b \leq c.$$

$a \multimap b$	0	$a \in (0, \infty)$	∞
0	∞	0	0
$b \in (0, \infty)$	∞	b/a	0
∞	∞	∞	∞

(5)

We sometimes abuse notation and write b/a for $a \multimap b$.

Now, on the poset $[0, \infty]$, consider **sum** $\oplus$, trivially extended to ∞ , and its De Morgan dual, **harmonic sum** [36]:

$$a \oplus^* b := (a^* \oplus b^*)^*. \quad (6)$$

Choosing $0 < p < \infty$, we can conjugate these operations by exponentiation⁴ to obtain **p -sums** and **harmonic p -sums**:

$$a \oplus^p b := (a^p \oplus b^p)^{1/p}, \quad a \oplus^{-p} b := (a^p \oplus^* b^p)^{1/p}.$$

These operations are all commutative and associative. The p -sums have unit 0 and absorbing element ∞ , while harmonic p -sums have unit ∞ and absorbing element 0. It is well-known that, as $p \rightarrow \infty$, these ‘soft’ operations converge pointwise to ‘hard’ ones, namely $\max$ ($\vee$) and $\min$ ($\wedge$) on $[0, \infty]$:

Lemma 2.1 (Additive Collapse).

$$a \oplus^p b \xrightarrow[p \rightarrow \infty]{} a \vee b, \quad a \oplus^{-p} b \xrightarrow[p \rightarrow \infty]{} a \wedge b, \quad (7)$$

PROOF. Classical, see e.g. [49]. □

Thus we can consider p as a *degree of hardness* (with its inverse $1/p$ being *softness* proper⁵), which governs, ultimately, how much idempotency we have renounced, since $2^{-1/p} \leq a/(a \oplus^p a)$ for $p < \infty$. Despite this difference, $\oplus^p$ and $\oplus^{-p}$ behave remarkably like their hard counterparts:

Lemma 2.2. *Let $0 < p \leq \infty$, then:*

(1) *for $a \leq a'$, we have $a \oplus^{-p} b \leq a' \oplus^{-p} b$, and $a \oplus^p b \leq a' \oplus^p b$;*

⁴Extended by $0^p = 0$ and $\infty^p = \infty$ for all p . We do not exponentiate either by 0 or ∞ .

⁵These are similar to ‘temperature’ and ‘coldness’ in statistical mechanics [11].

- (2) for all $a, b \in [0, \infty]$, $a \oplus^{-p} b \leq a \leq a \oplus^p b$;
 (3) for $p \leq q$, $a \oplus^{-p} b \leq a \oplus^{-q} b$, $a \oplus^q b \leq a \oplus^p b$; and thus $a \oplus^{-p} b \leq a \wedge b$ and $a \vee b \leq a \oplus^p b$;

The most important fact about p -sums and multiplication, however, is the following:

Lemma 2.3. *Conjunctive multiplication distributes over p -sum, i.e.:*

$$(a \otimes b) \oplus^p (a \otimes c) = a \otimes (b \oplus^p c). \quad (8)$$

Dually, disjunctive multiplication distributes over harmonic p -sum:

$$(a \otimes^* b) \oplus^{-p} (a \otimes^* c) = a \otimes^* (b \oplus^{-p} c). \quad (9)$$

Corollary 2.3.1.

$$c \multimap (a \oplus^{-p} b) = (c \multimap a) \oplus^{-p} (c \multimap b), \quad (a \oplus^p b) \multimap c = (a \multimap c) \oplus^{-p} (b \multimap c). \quad (10)$$

Lemma 2.4. *We also have*

$$0 \leq a \otimes^* 0 \quad \text{and} \quad (a \otimes^* b) \oplus^p (a \otimes^* c) = a \otimes^* (b \oplus^p c),$$

and dually

$$a \otimes \infty \leq \infty \quad \text{and} \quad a \otimes (b \oplus^{-p} c) = (a \otimes b) \oplus^{-p} (a \otimes c).$$

Corollary 2.4.1.

$$c \multimap (a \oplus^p b) = (c \multimap a) \oplus^p (c \multimap b), \quad (a \oplus^{-p} b) \multimap c = (a \multimap c) \oplus^p (b \multimap c). \quad (11)$$

As an anticipation of the logical meaning these operations will assume later, and in order to introduce the necessary terminology, we recommend revisiting [Table 1](#).

2.1 Varying hardness

All in all, for each $0 < p \leq \infty$, we get a structure we denote as $[0, \infty]_{\oplus^p, \otimes}$. Later, in [Section 5](#), we will introduce its abstract algebraic form—that of *softale*. For now, note that $[0, \infty]_{\oplus^q, \otimes} \cong [0, \infty]_{\oplus^p, \otimes}$ for every $0 < p, q < \infty$, where the isomorphism means that there is a bijection, given by the power map $(-)^{p/q}$, that commutes with duality, tensor, and p -sums. For $q = \infty$ this map is not bijective and only *lax* sum-preserving, i.e. $a^0 \oplus^p b^0 \leq (a \vee b)^0$.

Thus there really are two isomorphism classes of the structure we just presented, the **soft** one, epitomized by $[0, \infty]_{\oplus^1, \otimes}$, and a **hard** one, $[0, \infty]_{\oplus^\infty, \otimes} \equiv [0, \infty]_{\vee, \otimes}$. When it comes to practical verification, however, the choice of p influences the denotations and therefore the loss values, motivating our study for any value of p .

Finally, while the algebraic structure just described is particularly natural on $[0, \infty]$, $[-\infty, +\infty]^{\text{op}}$ supports precisely the same structure, as witnessed by the so-called *Napier's isomorphism* ([15, §2.2]) $-\log \dashv 1/\exp$. Through this isomorphism, the *Lawvere quantale* $\mathbb{L}$ [6, 41] can be identified with the restriction at $[1, \infty]$ of the hard (multiplicative) reals.

2.2 Qualitative truth

When using $[0, \infty]_{\oplus^p, \otimes}$ as the object of truth values, we will often feel the need to compare it back to the usual truth values **Prop**. There are various possibilities, that we call **qualitative modes**, with different desirable properties.⁶

In the **additive mode**, we consider $a = 0$ ‘false’ and everything else ‘true’. Thus the map casting from ‘quantitative’ to ‘qualitative’ truth is

$$0 < _ : [0, \infty] \longrightarrow \mathbf{Prop} \quad (12)$$

Notice this commutes with the conjunctive operations ($\otimes$ and $\oplus^{-p}$), which are both mapped to $\wedge$. Likewise $\oplus^p$ is mapped to $\vee$, but note $\otimes^*$ is not preserved. Rather, we can only say that $a > 0$ and $b > 0$ imply $a \otimes^* b > 0$ (so $\otimes^*$ laxly commutes with $\wedge$), and that $a \otimes^* b > 0$ implies $a > 0$ or $b > 0$ ($\otimes^*$ colaxly commutes with $\vee$). Negation is not preserved either way.

In the **multiplicative mode**, we pick 1 as threshold of truth and consider

$$1 \leq _ : [0, \infty] \longrightarrow \mathbf{Prop} \quad (13)$$

Conjunctive operations ($\otimes$ and $\oplus^{-p}$) laxly commute with $\wedge$, while disjunctive ones ($\otimes^*$ and $\oplus^p$) colaxly commute with $\vee$. When $p = \infty$, $\oplus^p$ and $\oplus^{-p}$ are preserved strongly. We also have $a > 1$ implies $1 \leq a^*$.

⁶Other choices are discussed in [15, §2.3].

3 Quantitative Calculi

We now fix $0 < p \leq \infty$, and thus $[0, \infty]_{\oplus^p, \otimes}$ as the ‘alethic theory’ (as introduced in [Section 1](#)) in which to ‘enrich’ our logic. We now define precisely how we make use of the structure of $[0, \infty]_{\oplus^p, \otimes}$ in a quantitative calculus.

3.1 Definitions

Let Φ be a **language**, i.e. a set of **formulae**. A **p -hard (quantitative) calculus** pertaining Φ deals with sequents $\Gamma \vdash \Delta$ where Γ, Δ are cedents $\Gamma, \Delta \dots$, i.e. finite multisets⁷ of formulae, including the empty one.

In a quantitative calculus, sequents are combined in **structures**, which are entities defined as follows

$$\mathcal{H} ::= \Gamma \vdash \Delta \mid \textcolor{red}{r} \mid \mathcal{H} \otimes \mathcal{H} \mid \mathcal{H} \otimes^* \mathcal{H} \mid \mathcal{H} \oplus^p \mathcal{H} \mid \mathcal{H} \oplus^{-p} \mathcal{H} \quad (14)$$

where $\Gamma \vdash \Delta$ ranges in the possible sequents in the language Φ while r ranges in $[0, \infty]$.

A structure of sequents involving no sequent is called **closed**, while a structure of sequents consisting of a single sequent is called **unary**. We remark that closed structures can be *evaluated* to real numbers, but *per se* are syntactic objects. We call the result of evaluating a closed structure $\mathcal{K}$ its **alethic value** or **validity**, and write it as $|\mathcal{K}|$.

A **rule** has the form $\frac{\mathcal{H}}{\mathcal{H}'}$. Its **premises** are the sequents appearing in $\mathcal{H}$ and its **conclusions** those appearing in $\mathcal{H}'$. It is an **axiom** if the top structure of sequents is closed. A set of rules $\mathcal{R}$ is a **calculus**.

Definition 3.1. A **derivation** in the calculus $\mathcal{R}$ is defined inductively as follows (where $*$ $\in \{\otimes, \otimes^*, \oplus^p, \oplus^{-p}\}$):

- (1) for each $\mathcal{H}$, the **identity derivation** $\frac{\mathcal{H}}{\mathcal{H}} \text{id}_{\mathcal{H}}$ is a derivation;
- (2) every instantiation of a rule in $\mathcal{R}$ is a derivation,
- (3) if $\frac{\mathcal{H}}{\mathcal{I}} \delta_1$ and $\frac{\mathcal{I}}{\mathcal{K}} \delta_2$ are derivations, so is $\frac{\mathcal{H}}{\mathcal{K}} \frac{\delta_1}{\delta_2}$

⁷Being multisets, we do not considered them ordered, and thus omit exchange rules from the calculi.

$$(4) \text{ if } \begin{array}{c} \mathcal{H}_1 \\ \vdots \\ \delta_1 \end{array} \text{ and } \begin{array}{c} \mathcal{H}_2 \\ \vdots \\ \delta_2 \end{array} \text{ are derivations, so is } \begin{array}{c} \mathcal{H}_1 * \mathcal{H}_2 \\ \vdots \\ \delta_1 * \delta_2 \end{array}$$

$$\begin{array}{c} \mathcal{K}_1 \\ \vdots \\ \delta_1 \end{array} \quad \begin{array}{c} \mathcal{K}_2 \\ \vdots \\ \delta_2 \end{array} \quad \begin{array}{c} \mathcal{K}_1 * \mathcal{K}_2 \\ \vdots \\ \delta_1 * \delta_2 \end{array}$$

We stress that a derivation can only mention finitely many rules.

Notation 3.2. A derivation δ of $\mathcal{H}$ is thus denoted as $\begin{array}{c} \vdots \\ \delta \\ \mathcal{H} \end{array}$ or $\begin{array}{c} \vdots \\ \delta' \\ \overline{\mathcal{H}} \end{array} R$ if we want to make the last rule explicit. It follows that a composite derivation in which the last rule of each operand is explicit can be written in either of two ways:

$$\frac{\begin{array}{c} \vdots \\ \delta'_1 * \delta'_2 \end{array}}{\begin{array}{c} \mathcal{H}_1 * \mathcal{H}_2 \\ \vdots \\ \delta_1 * \delta_2 \end{array}} \begin{array}{c} R_1 * R_2 \\ R \end{array} \quad \frac{\begin{array}{c} \vdots \\ \delta'_1 \end{array} \begin{array}{c} \vdots \\ \delta'_2 \end{array}}{\begin{array}{c} \mathcal{H}_1 * \mathcal{H}_2 \\ \vdots \\ \delta_1 * \delta_2 \end{array}} \begin{array}{c} R_1 * R_2 \\ R \end{array} \quad (15)$$

Even though this definition accommodates derivations with DAG structure, when a calculus is **counary** (i.e. all of its rules have conclusions given by unary structures), derivations are in the shape of forests. This will be the case for **pQLL**.

A **closed** derivation is a derivation where the top structure of sequents is closed. A **proof** is a closed derivation where the bottom structure of sequents is unary. The **validity** $|\delta|$ of a closed derivation δ is the alethic value of its top structure of sequents.

Example 3.3. Here is the computation of the validity of a proof in **1QLL**:

$$\left| \frac{\frac{1}{A \vdash A} \text{AX} \oplus^{-1} \frac{1}{A \vdash A} \text{AX}}{A \vee A \vdash A} \text{V}_L \right| = \left| \frac{1}{A \vdash A} \text{AX} \right| \oplus^{-1} \left| \frac{1}{A \vdash A} \text{AX} \right| = 1 \oplus^{-1} 1 = 1/2. \quad (16)$$

The **provability** $|\Gamma \vdash \Delta|_{\mathcal{R}}$ of a sequent $\Gamma \vdash \Delta$ in the calculus $\mathcal{R}$ is the supremum of the validity of its proofs:

$$|\Gamma \vdash \Delta|_{\mathcal{R}} := \bigvee \left\{ |\pi| \mid \begin{array}{c} \vdots \\ \pi \text{ closed} \\ \Gamma \vdash \Delta \end{array} \right\} \quad (\text{provability})$$

By replacing $\Gamma \vdash \Delta$ with a structure of sequents, we extend the above definition⁸. Observe that, as a consequence, any derivation from $\mathcal{H}$ to $\mathcal{H}'$ witnesses that $|\mathcal{H}| \leq |\mathcal{H}'|$.

⁸Observe there is a different way to extend the definition of provability to structures of sequents, by induction on their structure. For a counary calculus these two coincide, but not in general.

Thus the mere existence of a proof does not qualify a sequent as provable, since a sequent might admit proofs of provability 0 (**invalid** proofs); likewise, when a given sequent does not admit any proof, its provability is $\bigvee \emptyset = 0$.

Note, therefore, that to know the provability of a sequent it is not enough to find *a* proof, but potentially *all proofs* are necessary. For *pQLL* below, we prove cut-elimination and thus are able to *decide* the provability of a sequent, i.e. find in finite time its provability, by showing only finitely many proofs need to be checked, see [Theorem 4.3](#). In any case, in applications one is usually concerned with bounding below the provability of a sequent, rather than establishing its precise value, and for that a single proof suffices.

Finally, we say that a rule $\frac{\mathcal{H}}{\mathcal{H}'} R$ is **admissible** in a calculus $\mathcal{R}$ when $|\mathcal{K}|_{\mathcal{R}} = |\mathcal{K}|_{\mathcal{R} \cup \{R\}}$ for any structure of sequents $\mathcal{K}$. Usually this means there is an effective procedure which, given a closed derivation of $\mathcal{H}$ yields a closed derivation of $\mathcal{H}'$ of equal or greater validity. This is an important notion that arises variously in the rest of the paper, most prominently in [Theorem 4.1](#).

3.1.1 Hypersequent calculi & enriched calculi. Quantitative calculi are reminiscent of hypersequent calculi⁹, in which sequents are also organized in a further level of structure. Indeed, the fact that some fuzzy logics require more complex sequent structures has been observed many times [\[5, 17, 18\]](#). Let us show that our enriched setting coheres with (and generalises) that tradition. Formally, a hypersequent is just a (possibly empty) multiset of sequents, written

$$\mathcal{H} = \Gamma_1 \vdash \Delta_1 \mid \cdots \mid \Gamma_n \vdash \Delta_n \quad (17)$$

Rules in a hypersequent calculus have a conjunctive structure of hypersequents as premises, where a distinguished sequent is accompanied by a *hypercontext* (or *side sequents*), the $\mathcal{G}$ s below:

$$\frac{\mathcal{G} \mid \Gamma \vdash \Delta \quad \cdots \quad \mathcal{G}' \mid \Gamma' \vdash \Delta'}{\mathcal{G}'' \mid \Gamma'' \vdash \Delta''} R \quad (18)$$

We now comment on their link.

The above definitions can be easily generalized to alethic values other than $[0, \infty]_{\oplus^p, \otimes}$. It suffices to fix such an object $\mathcal{V}$ and some algebraic structure on it, and then a notion of **$\mathcal{V}$ -enriched calculus** ensues. Specifically, one can pick the usual truth values **Prop** and consider various structures on it. In sequent calculus the tacit choice of **(Prop, $\top$, $\wedge$)** is made: premises of rules are conjunctive structures of sequents and the only closed structure is $\top$.

