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Kink oscillation of a system of two braided magnetic flux tubes

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ABSTRACT

For the first time we study kink oscillations of braided coronal magnetic loops. We use the simplest model of a braided loop that consists just of two identical curved magnetic tubes. It is assumed that the plasma density is constant inside the tubes, and also in the surrounding plasma. To describe the kink oscillations we use the linear ideal magnetohydrodynamic equations in the cold plasma approximation. We introduce the geometrical parameter q characterizing the value of braiding, the greater q the stronger the braiding is. When $q = 0$ there is no braiding. In this case there are two modes of kink oscillations, fast and slow. We use the regular perturbation method to calculate the first corrections to the frequencies of these oscillations for small q. We found that these corrections are proportional to q^2. The dependence of the coefficients of proportionality on the ratio of densities inside and outside the tubes, and on the ratio of distance between the tubes to their cross-section radius is calculated numerically. We found that braiding reduces the frequency of fast oscillations and enhances the frequency of slow oscillations. The implication of obtained results for coronal seismology is discussed.

Key words: MHD – plasmas – waves – Sun: corona – Sun: oscillations.

1 INTRODUCTION

Now it is generally accepted that the solar corona is strongly magnetically structured. Magnetically structured plasma can support a variety of wave modes (e.g. B. Roberts 2019). The most common structures in the solar atmosphere are coronal magnetic loops. At the end of the previous century transverse oscillations of coronal loops were observed by the *Transition Region and Coronal Explorer* mission. These oscillations have been reported and interpreted as kink oscillations of magnetic flux tubes by M. J. Aschwanden et al. (1999) and V. M. Nakariakov et al. (1999). After that first observation there were numerous observations of the transverse oscillations of coronal magnetic loops (e.g. C. R. Goddard et al. 2016; A. Abedini 2018; A. Nechaeva et al. 2019). For the reviews of theory of kink oscillations of coronal loops, see e.g. M. S. Ruderman & R. Erdélyi (2009) and V. M. Nakariakov et al. (2021).

At present there are two competing models of coronal magnetic loops: monolithic and composite. Composite loops are assumed to consist of a few thin filaments. Sometimes it is observed that an array of two or more coronal loops oscillates simultaneously. If the loops are closely packet then they are similar to a composite loop. Most of the theoretical studies of coronal loop transverse oscillations used the model of a monolithic loop. The theory of transverse oscillations of composite loops is much more complex than that of transverse oscillations of monolithic loops. Hence, to develop the theory of composite loops the numerical studies are needed. However there is one exception when the analytical approach is possible. This is the theory of oscillations of loops consisting of just two filaments. The observation of collective oscillations of two parallel coronal magnetic loops was reported by R. Jain, R. A. Maurya & B. W. Hindman (2015). The problem of transverse oscillations of two parallel magnetic tubes was first studied numerically by M. Luna et al. (2008, hereafter Paper I) under the assumption that both tubes were identical. It was found in this paper that there are four modes of oscillations. These four modes can be separated into two pairs. The modes in the first pair are polarized along the line connecting the tube axes, while the modes in the second pair are polarized in the direction orthogonal to the line connecting the tube axes. In each pair one mode is symmetric and the other antisymmetric. The two tubes oscillate in phase in a symmetric mode, and in antiphase in an antisymmetric mode. Then T. Van Doorsselaere, M. S. Ruderman & D. Robertson (2008, hereafter Paper II) studied the same problem analytically. They reproduced the results obtained by M. Luna et al. (2008) and even generalized them to the case when the tubes are different. D. Robertson, M. S. Ruderman & Y. Taroyan (2010) investigated the oscillations of the two-tube system with the account of plasma stratification along the loops, while D. Robertson & M. S. Ruderman (2011) studied the resonant damping of these oscillations.

Recently the problem of transverse oscillations of two parallel magnetic loops was revisited. M. S. Ruderman & N. S. Petrukhin (2023) considered the effect of flow on the two-tube transverse oscillations. They calculated the dependence of frequencies of kink oscillations of a two-tube system on the flow velocity inside the tubes. It was shown that the effect of flow is fairly weak for typical parameters of coronal magnetic loops. However the effect

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of flow can be quite strong in the case of prominence threads. M. S. Ruderman & N. S. Petrukhin (2024) studied these oscillations in the case when the plasma in the magnetic tubes cools. The main result obtained in this paper is that plasma cooling results in the enhancement of both the oscillation frequency and amplitude. M. Shi et al. (2024) generalized the results obtained in Paper I and investigated the transverse oscillations of a two-tube system where the tubes have elliptic cross-sections. They studied the dependence of oscillation frequencies on the cross-section aspect ratios.

Sometimes composite coronal magnetic loops can be braided. Braiding can affect frequencies of their transverse oscillations. We aim to study transverse oscillations of a braided coronal loop consisting of just two filaments. The paper is organized as follows: In the next section we describe the equilibrium state and present the governing equations. In Section 3 we write the governing equations in the curvilinear non-orthogonal coordinates. In Section 4 we study the eigenmodes of kink oscillations in the case of weak braiding. Section 5 contains the summary of the results and our conclusions.

2 EQUILIBRIUM STATE AND GOVERNING EQUATIONS

We use the cold plasma approximation and neglect the plasma pressure in comparison with the magnetic pressure. We introduce cylindrical coordinates r, ϕ, z and consider the same force-free equilibrium magnetic field as in M. S. Ruderman (2015) and M. S. Ruderman & J. Terradas (2015). In accordance with this we take

$$B_z = \begin{cases} \sqrt{B_0^2 + 2A^2(R^2 - r^2)}, & r < R, \\ B_0, & r > R, \end{cases} \tag{1}$$

$$B_\phi = \begin{cases} Ar, & r < R, \\ R^2A/r, & r > R, \end{cases} \tag{2}$$

where $\boldsymbol{B} = (0, B_\phi, B_z)$ is the equilibrium magnetic field, and A and B_0 are positive constants. The equilibrium magnetic field is potential in the external region, while

$$\nabla \times \boldsymbol{B} = \frac{2A}{B_z}\boldsymbol{B}, \quad r < R. \tag{3}$$

It is assumed that the magnetic field lines are frozen in a dense photospheric plasma at $z = 0, L$. We consider a magnetic field line that crosses the plane $z = 0$ at the point with the cylindrical coordinates $\phi = \phi_0$ and $r = r_0 < R$. Its equation is

$$r = r_0, \quad z = (\phi - \phi_0)\sqrt{A^{-2}B_0^2 + 2R^2 - 2r_0^2}. \tag{4}$$