⁹Due to Pottinger [\[51\]](#) and Avron [\[4\]](#), see [\[48\]](#) for a modern account.

Hypersequent calculi can be seen as **Prop**-enriched calculi, but where **Prop** is now also considered with $\vee$. Then hypersequents become truly disjunctive expressions of sequents:

$$\mathcal{H} = \Gamma_1 \vdash \Delta_1 \vee \cdots \vee \Gamma_n \vdash \Delta_n \quad (19)$$

and rules are expressions in conjunctive normal form. In fact, most hypersequent calculi have rules whose premises all share the same hypercontext $\mathcal{G}$, in which case they reduce to ones of the form

$$\frac{\mathcal{G} \vee (\Gamma' \vdash \Delta' \wedge \cdots \wedge \Gamma' \vdash \Delta')}{\mathcal{G} \vee \Gamma'' \vdash \Delta''} R \quad (20)$$

When presented in this way the hypercontext can be dropped since, by our definition of derivation in a quantitative calculus, those can always be added back. This makes the translation from a sequent calculus completely trivial.

In Section 4.2, we will use these considerations to compare quantitative calculi such as *pQLL* with its hypersequent relatives.

3.2 The *pQLL* Calculus

We are finally ready to introduce the quantitative calculi this work is about, parametric in a choice of $0 < p \leq \infty$.

Notation 3.4. We deviate from traditional linear logic notation in denoting by $\top$ and $\perp$ the *additive* units and with **1** the (coinciding) *multiplicative* ones. We keep *tensor* as $\otimes$ and *par* as $\wp$ but use $\wedge/\vee$ for the additive connectives.

Definition 3.5. Fix a set of **propositional variables** V . A **formula of *pQLL*** with propositional variables in V is defined inductively as follows:

$$\begin{aligned} A ::= & a, a^\perp && ((\text{negated}) \text{ propositional variable } a \in V) \\ & | \mathbf{1} \mid A \otimes A \mid A \wp A && (\text{multiplicatives}) \\ & | \perp \mid \top \mid A \vee A \mid A \wedge A && (\text{additives}) \end{aligned}$$

Negation $(-)^{\perp}$ is a syntactic transformation defined by

$$\begin{aligned} (a)^{\perp} &\equiv a^{\perp}, & (a^{\perp})^{\perp} &\equiv a, & \mathbf{1}^{\perp} &\equiv \mathbf{1}, & \perp^{\perp} &\equiv \top, & \top^{\perp} &\equiv \perp, \\ (A \otimes B)^{\perp} &\equiv A^{\perp} \wp B^{\perp}, & (A \wp B)^{\perp} &\equiv A^{\perp} \otimes B^{\perp}, \\ (A \vee B)^{\perp} &\equiv A^{\perp} \wedge B^{\perp}, & (A \wedge B)^{\perp} &\equiv A^{\perp} \vee B^{\perp}, \end{aligned} \tag{21}$$

We also employ the abbreviation $A \multimap B \equiv A^{\perp} \wp B$.

p QLL is the calculus presented in Figure 3. For conceptual clarity it is laid out as a two-sided calculus, though there is a perfect duality which makes the left rules derivable from the right ones, and *vice versa*. This is due to the definition of negation we just gave and the rules $(-)^{\perp}_L/(-)^{\perp}_R$.

p QLL is conceived as an analogue of the usual (two-sided) sequent calculus of MALL (see Section 4.2) ‘in the alethic metatheory of the p -reals’—giving a **soft** version when $p < \infty$, or a **hard** one at $p = \infty$. Like in MALL, the rules for connectives are split into multiplicative and additive fragments (Figure 3c–3b).

The multiplicative fragment of p QLL is essentially identical to that of MALL, with the notable exception of being ‘isomix’ [21], in the sense that the units for $\otimes$ (tensor) and $\wp$ (par) coincide—this is implicit by having left and right introduction rules for the same constant $\mathbf{1}$, i.e. $\mathbf{1}_L$ and $\mathbf{1}_R$.

The additive fragment is also essentially identical to the corresponding rules in hypersequent calculi for MALL (see e.g. Figure 8), though being quantitative has its consequences. Let’s look at $\vee_R$: to introduce a disjunction $A \vee B$ on the right of the sequent, either A or B do not suffice—rather we need to *sum* a proof of A and one of B in the same context. The result is a suggestive ‘superposition principle’ for proofs involving additives, whereby we never make a choice of what we prove but we have obligations for all possible branches, which are then accumulated by sum (or dually, harmonic sum).

Remark 3.6. Note that this ‘disjunctive branching’ is optional in hypersequent MALL (Figure 8), since the usual rules inherited from the sequent calculus (in which only one premise in $\wedge_L$ and $\vee_R$ is sufficient) are interadmissible with these ones—see [48, Lemma 4.30]. This is also true for $p = \infty$ in this calculus (Lemma 4.5), but not in the soft versions.

As for the structural rules (Figure 3a), these also merit some attention. CUT and AX are standard. We note that, while AX is restricted to the introduction of unary cedents, by MIX (and EMP) we can easily extend it to arbitrary cedents. Note that EMP is derivable by cutting $\mathbf{1}_L$ against $\mathbf{1}_R$, though we keep it in the core calculus to avoid having to add it back in the cut-free calculus.

Finally, **EFQ** (*Ex Falso Quodlibet*) is also quite innovative, since usually adding ‘invalid’ rules to the calculus is not something considered desirable. Despite allowing to construct a proof of any sequent, proofs thus constructed have validity 0, thus we remain perfectly consistent (as proven later in [Theorem 4.2](#)).

Still, **EFQ** is far from useless! To see why, consider a proof like

$$\frac{\frac{1}{A \vdash A} \text{AX} \quad \frac{0}{A \vdash B} \text{EFQ}}{A \vdash A \vee B} \oplus^p \vee_L \quad (22)$$

Even though the branch for $A \vdash B$ does not contribute to the validity of the proof, the whole proof—which has positive validity—could not be written down if we did not have a proof for $A \vdash B$ ¹⁰. Predictably, this argument falls through at $p = \infty$ and there **EFQ** is eliminable, see [Lemma 4.6](#).

In fact, **EFQ** might be considered a catch-all axiom for external structural rules, as can be evidenced by the proof of the following:

Lemma 3.7 (Admissibility of the structural schema). *Consider the algebraic structure defined in (14), and suppose $t(x_1, \dots, x_n)$ and $s(x_1, \dots, x_n)$ be terms¹¹ of it. If $|t| \leq |s|$ pointwise when evaluated¹² in $[0, \infty]_{\oplus^p, \otimes}$, then the rule*

$$\frac{t(\Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n)}{s(\Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n)} \quad (\text{str})$$

*is admissible in any p -hard counary calculus admitting **EFQ**.*

PROOF. Consider a closed derivation δ ending in **str**. Since the calculus is counary (all rules end in a sequent), the derivation above **str** will be given by a combination of proofs of each of its premises, which in turn correspond to a variable occurrence in t . Thus, for each variable x_i in the context of t , we define π_i to be the proof of maximal validity amongst those that insist (in δ) on an occurrence of $\Gamma_i \vdash \Delta_i$ in the premise of **str**—note that $x * y \leq (x \vee y) * (x \vee y)$ for every connective $*$ $\in \{\otimes, \otimes^*, \oplus^p, \oplus^{-p}\}$. If x_i does not appear in t , then $\pi_i = \text{EFQ}$. Excising the derivation above **str** and replacing it with the derivation $s(\pi_1, \dots, \pi_n)$, we obtain a new derivation with validity $s(|\pi_1|, \dots, |\pi_n|) \geq t(|\pi_1|, \dots, |\pi_n|) \geq |\delta|$, as desired. $\square$

¹⁰Note that A and B are metavariables, thus the actual validity of $A \vdash A \vee B$ might differ. Still, there are instances of sequents $A \vdash B$ which are not provable otherwise, e.g. $a \vdash b$ for a, b propositional variables, and thus the argument still applies.

¹¹Thus, rather than ‘structures of sequents’, t and s are constructed out of a countable supply of fresh variables—rather than sequents—and the same red connectives considered in (14).

¹²Here we mean that, for all $a_1, \dots, a_n \in [0, \infty]$, we can prove $|t|(a_1, \dots, a_n) \leq |s|(a_1, \dots, a_n)$ in the ambient theory where the structure of the alethic values live.

This result effectively grants us the use of any of the properties of $\otimes$, $\otimes^*$, $\oplus^p$, and $\oplus^{-p}$ in $[0, \infty]_{\oplus^p, \otimes}$ at the level of structures of sequents, since any such ‘generalized’ derivation is in fact rewritable to a proper derivation, as illustrated by the proof.

Its significance is in granting *pQLL* (and other calculi to which it applies) the same ergonomics that traditional sequent calculi enjoy, in which we do not worry about discharging premises in a particular order. For instance, we can show that the following ‘external structural rules’ familiar from hypersequent calculi [48, Chapter 4] are admissible:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \quad \oplus^p \quad \Gamma' \vdash \Delta'} \text{EW} \quad \frac{\Gamma \vdash \Delta \quad \oplus^{-p} \quad \Gamma' \vdash \Delta'}{\Gamma \vdash \Delta} \text{ECOW} \quad \frac{\Gamma \vdash \Delta \quad \bullet \quad \Gamma' \vdash \Delta'}{\Gamma' \vdash \Delta' \quad \bullet \quad \Gamma \vdash \Delta} \text{EE} \bullet \quad (23)$$

for $\bullet \in \{\otimes, \otimes^*, \oplus^p, \oplus^{-p}\}$

Example 3.8. To illustrate the proof of Lemma 3.7, consider EW, which corresponds to taking $t(x_1) = x_1$ and $s(x_1, x_2) = x_1 \oplus^p x_2$: indeed $a \leq a \oplus^p b$ for all $a, b \in [0, \infty]$. A closed derivation $\delta = \frac{\pi}{\text{EW}}$ is depicted below left. Following the recipe in the proof, we set $\pi_1 := \pi$ (the unique premise insisting on x_1), and $\pi_2 := \text{EFQ}$ since x_2 does not occur in t . Excising the application of EW and replacing it with $s(\pi_1, \pi_2) = \pi_1 \oplus^p \pi_2$, we obtain the rewritten derivation below right whose validity is $|\pi| \oplus^p 0 = |\pi| = |\delta|$, as desired.

$$\frac{\frac{\vdots \pi}{\Gamma \vdash \Delta}}{\Gamma \vdash \Delta \quad \oplus^p \quad \Gamma' \vdash \Delta'} \text{EW} \quad \mapsto \quad \frac{\frac{\vdots \pi}{\Gamma \vdash \Delta} \quad \frac{0}{\Gamma' \vdash \Delta'} \text{EFQ}}{\Gamma \vdash \Delta \quad \oplus^p \quad \Gamma' \vdash \Delta'} \text{id} \quad (24)$$

4 Proof-theoretic and computational properties

In this section, we discuss the common results traditionally expected from a sequent system, and explain the key technical challenges that arise in our journey towards a combination of *linearity* and *quantitativity* of the truth values.

4.1 Cut-elimination and its consequences

Cut-elimination is a desirable property of sequent calculi as it affords the subformula property, the consistency, and even, in some cases, completeness [14] of the logic in question. Because of this, any new logic introduced in the last century is subject to the test, and so is ours:

Theorem 4.1 (Cut-elimination). *Let $pQLL^{\text{cut}}$ be the calculus obtained by removing CUT from $pQLL$. Then $|\Gamma \vdash \Delta|_{pQLL} = |\Gamma \vdash \Delta|_{pQLL^{\text{cut}}}$. In other words, $pQLL^{\text{cut}}$ admits CUT.*

PROOF. Since $p\text{QLL}^{\text{CUT}} \subseteq p\text{QLL}$ (as rule-sets), then clearly provability for the latter cannot be smaller than provability for the first (there are more proofs in $p\text{QLL}$, which means the supremum of (provability) runs over a larger set). Conversely, in [Appendix B](#), we describe a proof transformation from $p\text{QLL}$ to $p\text{QLL}^{\text{CUT}}$ that is non-decreasing in validity. Our proof method is very classical, and follows Melliès' [47] and the Linear Logic Handbook's [52]¹³. The reader can read it in full in [Appendix B](#). Here we sketch the rough idea: given a proof

$$\frac{\begin{array}{c} \vdots \pi \\ \Gamma \vdash A, \Delta \quad \otimes \quad \Gamma', A \vdash \Delta' \end{array}}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT}$$

we rewrite it to a proof π' with the same conclusion $\Gamma, \Gamma' \vdash \Delta, \Delta'$, such that either of the three conditions on applications of **CUT** in π' holds: their number decreases, or they appear at a lower depth in the proof, or the cut formulae pertain to a structurally smaller sequent. We also show that in this case $|\pi| \leq |\pi'|$.

This defines a deterministic, terminating, validity non-decreasing, proof transformation, as required. $\square$

This cut-elimination result offers support for our general agenda—to show that the proposed quantitative upgrade to the sequent calculus is (1) non-trivial and (2) natural from the point of view of preceding proof-theoretic tradition.

For (1), note that our cut-elimination result relies on the notion of proof validity understood in a quantitative way, and directly reasons about quantitative validity of proofs that are subject to cut-elimination. To the best of our knowledge, this is the first result of this kind, as all prior cut-elimination proofs for fuzzy logic (given in e.g. [48]), while relying on quantitative formula validity, still had a Boolean validity of sequents and proofs at the alethic level.

For (2), notice that, despite a seemingly radical revision of the prior tradition, the structure of the inductive argument and the termination proof largely follows a classical scheme.

We observe, moreover, that in most cases that do not involve additives ($\vee$ and $\wedge$) we can in fact establish that $|\pi| = |\pi'|$. In the other cases, proof validity actually increases, showing that the quantitative upgrade did not break the key structural principles that any cut-elimination result relies on. In fact, this result generalises cut elimination for MALL, where only one proof validity is possible: $|\pi| = |\pi'| = 1$.

¹³We followed the latter in our reconstruction of the termination argument.

As an illustration of such matters, consider the following exemplar case from our cut-elimination proof. We are aiming to show that the proof that introduces $A \vee B$ as a cut formula:

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma \vdash A, \Delta} \oplus^p \quad \frac{\vdots \pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \vee B, \Delta} \vee_R \quad \frac{\frac{\frac{\vdots \pi_3}{\Gamma', A \vdash \Delta'} \oplus^{-p} \quad \frac{\vdots \pi_4}{\Gamma', B \vdash \Delta'}}{\Gamma', A \vee B \vdash \Delta'} \vee_L}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT}$$

can be transformed into a proof with the same root sequent and an application of **CUT** on a structurally smaller formula. We see that, from the above, we can obtain either of these two proofs:

$$\frac{\frac{\vdots \pi_1}{\Gamma \vdash A, \Delta} \otimes \quad \frac{\vdots \pi_3}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT} \quad \frac{\frac{\vdots \pi_2}{\Gamma \vdash B, \Delta} \otimes \quad \frac{\vdots \pi_4}{\Gamma', B \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT}$$

Following the standard inductive argument, we obtain that in either of these two cases, the **CUT** is applied on a structurally smaller formula, that is A or B , instead of $A \vee B$. This is analogous to the additive disjunction case in Melliès in [47]. However, we are not done—we also need to show that we can guarantee that the proof transformation increases proof validity! To proceed, we need to prove that

$$(|\pi_1| \oplus^p |\pi_2|) \otimes (|\pi_3| \oplus^{-p} |\pi_4|) \leq (|\pi_1| \otimes |\pi_3|) \vee (|\pi_2| \otimes |\pi_4|),$$

which holds in the reals (**Lemma A.3**). Thus we can, in a deterministic way, pick the option of maximal validity (and, by convention, the one involving A if both validities are equal)¹⁴.

Thus we can guarantee that our proof transformation will construct a proof that has equal or greater validity than the one we started from. This completes our argument.

Cut-elimination still grants the usual consequences, namely, the subformula property and consistency.

Theorem 4.2 (Consistency). *For every $0 < p \leq \infty$, $p\text{QLL}$ is consistent, i.e. there are sequents whose provability is 0.*

PROOF. In $p\text{QLL}^{\text{CUT}}$, $\vdash \perp$ can only possibly be proved via **EFQ**, which has validity 0. $\square$

¹⁴Note that we can't 'keep both proofs' by putting them together using **EC** since $(a \oplus^p b) \otimes (c \oplus^{-p} d) \not\leq 2^{-1/p} \otimes ((a \otimes c) \oplus^p (b \otimes d))$. Still, the proof transformation that makes **EC** admissible (described in **Lemma 3.7**) acts exactly by choosing the maximal validity option, as we do here.