Now we introduce bipolar coordinates σ and τ in the $z = 0$ plane (see Fig. 1). The bipolar coordinates (σ, τ) in the $z = 0$ are related to Cartesian coordinates (x_0, y_0) by

$$x_0 = \frac{a \sinh \tau}{\cosh \tau - \cos \sigma}, \quad y_0 = \frac{a \sin \sigma}{\cosh \tau - \cos \sigma}, \tag{5}$$

where a is a positive constant with the dimension of length satisfying $a < R$. The bipolar coordinates are also related to polar coordinates by

$$r^2 = x_0^2 + y_0^2 = a^2 \frac{\cosh \tau + \cos \sigma}{\cosh \tau - \cos \sigma}, \quad \tan \phi_0 = \frac{y_0}{x_0} = \frac{\sin \sigma}{\sinh \tau}. \tag{6}$$

Next we consider the dimensionless length along a magnetic field line s given by

$$s = \frac{z}{L}\sqrt{\frac{B_0^2 + A^2(2R^2 - r^2)}{B_0^2 + 2A^2(R^2 - r^2)}}. \tag{7}$$

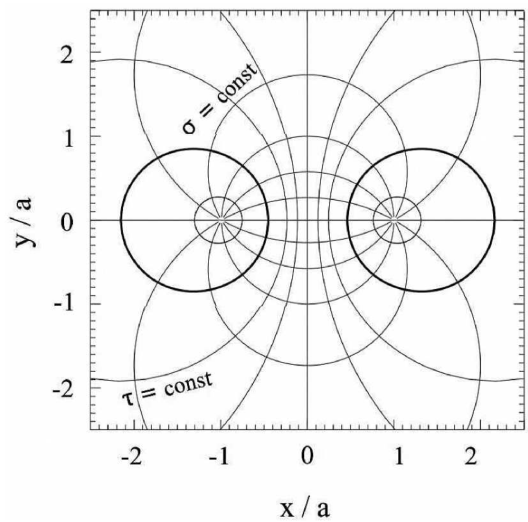

Figure 1. Bipolar coordinates.

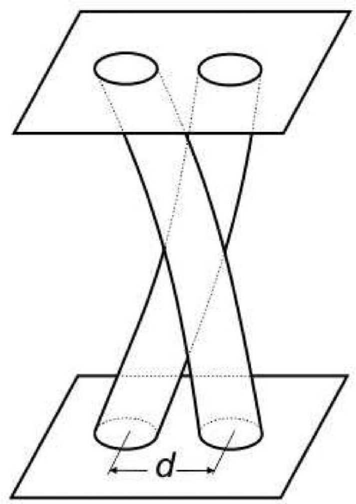

Figure 2. Sketch of the equilibrium; $d = 2x_c$ is the distance between the tube axes. This figure corresponds to $q = \pi$.

As a result we obtain the curvilinear coordinated σ, τ, s defined in the cylinder $r < R$. We restrict the interval of variation of τ by the condition that $r < R$ for all values of $\sigma = 0$. Since r takes its maximum at $\sigma = 0$ for fixed τ this condition reduces to

$$\cosh \tau > \frac{R^2 + a^2}{R^2 - a^2}. \tag{8}$$

We consider two circles in the $z = 0$ plane defined by equations $\tau = \tau_0$ and $\tau = -\tau_0$, where τ_0 satisfies the inequality (8). The radii of these circles are equal to r_c and they are centred at $\pm x_c$, where

$$x_c = a \coth \tau_0, \quad r_c = \frac{a}{\sinh \tau_0}. \tag{9}$$

We also consider the magnetic field lines that cross these circles at the $z = 0$ plane. These lines constitute the surfaces of two curvilinear cylinders that mimic two braided coronal magnetic loops. The volumes inside these curvilinear cylinders are defined by $|\tau| > \tau_0$, while the domain outside the cylinders is defined by $|\tau| < \tau_0$. The sketch of the equilibrium is shown in Fig. 2. We assume that the plasma density is constant inside the cylinders and outside, so

$$\rho = \begin{cases} \rho_i, & |\tau| > \tau_0, \\ \rho_e, & |\tau| < \tau_0, \end{cases} \tag{10}$$

where $\rho_e < \rho_i$. The governing equations for perturbations are

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \frac{1}{\mu_0}[(\nabla \times \boldsymbol{b}) \times \boldsymbol{B} + (\nabla \times \boldsymbol{B}) \times \boldsymbol{b}], \tag{11}$$
$$\boldsymbol{b} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}), \tag{12}$$

where $\boldsymbol{\xi}$ is the plasma displacement and $\boldsymbol{b}$ the magnetic field perturbation. To finalize the problem formulation we impose the boundary conditions. At the surfaces of curvilinear cylinders the normal component of the plasma displacement and the perturbation of the magnetic pressure must be continuous,

$$[[\,\xi_n\,]] = 0, \quad [[\,P\,]] = 0 \quad \text{at} \ \ \tau = \pm\tau_0, \tag{13}$$

where the double brackets indicate the jump of a quantity across the boundary, ξ_n is the normal component of the plasma displacement, $P = \boldsymbol{B} \cdot \boldsymbol{b}/\mu_0$ the perturbation of magnetic pressure, and μ_0 the magnetic permeability of free space. Since we assume that the magnetic field lines are frozen in the dense photospheric plasma it follows that

$$\boldsymbol{\xi} = 0 \quad \text{at} \ \ z = 0, \ L. \tag{14}$$

3 TRANSFORMATION OF GOVERNING EQUATIONS TO CURVILINEAR COORDINATES

We now transform equations (11) and (12), and the boundary conditions (13) and (14) to curvilinear coordinates. We assume that the whole system is thin and take $R/L = \epsilon \ll 1$. We introduce the 'slow' time $T = \epsilon t$, and also assume that the pitch (the distance between two neighbouring coils) of a magnetic field line crossing the $z = 0$ plane at the point (r_0, ϕ_0) with $r_0 < R$ is of the order of L. This implies that $B_\phi/B_0 = \mathcal{O}(\epsilon)$, and consequently $AR/B_0 = \mathcal{O}(\epsilon)$. In accordance with this we write

$$A = \frac{\epsilon q B_0}{R}, \tag{15}$$

where q is a quantity smaller or of the order of unity. The geometrical interpretation of this quantity is the following. We connect the centres of the two tubes in the plane $z = 0$ by a straight line. Now we consider the cross-sections of the two-tube system by a plane $z = z_0$. When z_0 increases the line connecting the centres of the two tube cross-sections starts to rotate counter-clockwise if $q > 0$ and clockwise if $q < 0$. The angle between the initial position of the line at $z_0 = 0$ and its final position at $z_0 = L$ is equal to q.