Theorem 4.3 (Decidability). *For any $p\text{QLL}$ sequent $\Gamma \vdash \Delta$, the supremum defining provability is in fact a maximum, i.e. there is a proof π that realizes its provability.*

PROOF. The subformula property means there are only a finite number of distinct proofs; see Appendix B.6 for full proof. $\square$

The main take-away message of this section is that, despite the addition of the quantitative proof validity, the calculus still retains the constructive nature of the sequent calculus. As further evidence of this fact, Bailitis et al. [12] have recently formalised QLL in Rocq, following closely the formalisation by Laurent [40], and adding the validity calculation via an ad-hoc recursive function, rather than as a part of the inductive relation that defines the calculus rules. As a result, the cut-elimination proof by Bailitis follows a structurally identical argument to that of Laurent, modulo an additional argument concerning the increase of validity in the cut-free proofs.

4.2 Hard and qualitative $p\text{QLL}$

$p\text{QLL}$ is related to MALL in two different ways. The first is in a *structural* way: ‘qualitatively’, $p\text{QLL}$ is equivalent to (the isomix¹⁵ variant of) the hypersequent calculus for MALL . The second way is ‘in the limit’: as mentioned in Section 2, at $p = \infty$ the additive operations in $[0, \infty]_{\oplus, \otimes}$ collapse to their ‘hard’ variants, i.e. max and min. The corresponding logic, ∞QLL (“hard QLL ”), is thus even closer to MALL , due to the fact that now the rules for additives are interadmissible with the classical ones.

To unburden proofs of inessential bureaucracy, we adopt the same language for formulae of both calculi. We thus liberally confuse the sequents of both calculi¹⁶.

4.2.1 Qualitative $p\text{QLL}$. By ‘qualitative $p\text{QLL}$ ’ we mean the calculus $p\text{QLL}^{>0}$ obtained by replacing every $\otimes$, $\oplus^{-p}$ appearing in its rules with $\wedge$ and every $\oplus^p$ with $\vee$.

In Appendix C, the reader can find a detailed argument leading to the following theorem—the crucial observations have already been made in Section 3.1.1.

Theorem 4.4. *Let $\Gamma \vdash \Delta$ be a sequent in qualitative $p\text{QLL}$, then*

$$\Gamma \vdash \Delta \text{ provable in } \text{HMALL} \Leftrightarrow |\Gamma \vdash \Delta|_{p\text{QLL}} > 0.$$

¹⁵i.e. $\text{MALL} + \text{MIX} + \text{EMP}$ (which in particular makes the units for tensor and par coincide), terminology from [21].

¹⁶The reader may find it helpful to consult Figures 6 to 8 in Appendix C, laying out a sequent calculus and a hypersequent calculus for MALL (HMALL), as well as the qualitative version of $p\text{QLL}$.

4.2.2 Hard QLL. We now turn to ∞QLL : this calculus is still quantitative, unlike $p\text{QLL}^{>0}$, but validity of proofs will always be either 0, 1, or ∞ , unless axioms of a different validity are introduced. Indeed, the additives at $p = \infty$ are ‘hard’ and thus expected to behave as usual.

Lemma 4.5. *The rules below are admissible in ∞QLL and, vice versa, adding these to $\infty\text{QLL}^{\vee_R^\infty}$ make $\vee_R^\infty$ admissible again.*

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee_R^\infty 1 \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee_R^\infty 2$$

PROOF. Note that one can derive these two rules from $\vee_R^p$ even for finite p , by using **EW** to introduce the neglected branch. Conversely, suppose given $\vee_R^\infty 1$ and $\vee_R^\infty 2$, now a derivation involving

$$\frac{\begin{array}{c} \vdots \pi_A \\ \Gamma \vdash A, \Delta \end{array} \oplus^\infty \begin{array}{c} \vdots \pi_B \\ \Gamma \vdash B, \Delta \end{array}}{\Gamma \vdash A \vee B, \Delta} \vee_R^\infty \quad (25)$$

will have validity at least $|\pi_A| \vee |\pi_B|$, which is the same lower bound we get from the unary rules. Note this reasoning only works for $p = \infty$. $\square$

The above obviously dualize to the case of additive conjunction.

We can also show that:

Lemma 4.6 (EFQ-elimination). *Let $\infty\text{QLL}^{\text{EFQ}}$ be the calculus obtained by removing **EFQ** from ∞QLL . Then*

$$|\Gamma \vdash \Delta|_{\infty\text{QLL}} = |\Gamma \vdash \Delta|_{\infty\text{QLL}^{\text{EFQ}}}, \quad (26)$$

*in other words, $\infty\text{QLL}^{\text{EFQ}}$ admits **EFQ**.*

PROOF. Let $\infty\text{QLL}'$ be the calculus obtained by replacing $\vee_R^\infty$ and $\wedge_L^\infty$ with their unary versions. As proven in Lemma 4.5, $|\Gamma \vdash \Delta|_{\infty\text{QLL}} = |\Gamma \vdash \Delta|_{\infty\text{QLL}'}$. As discussed previously, removing **EFQ** is only problematic in the presence of rules involving additive disjunctive structures as their premise. However, $\infty\text{QLL}'$ has no such rules. $\square$

Thus ∞QLL is even more similar to **MALL** than the soft calculi, and in fact we have a stronger conservativity theorem:

Theorem 4.7. *Let $\Gamma \vdash \Delta$ be a sequent in **MALL** or ∞QLL , then*

$$\Gamma \vdash \Delta \text{ provable in } \text{MALL} \Leftrightarrow |\Gamma \vdash \Delta|_{\infty\text{QLL}} \geq 1. \quad (27)$$

PROOF. See [Appendix C](#). □

We could summarize the section as follows: in the ‘additive’ mode of truth (i.e. being strictly positive), $p\text{QLL}$ is equivalent to MALL , while in the ‘multiplicative’ mode (i.e. being equal or greater than 1) only $[0, \infty]_{\oplus, \otimes}$ (the Lawvere quantale) recovers MALL .

Joining [Theorem 4.4](#) and [Theorem 4.7](#) with the well-known fact that HMALL is equivalent to MALL , we have that if $\Gamma \vdash \Delta$ is a sequent in either MALL or $p\text{QLL}$ or ∞QLL , then $\Gamma \vdash \Delta$ provable in MALL iff $|\Gamma \vdash \Delta|_{\infty\text{QLL}} \geq 1$ iff $|\Gamma \vdash \Delta|_{p\text{QLL}} > 0$.

4.2.3 Approximation. A major motivation for the choice of $\oplus^p / \oplus^{-p}$ as semantics for soft additives is that the parameter p can be used to tune the detachment from hard additives ([Lemma 2.1](#)). Here we reap the logical consequences of this fact.

Write as $(-)^p$ the obvious translation of proofs from 1QLL (or really, any $q\text{QLL}$ for $q < \infty$) to $p\text{QLL}$.

Theorem 4.8 (Approximation Theorem). *Let $\Gamma \vdash \Delta$ be a $p\text{QLL}$ sequent, then*

$$|\Gamma \vdash \Delta|_{p\text{QLL}} \xrightarrow{p \rightarrow \infty} |\Gamma \vdash \Delta|_{\infty\text{QLL}}. \quad (28)$$

PROOF. Let π be a proof of $\Gamma \vdash \Delta$ in 1QLL , and consider $|\pi^p|$ as a function of p . Since it is an expression involving $\oplus^p$, $\oplus^{-p}$, and $\otimes$ only, it is continuous on the interior of $[0, \infty]$. Thus assume the top structure of sequents in π , call it $\mathcal{H}$, does not involve either 0 or ∞ —by continuity and [Lemma 2.1](#), we get the desired claim. Now suppose instead $\mathcal{H}$ involves just 0 and ∞ , then $|\pi^p|$ is constant and hence the claim is met. The remaining cases trivially reduce to these two. □

4.2.4 A Resource Interpretation. Finally, we want to hint at an intuitive ‘interpretation’ of QLL . Linear Logic has a *resource interpretation* whereby a sequent $\Gamma \vdash A$ asserts the convertibility of the resources in Γ to the resource A . Quantitative Linear Logic, especially 1QLL , refines this interpretation by quantifying convertibility: the provability of a sequent $\Gamma \vdash A$ represents a *conversion rate* from the resources in Γ to A . For example, $|1\text{€} \vdash \text{Pizza}| = 5$ means a pizza costs 5€. The $p\text{QLL}$ calculus rules can then be interpreted as a calculus of conversion rates, e.g. if $|1\text{€} \vdash \text{Soda}| = 4$ then, by $\vee_{\text{R}}^1$, $|1\text{€} \vdash \text{Pizza} \vee \text{Soda}| = 9$. Clearly, if the conversion rate from A to B is > 0 , then we are right to say A is convertible to B —this is [Theorem 4.4](#).

5 Softales

Following [41], the algebraic/categorical semantics of *pQLL* takes place in a kind of enriched preorder we call *softale*. These are $[0, \infty]_{\otimes}$ -enriched preorders equipped with a monoidal structure, a suitable duality, and having all ‘soft p -joins’. In order to explain the definition, we start by recalling some standard notions of enriched category theory [39].

Definition 5.1. A $[0, \infty]_{\otimes}$ -**preorder** $\mathcal{A} = (A, \sqsubseteq)$ is a set A equipped with a function $\sqsubseteq : A \times A \rightarrow [0, \infty]$ satisfying for all $a, b, c \in A$,

$$1 \leq (a \sqsubseteq a), \quad (\text{reflexivity})$$

$$(a \sqsubseteq b) \otimes (b \sqsubseteq c) \leq (a \sqsubseteq c). \quad (\text{transitivity})$$

A **map** or **functor** of $[0, \infty]_{\otimes}$ -**preorders** $F : \mathcal{A} \rightarrow \mathcal{B}$ is a function between their underlying sets such that for all $a, a' \in A$,

$$(a \sqsubseteq a') \leq (Fa \sqsubseteq Fa'). \quad (29)$$

We denote all enriched order relations as $\sqsubseteq$, since their arguments suffice to disambiguate. Every $[0, \infty]_{\otimes}$ -preorder has two associated **underlying preorders**: the **multiplicative** one $\mathcal{A}_{\geq 1} = (A, \sqsubseteq_{\geq 1})$ and the **additive** one, defined as

$$a \sqsubseteq_{\geq 1} b \iff 1 \leq (a \sqsubseteq b), \quad a \sqsubseteq_{>0} b \iff 0 < (a \sqsubseteq b). \quad (30)$$

The first is the one enriched category theory defaults to. We say $a, b \in \mathcal{A}$ are **multiplicatively** or **additively isomorphic** and write $a \cong_{\geq 1} b$ or $a \cong_{>0} b$ when they are equivalent in the respective underlying preorder.

Definition 5.2 (Softale). A $(p\text{-})$ **softale** is a $[0, \infty]_{\otimes}$ -enriched preorder S equipped with

- (1) a **multiplicative structure** consisting of a **unit** element $1 \in S$ and a **tensor** map $\bullet : S \times S \rightarrow S$ satisfying

$$(a \sqsubseteq b) \otimes (c \sqsubseteq d) \leq (a \bullet c) \sqsubseteq (b \bullet d), \quad (31)$$

$$a \bullet 1 \cong_{\geq 1} a, \quad (a \bullet b) \bullet c \cong_{\geq 1} a \bullet (b \bullet c), \quad a \bullet b \cong_{\geq 1} b \bullet a, \quad (32)$$

and a **duality** $(-)^{\perp} : S \rightarrow S$ satisfying

$$1^{\perp} = 1, \quad a^{\perp\perp} = a, \quad (33)$$

$$(a \sqsubseteq b) = (b^{\perp} \sqsubseteq a^{\perp}), \quad (34)$$

$$(a \bullet b \sqsubseteq c^{\perp}) = (a \sqsubseteq (b \bullet c)^{\perp}); \quad (35)$$

(2) and an **additive structure**, consisting of a **bottom** element $\perp \in S$, and a choice of **p -join** $a \vee b$ for all $a, b \in S$, satisfying

$$(\perp \sqsubseteq a) = \infty, \quad (36)$$

$$(a \sqsubseteq c) \oplus^{-p} (b \sqsubseteq c) \leq (a \vee b \sqsubseteq c), \quad (37)$$

$$(c \sqsubseteq a) \oplus^p (c \sqsubseteq b) \leq (c \sqsubseteq a \vee b), \quad (38)$$

A **(strict) map of softales** $F : S \rightarrow \mathcal{T}$ is a map of $[0, \infty]_{\otimes}$ -preorders such that

$$1 = F1, \quad Fa \bullet Fb = F(a \bullet b), \quad F(a^{\perp}) = (Fa)^{\perp}, \quad \perp = F\perp, \quad Fa \vee Fb = F(a \vee b). \quad (39)$$

Softales and their maps form a category named **Sftl**.

Remark 5.3. The multiplicative structure on a softale, by itself, could be classified as a **$*$ -autonomous $[0, \infty]_{\otimes}$ -preorder**, and indeed induces a $*$ -autonomous structure on the underlying (Set-)preorder. Just as in the Set-enriched case (see [13]), we have an induced monoidal product **par**: $a \circ b := (a^{\perp} \bullet b^{\perp})^{\perp}$. Moreover, from (35) and then (33) we can prove

$$1 \leq (a \bullet (b \circ c) \sqsubseteq (a \bullet b) \circ c), \quad 1 \leq (a \bullet b \sqsubseteq a \circ b) \quad (40)$$

where the last is the so-called ‘mix’ property [21]. This makes it *closed* too, in the sense that $b \rightarrow c := b^{\perp} \bullet c$ is a residuation in the enriched sense, that is

$$(a \bullet b \sqsubseteq c) = (a \sqsubseteq b \rightarrow c). \quad (41)$$

Remark 5.4. We let our definition of p -joins be inspired by the calculus, but also by the following observation on the ‘categorical’ definition of joins in the hard case. Usually, the definition by representability

$$(a \sqsubseteq c) \wedge (b \sqsubseteq c) \iff (a \vee b) \sqsubseteq c \quad (\text{representability})$$

is equivalent to the one given as a universal property, which can be split in two statements:

$$(a \sqsubseteq c) \wedge (b \sqsubseteq c) \implies (a \vee b) \sqsubseteq c, \quad (\text{minimality})$$

$$(c \sqsubseteq a) \vee (c \sqsubseteq b) \implies c \sqsubseteq (a \vee b). \quad (\text{majorization})$$

This equivalence, however, breaks in the quantitative setting, since to prove **majorization** from **representability** one crucially needs $\wedge$ to distribute over $\vee$, and that does not apply to their soft versions¹⁷. Conversely, given **majorization** and **minimality** one relies on the idempotence of $\wedge$ to conclude **representability**.

Just as we defined $\circ$, one can define p -**meets** for a softale by setting $a \wedge b \equiv (a^\perp \vee b^\perp)^\perp$. Clearly they satisfy dual properties to (37)–(38).

When (37) and (38) hold in the converse direction too we say the additives are **representable**, and note they make the multiplicative underlying preorder a lattice in the traditional sense.

Example 5.5 (Multiplicative reals). The structure on $[0, \infty]$ we sketched in Section 2 is that of a softale, where $\sqsubseteq := -\circ$, $\bullet := \otimes$, $(-)^\perp := (-)^*$, $\circ \equiv \otimes^*$, $\vee := \oplus^p$, and $\wedge \equiv \oplus^{-p}$. This softale has representable additives, as seen in Corollaries 2.3.1 and 2.4.1.

Example 5.6 (Varying hardness). Observe that every $[0, \infty]_{\oplus^q, \otimes}$ can become a 1-softale $[0, \infty]_{\oplus^q, \otimes}^{1/q}$ whose preorder is $a \multimap^{1/q} b := (a \multimap b)^{1/q}$, including when $q = \infty$. However, we have $[0, \infty]_{\oplus^1, \otimes} \cong [0, \infty]_{\oplus^q, \otimes}^{1/q}$ as 1-softales only when $q < \infty$.

Example 5.7 (Involutive commutative residuated lattices). Every commutative residuated lattice $(R, \preceq, 1, \odot, \vee)$ such that $a^\perp \equiv a \rightarrow 1$ is involutive and satisfies (34), is a p -softale, by setting $a \sqsubseteq b = 0$ if $a \preceq b$ and ∞ otherwise.