We introduce the contravariant components of vectors $\boldsymbol{\xi}$ and $\boldsymbol{b}$ with respect to the curvilinear coordinates defined by

$$\boldsymbol{\xi} = \xi_\sigma \boldsymbol{\epsilon}_\sigma + \xi_\tau \boldsymbol{\epsilon}_\tau + \xi_s \boldsymbol{\epsilon}_s, \quad \boldsymbol{b} = b_\sigma \boldsymbol{\epsilon}_\sigma + b_\tau \boldsymbol{\epsilon}_\tau + b_s \boldsymbol{\epsilon}_s, \tag{16}$$

where $(\boldsymbol{\epsilon}_\sigma, \boldsymbol{\epsilon}_\tau, \boldsymbol{\epsilon}_s)$ is the covariant basis of the curvilinear coordinates. The expressions of these vectors in terms of unit vectors of the auxiliary Cartesian coordinates are given in Appendix A. We introduce the parallel component of $\boldsymbol{\xi}$ defined by

$$\xi_\parallel = \frac{\boldsymbol{B} \cdot \boldsymbol{\xi}}{B}. \tag{17}$$

It immediately follows from equations (3) and (11) that $\xi_\parallel = 0$. Taking the scalar product of equation (12) with $\boldsymbol{B}$ and using the identity

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v} \tag{18}$$

we obtain

$$P = -\frac{1}{\mu_0} \nabla \cdot (B^2 \boldsymbol{\xi}). \tag{19}$$

Using equation (3) and the fact that $\boldsymbol{B} \parallel \boldsymbol{\epsilon}_s$ we obtain the relations

$$\boldsymbol{B} = \frac{aB}{L} \boldsymbol{\epsilon}_s, \quad B_z = B_0\big[1 + \mathcal{O}(\epsilon^2)\big],$$
$$B = B_0\big[1 + \mathcal{O}(\epsilon^2)\big], \quad \nabla \times \boldsymbol{B} = \frac{2\epsilon q}{R} \boldsymbol{B}\big[1 + \mathcal{O}(\epsilon^2)\big]. \tag{20}$$

Using equations (20) and (B6) yields

$$\boldsymbol{\xi} \times \boldsymbol{B} = B_0(\xi_\tau \boldsymbol{\epsilon}_\sigma - \xi_\sigma \boldsymbol{\epsilon}_\tau)\big[1 + \mathcal{O}(\epsilon^2)\big]. \tag{21}$$

With the aid of equations (B2)–(B5) we obtain

$$\boldsymbol{\epsilon}_\sigma \cdot (\boldsymbol{\xi} \times \boldsymbol{B}) = \frac{B_0 \xi_\tau [1 + \mathcal{O}(\epsilon^2)]}{(\cosh\tau - \cos\sigma)^2}, \tag{22}$$
$$\boldsymbol{\epsilon}_\tau \cdot (\boldsymbol{\xi} \times \boldsymbol{B}) = -\frac{B_0 \xi_\sigma [1 + \mathcal{O}(\epsilon^2)]}{(\cosh\tau - \cos\sigma)^2}, \tag{23}$$
$$\boldsymbol{\epsilon}_s \cdot (\boldsymbol{\xi} \times \boldsymbol{B}) = \frac{qB_0(v\xi_\sigma + u\xi_\tau)}{(\cosh\tau - \cos\sigma)^2}\big[1 + \mathcal{O}(\epsilon^2)\big], \tag{24}$$

where

$$u = \frac{\cos\sigma \sinh\tau + sq \sin\sigma \cosh\tau}{(\cosh\tau - \cos\sigma)^2}, \tag{25}$$
$$v = \frac{\sin\sigma \cosh\tau - sq \cos\sigma \sinh\tau}{(\cosh\tau - \cos\sigma)^2}. \tag{26}$$

Using equations (22)–(24), (A20), (A28), and (B1) we obtain

$$b_\sigma = \frac{\epsilon B_0}{R}\left[\frac{\partial \xi_\sigma}{\partial s} + q(\cosh\tau - \cos\sigma)^2 \frac{\partial(v\xi_\sigma + u\xi_\tau)}{\partial\tau}\right] \times \big[1 + \mathcal{O}(\epsilon^2)\big], \tag{27}$$
$$b_\tau = \frac{\epsilon B_0}{R}\left[\frac{\partial \xi_\tau}{\partial s} - q(\cosh\tau - \cos\sigma)^2 \frac{\partial(v\xi_\sigma + u\xi_\tau)}{\partial\sigma}\right] \times \big[1 + \mathcal{O}(\epsilon^2)\big]. \tag{28}$$

Using equations (3), (15), (20), and (B6) yields

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{b} = -\frac{2\epsilon q B_0}{R}(b_\tau \boldsymbol{\epsilon}_\sigma - b_\sigma \boldsymbol{\epsilon}_\tau). \tag{29}$$

Using equations (20) and (A16) we reduce the condition $\xi_\parallel = 0$ to $\xi_s = \mathcal{O}(\epsilon^2)$.

Now we express b_s in terms of b_σ, b_τ, and P. Using equation (20) we obtain

$$P = \frac{aB}{\mu_0 L}\big[b_\sigma(\boldsymbol{\epsilon}_\sigma \cdot \boldsymbol{\epsilon}_s) + b_\sigma(\boldsymbol{\epsilon}_\tau \cdot \boldsymbol{\epsilon}_s) + a^{-2} b_s L^2\big]. \tag{30}$$

With the aid of equations (B4) and (B5) we obtain from this equation

$$b_s = \frac{a\mu_0 P}{BL} - q\epsilon^2(ub_\sigma - vb_\tau)\big[1 + (\epsilon^2)\big]. \tag{31}$$

Using equation (A28) we obtain

$$(\nabla \times \boldsymbol{b})_\sigma = \frac{1}{L\Delta}\left(\frac{\partial(\boldsymbol{b} \cdot \boldsymbol{\epsilon}_s)}{\partial\tau} - \frac{\partial(\boldsymbol{b} \cdot \boldsymbol{\epsilon}_\tau)}{\partial s}\right),$$
$$(\nabla \times \boldsymbol{b})_\tau = \frac{1}{L\Delta}\left(\frac{\partial(\boldsymbol{b} \cdot \boldsymbol{\epsilon}_\sigma)}{\partial s} - \frac{\partial(\boldsymbol{b} \cdot \boldsymbol{\epsilon}_s)}{\partial\sigma}\right), \tag{32}$$

where Δ is defined by equation (A20). Using equation (20) and the definition of P we obtain

$$\frac{\partial(\boldsymbol{b}\cdot\boldsymbol{\epsilon}_s)}{\partial\sigma} = \frac{L\mu_0}{aB}\left(\frac{\partial P}{\partial\sigma} - \frac{P}{B}\frac{\partial B}{\partial\sigma}\right), \quad \frac{\partial(\boldsymbol{b}\cdot\boldsymbol{\epsilon}_s)}{\partial\tau} = \frac{L\mu_0}{aB}\left(\frac{\partial P}{\partial\tau} - \frac{P}{B}\frac{\partial B}{\partial\tau}\right). \tag{33}$$

Using equations (31) and (B1)–(B5) yields

$$\boldsymbol{b}\cdot\boldsymbol{\epsilon}_\sigma = \left[\frac{b_\sigma}{(\cosh\tau-\cos\sigma)^2} + \frac{qa\mu_0 uP}{LB_0}\right]\left[1+(\epsilon^2)\right], \tag{34}$$

$$\boldsymbol{b}\cdot\boldsymbol{\epsilon}_\tau = \left[\frac{b_\tau}{(\cosh\tau-\cos\sigma)^2} - \frac{qa\mu_0 vP}{LB_0}\right]\left[1+(\epsilon^2)\right]. \tag{35}$$

Using equations (1) and (2) we obtain

$$\frac{1}{B}\frac{\partial B}{\partial\sigma} = -\frac{\epsilon^2 q^2 B_0^2}{2B^2R^2}\frac{\partial r^2}{\partial\sigma}, \quad \frac{1}{B}\frac{\partial B}{\partial\tau} = -\frac{\epsilon^2 q^2 B_0^2}{2B^2R^2}\frac{\partial r^2}{\partial\tau}. \tag{36}$$

Now using equations (20), (29), (32)–(36), and (A28) we obtain that the σ and τ-component of equation (11) are given by

$$\epsilon^2\rho\frac{\partial^2\xi_\sigma}{\partial T^2} = \frac{B}{\Delta}\left[\frac{\partial}{\partial s}\left(\frac{\epsilon\Delta b_\sigma}{\mu_0 R} + \frac{q\epsilon^2 uP}{aB_0}\right) - \frac{1}{aB}\left(\frac{\partial P}{\partial\sigma} - \frac{P}{B}\frac{\partial B}{\partial\sigma}\right)\right]\left[1+(\epsilon^2)\right] - \frac{2\epsilon qB_0 b_\tau}{\mu_0 R}, \tag{37}$$

$$\epsilon^2\rho\frac{\partial^2\xi_\tau}{\partial T^2} = \frac{B}{\Delta}\left[\frac{\partial}{\partial s}\left(\frac{\epsilon\Delta b_\tau}{\mu_0 R} + \frac{q\epsilon^2 vP}{aB_0}\right) - \frac{1}{aB}\left(\frac{\partial P}{\partial\tau} - \frac{P}{B}\frac{\partial B}{\partial\tau}\right)\right]\left[1+(\epsilon^2)\right] + \frac{2\epsilon qB_0 b_\sigma}{\mu_0 R}. \tag{38}$$

It follows from equations (27)–(29) that all terms in equation (37) but one proportional to $\partial P/\partial\sigma$ are of the order of ϵ^2. Similarly all terms in equation (38) but one proportional to $\partial P/\partial\tau$ are of the order of ϵ^2. This observation implies that P must be of the order of ϵ^2. In accordance with this we introduce the scaled magnetic pressure perturbation $Q = \epsilon^{-2}P$. From now on we derive all equations only in the leading order approximation with respect to ϵ. Then using equations (27), (28), and (36) we transform equations (37) and (38) to

$$\frac{\partial^2\xi_\sigma}{\partial T^2} - \frac{V_A^2}{R^2}\frac{\partial^2\xi_\sigma}{\partial s^2} = -\frac{1}{a\rho\Delta}\frac{\partial Q}{\partial\sigma} - \frac{qV_A^2}{R^2}\left[2\frac{\partial\xi_\tau}{\partial s} - \frac{1}{\Delta}\frac{\partial^2(v\xi_\sigma+u\xi_\tau)}{\partial\tau\partial s} - \frac{2q}{\Delta}\frac{\partial(v\xi_\sigma+u\xi_\tau)}{\partial\sigma}\right], \tag{39}$$

$$\frac{\partial^2\xi_\tau}{\partial T^2} - \frac{V_A^2}{R^2}\frac{\partial^2\xi_\tau}{\partial s^2} = -\frac{1}{a\rho\Delta}\frac{\partial Q}{\partial\tau} + \frac{qV_A^2}{R^2}\left[2\frac{\partial\xi_\sigma}{\partial s} - \frac{1}{\Delta}\frac{\partial^2(v\xi_\sigma+u\xi_\tau)}{\partial\sigma\partial s} + \frac{2q}{\Delta}\frac{\partial(v\xi_\sigma+u\xi_\tau)}{\partial\tau}\right], \tag{40}$$

where $V_A^2 = B_0^2/(\mu_0\rho)$ and now Δ is given by the approximate expression

$$\Delta = \frac{1}{(\cosh\tau-\cos\sigma)^2}. \tag{41}$$

With the aid of equation (A22) we reduce equation (37) in the main order approximation to

$$\frac{\partial(\Delta\xi_\sigma)}{\partial\sigma} + \frac{\partial(\Delta\xi_\tau)}{\partial\tau} = 0. \tag{42}$$

Using cross-differentiating to eliminate Q from equations (39) and (40) we obtain with the aid of equation (42)

$$\left(\frac{\partial^2}{\partial T^2} - \frac{V_A^2}{R^2}\frac{\partial^2}{\partial s^2}\right)\left(\frac{\partial(\Delta\xi_\sigma)}{\partial\tau} - \frac{\partial(\Delta\xi_\tau)}{\partial\sigma}\right) = \frac{qV_A^2}{R^2}\left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2}\right)\frac{\partial(v\xi_\sigma+u\xi_\tau)}{\partial s}. \tag{43}$$

It follows from equation (42) that ξ_σ and ξ_τ can be expressed in terms of the flux function ψ as

$$\xi_\sigma = \frac{1}{\Delta}\frac{\partial\psi}{\partial\tau}, \quad \xi_\tau = -\frac{1}{\Delta}\frac{\partial\psi}{\partial\sigma}. \tag{44}$$