Conversely, we have an analogous of the results we have seen in Section 4.2:

Proposition 5.8. *The additive underlying preorder of a p -softale and the multiplicative underlying preorder of an ∞ -softale are always involutive commutative residuated lattices.¹⁸*

PROOF. By the above considerations, $\mathcal{S}_{\geq 1}$ would clearly be commutative residuated monoidal (and the same applies to $\mathcal{S}_{>0}$, since laws holding at 1 hold positively too), as well as have joins in the traditional sense. The duality $(-)^\perp$ coincides with $a \rightarrow 1$, making this involutive too. $\square$

¹⁷Specifically, the needed direction is $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$, whereas in $[0, \infty]_{\oplus^p, \otimes}$ is the other one which holds.

¹⁸Such terms are defined in [19, Chapter IV].

Recall (e.g. from [39]) that for $\mathcal{A}, \mathcal{B}$ $[0, \infty]_{\otimes}$ -enriched, the **preorder of maps** $\mathcal{B}^{\mathcal{A}}$ is again $[0, \infty]_{\otimes}$ -enriched, with order

$$F \Rightarrow G \equiv \bigwedge_{a \in A} (Fa \sqsubseteq Ga). \quad (42)$$

Example 5.9. Let $\mathcal{A}$ an enriched order, consider the order of maps $[0, \infty]^{\mathcal{A}}$. This is always a p -softale with all the structure defined pointwise. The key observation is that, for any binary operation $*$ on $[0, \infty]$, ' $\wedge * \wedge \leq \wedge *$ ' since the latter corresponds to taking the infimum over a smaller set.

Interestingly, $[0, \infty]^{\mathcal{A}}$ does not admit a softale structure given by convolution¹⁹: in fact $1 \sqsubseteq -$ is not a valid unit since it does not coincide with its dual, namely $(- \sqsubseteq 1)^*$. Note also that $[0, \infty]^{\mathcal{A}}$ does not have representable additives because infima do not commute with (harmonic) p -sums.

5.1 Model theory

We prove soundness in $pQLL$ by showing that the axioms of softales make each rule of $pQLL$ into a valid reasoning scheme within a softale. We then show completeness by constructing *classifying softales*, that is, softales of $pQLL$ formulae and whose enriched order is given by *provability* of sequents. Finally, we discuss a version of completeness for $[0, \infty]$, limited to a certain kind of theories which arise in applications to neural network verification.

5.1.1 Soundness and completeness. Fix a p -softale $\mathcal{S}$. A *cedent* in $\mathcal{S}$ is a multiset $\Gamma = \{A_1, \dots, A_n\}$ of elements of $\mathcal{S}$. We write $\Gamma^\bullet$ for the element $A_1 \bullet \dots \bullet A_n$, and Γ° for $A_1 \circ \dots \circ A_n$. The empty cedent reduces to 1 in both ways. A *sequent* in $\mathcal{S}$ is an inequality expression of the form $\Gamma^\bullet \sqsubseteq \Delta^\circ$ for Γ and Δ cedents in $\mathcal{S}$. A *structure of sequents* in $\mathcal{S}$ is formed as per the grammar in (14), by replacing sequents with sequents in $\mathcal{S}$.

The *semantic validity of a sequent* in $\mathcal{S}$, written $\Gamma^\bullet \sqsubseteq \Delta^\circ$, is just its value as assigned by the enriched order relation $\sqsubseteq$ of $\mathcal{S}$. This extends to structures of sequents in $\mathcal{S}$ by induction, yielding a semantic valuation $|\cdot|_{\mathcal{S}}$.

Proposition 5.10 (Soundness). *For every rule $\frac{\mathcal{H}}{\mathcal{H}'}$ of $pQLL$, $|\mathcal{H}|_{\mathcal{S}} \leq |\mathcal{H}'|_{\mathcal{S}}$.*

PROOF. The proof is rather straightforward as the algebraic properties of softales are directly related to the rules of $pQLL$. We cover the argument in detail in [Appendix D](#). $\square$

¹⁹In the sense of Day, see [27].

Furthermore, we can get completeness by exhibiting a ‘syntactic softale’ construction, simply by constructing their order relations out of the rules of $p\text{QLL}$ and potentially additional axioms.

Definition 5.11. A $p\text{QLL}$ theory $\mathbb{T}$ is a choice of atomic propositions V plus a set of **axioms** given by rules of the form $\frac{r}{\Gamma \vdash \Delta}$.

Definition 5.12. Let $\mathbb{T}$ be a $p\text{QLL}$ theory, its **classifying p -softale** $\text{LT}^p[\mathbb{T}]$ is defined as follows:

- (1) its elements are the formulae of Definition 3.5 with the atomic propositions of $\mathbb{T}$,
- (2) its order relation is defined by provability in the theory

$$\varphi \Rightarrow \psi := |\varphi \vdash_{\mathbb{T}} \psi|, \quad (43)$$

where the right hand side denotes the **provability** of a sequent;

- (3) its multiplicative and additive structures are given by the connectives of $p\text{QLL}$, i.e. respectively by $\otimes$, $(-)^*$, and $\vee$.

The proof of the necessary laws is in Appendix E.

Definition 5.13. Any map $\mathcal{M} : \text{LT}^p[\mathbb{T}] \rightarrow \mathcal{S}$ exhibits a **model** (or *sound interpretation*) of $\mathbb{T}$ in the p -softale $\mathcal{S}$, meaning syntactic validity (provability) bounds below semantic validity:

$$|\varphi \vdash \psi| \leq (\mathcal{M}(\varphi) \sqsubseteq \mathcal{M}(\psi)). \quad (44)$$

In particular a model satisfies the axioms of $\mathbb{T}$, i.e. $r \leq (\mathcal{M}(\Gamma) \sqsubseteq \mathcal{M}(\Delta))$ for every axiom $\frac{r}{\Gamma \vdash \Delta}$ in $\mathbb{T}$.

The classifying softale $\text{LT}^p[\mathbb{T}]$ enjoys the following universal property: for every $\mathcal{M}$ as above that satisfies the axioms of $\mathbb{T}$ there exists a unique lift $\hat{\mathcal{M}}$ (defined identically to $\mathcal{M}$ on formulae):

$$\begin{array}{ccc} \text{LT}^p[V] & \xrightarrow{\mathcal{M}} & \mathcal{S} \\ \downarrow & \nearrow \exists! \hat{\mathcal{M}} & \\ \text{LT}^p[\mathbb{T}] & & \end{array} \quad (45)$$

When $\mathbb{T}$ is the empty theory on a set of atoms V , a model $\text{LT}^p[V] \rightarrow \mathcal{S}$ is completely determined by its value on propositional variables, and *vice versa*: on compound formulae the value of $\mathcal{M}$ is forced by preservation of the connectives, while (44) ensures that this assignment is enriched functorial. Thus we get:

Proposition 5.14. *The classifying p -softales of empty theories are free over sets, i.e.*

$$\mathbf{Sfl}(\mathbf{LT}^p[V], S) \cong \mathbf{Set}(V, S). \quad (46)$$

Therefore for any valuation $v : V \rightarrow S$ of the propositional variables, we get a unique model $\llbracket - \rrbracket_v$ that exhibits soundness in the sense of provability.

Corollary 5.14.1 (Completeness). *$p\mathbf{QLL}$ is sound and complete for softales, i.e. for every $p\mathbf{QLL}$ sequent $\varphi \vdash \psi$, any p -softale and any valuation $v : V \rightarrow S$, we have $|\varphi \vdash \psi| \leq (\llbracket \varphi \rrbracket_v \sqsubseteq \llbracket \psi \rrbracket_v)$, and in fact the left hand side is a minimum:*

$$\bigwedge_{S \in \mathbf{Sfl}} \bigwedge_{v : V \rightarrow S} (\llbracket \varphi \rrbracket_v \sqsubseteq \llbracket \psi \rrbracket_v) = |\varphi \vdash \psi|. \quad (47)$$

5.2 Grounded completeness

Despite its technical elegance, $p\mathbf{QLL}$ falls short of having a complete semantics in $[0, \infty]_{\oplus p, \otimes}$. However, for our goals, a milder form of completeness suffices, which is satisfied by some theories for the following extension of $p\mathbf{QLL}$.

Definition 5.15. Let $p\mathbf{QLL}^*$ be the quantitative calculus obtained by adding the following rule to $p\mathbf{QLL}$:

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \otimes^* \quad \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{MIX}^* \quad (48)$$

Correspondingly, call a p -softale **mix**^{*} when it satisfies

$$(a \sqsubseteq b) \otimes^* (c \sqsubseteq d) \leq (a \bullet c) \sqsubseteq (b \bullet d); \quad (49)$$

All the definitions we gave in the previous section apply trivially to $p\mathbf{QLL}^*$ too (note also that $p\mathbf{QLL}^*$ enjoys the same proof-theoretic properties as $p\mathbf{QLL}$, including cut-elimination). Moreover, **Proposition 5.10** still applies (note how soundness of **MIX**^{*} can be proven just like that of **MIX** by swapping (31) for (49)), and therefore we have:

Proposition 5.16. *$p\mathbf{QLL}^*$ theories admit a classifying **mix**^{*} p -softale construction, and it is sound and complete in **mix**^{*} p -softales.*

We follow the usual terminology and say that a formula is *grounded* if it does not contain propositional variables. Similarly, in $p\mathbf{QLL}^*$, we consider theories defined via a set of atoms V

together with a valuation $v : V \rightarrow [0, \infty]$ that is used to axiomatically ground such atoms. The **grounded $p\text{QLL}^*$ -theory** $\mathbb{T}_v$ is defined by the following axiom schema:

$$\frac{v(a)}{\vdash a} a_R \quad \frac{v(a)^*}{a \vdash} a_L \quad \text{for all } a \in V. \quad (50)$$

We denote its classifying mix^{*} softale as $\text{LT}^p[V, v]$.

Example 5.17 (Bayesian probability). As a natural example, we consider the grounded theory associated to a finite set Ω of *outcomes*, endowed with a probability density $p : \Omega \rightarrow [0, 1]$ —we claim this is a logic environment for Bayesian probability.

Indeed, every event $A \subseteq \Omega$ corresponds to a formula in this theory, namely $\bigvee_{a \in A} \ulcorner a \urcorner$ (where $\ulcorner a \urcorner$ means a considered as constant of the theory), which we denote²⁰ again by A . Every sequent $A \vdash B$ where A, B are such propositions, corresponds then to the relative probability of B compared to A , i.e. the *odds* of B to A .²¹ In particular, in such a theory the sequents $A \vdash A \cap B$ evaluate to $p(B \mid A)$:

$$\frac{\frac{\frac{\bigoplus_{b \in A \cap B} \bigoplus_{a \in A}^* p(a)^* \otimes p(b)}{\bigoplus_{b \in A \cap B} \bigoplus_{a \in A}^* (a \vdash \otimes \vdash b)} \bigoplus_b \bigoplus_a^* (a_L \otimes b_R)}{\bigoplus_{b \in A \cap B} \bigoplus_{a \in A}^* a \vdash b} \bigoplus_b \bigoplus_a^* \text{MIX}}{\frac{\bigoplus_{b \in A \cap B} \bigoplus_{a \in A}^* a \vdash b}{\bigoplus_{b \in A \cap B} A \vdash b} \bigoplus_b \vee_L^1} \bigoplus_b \vee_L^1 \quad (51)$$

and this proof has validity

$$\frac{\bigoplus_{b \in A \cap B} p(b)}{\bigoplus_{a \in A} p(a)} = \frac{p(A \cap B)}{p(A)} = p(B \mid A). \quad (52)$$

Note when $p(A) = 0$, 1QLL^* says $p(B \mid A) = \infty$ when $p(B) > 0$ and 0 otherwise.

For grounded theories, we have the following completeness theorem:

Theorem 5.18 (Grounded Completeness). *Let φ be a $p\text{QLL}$ formula and let $v : V \rightarrow [0, \infty]$ be a grounding, and let $\llbracket - \rrbracket_v : \text{LT}^p[V, v] \rightarrow [0, \infty]$ be the unique model of $\mathbb{T}_v$ induced by v . Then*

$$\llbracket \varphi \rrbracket_v = \left| \vdash_{\mathbb{T}_v} \varphi \right|. \quad (53)$$

²⁰This encoding is not perfect, since A is considered a multiset in this way. A first-order encoding is much more satisfactory.

²¹This encoding of Bayesian probability in line with the conversion rate interpretation seen in [Section 4.2.4](#), since *odds* can be conceived as a conversion rate between bets and payoffs (see [\[28\]](#)).

PROOF. Since we have soundness already, it suffices to prove the left hand side is $\leq$ than the right hand side. The result follows from an easy induction on φ which the reader can find in [Appendix F](#). In particular, the grounding schema (50) is used for the base case when φ is an atom or the dual of an atom, while MIX^* is used to prove the induction step for $\varphi = \varphi_1 \wp \varphi_2$. $\square$

6 QLL in neuro-symbolic learning and verification

In this section, we finally deploy our theoretical framework in the context of neuro-symbolic learning, and specifically property-driven training. We start with an opinionated account of property-driven training of neural models.

We assume to be training a parameterised model $f_\theta : X \rightarrow Y$ with trainable parameters $\theta \in \Theta$, where $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$ and $\Theta = \mathbb{R}^v$ are task-specific smooth manifolds. In the following, we conceive of f_θ as a classifier, so that we think of n as a *set of classes* or *labels*, and each $f_\theta(x)_i$ as a logit assigned to class $i \in n$.

We are also given a logical property specifying the desired behaviour of f_θ on certain regions of the input space. Here, we always assume the property to have the following form

$$\forall x \in X, a \leq x \leq b \implies \varphi(x) \quad (54)$$

where a, b , and φ may depend on a choice of point $(\hat{x}, \hat{y}) \in \mathcal{D}$ in a sample of the training data. The **robustness** property we presented in the introduction is an example of this.

We call φ the **specification** proper, and we conceive it as a proposition in a propositional theory parameterized by $\hat{x}, \hat{y}, x$ and f_θ that we describe in [Section 6.1](#) below.

Common practice in machine learning [\[31, 32, 53\]](#) broadly follows the work of Fischer et al. [\[30\]](#) in incorporating (54) into the optimisation objective as follows:

$$\operatorname{argmin}_\theta \mathbb{E}_{(\hat{x}, \hat{y}) \sim \mathcal{D}} \left[\mathcal{L}_{\text{pred}}(f_\theta(\hat{x}), \hat{y}) \cap^1 \lambda \sup_{a \leq x \leq b} \llbracket \varphi \rrbracket(f_\theta(x), \hat{y}) \right], \quad (55)$$

where (1) $\mathcal{D} \subseteq X \times Y$ is a sample of the data distribution, (2) $\mathcal{L}_{\text{pred}}$ is any standard loss function concerning the prediction task, (3) $\cap^1$ is an operation we define and motivate below (usually it's just +), and (4) $\llbracket - \rrbracket$ is the logic-specific semantic function that interprets the property φ as a real-valued function.

The inner supremum, computed via Projected Gradient Descent (PGD) [\[24, 50\]](#), makes the optimisation *adversarial*: at each step the optimiser searches for a worst-case violator of $\llbracket \varphi \rrbracket$

in $H = \{a \leq x \leq b\}$, and then nudges θ so as to reduce the violation. The latter part of the optimisation objective is responsible for “property-driven training”.

The function $\llbracket - \rrbracket$ is where a differentiable logic comes in, ‘compiling’ the specification language to (differentiable) real-valued functions. As discussed in [Section 1](#), the state of the art treats this compiler as a heuristic: each choice of $\llbracket - \rrbracket$ comes with its own ad-hoc justification, and none with a theorem relating the *value* of $\llbracket \varphi \rrbracket$ to the logical status of φ .

Rather, we consider $\llbracket - \rrbracket$ to be a *model* ([Definition 5.13](#)), so that we can directly tie the value of $\llbracket \varphi \rrbracket$ to a notion of logical satisfaction of a certain theory, namely the theory of neuro-symbolic specifications at hand. This is adequacy ([Figure 1](#)). By [Theorem 5.18](#), we can furthermore equate satisfaction in the model with provability in the theory—a *logical* form of adequacy.

The rest of this section constructs such a result in the framework of [QLL](#).