Substituting these expressions into equation (43) and using equations (25), (26), and (41) yields

$$\frac{\partial^2\Psi}{\partial\sigma^2} + \frac{\partial^2\Psi}{\partial\tau^2} = 0, \tag{45}$$

$$\Psi = \frac{\partial^2\psi}{\partial T^2} - \frac{V_A^2}{R^2}\frac{\partial^2\psi}{\partial s^2} + \frac{qV_A^2}{R^2}\frac{\partial}{\partial s}\left[\cos\sigma\sinh\tau\left(\frac{\partial\psi}{\partial\sigma} + sq\frac{\partial\psi}{\partial\tau}\right) + \sin\sigma\cosh\tau\left(sq\frac{\partial\psi}{\partial\sigma} - \frac{\partial\psi}{\partial\tau}\right)\right]. \tag{46}$$

Next we rewrite the boundary conditions in terms of flux function. In the leading order approximation $z = L$ corresponds to $s = 1$. Then it follows from equations (14) and (44) that ψ = const at $s = 0, 1$. Since ψ is defined with the accuracy up to an additive function of s we can always make

$$\psi = 0 \quad \text{at} \quad s = 0, 1. \tag{47}$$

To rewrite the first boundary condition in equation (13) in terms of ψ we need to obtain the expression for the unit vector normal to the tube surface. Since the tube surface it defined by the equation τ = const it follows that vectors $\boldsymbol{\epsilon}_\sigma$ and $\boldsymbol{\epsilon}_s$ are tangent to this surface. Then the vector $\boldsymbol{\epsilon}_s\times\boldsymbol{\epsilon}_\sigma$ is normal to the tube surface. Using equations (B2) and (B6) we obtain that in the leading order approximation the unit vector normal to the tube surface is

$$\boldsymbol{n} = (\cosh\tau-\cos\sigma)\boldsymbol{\epsilon}_\tau. \tag{48}$$

Then using the estimate $\xi_s = \mathcal{O}(\epsilon^2)$ and equations (B2), (B3), and (B5) we obtain that in the leading order approximation

$$\xi_n = \frac{\xi_\tau}{\cosh\tau-\cos\sigma}. \tag{49}$$

Since the denominator in this expression is continuous at the tube boundary we can write the first boundary condition in equation (13) as

$$[[\,\psi\,]] = 0 \quad \text{at} \quad \tau = \pm\tau_0. \tag{50}$$

When deriving this boundary condition we used equation (44) and the fact that it is valid for any σ and thus can be integrated with respect to σ.

Finally we transform the second boundary condition in equation (13). Since this boundary condition is valid for any σ we can differentiate it with respect to σ. Then we can use equations (37) and (44) to express it in terms of ψ. As a result we obtain

$$\left[\left[\rho\frac{\partial\Psi}{\partial\tau} + \frac{2q\rho V_A^2}{R^2}\frac{\partial}{\partial\sigma}\left((qs\cos\sigma\sinh\tau - \sin\sigma\cosh\tau)\frac{\partial\psi}{\partial\tau}\right)\right]\right] = 0 \quad \text{at} \quad \tau = \pm\tau_0. \tag{51}$$

When deriving this boundary condition we used the fact that ψ and ρV_A^2 are continuous at $\tau = \pm\tau_0$.

Equations (45) and (46) together with the boundary conditions given by equations (47), (50), and (51) are used in the next section to study the eigenmodes of kink oscillations.

4 EIGENMODES OF KINK OSCILLATIONS

In this section we study the eigenmodes of kink oscillations. To do this we take $\psi \propto \exp(-i\Omega T)$. Then equation (46) reduces to

$$\Psi = -\Omega^2\psi - \frac{V_A^2}{R^2}\frac{\partial^2\psi}{\partial s^2} + \frac{qV_A^2}{R^2}\frac{\partial}{\partial s}\left[\cos\sigma\sinh\tau\left(\frac{\partial\psi}{\partial\sigma} + sq\frac{\partial\psi}{\partial\tau}\right) + \sin\sigma\cosh\tau\left(sq\frac{\partial\psi}{\partial\sigma} - \frac{\partial\psi}{\partial\tau}\right)\right]. \tag{52}$$

4.1 No braiding

We start the analysis of eigenmodes of kink oscillations from the case where the two magnetic tubes are not braided and take $q = 0$. This is a particular case of the analysis presented in Paper II. The dispersion equation determining the eigenfrequencies of kink oscillations was derived in Paper II directly from the full set of linear ideal magnetohydrodynamic (MHD) equations. Here we derive it using the equations and boundary conditions for the single function ψ. Comparison with the results obtained in Paper II will serve as the verification of equations and boundary conditions for this function. It will also serve as the first step of the analysis of the braiding effect on the oscillation frequencies. We only consider the modes fundamental in the axial direction. In accordance with this and equation (47) we take $\psi \propto \sin(\pi s)$. Then equation (46) reduces to

$$\Psi = -\left(\Omega^2 - \frac{\pi^2 V_A^2}{R^2}\right)\psi. \tag{53}$$

Substituting this expression in equations (45) and (51) we obtain

$$\frac{\partial^2\psi}{\partial\sigma^2} + \frac{\partial^2\psi}{\partial\tau^2} = 0, \tag{54}$$

$$\left[\left[\rho\left(\Omega^2 - \frac{\pi^2 V_A^2}{R^2}\right)\frac{\partial\psi}{\partial\tau}\right]\right] = 0 \quad \text{at} \quad \tau = \pm\tau_0. \tag{55}$$

Perturbations must decay far from the tube, that is at $r \to \infty$. It follows from equation (6) that $r \to \infty$ corresponds to $\sigma \to 0$ and $\tau \to 0$. Then it follows from equation (44) that

$$\psi \to \psi_0 \quad \text{at} \quad \sigma \to 0 \quad \text{and} \quad \tau \to 0. \tag{56}$$

where ψ_0 is a constant. Subtracting $\psi_0\sin(\pi s)$ from ψ we make $\psi \to 0$ while ψ still satisfies equation (47). We only consider kink oscillations. Following Paper II we take the solution to equation (54) given by

$$\psi = \Phi(\tau)\cos(\sigma - \sigma_0) - \Phi(0)\cos(\sigma_0), \tag{57}$$

where $\Phi(\tau)$ is the function to be determined. Substituting equation (57) into equation (54) yields

$$\frac{d^2\Phi}{d\tau^2} - \Phi = 0. \tag{58}$$

It follows that $\tau \to \infty$ corresponds to $y \to 0$ and $x \to a$, while $\tau \to -\infty$ corresponds to $y \to 0$ and $x \to -a$. Since the solution must be regular at the points with Cartesian coordinates $(a, 0)$ and $(-a, 0)$ we conclude that it must be bounded as $\tau \to \pm\infty$. The general solution to equation (58) satisfying this condition is