6.1 Specifications as formulae in a theory

The property (54) is really a first-order sequent, so it would ideally be accounted for in the logic via first-order facilities, but given a propositional calculus, we content ourselves with accounting for it externally: the adversarial search for $x \in H$ described above discharges the quantifier, and we are left with the propositional specification φ , instantiated at concrete reals. These are the formulae of the following [pQLL](#)^{*}-theory:

Definition 6.1. The **theory of neuro-symbolic specifications** for f_θ at $(\hat{x}, \hat{y}) \in \mathcal{D}$ and $x \in X$, is the grounded theory with set of propositional variables

$$K = \{\ulcorner \hat{y}_1 \urcorner, \dots, \ulcorner \hat{y}_n \urcorner, \ulcorner y_1 \urcorner, \dots, \ulcorner y_n \urcorner\} \quad (56)$$

and induced by the valuation

$$\llbracket \ulcorner \hat{y}_i \urcorner \rrbracket_p = \hat{y}_i, \quad \llbracket \ulcorner y_i \urcorner \rrbracket_p = e^{-f_\theta(x)_i}. \quad (57)$$

This amounts to the rules of [pQLL](#) plus [MIX](#)^{*} plus the following grounding schema

$$\frac{\hat{y}_i}{\vdash \ulcorner \hat{y}_i \urcorner} \quad \frac{e^{-f_\theta(x)_i}}{\vdash \ulcorner y_i \urcorner} \quad \frac{\hat{y}_i^*}{\vdash \ulcorner \hat{y}_i \urcorner^\perp} \quad \frac{e^{f_\theta(x)_i}}{\vdash \ulcorner y_i \urcorner^\perp} \quad (\text{grounding schema})$$

We denote such theory by $\mathbb{T}_{x, \hat{y}, f_\theta}$.

The propositional variables $\lceil y_i \rceil$ represent the outputs of the network, while $\lceil \hat{y}_i \rceil$ represent the labels. The key idea here is to interpret a classifier’s outputs and labels as quantitative propositions in their own right: the atoms of the theory are not statements *about* the network, they are the network’s outputs, read logically.

6.2 Losses as elements of a model

Loss functions for neural models are most commonly given by *additive* quantities: the predictions coming out of classifiers are logits (logarithms of unnormalized probabilities), and so is the cross-entropy used in the prediction part of the loss—with this attitude, $[-\infty, +\infty]$ is commonly known as *logspace* [10].

Therefore, when augmenting a loss function with a logical term $\llbracket \varphi \rrbracket$ as in (55), we need to target the same ‘units’—this is our *naturality* principle from Section 1, guiding our choice of semantics.

We therefore adopt the *additive* semantics of QLL, which is the family of softales defined by the data in Table 2. Note this *semantic* ‘additive’ vs ‘multiplicative’ distinction is unrelated to the *proof-theoretic* one involving the connectives. The semantic distinction concerns which arithmetic operation naturally aggregates such quantities, specifically when the objects they pertain to are combined ‘independently’ (e.g. the entropy of independent random variables, the energy of non-interacting physical systems, etc.).

Clearly, additive and multiplicative reals form isomorphic softales for any choice of p , as witnessed by *Napier’s isomorphism* $-\log \dashv 1/\exp$: conjunctive multiplication $\otimes$ becomes ordinary addition, $\oplus^p$ becomes ‘harmonic’ log-sum-exp, and $\oplus^{-p}$ becomes plain log-sum-exp. To avoid confusion with Table 1, let us use a different notation for additive $\oplus^p$ and $\oplus^{-p}$, and write instead $\cap^p$ and $\cup^p$:

$$a \cap^p b := -\frac{1}{p} \log(e^{-pa} + e^{-pb}), \quad a \cup^p b := \frac{1}{p} \log(e^{pa} + e^{pb}). \quad (58)$$

Note that this isomorphism is order-reversing, meaning additive quantities will be ‘more true’ as they get minimised, and this indeed matches how training losses are interpreted.

Remark 6.2 (Analytical considerations). Crucially, for finite p , the operations are still smooth, strictly entrywise monotone, and have non-vanishing partial derivatives:

$$\partial_x(x \cap^p y) = \frac{e^{py}}{e^{px} + e^{py}}, \quad \partial_x(x \cup^p y) = \frac{e^{px}}{e^{px} + e^{py}}. \quad (59)$$

	polarity	additive	multiplicative
duality $a^* := -a$	positive	$\perp := +\infty$ $a \cup^p b \xrightarrow{p \rightarrow \infty} a \cup^\infty b := \min(a, b)$	$1 := 0$ $a + b \quad -\infty + \infty := +\infty$
	negative	$\top := -\infty$ $a \cap^p b \xrightarrow{p \rightarrow \infty} a \cap^\infty b := \max(a, b)$	$1 := 0$ $a +^- b \quad -\infty +^- \infty := -\infty$

implication: $a \multimap b = a^* +^- b$, thus $b - a$, **enriched order:** $a \sqsubseteq b := e^{-(a \multimap b)}$,

Table 2. The family of softales we collectively refer to as the **additive reals** $[-\infty, +\infty]_{\cup^p, \oplus}$, analogue of Table 1.

The price for this—loss of idempotency—is precisely quantified by Derivation 16: idempotency can be paid for by a factor of $\frac{1}{p} \log 2$, a cost which vanishes as $p \rightarrow \infty$ and is only relevant in the proof-theory, where this is addressed otherwise.

An independent experimental evaluation of this version of QLL against the existing differentiable logics is given by Flinkow et al. [31]—we refer the reader there for the particulars from the optimisation point of view.

6.3 Adequacy and completeness: loss is satisfaction is provability

Every theory of neuro-symbolic specifications has then a canonical interpretation in $[-\infty, +\infty]_{\cup^p, \oplus}$, induced by the valuation of propositional variables

$$\llbracket \ulcorner \hat{y}_i \urcorner \rrbracket_p = -\log \hat{y}_i, \quad \llbracket \ulcorner y_i \urcorner \rrbracket_p = f_\theta(x)_i. \quad (60)$$

For each $0 < p \leq \infty$, we thus get a model we can suggestively denote as

$$\hat{x}, \hat{y}, x, f_\theta \models^p - : \mathbf{LT}^p[\mathbb{T}_{x, \hat{y}, f_\theta}] \rightarrow [-\infty, +\infty]_{\cup^p, \oplus} \quad (61)$$

Definition 6.3. The **quantitative satisfaction** of a specification φ by f_θ at $(\hat{x}, \hat{y})$ and $a \leq x \leq b$ is the semantic validity of $\vdash \varphi$ according to the canonical interpretation of $\mathbf{LT}^p[\mathbb{T}_{x, \hat{y}, f_\theta}]$ in the additive reals:

$$\hat{x}, \hat{y}, x, f_\theta \models^p \varphi = 1 \sqsubseteq \llbracket \varphi \rrbracket_p = e^{-\llbracket \varphi \rrbracket_p}. \quad (62)$$

This is adequacy (Figure 1): every value of the loss $\llbracket \varphi \rrbracket$ has an interpretation in terms of satisfaction by a model. By Theorem 5.18, we can also conclude a logical adequacy result for property-driven training:

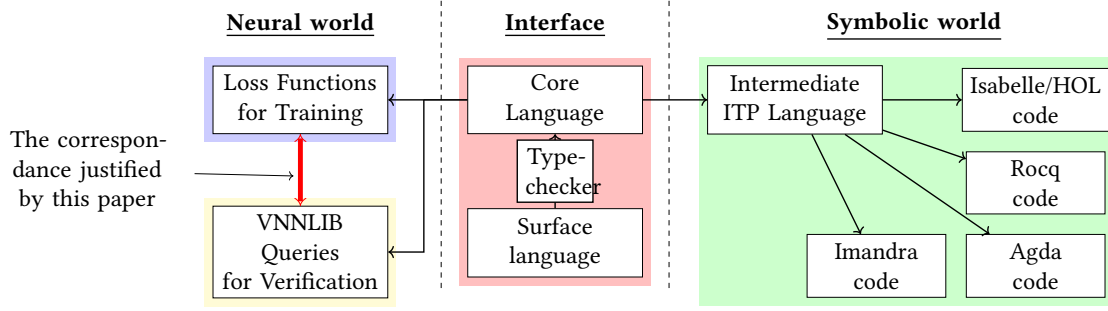


Fig. 4. The architecture of **Vehicle**: a surface language in which neural network specifications are written and type-checked, and then connected to both “neural” and “symbolic” backends.

Theorem 6.4. Given f_θ as above, given $(\hat{x}, \hat{y}) \in \mathcal{D}$ and for each $a \leq x \leq b$, we have

$$\hat{x}, \hat{y}, x, f_\theta \models^p \varphi = |\vdash_{\mathbb{T}_{x, \hat{y}, f_\theta}}^p \varphi|. \quad (63)$$

These results finally allow us to (1) interpret every value of the loss logically, not just the extremal ones, and (2) formally motivate logical loss minimisation as a way to *improve* satisfaction of the specification, since $\llbracket \varphi \rrbracket_p$ decreases as $\hat{x}, \hat{y}, x, f_\theta \models^p \varphi$ increases.

Moreover, the theorem holds for every p , thus working both for the soft semantics used at training time and for the hard semantics which can be used at test time to give $\hat{x}, \hat{y}, x, f_\theta \models^\infty \varphi$ a more traditional logical interpretation. In fact, by [Theorem 4.7](#) (and [Proposition 5.8](#)), we can also deduce a traditional, binary, satisfaction result: $\hat{x}, \hat{y}, x, f_\theta \models \varphi$ if and only if $|\vdash_{\mathbb{T}_{x, \hat{y}, f_\theta}}^p \varphi| > 0$.

7 Conclusions, related and future work

We have set out to supply logical meaning to neuro-symbolic methods by rethinking the fundamentals—calculi, proof validity, and satisfaction—in a quantitative way. We hope that our results invite the community to reconceive differentiable logics not as numerical heuristics, but as bona fide logics in their own right.

From a practical perspective, the long-term goal of this work is to give a formal account of why property-driven training succeeds. More precisely, we seek to characterise the relationship between minimising the differentiable loss generated from a logical specification and satisfying that specification after training. Although this correspondence motivates much of the existing literature on differentiable logics, it has largely remained implicit.

```

1  @parameter
2  p : Real
3
4  qllAdditive : DifferentiableTensorLogic
5  qllAdditive =
6  { trueElement      = -infinity
7    , falseElement   = infinity
8    , pointwiseNegation = \x -> -x
9    , pointwiseConjunction = \{dims} x y ->
10      const (1/p) dims *
11        log(exp(const p dims * x) +
12              exp(const p dims * y))
13    , pointwiseDisjunction = \{dims} x y ->
14      const (1/p) dims *
15        log(exp(const (-p) dims * x) +
16              exp(const (-p) dims * y))
17    , pointwiseLessEqualThan = \x y -> x - y
18    , pointwiseEqual = \x y ->
19      max (x - y) (y - x)
20    , pointwiseNotEqual = \x y ->
21      - max (x - y) (y - x)
22    , ...
23  }
24
25  type Image = Tensor Real [28, 28]
26  type Label = Index 10
27
28  validImage : Image -> Bool
29  validImage x = forall i j . 0 <= x ! i ! j <= 1
30
31  @network
32  classifier : Image -> Tensor Real [10]
33
34  advises : Image -> Label -> Bool
35  advises x i =
36    forall j . classifier x ! i >= classifier x
37      <=> ! j
38
39  @parameter
40  epsilon : Real
41
42  boundedByEpsilon : Image -> Bool
43  boundedByEpsilon x =
44    forall i j . -epsilon <= x ! i ! j <=
45      epsilon
46
47  robustAround : Image -> Label -> Bool
48  robustAround image label = forall
49    <=> perturbation .
50      boundedByEpsilon perturbation and
51      validImage (image + perturbation) =>
52      advises (image + perturbation) label
53
54  @dataset
55  image : Vector Image n
56
57  @dataset
58  label : Vector Label n
59
60  @property
61  robust : Bool
62  robust = forall i .
63    robustAround (image ! i) (label ! i)

```

Listing 1. A *Vehicle* specification expressing the *robustness* property for MNIST task. It starts by specifying the additive semantics of *QLL* (Equation (58)). Here $x ! i$ means accessing the i -th row of the tensor x .

Vehicle [25, 26] provides a concrete illustration of why such a correspondence is valuable. **Vehicle** unifies property-driven training, neural network verification, and interactive theorem proving through a single dependently typed specification language, as illustrated in Figure 4. A specification is written and type-checked *once*, before being compiled to loss functions for training, to satisfiability queries for neural network verifiers [38, 55], and to code for interactive theorem provers. Listing 1 shows a **Vehicle** specification of the classic robustness property introduced in Section 1.1. Using additive *pQLL*, which is itself expressible in the **Vehicle** DSL (lines 1-23), this specification can be used to train a neural network and then verify that the resulting network satisfies it. Our results provide the first crucial step towards providing a formal justification for why success in the former compilation target should assist success in the latter. The red arrow in Figure 4 is a place where Theorem 6.4 can be applied to give a such a justification.

However, the present work is restricted to the propositional fragment of the language. Consequently, it naturally handles quantification over finite index sets, such as the quantifiers on Lines 29, 36, 43, 59 of [Listing 1](#), which are simply syntactic sugar for finite conjunctions. In contrast, the quantifier on Line 46 ranges over the continuous space $\mathbb{R}^{28 \times 28}$ and therefore lies beyond the scope of the current theory. The natural next step is therefore to extend *pQLL* with first-order quantification. We envisage *soft* quantifiers interpreted as integration over probability spaces, following the ideas of [15]. In the longer term, we view *pQLL* as the foundation of a quantitative metatheory for increasingly expressive neuro-symbolic specification languages, ultimately encompassing the full range of compilation targets supported by systems such as **Vehicle**. Initial evidence suggests that such an approach will have practical as well as theoretical benefits: a recent, yet to be reviewed, empirical study [31] has shown that deploying *pQLL* connectives as part of standard neuro-symbolic Python libraries that implement the optimisation objective of [Equation \(55\)](#) outperforms other differentiable logics as measured by the verifiability of the trained network.

Even at the propositional level, many questions remain open. From a linear logic perspective, we would like to study normal forms for *pQLL* proofs (‘quantitative proof nets’), understand how the exponential modalities fit into the picture, and probe potential connections with differential linear logic [29]. Closer to *pQLL* itself, the most pressing puzzle is the status of *MIX** as a logical rule, which hints at a duality theory at the sequent level that we have yet to uncover; questions of completeness likewise remain, alongside the further exploration of links with fuzzy logics and hypersequent calculi, and a clarification of the peculiar ‘additive universal properties’ we only touched upon in [Section 5](#).

Ultimately, the programme we envision requires work on all three sides of the logic-computation-categories trinity. Indeed, while Lawvere tackled the latter aspect and this work begins to approach the first, we are unaware of a computational reading of these ideas.

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A Algebraic lemmata

Lemma A.1 (Arithmetic of generalized fractions). *For all $a, b, c, d \in [0, \infty]$:*

- (1) $1 \leq a \multimap a$,
- (2) $(b \multimap a) \otimes (c \multimap b) \leq (c \multimap a)$,
- (3) $(b \multimap a) \otimes (d \multimap c) \leq (b \otimes d) \multimap (a \otimes c)$,
- (4) $a \multimap (b \multimap c) = (a \otimes b) \multimap c$.

PROOF. For (1), start from $1 \otimes a \leq a^{**}$ and apply the defining property of $(-)^*$ to get $1 \leq (a \otimes a^*)^* = (a^* \otimes^* a) = a \multimap a$. Claim (2)–(4) hold in any symmetric monoidal closed category. $\square$

Remark A.2. Note the above inequalities are all equalities as soon as the numbers involved are finite and non-zero.

Lemma A.3. *Let $a, b, c, d \in [0, \infty]$, $0 < p \leq \infty$. Then*

$$(a \oplus^p b) \otimes (c \oplus^{-p} d) \leq (a \otimes c) \vee (b \otimes d). \quad (64)$$

PROOF. For simplicity, and without risk of ambiguity, below we write $\otimes$ as juxtaposition, i.e. we default to conjunctive multiplication as our multiplication of choice. We also assume, without loss of generality, that $ac = ac \vee bd$. We proceed by case disjunction.

If $a, b, c, d \in (0, \infty)$, then:

$$\begin{aligned}
 & bd \leq ac \\
 \iff & b^p d^p \leq a^p c^p \\
 \iff & b^p \leq a^p c^p d^{-p} \\
 \iff & a^p + b^p \leq a^p + a^p c^p d^{-p} \\
 \iff & a^p + b^p \leq a^p c^p (c^{-p} + d^{-p}) \\
 \iff & (a^p + b^p) (c^{-p} + d^{-p})^{-1} \leq a^p c^p \\
 \iff & (a^p + b^p)^{1/p} (c^{-p} + d^{-p})^{-1/p} \leq ac \\
 \iff & (a \oplus^p b)(c \oplus^{-p} d) \leq ac
 \end{aligned} \quad (65)$$

If $a = 0$ or $c = 0$, since $ac \geq bd$, it follows that $b = 0$ or $d = 0$. We need to prove that $(a \oplus^p b)(c \oplus^{-p} d) = 0$.

- If $a = 0$ and $b = 0$ then $a \oplus^p b = 0$ and 0 is absorbing for $\otimes$.

- If $a = 0$ and $d = 0$ then $c \oplus^{-p} d = 0$ as 0 is absorbing for $\oplus^{-p}$. Then we conclude using that 0 is absorbing for $\otimes$.

By symmetry, the same proof works for $c = 0$.

If $b = 0$ or $d = 0$ then either $a \oplus^p b = 0$ or $c \oplus^{-p} d = 0$, so that $(a \oplus^p b) \otimes (c \oplus^{-p} d) = 0$.

If $a = \infty$ or $c = \infty$, since $a = 0$ and $c = 0$ have already been treated, we consider here that either $a = \infty$ and $c > 0$ (∞ included) or the converse way. This rules out $0 \otimes \infty = 0$, and forces $ac = \infty$, from which we conclude.

If $b = \infty$ or $d = \infty$, the case $d = 0$ has already been treated. So we can assume $d > 0$, but then $bd = \infty$. Since $ac \geq bd$ by hypothesis, $ac = \infty$ and the statement holds. $\square$

B Cut-elimination

As $p\text{QLL}$'s closest relative is multiplicative-additive linear logic [34], our cut-elimination proof resembles the cut-elimination proof given for linear logic in [47] by Melliès. Note that Melliès' presentation is for *intuitionistic* linear logic, so that a sequent always has a single formula as consequence. We have carefully adapted the argument to $p\text{QLL}$, in which multiple formulas are allowed in the conclusion of a sequent. The Linear Logic Handbook [52], in which another cut-elimination proof is provided for classical linear logic (but in a single-sided sequent style), has been an inspiration for this adaptation in several cases.

B.1 The General Setup

Consider a proof π ending in **CUT**:

$$\frac{\frac{\vdots \pi_1}{\Gamma \vdash A, \Delta} R_1 \quad \frac{\vdots \pi_2}{\Gamma', A \vdash \Delta'} R_2}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT} \quad (66)$$

Here A is called the **cut formula**, and the formulae appearing in the other cedents of the premises are thus **non-cut**. Since $p\text{QLL}$ is counary (recall Section 3), we know π does indeed decompose into two proofs as depicted above. The sequent $\Gamma \vdash A, \Delta$ is called the **conclusion** while $\Gamma', A \vdash \Delta'$ is the **hypothesis**. We thus say that this proof presents ' R_1 vs R_2 '.

Here we present an inductive argument that proceeds by rewriting π so as to make **CUT** either appear deeper in the proof or concerning 'smaller' cut formulae. Formally, for each occurrence

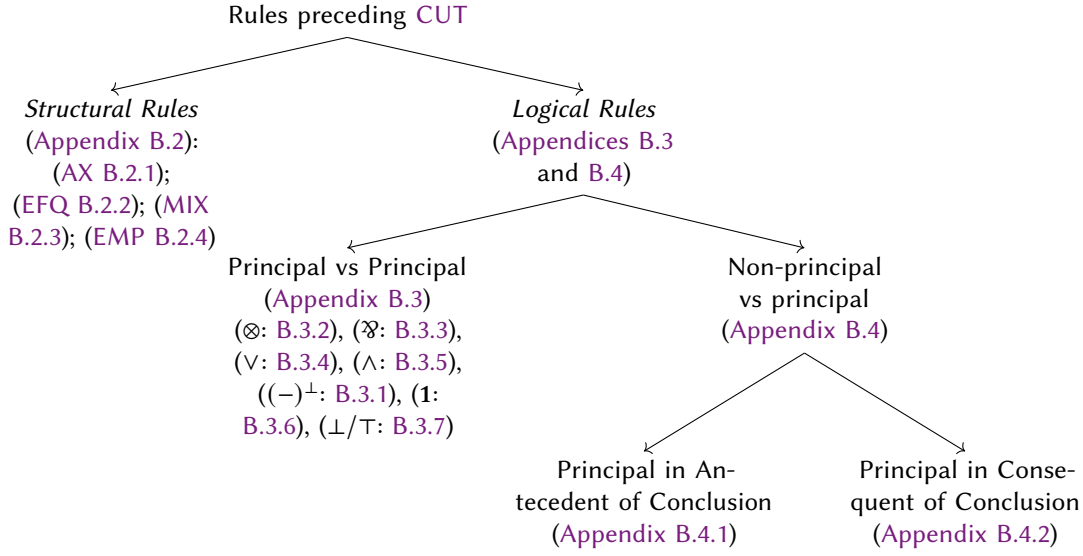


Fig. 5. The structure of the cut-elimination cases, with active hyperreferences to the relevant subsections.

c of a **CUT** in π , we define its **depth** $\text{depth}(c)$ as the number of rules from the leaves of π to c , and its **rank** $\text{rank}(c)$ as the number of connectives in the cut formula.

In line with other linear logic approaches to cut-elimination [47, 52], our inductive argument proceeds by induction on the depth of the **CUT** and the formula rank.

The cases to analyze are distinguished by whether R_1 and R_2 are structural (i.e. from Figure 3a) or non-structural. In the former case, we will proceed, in Appendix B.2, by showing that π can be rewritten to a π' that either contains fewer **CUT**s or where the **CUT**s decrease their depth. In the latter case, we must additionally reason by induction on the formula rank. Recall that a formula in the conclusion of a rule (from e.g. $p\text{QLL}$) is **principal** when it does not occur in the cedents of its premises—note that, by definition, non-structural rules (Figures 3b and 3c) have precisely one principal formula.²² This will be the formula whose rank decreases in this case of rewriting.

Thus, we first consider cases when the cut formula appears as principal formula in either R_1 or R_2 (Appendix B.3), and then when a non-cut formula appears as the principal formula in either the hypothesis or conclusion (Appendix B.4). The resulting case structure is shown in Figure 5, with references to our subsections detailing the proof.

Finally, we give a more rigorous termination argument in Appendix B.5.

²²Note all the formulae in the conclusion of **EFQ** are principal. However, for the purposes of cut-elimination, we still treat this rule as a structural one in Section B.2.2.

B.2 Structural rules

For these cases, we consider rules applied on the left branch. By admissibility of **EE**, the other case is completely analogous.

As for termination, note that for [Sections B.2.1, B.2.2 and B.2.4](#) the number of **CUT** applications decreases from the original proof to its transformed version, while for [Section B.2.3](#) the depth of the **CUT** application decreases.

*B.2.1 **AX** vs hypothesis. Rewrite*

$$\frac{\frac{1}{A \vdash A} \text{ **AX** } \quad \frac{\vdots \pi}{\Gamma, A \vdash \Delta} \otimes}{\Gamma, A \vdash \Delta} \text{ **CUT**}$$

to

$$\frac{\vdots \pi}{\Gamma, A \vdash \Delta}$$

Validity is preserved since $1 \otimes |\pi| = |\pi|$.

*B.2.2 **EFQ** vs hypothesis. Rewrite*

$$\frac{\frac{0}{\Gamma \vdash A, \Delta} \text{ **EFQ** } \quad \frac{\vdots \pi}{\Gamma', A \vdash \Delta'} \otimes}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ **CUT**}$$

to

$$\frac{0}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ **EFQ**}$$

Validity is preserved as $0 \otimes |\pi| = 0$.

*B.2.3 **MIX** vs hypothesis. Rewrite*

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma'_1 \vdash \Delta'_1} \otimes \frac{\vdots \pi_2}{\Gamma'_2 \vdash \Delta'_2, A} \text{ **MIX** } \quad \frac{\vdots \pi_3}{\Gamma, A \vdash \Delta} \otimes}{\Gamma'_1, \Gamma'_2 \vdash A, \Delta'_1, \Delta'_2} \text{ **CUT**}}{\Gamma, \Gamma'_1, \Gamma'_2 \vdash \Delta, \Delta'_1, \Delta'_2}$$

to

$$\frac{\frac{\frac{\vdots \pi_2}{\Gamma'_2 \vdash \Delta'_2, A} \otimes \frac{\vdots \pi_3}{\Gamma, A \vdash \Delta}}{\Gamma, \Gamma'_2 \vdash \Delta, \Delta'_2} \text{CUT} \otimes \frac{\vdots \pi_1}{\Gamma'_1 \vdash \Delta'_1}}{\Gamma, \Gamma'_1, \Gamma'_2 \vdash \Delta, \Delta'_1, \Delta'_2} \text{MIX}$$

Then $(|\pi_1| \otimes |\pi_2|) \otimes |\pi_3| \leq |\pi_1| \otimes (|\pi_2| \otimes |\pi_3|)$ holds by associativity of $\otimes$. The case in which A appears in the other branch of **MIX** is analogous.

B.2.4 EMP vs hypothesis. This case is vacuous since the conclusion of **EMP** does not contain any formula, and thus cannot be the premise of a **CUT**.

B.3 Principal formula vs principal formula

In all the cases of this subsection (except for **B.3.6** and **B.3.7**), our proof shows that the rank of the cut formula decreases from the original proof to its transformed version. In those last two cases, we reduce the number of cut occurrences.

B.3.1 Dualities (rules $(-)_R^\perp$ and $(-)_L^\perp$). Rewrite

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma, A \vdash \Delta}}{\Gamma \vdash A^\perp, \Delta} (-)_R^\perp \otimes \frac{\frac{\frac{\vdots \pi_2}{\Gamma \vdash A, \Delta'}}{\Gamma', A^\perp \vdash \Delta'} (-)_L^\perp}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT}$$

to

$$\frac{\frac{\vdots \pi_2}{\Gamma' \vdash A, \Delta'} \otimes \frac{\vdots \pi_1}{\Gamma, A \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT}$$

We know that $|\pi_1| \otimes |\pi_2| = |\pi_2| \otimes |\pi_1|$ since $\otimes$ is commutative.

B.3.2 Tensor (rules $\otimes_L$ and $\otimes_R$). Rewrite

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma \vdash A, \Delta} \otimes \frac{\vdots \pi_2}{\Gamma' \vdash B, \Delta'}}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \otimes_R \frac{\frac{\vdots \pi_3}{\Gamma'' \vdash A, B \vdash \Delta''}}{\Gamma'', A \otimes B \vdash \Delta''} \otimes_L}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''} \text{CUT}$$

to

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma \vdash A, \Delta} \otimes \frac{\frac{\frac{\vdots \pi_2}{\Gamma' \vdash B, \Delta'} \otimes \frac{\vdots \pi_3}{\Gamma'', A, B \vdash \Delta''}}{\Gamma', \Gamma'', A \vdash \Delta', \Delta''} \text{CUT}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''} \text{CUT}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''} \text{CUT}$$

Then $(|\pi_1| \otimes |\pi_2|) \otimes |\pi_3| = |\pi_1| \otimes (|\pi_2| \otimes |\pi_3|)$, by associativity of tensor.

B.3.3 Par (rules $\wp_L$ and $\wp_R$). Follows by [Appendix B.3.1](#) and [Appendix B.3.2](#), since, by duality, a cut along $A \wp B$ corresponds to a cut along $A^\perp \otimes B^\perp$. Indeed the proof:

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma \vdash \Delta, A, B} \wp_R \otimes \frac{\frac{\frac{\vdots \pi_2}{\Gamma', A \vdash \Delta'} \otimes \frac{\vdots \pi_3}{\Gamma'', B \vdash \Delta''}}{\Gamma', \Gamma'', A \wp B \vdash \Delta', \Delta''} \wp_L}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''} \text{CUT}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''} \text{CUT}$$

can be rewritten to the following proof, which falls into the cases previously covered:

$$\frac{\frac{\frac{\frac{\vdots \pi_2}{\Gamma', A \vdash \Delta'} \text{ } (-)_R^\perp \otimes \frac{\frac{\vdots \pi_3}{\Gamma'', B \vdash \Delta''} \text{ } (-)_R^\perp}{\Gamma', \Gamma'' \vdash A^\perp \otimes B^\perp, \Delta', \Delta''} \otimes_R \frac{\frac{\frac{\vdots \pi_1}{\Gamma \vdash \Delta, A, B} \text{ } (-)_L^\perp}{\Gamma, B^\perp \vdash \Delta, A} \text{ } (-)_L^\perp}{\Gamma, A^\perp \otimes B^\perp \vdash \Delta} \otimes_L}{\Gamma, \Gamma', \Gamma'' \vdash \Delta', \Delta'', \Delta} \text{CUT}$$

B.3.4 Soft Disjunction (rules $\vee_L$ and $\vee_R$). We have:

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma \vdash A, \Delta} \oplus^p \frac{\vdots \pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \vee B, \Delta} \vee_R \otimes \frac{\frac{\frac{\vdots \pi_3}{\Gamma', A \vdash \Delta'} \oplus^{-p} \frac{\vdots \pi_4}{\Gamma', B \vdash \Delta'}}{\Gamma', A \vee B \vdash \Delta'} \vee_L}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT}$$

There are two possible cut-elimination strategies here.

Choice A. Rewrite to:

$$\frac{\frac{\vdots \pi_1}{\Gamma \vdash A, \Delta} \otimes \frac{\vdots \pi_3}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT}$$

Choice B. Rewrite to:

$$\frac{\begin{array}{c} \vdots \pi_2 \\ \Gamma \vdash B, \Delta \end{array} \otimes \begin{array}{c} \vdots \pi_4 \\ \Gamma', B \vdash \Delta' \end{array}}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT}$$

We showed in [Lemma A.3](#) that:

$$(|\pi_1| \oplus^p |\pi_2|) \otimes (|\pi_3| \oplus^{-p} |\pi_4|) \leq (|\pi_1| \otimes |\pi_3|) \vee (|\pi_2| \otimes |\pi_4|).$$

Thus perform the rewrite of maximal validity (and, by convention, the one involving A if both validities are equal). Note this is a deterministic rewrite strategy.

B.3.5 Soft Conjunction (rules $\wedge_L$ and $\wedge_R$). Follows by [Appendix B.3.1](#) and [Appendix B.3.4](#), since a cut along $A \wedge B$ corresponds to a cut along $A^\perp \vee B^\perp$ (analogously to [Appendix B.3.3](#)).

B.3.6 Multiplicative unit (1_L and 1_R). Due to the context restrictions of 1_L and 1_R , there is only one possible such proof:

$$\frac{\frac{1}{\vdash 1} \quad 1_R \quad \otimes \quad \frac{1}{1 \vdash} \quad 1_L}{\vdash} \text{CUT}$$

Rewrite this to

$$\frac{1}{\vdash} \text{EMP}$$

Clearly $1 \otimes 1 = 1$, and the number of cut applications decreases.

B.3.7 Additive units ($\perp_L$ and $\top_R$). Since $\perp$ (resp. $\top$) only has an introduction rule for the left (resp. right) side of a sequent, the only way for a **CUT** to use $\top$ or $\perp$ as cut formula is to rely on **EFQ** to cover the missing introduction rule. Then we proceed as in ([Section B.2.2](#)), by rewriting the cut to a **EFQ** rule introducing the same conclusion, and this is validity-preserving (always with validity 0).

B.4 Non-principal formula vs principal formula

The final case we consider is when the cut formula is not principal in both of the rules applied to the premises of the **CUT**. Let us use the usual terminology and say that in $\Gamma \vdash \Delta$, the cedent Γ is the antecedent and Δ is the consequent. This case splits into four subcases, depending on:

- whether the principal formula appears in the conclusion or in the hypothesis of the **CUT** rule,

- and whether the principal formula is in the antecedent or the consequent of that conclusion/hypothesis.

We treat all the cases in which this principal formula belongs to the conclusion (left premise) of the **CUT**. The cases when the principal formula belongs to the **CUT**'s hypothesis are symmetric.

B.4.1 Principal Formula is in the antecedent of the conclusion of the **CUT.** In this subsection, we consider the following cases when a principal formula $A \bullet B$ is contained in the antecedent of the conclusion of the **CUT**. In all cases of this section, the proof transformation decreases the depth of the cut.