$$\Phi = \begin{cases} C_- e^{\tau}, & \tau < -\tau_0, \\ C_1 e^{\tau} + C_2 e^{-\tau}, & |\tau| < \tau_0, \\ C_+ e^{-\tau}, & \tau > \tau_0, \end{cases} \tag{59}$$

where C_-, C_+, C_1, and C_2 are constants. It follows from the boundary condition given by equation (50) that $\Psi(\tau)$ must be continuous at $\tau = \pm\tau_0$. This condition yields

$$C_- = C_1 + C_2 e^{2\tau_0}, \quad C_+ = C_1 e^{2\tau_0} + C_2. \tag{60}$$

Substituting equation (57) into equation (55) and using equations (59) and (60) we obtain

$$\begin{aligned} &\Omega^2(\rho_i - \rho_e)C_1 + (\rho_i + \rho_e)\left(\Omega^2 - \frac{\pi^2 C_k^2}{R^2}\right)C_2 e^{2\tau_0} = 0, \\ &(\rho_i + \rho_e)\left(\Omega^2 - \frac{\pi^2 C_k^2}{R^2}\right)C_1 e^{2\tau_0} + \Omega^2(\rho_i - \rho_e)C_2 = 0, \end{aligned} \tag{61}$$

where C_k is the kink speed defined by

$$C_k^2 = \frac{2\rho V_A^2}{\rho_i + \rho_e}. \tag{62}$$

The condition of existence of non-trivial solutions to this system of equations for C_1 and C_2 gives the dispersion equation

$$\Omega^2(\rho_i - \rho_e) = \pm e^{2\tau_0}(\rho_i + \rho_e)\left(\Omega^2 - \frac{\pi^2 C_k^2}{R^2}\right). \tag{63}$$

This equation coincides with the dispersion equation derived by Paper II if we take in the latter equal tube radii and equal plasma densities in the tubes. The two solutions to equation (63) are

$$\Omega_\pm^2 = \frac{\pi^2 C_k^2(\rho_i + \rho_e)}{R^2[(\rho_i + \rho_e) \mp (\rho_i - \rho_e)e^{-2\tau_0}]}. \tag{64}$$

We see that there are two different eigenfrequencies of kink oscillations, while it was found in the numerical study presented in Paper I that there are four different eigenfrequencies. To describe the results obtained in Paper I we consider the components of the plasma displacement in the auxiliary Cartesian coordinates, ξ_x and ξ_y. The four wave modes found in Paper I have been denoted as A_x, A_y, S_x, and S_y. In A_x and S_x modes the oscillations are polarized in the x-direction, and A_y and S_y modes are polarized in the y-direction. The A_x and A_y modes are antisymmetric, that is, the two tubes are displaced in the opposite directions. The S_x and S_y mode are symmetric, that is the two tubes are displaced in the same direction. We denote the oscillation frequencies of the four modes as Ω_{Ax}, Ω_{Ay}, Ω_{Sx}, and Ω_{Sy}. It is found in Paper I that $\Omega_{Sx} < \Omega_{Ay} < \Omega_{Sy} < \Omega_{Ax}$. It was explained in Paper II that the discrepancy between the results obtained in Paper I and Paper II is related with the following. In Paper I, the exact numerical solution to the problem is presented. In contrast the analytical solution given in Paper II is approximate and obtained in the leading order approximation with respect to ϵ. In this approximation $\Omega_{Ay} - \Omega_{Sx} = \mathcal{O}(\epsilon^2)$ and $\Omega_{Ax} - \Omega_{Sy} = \mathcal{O}(\epsilon^2)$, so in the approximate solution $\Omega_{Ay} = \Omega_{Sx} = \Omega_-$ and $\Omega_{Ax} = \Omega_{Sy} = \Omega_+$. Hence, in the analytical solution the mode with the frequency Ω_- is degenerated and the polarization of oscillations with this frequency can be polarized in any direction. The same is true for oscillations with the frequency Ω_+. It is shown in Paper II that we have to choose $\sigma_0 = 0$ to obtain oscillations polarized in the x-direction, and $\sigma_0 = \pi/2$ to obtain oscillations polarized in the y-direction. Hereafter, we call Ω_+ and Ω_- the frequencies of the fast and slow oscillations, respectively.

4.2 Weak braiding

In this section we consider the case where the braiding is weak, $|q| \ll 1$. We look for the solution to the eigenvalue problem in the form

$$\psi = \psi_1 + q\psi_2 + q^2\psi_3 + \ldots, \quad \Omega = \Omega_1 + q\Omega_2 + q^2\Omega_3 + \ldots \tag{65}$$

4.2.1 First-order approximation

We take $\Omega_1 = \Omega_+$ or $\Omega_1 = \Omega_-$. We also take

$$\psi_1 = [\Phi(\tau)\cos(\sigma-\sigma_0) - \Phi(0)\cos(\sigma_0)]\sin(\pi s), \tag{66}$$

with $\Phi(\tau)$ is defined by equations (59)–(61). Since we consider a linear problem we can take the value of one of the constants in equation (59) arbitrarily. We take

$$C_1 = \frac{R^3\Omega_1^2(\rho_i-\rho_e)}{\rho V_A^2}. \tag{67}$$

It follows from equations (60) and (61) that

$$C_2 = \mp C_1, \quad C_- = C_1(1 \mp e^{2\tau_0}), \quad C_+ = C_1(e^{2\tau_0} \mp 1), \tag{68}$$

where the upper and lower signs in these expressions correspond to $\Omega_1 = \Omega_+$ and $\Omega_2 = \Omega_-$, respectively.

4.2.2 Second-order approximation

Substituting equation (65) into equation (52) yields

$$\begin{aligned}\Psi = \frac{V_A^2}{R^2}\bigg\{ & -k^2\psi_1 - \frac{\partial^2\psi_1}{\partial s^2} - q\left(k^2\psi_2 + \frac{\partial^2\psi_2}{\partial s^2}\right) \\ & -\frac{2qkR}{V_A}\Omega_2\psi_1 + q\frac{\partial}{\partial s}\bigg(\frac{\partial\psi_1}{\partial\sigma}\cos\sigma\sinh\tau \\ & -\frac{\partial\psi_1}{\partial\tau}\sin\sigma\cosh\tau\bigg) - q^2\left(k^2\psi_3 + \frac{\partial^2\psi_3}{\partial s^2}\right) \\ & -q^2\left(\frac{R^2\Omega_2^2}{V_A^2} + \frac{2kR}{V_A}\Omega_3\right)\psi_1 + q^2\frac{2kR}{V_A}\Omega_2\psi_2 \\ & +q^2\frac{\partial}{\partial s}\bigg[\frac{\partial\psi_2}{\partial\sigma}\cos\sigma\sinh\tau - \frac{\partial\psi_2}{\partial\tau}\sin\sigma\cosh\tau \\ & +s\bigg(\frac{\partial\psi_1}{\partial\tau}\cos\sigma\sinh\tau + \frac{\partial\psi_1}{\partial\sigma}\sin\sigma\cosh\tau\bigg)\bigg]\bigg\} + \mathcal{O}(q^3), \end{aligned} \tag{69}$$

where $k = \Omega_1 R/V_A$. In this section we only use the terms of the order of q in this expression. However we also kept the terms of the order of q^2 having in mind that they will be used in the third-order approximation. Substituting equation (69) into equation (45) and collecting terms of the order of q we obtain

$$\begin{aligned}\left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2}\right)\bigg[& k^2\psi_2 + \frac{\partial^2\psi_2}{\partial s^2} \\ & +\pi\cos(\pi s)\bigg(\Phi\cos\sigma\sin(\sigma-\sigma_0)\sinh\tau \\ & +\frac{d\Phi}{d\tau}\sin\sigma\cos(\sigma-\sigma_0)\cosh\tau\bigg)\bigg] = 0. \end{aligned} \tag{70}$$

Collecting terms of the order of q in equation (50) yields

$$[[\,\psi_2\,]] = 0 \quad \text{at} \quad \tau = \pm\tau_0. \tag{71}$$

Substituting equation (69) into equation (51) and collecting terms of the order of q we obtain that the following boundary condition is satisfied at $\tau = \pm\tau_0$:

$$\begin{aligned}\bigg[\bigg[& k^2\frac{\partial\psi_2}{\partial\tau} + \frac{\partial^3\psi_2}{\partial\tau\partial s^2} + \frac{2kR}{V_A}\Omega_2\frac{\partial\psi_1}{\partial\tau} + 2\frac{\partial}{\partial\sigma}\left(\frac{\partial\psi_1}{\partial\tau}\sin\sigma\cosh\tau\right) \\ & -\frac{\partial^2}{\partial\tau\partial s}\left(\frac{\partial\psi_1}{\partial\sigma}\cos\sigma\sinh\tau - \frac{\partial\psi_1}{\partial\tau}\sin\sigma\cosh\tau\right)\bigg]\bigg] = 0. \end{aligned} \tag{72}$$

We look for the solution to equation (70) in the form $\psi_2 = \psi_{2p} + \psi_{2c}$, where

$$\begin{aligned} k^2\psi_{2p} + \frac{\partial^2\psi_{2p}}{\partial s^2} = -\pi\cos(\pi s)\bigg(& \Phi\cos\sigma\sin(\sigma-\sigma_0)\sinh\tau \\ & +\frac{d\Phi}{d\tau}\sin\sigma\cos(\sigma-\sigma_0)\cosh\tau\bigg), \end{aligned} \tag{73}$$

and ψ_{2c} satisfies the equation

$$\left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2}\right)\left(k^2\psi_{2c} + \frac{\partial^2\psi_{2c}}{\partial s^2}\right) = 0. \tag{74}$$

Both ψ_{2p} and ψ_{2c} must satisfy the boundary conditions

$$\psi_{2p} = \psi_{2c} = 0 \quad \text{at} \quad s = 0, 1. \tag{75}$$

Using equation (73) we transform equation (72) to

$$\begin{aligned}\bigg[\bigg[& k^2\frac{\partial\psi_{2c}}{\partial\tau} + \frac{\partial^3\psi_{2c}}{\partial\tau\partial s^2} + \frac{2kR}{V_A}\Omega_2\frac{\partial\psi_1}{\partial\tau} \\ & +2\frac{\partial}{\partial\sigma}\left(\frac{\partial\psi_1}{\partial\tau}\sin\sigma\cosh\tau\right)\bigg]\bigg] = 0 \quad \text{at} \quad \tau = \pm\tau_0. \end{aligned} \tag{76}$$

The solution to equation (73) satisfying the boundary condition given by equation (75) is

$$\begin{aligned}\psi_{2p} = & \frac{\pi}{\pi^2 - k^2}\left(\cos(\pi s) - \frac{\sin[k(1/2-s)]}{\sin(k/2)}\right) \\ & \times\bigg(\Phi\cos\sigma\sin(\sigma-\sigma_0)\sinh\tau \\ & +\frac{d\Phi}{d\tau}\sin\sigma\cos(\sigma-\sigma_0)\cosh\tau\bigg). \end{aligned} \tag{77}$$

We note that $\psi_{2p} \to 0$ as $\sigma \to 0$ and $\tau \to 0$, however ψ_{2p} is not continuous at $\tau = \pm\tau_0$ and it does not satisfy equation (72). We can look for the solution to equation (74) satisfying the boundary condition equation (75) in the form of Fourier expansion

$$\begin{aligned}\psi_{2c} = \sum_{n=1}^{\infty}\sin(\pi n s)\bigg(& \sum_{m=1}^{\infty}[h_{nm}^c(\tau)\cos(m\sigma) \\ & +h_{nm}^s(\tau)\sin(m\sigma) - h_{nm}^c(0)] + h_n(\tau)\bigg), \end{aligned} \tag{78}$$

where $h_n(\tau)$ is a piece-wise constant function, $h_n(\tau) = h_{n-}$ for $\tau < -\tau_c$, $h_n(\tau) = 0$ for $|\tau| < \tau_c$, and $h_n(\tau) = h_{n+}$ for $\tau > \tau_c$. It is straightforward to see that ψ_{2c} satisfies the condition $\psi_{2c} \to 0$ as $\sigma \to 0$ and $\tau \to 0$. Substituting equation (77) into equation (74) yields

$$\frac{d^2 h_{nm}^{c,s}}{d\tau^2} - m^2 h_{nm}^{c,s} = 0. \tag{79}$$

The general solution to this equation satisfying the regularity condition at $\tau \to \pm\infty$ is

$$h_{nm}^{c,s}(\tau) = \begin{cases} C_{nm-}^{c,s}e^{m\tau}, & \tau < -\tau_0, \\ C_{nm1}^{c,s}e^{m\tau} + C_{nm2}^{c,s}e^{-m\tau}, & |\tau| < \tau_0, \\ C_{nm+}^{c,s}e^{-m\tau}, & \tau > \tau_0, \end{cases} \tag{80}$$

where $C^{c,s}_{nm-}$, $C^{c,s}_{nm+}$, $C^{c,s}_{nm1}$, and $C^{c,s}_{nm2}$ are constants. We also can expand ψ_{2p} in the Fourier series. Using equation (77) we obtain