B.4.1.1 Rule $\otimes_L$. Rewrite

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma, C, D \vdash A, \Delta}}{\Gamma, C \otimes D \vdash A, \Delta} \otimes_L \quad \frac{\vdots \pi_2}{\Gamma', A \vdash \Delta'} \otimes}{\Gamma, C \otimes D, \Gamma' \vdash \Delta, \Delta'} \text{CUT}$$

to

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma, C, D \vdash A, \Delta} \otimes \quad \frac{\vdots \pi_2}{\Gamma', A \vdash \Delta'}}{\Gamma, C, D, \Gamma' \vdash \Delta, \Delta'} \text{CUT}}{\Gamma, C \otimes D, \Gamma' \vdash \Delta, \Delta'} \otimes_L$$

B.4.1.2 Rule $\wp_L$. Without loss of generality, suppose the cut formula comes from the left premise of $\wp_L$. Rewrite:

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma_1, C \vdash A, \Delta_1} \otimes \quad \frac{\vdots \pi_2}{\Gamma_2, D \vdash \Delta_2}}{\Gamma_1, C \wp D, \Gamma_2 \vdash A, \Delta_1, \Delta_2} \wp_L \quad \frac{\vdots \pi_3}{\Gamma', A \vdash \Delta'} \otimes}{\Gamma_1, C \wp D, \Gamma_2, \Gamma' \vdash \Delta_1, \Delta_2, \Delta'} \text{CUT}$$

to

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma_1, C \vdash A, \Delta_1} \otimes \quad \frac{\vdots \pi_3}{\Gamma', A \vdash \Delta'}}{\Gamma_1, C, \Gamma' \vdash \Delta_1, \Delta'} \text{CUT} \quad \frac{\vdots \pi_2}{\Gamma_2, D \vdash \Delta_2} \otimes}{\Gamma_1, C \wp D, \Gamma_2, \Gamma' \vdash \Delta_1, \Delta_2, \Delta'} \wp_L$$

We have $(|\pi_1| \otimes |\pi_2|) \otimes |\pi_3| = (|\pi_1| \otimes |\pi_3|) \otimes |\pi_2|$ by associativity and commutativity of $\otimes$.

B.4.1.3 Rule $\vee_L$. Rewrite

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma, A \vdash C, \Delta} \oplus^{-p} \quad \frac{\vdots \pi_2}{\Gamma, B \vdash C, \Delta}}{\Gamma, A \vee B \vdash C, \Delta} \vee_L \quad \frac{\vdots \pi_3}{\Gamma', C \vdash \Delta'}}{\Gamma, A \vee B, \Gamma' \vdash \Delta, \Delta'} \text{CUT}$$

to

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma, A \vdash C, \Delta} \otimes \quad \frac{\vdots \pi_3}{\Gamma', C \vdash \Delta'}}{\Gamma, A, \Gamma' \vdash \Delta, \Delta'} \text{CUT} \quad \oplus^{-p} \quad \frac{\frac{\frac{\vdots \pi_2}{\Gamma, B \vdash C, \Delta} \otimes \quad \frac{\vdots \pi_3}{\Gamma', C \vdash \Delta'}}{\Gamma, B, \Gamma' \vdash \Delta, \Delta'} \text{CUT}}{\Gamma, A \vee B, \Gamma' \vdash \Delta, \Delta'} \vee_L$$

Validity is preserved by [Lemma 2.4](#).

B.4.1.4 Rule $\wedge_L$. Rewrite

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma, A \vdash C, \Delta} \oplus^p \quad \frac{\vdots \pi_2}{\Gamma, B \vdash C, \Delta}}{\Gamma, A \wedge B \vdash C, \Delta} \wedge_L \quad \frac{\vdots \pi_3}{\Gamma', C \vdash \Delta'}}{\Gamma, A \wedge B, \Gamma' \vdash \Delta, \Delta'} \text{CUT}$$

to

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma, A \vdash C, \Delta} \otimes \quad \frac{\vdots \pi_3}{\Gamma', C \vdash \Delta'}}{\Gamma, A, \Gamma' \vdash \Delta, \Delta'} \text{CUT} \quad \oplus^p \quad \frac{\frac{\frac{\vdots \pi_2}{\Gamma, B \vdash C, \Delta} \otimes \quad \frac{\vdots \pi_3}{\Gamma', C \vdash \Delta'}}{\Gamma, B, \Gamma' \vdash \Delta, \Delta'} \text{CUT}}{\Gamma, A \wedge B, \Gamma' \vdash \Delta, \Delta'} \wedge_L$$

Validity is preserved by [Lemma 2.3](#).

B.4.1.5 Rule $(-)_L^\perp$. Rewrite

$$\frac{\frac{\vdots \pi_1}{\Gamma \vdash A, B, \Delta} \quad \frac{\vdots \pi_2}{\Gamma', B \vdash \Delta'} \quad (-)_L^\perp \quad \otimes}{\Gamma, A^\perp \vdash B, \Delta} \quad \text{CUT}$$

to

$$\frac{\frac{\vdots \pi_1}{\Gamma \vdash B, A, \Delta} \quad \frac{\vdots \pi_2}{\Gamma', B \vdash \Delta'} \quad \otimes}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'} \quad \text{CUT} \quad (-)_L^\perp$$

B.4.1.6 Rule 1_L . This rule's conclusion has no side cedents, therefore it cannot appear as non-principal, and this case is vacuous.

B.4.1.7 Rule $\perp_L$. Rewrite

$$\frac{\frac{\infty}{\Gamma, \perp \vdash A, \Delta} \quad \frac{\vdots \pi}{\Gamma', A \vdash \Delta'} \quad \perp_L \quad \otimes}{\Gamma, \perp, \Gamma' \vdash \Delta, \Delta'} \quad \text{CUT}$$

to

$$\frac{\infty}{\Gamma, \perp, \Gamma' \vdash \Delta, \Delta'} \quad \perp_L$$

Validity is non-decreasing since $\infty \otimes |\pi| \leq \infty$, and the number of cut occurrences decreases.

B.4.2 Principal Formula is in the consequent of the conclusion of the CUT . We consider now the case in which the cut formula is again not principal in *both* premises of the CUT , but now we focus on a principal formula that appears as the consequent of the conclusion of the CUT rule.

This case is perfectly dual to the previous one.

Indeed, consider the following case of induction, where CUT needs to be commuted with a right introduction rule for some logical connective we denote by $\bullet$:

$$\frac{\frac{\vdots \pi_1}{\Gamma \vdash \Delta, C \bullet D, A} \quad \frac{\vdots \pi_2}{\Gamma', A \vdash \Delta'} \quad \bullet_R \quad \otimes}{\Gamma, \Gamma' \vdash \Delta, \Delta', C \bullet D} \quad \text{CUT}$$

This can be rewritten to the following proof, which falls into the cases previously covered:

$$\frac{\frac{\frac{\vdots \pi_3}{\Gamma, C^\perp \bullet^\perp D^\perp \vdash \Delta, A} \bullet_R^\perp \quad \frac{\vdots \pi_2}{\Gamma', A \vdash \Delta'} \otimes}{\Gamma, \Gamma', C^\perp \bullet^\perp D^\perp \vdash \Delta, \Delta'} \text{CUT}}{\Gamma, \Gamma' \vdash \Delta, \Delta', C \bullet D} (-)_R^\perp$$

To define π_3 , note that $\bullet_R^\perp$ has the same amount of hypotheses as $\bullet_L$ does. In the unary case, π_3 is obtained by applying $(-)_L^\perp$ twice (to C and D), and then composing with π_1 . In the binary case, π_1 has for conclusion hypersequent a tensor of two sequents, so that $\pi_1 = \pi_{1L} \bullet \pi_{1R}$ for the appropriate $\bullet$ used by the rule under study. Then define π_{3L} (resp. π_{3R}) by applying $(-)_L^\perp$ twice (to C and D), and then composing with π_{1L} (resp. π_{1R}). Remark that $\pi_3 = \pi_{3L} \bullet \pi_{3R}$ provides a proof of the same validity.

B.5 The Termination argument

Let us use the notation from [Appendix B.1](#). For a proof π , define the metrics $\text{met}(\pi) = (r_m, n_m, n, d)$ such that if π is cut-free, then $\text{met}(\pi) = (0, 0, 0, 0)$, and else:

- (1) r_m is the maximal rank of the cut formula of a cut occurrence in π ,
- (2) n_m is the number of cut occurrences of maximal rank in π ,
- (3) n is the total number of cut occurrences in π ,
- (4) d is the sum of the depths of all cut occurrences in π .

We consider the lexicographic order over such tuples, which is well-founded. Now every step of the cut-elimination process is strictly decreasing for this order, so that the process needs to terminate. Note that the process is defined for every possible case involving a cut occurrence, so it can only halt on a cut-free proof: else, a rewrite could be applied.

B.6 Decidability

Recall the definition of [provability](#). As observed above, the cut-elimination process is validity non-decreasing and thus we know the provability of a sequent, if it is realized, can be realized by a [CUT](#)-free proof. Note moreover that:

- (1) Explicit dualities in [pQLL](#) (introduced via the rules $(-)_L^\perp$ and $(-)_R^\perp$) can be eliminated without affecting validity, so that we can restrict our attention to derivations of $\Gamma \vdash \Delta$ that do not use [CUT](#) nor $(-)_L^\perp$ nor $(-)_R^\perp$. Call these **reduced**.

- (2) We can, again without change in validity, remove any **MIX** which uses the empty sequent in one of its branches.
- (3) By inspecting the remaining rules and cases, we observe a form of subformula property: each premise of a rule, if any, is of strictly smaller complexity (understood in the traditional sense as the sum of the number of logical connectives in each formula, over all the formulas appearing in contexts on both sides of the turnstile).

Due to this subformula property, reduced proofs are in finite number, and they can be effectively enumerated. The validity of each **pQLL** derivation is computable up to any desired precision by real arithmetic.²³ It follows that provability is realized by at least one proof (since there are finitely many proofs of potential maximal validity), and that proof is computable via the arguments just raised.

C Equivalence of **MALL** and ∞ **QLL**

C.1 Proof of Theorem 4.4

To show that qualitative **pQLL** is equivalent to **HMALL** we turn to the observations of Section 3.1.1 regarding hypersequent and **Prop**-enriched calculi. In particular, it is straightforward to translate proof rules from a hypersequent calculus to admissible proof rules in a **Prop**-enriched one admitting **EFQ**, following the same procedure outlined in Lemma 3.7. Specifically, given proofs π_i as below left, any of them either proves $\mathcal{G}$ or $\mathcal{H}_i$. Thus define $\pi_{\mathcal{G}}$ to be any of those proving $\mathcal{G}$, or **EFQ** in case there is none, and likewise for $\pi_{\mathcal{H}_i}$. We thus obtain a proof as below right:

$$\frac{\begin{array}{c} \vdots \pi_1 \\ (\mathcal{G} \vee \mathcal{H}_1) \end{array} \wedge \cdots \wedge \begin{array}{c} \vdots \pi_n \\ (\mathcal{G} \vee \mathcal{H}_n) \end{array}}{\mathcal{G} \vee \mathcal{K}} \mapsto \frac{\begin{array}{c} \vdots \pi_{\mathcal{G}} \\ \mathcal{G} \end{array} \vee \begin{array}{c} \vdots \pi_{\mathcal{H}_1} \wedge \cdots \wedge \pi_{\mathcal{H}_n} \\ (\mathcal{H}_1 \wedge \cdots \wedge \mathcal{H}_n) \end{array}}{\mathcal{G} \vee \mathcal{K}}$$

²³ Observe, in particular, that it is trivial to decide whether the validity of a proof is non-zero, since such validity is given as a finite expression involving only 0, 1, and ∞ whose positivity can be checked inductively.

(a) Structural Rules

$$\frac{}{A \vdash A} \text{AX} \quad \frac{}{\vdash} \text{EMP}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT} \quad \frac{\Gamma \vdash \Delta \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{MIX}$$

(b) Rules for Multiplicatives

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \otimes_L \quad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \wp B, \Delta} \wp_R$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \wp B \vdash \Delta, \Delta'} \wp_L \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \otimes_R$$

$$\frac{}{1 \vdash} 1_L \quad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^\perp \vdash \Delta} \perp_L \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} \perp_R \quad \frac{}{\vdash 1} 1_R$$

(c) Rules for Additives

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee_L \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge_R$$

$$\frac{\Gamma, A_i \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta} \wedge_{Li} \quad i = 1, 2 \quad \frac{\Gamma \vdash A_i, \Delta}{\Gamma \vdash A_1 \vee A_2, \Delta} \vee_{Ri}$$

$$\frac{}{\Gamma, \perp \vdash \Delta} \perp_L \quad \frac{}{\Gamma \vdash \top, \Delta} \top_R$$

Fig. 6. *MALL*

Vice versa, given proofs in a **Prop**-enriched calculus like qualitative *pQLL*, we define

$$\frac{\frac{\vdots \pi_1 \quad \vdots \pi_2}{\mathcal{H}_1 \vee \mathcal{H}_2} \quad \mathcal{K}}{\mathcal{K}} \mapsto \frac{\frac{\vdots \pi_2 \mid \pi_2}{\emptyset \mid \mathcal{H}_1 \mid \mathcal{H}_2} \quad \mathcal{K}}{\emptyset \mid \mathcal{K}} \quad (67)$$

$$\frac{\frac{\vdots \pi_1 \quad \vdots \pi_2}{\mathcal{H}_1 \wedge \mathcal{H}_2} \quad \mathcal{K}}{\mathcal{K}} \mapsto \frac{\frac{\vdots \pi_2 \quad \vdots \pi_2}{\emptyset \mid \mathcal{H}_1 \quad \emptyset \mid \mathcal{H}_2} \quad \mathcal{K}}{\emptyset \mid \mathcal{K}}$$

The language of *MALL* is, up to relabeling of connectives, the same as the one introduced in Definition 3.5 for $p = \infty$. We can then easily draw a correspondence between the rules of one and the other (compare Figure 7 and Figure 8), with a few exceptions to highlight:

(a) Structural Rules

$$\frac{T}{A \vdash A} \text{AX} \quad \frac{T}{\vdash} \text{EMP} \quad \frac{F}{\Gamma \vdash \Delta} \text{EFQ}$$

$$\frac{\Gamma \vdash A, \Delta \quad \wedge \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT} \quad \frac{\Gamma \vdash \Delta \quad \wedge \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{MIX}$$

(b) Rules for Multiplicatives

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \otimes_L \quad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \wp B, \Delta} \wp_R$$

$$\frac{\Gamma, A \vdash \Delta \quad \wedge \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \wp B \vdash \Delta, \Delta'} \wp_L \quad \frac{\Gamma \vdash A, \Delta \quad \wedge \quad \Gamma', \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \otimes_R$$

$$\frac{T}{1 \vdash} 1_L \quad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^\perp \vdash \Delta} \perp_L \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} \perp_R \quad \frac{T}{\vdash 1} 1_R$$

(c) Rules for Additives

$$\frac{\Gamma, A \vdash \Delta \quad \wedge \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee_L^p \quad \frac{\Gamma \vdash A, \Delta \quad \wedge \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge_R^p$$

$$\frac{\Gamma, A \vdash \Delta \quad \vee \quad \Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge_L^p \quad \frac{\Gamma \vdash A, \Delta \quad \vee \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee_R^p$$

$$\frac{T}{\Gamma, \perp \vdash \Delta} \perp_L \quad \frac{T}{\Gamma \vdash \top, \Delta} \top_R$$

Fig. 7. The qualitative version of $pQLL$.

- (1) EFQ in qualitative $pQLL$ actually translates to external weakening in the hypersequent calculus, with $\mathcal{G} = \emptyset$,
- (2) *vice versa*, the analogue of [Lemma 3.7](#) for **Prop**-enriched calculi makes the translations of external weakening and contraction admissible in quantitative $pQLL$.

Thus we obtain [Theorem 4.4](#).

C.2 Proof of [Theorem 4.7](#)

The claim boils down to a correspondence between rules that allows us to translate proofs directly between the two systems. Such a correspondence is mostly evident by comparing [Figure 3](#) with [Figure 6](#).

$$\begin{array}{c}
\text{(a) Internal Structural Rules} \\
\frac{}{\mathcal{G} \mid A \vdash A} \text{AX} \quad \frac{}{\mathcal{G} \mid \vdash} \text{EMP} \\
\\
\frac{\mathcal{G} \mid \Gamma \vdash A, \Delta \quad \mathcal{G} \mid \Gamma', A \vdash \Delta'}{\mathcal{G} \mid \Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT} \quad \frac{\mathcal{G} \mid \Gamma \vdash \Delta \quad \mathcal{G} \mid \Gamma' \vdash \Delta'}{\mathcal{G} \mid \Gamma, \Gamma' \vdash \Delta, \Delta'} \text{MIX} \\
\\
\text{(b) External Structural Rules} \\
\frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}} \text{EW} \quad \frac{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H}} \text{EC} \\
\\
\text{(c) Rules for Multiplicatives} \\
\frac{\mathcal{G} \mid \Gamma, A, B \vdash \Delta}{\mathcal{G} \mid \Gamma, A \otimes B \vdash \Delta} \otimes_L \quad \frac{\mathcal{G} \mid \Gamma \vdash A, B, \Delta}{\mathcal{G} \mid \Gamma \vdash A \wp B, \Delta} \wp_R \\
\\
\frac{\mathcal{G} \mid \Gamma, A \vdash \Delta \quad \mathcal{G} \mid \Gamma', B \vdash \Delta'}{\mathcal{G} \mid \Gamma, \Gamma', A \wp B \vdash \Delta, \Delta'} \wp_L \quad \frac{\mathcal{G} \mid \Gamma \vdash A, \Delta \quad \mathcal{G} \mid \Gamma' \vdash B, \Delta'}{\mathcal{G} \mid \Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \otimes_R \\
\\
\frac{}{\mathcal{G} \mid 1 \vdash} 1_L \quad \frac{\mathcal{G} \mid \Gamma \vdash A, \Delta}{\mathcal{G} \mid \Gamma, A^\perp \vdash \Delta} \perp_L \quad \frac{\mathcal{G} \mid \Gamma, A \vdash \Delta}{\mathcal{G} \mid \Gamma \vdash A^\perp, \Delta} \perp_R \quad \frac{}{\mathcal{G} \mid \vdash 1} 1_R \\
\\
\text{(d) Rules for Additives} \\
\frac{\mathcal{G} \mid \Gamma, A \vdash \Delta \quad \mathcal{G} \mid \Gamma, B \vdash \Delta}{\mathcal{G} \mid \Gamma, A \vee B \vdash \Delta} \vee_L \quad \frac{\mathcal{G} \mid \Gamma \vdash A, \Delta \quad \mathcal{G} \mid \Gamma \vdash B, \Delta}{\mathcal{G} \mid \Gamma \vdash A \wedge B, \Delta} \wedge_R \\
\\
\frac{\mathcal{G} \mid \Gamma, A \vdash \Delta \mid \Gamma, B \vdash \Delta}{\mathcal{G} \mid \Gamma, A \wedge B \vdash \Delta} \wedge_L \quad \frac{\mathcal{G} \mid \Gamma \vdash A, \Delta \mid \Gamma \vdash B, \Delta}{\mathcal{G} \mid \Gamma \vdash A \vee B, \Delta} \vee_R \\
\\
\frac{}{\mathcal{G} \mid \Gamma, \perp \vdash \Delta} \perp_L \quad \frac{}{\mathcal{G} \mid \Gamma \vdash \top, \Delta} \top_R
\end{array}$$

Fig. 8. The hypersequent calculus for isomix MALL (adapted from [48, Fig 4.1]), *HMALL*.

Indeed, when going from *MALL* to ∞ QLL, the only sticking point is the admissibility of the additive fragment of the former in the latter, which we established in Lemma 4.5. As for the converse, note that a proof in ∞ QLL can only have 0, 1, or ∞ as validities. Moreover, a proof translated from *MALL* does not use EFQ, and thus has validity greater or equal than 1, thus proving one half of the claim.

Conversely, let P be a ∞ QLL proof of $\Gamma \vdash \Delta$ of validity greater than 1. By Lemma 4.5 and Lemma 4.6, we can assume P is EFQ-free and only uses the unary versions of $\vee_R^\infty$ and $\wedge_L^\infty$. Then

the translation of rules in this direction is straightforward, once noted that, since $a \otimes b = 0$ iff either $a = 0$ or $b = 0$, the rules of ∞QLL involving $\otimes$ can be replaced as-is with those in MALL .

D Soundness

We prove now [Proposition 5.10](#).

For brevity and without loss of generality, when a cedent $\Gamma^\bullet$ or Δ° has to appear, we replace it by a single variable for an element of $\mathcal{S}$.

We start by noting that our translation of cedents is well-posed (that is, respects multiset structure) thanks to the unitality, associativity, and commutativity axioms, that is [\(32\)](#). Likewise, we can use 1 on both sides because $1^{\perp\perp} = 1$, that is [\(33\)](#). We thus employ these facts tacitly below.

We start from the following rules, which together with the axioms for involutivity [\(33\)](#) and duality [\(34\)](#) allow us to cut the subsequent cases in half.

$(-)_R^\perp$ is sound by [\(35\)](#): $(a, b \sqsubseteq c) = (a \sqsubseteq (b \bullet c^\perp)^\perp) = (a \sqsubseteq b^\perp \circ c)$.

$(-)_L^\perp$ holds by duality, whose application we illustrate here once and for all:

$$(a \sqsubseteq b \circ c) \stackrel{(34)}{=} (b \bullet c \sqsubseteq a) \stackrel{(-)_R^\perp}{=} (c \sqsubseteq b^\perp \circ a) \stackrel{(34)}{=} (a \bullet b \sqsubseteq c) \quad (68)$$

Note the use of commutativity as well as the fact $a \circ b = (a^\perp \bullet b^\perp)^\perp$.

We then have:

AX $1 \leq (a \sqsubseteq a)$ by [\(reflexivity\)](#) for enriched categories.

EMP, 1_L , and 1_R These all amount (thanks also to $1^\perp = 1$ from [\(33\)](#)) to checking that $1 \leq (1 \sqsubseteq 1)$, which holds by [\(reflexivity\)](#).

EFQ This rule corresponds to the trivial fact $0 \leq (a \sqsubseteq b)$.

CUT After moving the context to the outer cedents, it is enough to check that $(a \sqsubseteq b) \otimes (b \sqsubseteq c) \leq (a \sqsubseteq c)$, which holds by [\(transitivity\)](#).

MIX This follows from [\(31\)](#) and [\(40\)](#) (right):

$$\begin{aligned} (a \sqsubseteq b) \otimes (a' \sqsubseteq b') \otimes 1 &\stackrel{(31) \otimes (40)}{\leq} ((a \bullet a') \sqsubseteq (b \bullet b')) \otimes ((b \bullet b') \sqsubseteq (b \circ b')) \\ &\leq (a \bullet a') \sqsubseteq (b \circ b'). \end{aligned} \quad (69)$$

$\otimes_L$ and $\wp_R$ hold by definition,

$\otimes_R$ and $\wp_L$ It suffices to prove the first as the second is formally dual, and, once the context is all moved on the left side, the first is directly [\(31\)](#).

$\vee_L$ and $\wedge_R$ It suffices to prove the first as the second is formally dual, and the first is directly (37). Specifically, we move all the context on the right then invoke (37).

$\wedge_L$ and $\vee_R$ It suffices to prove the first as the second is formally dual, and the first is directly (38).

$\perp_L$ and $\top_R$ It suffices to prove the first as the second is formally dual, and the first is directly (36).

E Construction of the classifying softale

Here we prove that Definition 5.12 is well-posed.

We start by observing $\text{LT}^p[\mathbb{T}]$ is a $[0, \infty]_\otimes$ -preorder. Indeed, reflexivity is granted by AX and transitivity by CUT . The way this is proven is paradigmatic for all the other inequations below, namely by exhibiting a derivation starting from the left-hand side and ending in the right-hand side, e.g.

$$1 \leq |A \vdash A| \iff \frac{1}{A \vdash A} \text{AX}$$

$$|A \vdash B| \otimes |B \vdash C| \leq |A \vdash C| \iff \frac{A \vdash B \quad B \vdash C}{A \vdash C} \text{CUT}$$

We now similarly prove each structural equation (the items below are numbered by the equation they prove in Definition 5.2).

(31) $|A \vdash B| \otimes |D \vdash C| \leq |A \otimes D \vdash B \otimes C|$ is witnessed by the following derivation:

$$\frac{\frac{A \vdash B \quad D \vdash C}{A \otimes D \vdash B, C} \otimes_R}{A \otimes D \vdash B \otimes C} \otimes_L \quad (70)$$

(32) We must prove $1 \leq |A \otimes 1 \vdash A|$ and $1 \leq |A \vdash A \otimes 1|$:

$$\frac{\frac{\frac{1}{A \vdash A} \text{AX} \quad \frac{1}{1 \vdash 1} \text{1}_L}{A, 1 \vdash A} \otimes}{A \otimes 1 \vdash A} \text{MIX} \quad \frac{\frac{1}{A \vdash A} \text{AX} \quad \frac{1}{1 \vdash 1} \text{1}_R}{A \vdash A \otimes 1} \otimes_R \quad (71)$$

The associativity and commutativity of $\otimes$ are witnessed by the same proofs that witness them for MALL . Here are the two directions of associativity, both witnessed to have validity

greater than 1:

$$\begin{array}{c}
 \frac{\frac{1}{A \vdash A} \text{AX}}{\frac{1}{A \vdash A} \text{AX} \otimes \frac{\frac{1}{C \vdash C} \text{AX} \otimes \frac{1}{B \vdash B} \text{AX}}{B, C \vdash B \otimes C} \otimes_R} \otimes_R \\
 \frac{A, B, C \vdash A \otimes (B \otimes C)}{A \otimes B, C \vdash A \otimes (B \otimes C)} \otimes_L \\
 \frac{A \otimes B, C \vdash A \otimes (B \otimes C)}{(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)} \otimes_L
 \end{array} \quad (72)$$

$$\begin{array}{c}
 \frac{\frac{1}{A \vdash A} \text{AX} \otimes \frac{1}{B \vdash B} \text{AX}}{A, B \vdash A \otimes B} \otimes_R \otimes \frac{\frac{1}{C \vdash C} \text{AX}}{C \vdash C} \otimes_R \\
 \frac{A, B, C \vdash (A \otimes B) \otimes C}{A, B \otimes C \vdash (A \otimes B) \otimes C} \otimes_L \\
 \frac{A, B \otimes C \vdash (A \otimes B) \otimes C}{A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C} \otimes_L
 \end{array} \quad (73)$$

Commutativity is witnessed in both directions by the same proof:

$$\begin{array}{c}
 \frac{\frac{1}{A \vdash A} \text{AX} \otimes \frac{1}{B \vdash B} \text{AX}}{B, A \vdash B \otimes A} \otimes_R \\
 \frac{A, B \vdash B \otimes A}{A \otimes B \vdash B \otimes A} \otimes_L
 \end{array} \quad (74)$$

where we freely reordered the antecedent by means of our convention that cedents are not considered to have an order (see the *incipit* of Section 3.1).

(33) holds definitionally, recall (21),

(34) $|A \vdash B| = |B^\perp \vdash A^\perp|$ by $(-)_L^\perp$ and $(-)_R^\perp$,

(35) $|A \otimes B \vdash C^\perp| = |A \vdash (B \otimes C)^\perp|$ is witnessed by the proofs

$$\begin{array}{c}
 \frac{\frac{A, B \vdash C^\perp}{A, B, C \vdash} (-)_L^\perp}{A, B \otimes C \vdash} \otimes_L \\
 \frac{A, B \otimes C \vdash}{A \vdash (B \otimes C)^\perp} (-)_R^\perp
 \end{array}
 \quad
 \begin{array}{c}
 \frac{\frac{1}{B \vdash B} \text{AX} \otimes \frac{1}{C \vdash C} \text{AX}}{B, C \vdash B \otimes C} \otimes_R \otimes \frac{A \vdash (B \otimes C)^\perp}{A, B \otimes C \vdash} (-)_L^\perp \\
 \frac{A, B, C \vdash}{A, B \vdash C^\perp} (-)_R^\perp \text{CUT}
 \end{array} \quad (75)$$

(36) $|\perp \vdash A| = \infty$ is directly witnessed by $\perp_L$.

(37) and (38) are proven by $\forall_R$ and $\forall_L$, respectively.

F Proof of Grounded Completeness (Theorem 5.18)

In the following, we fix the valuation $v : V \rightarrow [0, \infty]$ and work in the grounded theory $\mathbb{T}_v$, so that $\vdash$ will mean entailment in that theory.

We start by noting that MIX^* does not interfere with cut-elimination: inspecting the proof at Appendix B for the cases involving MIX , one can note that Equation (3) suffices to make the analogous proof transformations go through for MIX^* .

The next tool is the following observation:

Lemma F.1. *For all formulae A, B in the language of $\mathbb{T}_v$,*

$$|A \vdash B|^* \leq |B \vdash A| \quad (76)$$

PROOF. We start by observing that²⁴ $|A \vdash B| = |\vdash A^\perp, B| = |\vdash A^\perp \wp B|$, thus without loss of generality we can prove $|A \vdash| \geq |\vdash A|^*$. Moreover, we also have $|A \vdash| = |\vdash A^\perp|$, and thus proving the inequality for A proves it for $A^\perp$ too. Therefore, without loss of generality, we can assume A is positive (i.e. made of propositional variables, $\perp$, $\mathbf{1}$, $\vee$, and $\otimes$).

We prove the theorem by induction on the structure of A .

For constants, we show that for every proof π of $A \vdash$ there is a proof $\bar{\pi}$ of $\vdash A$ such that $|\pi| \geq |\bar{\pi}|^*$. It follows that $|A \vdash| \geq |\pi| \geq |\bar{\pi}|^* \geq |\vdash A|^*$. This is how we get $\bar{\pi}$:

$$\frac{v(a)}{a^\perp \vdash} a_R \quad \mapsto \quad \frac{v(a)^*}{\vdash a^\perp} a_L \quad (77)$$

$$\frac{v(a)^*}{a \vdash} a_L \quad \mapsto \quad \frac{v(a)}{\vdash a} a_R \quad (78)$$

$$\frac{\infty}{\perp \vdash} \perp_L \quad \mapsto \quad \frac{0}{\vdash \perp} \text{EFQ} \quad (79)$$

$$\frac{1}{\mathbf{1} \vdash} \mathbf{1}_L \quad \mapsto \quad \frac{1}{\vdash \mathbf{1}} \mathbf{1}_R \quad (80)$$

For $\otimes$, we first observe that $|A \otimes B \vdash| = |A, B \vdash|$ by invertibility of $\otimes_L$. We then have

$$|A \otimes B \vdash| \geq |A \vdash| \otimes^* |B \vdash| \geq |\vdash A|^* \otimes^* |\vdash B|^*, \quad (81)$$

²⁴This holds because $\wp_R$ is invertible. Its inverse is obtained by cutting $\vdash A \wp B$ with $A \wp B \vdash A, B$.

with the first inequality witnessed by MIX^* and the second being the induction hypothesis. Now since $|\vdash A \otimes B| \geq |\vdash A| \otimes |\vdash B|$ as witnessed by $\otimes_R$, we have

$$|\vdash A|^* \otimes^* |\vdash B|^* = (|\vdash A| \otimes |\vdash B|)^* \geq |\vdash A \otimes B|^*. \quad (82)$$

Similarly, $|A \vee B| \geq |A| \oplus |B|$ by $\vee_L$, and thus

$$|A \vee B| \geq |\vdash A|^* \oplus^* |\vdash B|^*. \quad (83)$$

Now since $|\vdash A \vee B| \geq |\vdash A| \oplus |\vdash B|$ because of $\vee_R$, we have

$$|\vdash A|^* \oplus^* |\vdash B|^* = (|\vdash A| \oplus |\vdash B|)^* \geq |\vdash A \vee B|^*. \quad (84)$$

□

PROOF OF THEOREM 5.18. By induction on φ :

- (1) $\varphi = 1$, then $1 \leq 1$,
- (2) $\varphi = \perp$, then 0 is smaller than anything,
- (3) $\varphi = \top$, then $\infty \leq \infty$,
- (4) $\varphi = a$ propositional variable, then by definition of $\llbracket a \rrbracket_v$ and the subformula property,

$$v(a) = \llbracket a \rrbracket_v = |\vdash^{\text{T}_v} a| = v(a), \quad (85)$$

- (5) $\varphi = a^\perp$ —same as above.
- (6) $\varphi = \psi^\perp$, then $\llbracket \psi^\perp \rrbracket_v = \llbracket \psi \rrbracket_v^* = |\vdash^{\text{T}_v} \psi|^*$ and by Lemma F.1 $|\vdash^{\text{T}_v} \psi|^* \leq |\vdash^{\text{T}_v} \psi^\perp|$.
- (7) $\varphi = \alpha * \beta$ for $*$ $\in \{\otimes, \wp, \wedge, \vee\}$, we have

$$\llbracket \alpha * \beta \rrbracket_v = \llbracket \alpha \rrbracket_v * \llbracket \beta \rrbracket_v \leq |\vdash^{\text{T}_v} \alpha| * |\vdash^{\text{T}_v} \beta| \leq |\vdash^{\text{T}_v} \alpha * \beta| \quad (86)$$

Here the last inequality is induced by the right introduction rule of $\otimes, \vee, \wedge$ for those connectives, and by MIX^* for $\wp$.

□