$$\psi_{2p} = \sum_{n=1}^{\infty} g_n \sin(\pi ns)\left[\left(\frac{d\Phi}{d\tau}\cosh\tau + \Phi\sinh\tau\right)(\cos\sigma_0\sin 2\sigma - \sin\sigma_0\cos 2\sigma) + \left(\frac{d\Phi}{d\tau}\cosh\tau - \Phi\sinh\tau\right)\sin\sigma_0)\right], \tag{81}$$

where

$$g_n = \frac{n[1+(-1)^n]}{(n^2-1)(\pi^2 n^2 - k^2)}, \tag{82}$$

for $n > 1$, and $g_1 = 0$. Now we write the Fourier expansion for ψ_2 similar to that given by equation (78) for ψ_{2c}, however with $w^{c,s}_{nm}$ substituted for $h^{c,s}_{nm}$, and w_n substituted for h_n. Using equations (78) and (81) we obtain

$$\begin{aligned} &w^{c,s}_{nm} = h^{c,s}_{nm} \quad \text{for} \quad m = 1 \quad \text{and} \quad m \geq 3, \\ &w^c_{n2} = h^c_{n2} - g_n\left(\frac{d\Phi}{d\tau}\cosh\tau + \Phi\sinh\tau\right)\sin\sigma_0, \\ &w^s_{n2} = h^s_{n2} + g_n\left(\frac{d\Phi}{d\tau}\cosh\tau + \Phi\sinh\tau\right)\cos\sigma_0, \\ &w_n = h_n + g_n\left(\frac{d\Phi}{d\tau}\cosh\tau - \Phi\sinh\tau\right)\sin\sigma_0. \end{aligned} \tag{83}$$

With the aid of equations (59), (60), and (80) we obtain from the condition of continuity of ψ_2 at $\tau = \pm\tau_0$ that

$$\begin{aligned} &C^{c,s}_{nm-} = C^{c,s}_{nm1} + C^{c,s}_{nm2} e^{2m\tau_0}, \quad \text{for} \quad m = 1 \text{ and } m \geq 3, \\ &C^{c,s}_{nm+} = C^{c,s}_{nm1} e^{2m\tau_0} + C^{c,s}_{nm2}, \end{aligned} \tag{84}$$

$$\begin{aligned} C^c_{n2-} &= C^c_{n21} + C^c_{n22} e^{4\tau_0} + [C_1(g_{ni} - g_{ne}) \\ &\quad + C_2 e^{2\tau_0}(g_{ni} + e^{2\tau_0} g_{ne})]\sin\sigma_0, \\ C^s_{n2-} &= C^s_{n21} + C^s_{n22} e^{4\tau_0} - [C_1(g_{ni} - g_{ne}) \\ &\quad + C_2 e^{2\tau_0}(g_{ni} + e^{2\tau_0} g_{ne})]\cos\sigma_0, \end{aligned} \tag{85}$$

$$\begin{aligned} C^c_{n2+} &= C^c_{n21} e^{4\tau_0} + C^c_{n22} - [C_1 e^{2\tau_0}(g_{ni} + e^{2\tau_0} g_{ne}) \\ &\quad + C_2(g_{ni} - g_{ne})]\sin\sigma_0, \\ C^s_{n2+} &= C^s_{n21} e^{4\tau_0} + C^s_{n22} + [C_1 e^{2\tau_0}(g_{ni} + e^{2\tau_0} g_{ne}) \\ &\quad + C_2(g_{ni} - g_{ne})]\cos\sigma_0, \end{aligned} \tag{86}$$

$$\begin{aligned} h_{n-} &= -[C_1(g_{ni} - g_{ne}) + C_2(e^{2\tau_0} g_{ni} + g_{ne})]\sin\sigma_0, \\ h_{n+} &= [C_1(e^{2\tau_0} g_{ni} + g_{ne}) + C_2(g_{ni} - g_{ne})]\sin\sigma_0. \end{aligned} \tag{87}$$

Next we use the boundary condition given by equation (76). We substitute equation (78) into equation (76) to obtain the expression for ψ_{2c} and Ω_2. When doing so we use equations (59), (60), (66), (68), (80), (84), and (85).

We start by calculating the coefficients $h^c_{nm}(\tau)$ and $h^s_{nm}(\tau)$ for $m \geq 3$. We obtain

$$\begin{aligned} &(k_i^2 - k_e^2)C^{c,s}_{nm1} + (k_i^2 + k_e^2 - 2\pi^2 n^2)e^{2m\tau_0} C^{c,s}_{nm2} = 0, \\ &(k_i^2 + k_e^2 - 2\pi^2 n^2)e^{2m\tau_0} C^{c,s}_{nm1} + (k_i^2 - k_e^2)C^{c,s}_{nm2} = 0. \end{aligned} \tag{88}$$

This is the system of linear homogeneous algebraic equations for $C^{c,s}_{nm1}$ and $C^{c,s}_{nm2}$. Using the expression for k and equation (64) we obtain that the determinant of this system is equal to zero when one of the two equations is satisfied,

$$\begin{aligned} \frac{\rho_i - \rho_e}{\rho_i + \rho_e} &= -\frac{(n^2-1)e^{2m\tau_0}}{1 \mp n^2 e^{2(m-1)\tau_0}}, \\ \frac{\rho_i - \rho_e}{\rho_i + \rho_e} &= \frac{(n^2-1)e^{2m\tau_0}}{1 \pm n^2 e^{2(m-1)\tau_0}}, \end{aligned} \tag{89}$$

where, as before, the upper and lower signs in these expressions correspond to $\Omega_1 = \Omega_+$ and $\Omega_1 = \Omega_-$, respectively. Since we assume that $\rho_i > \rho_e$ it is obvious that neither of these relations with the lower signs can be satisfied. One of the two relations with the upper sign can be satisfied for particular values of parameters. However, a small variation of any parameter destroys the equality. This implies that the case where the determinant is zero is structurally unstable and can be neglected. In the generic case the determinant is nonzero and consequently the system of equations (88) has only the trivial solution, $C^{c,s}_{nm1} = C^{c,s}_{nm2} = 0$. Hence the expression for ψ_{2c} does not contain the terms proportional to $\cos(m\sigma)$ and $\sin(m\sigma)$ with $m \geq 3$.

Now we take $m = 1$ and $n = 1$, and collect the terms proportional to $\sin(\pi s)\cos\sigma$ and $\sin(\pi s)\sin\sigma$ in equation (76). This yields

$$\begin{aligned} &(k_i^2 - k_e^2)C^{c,s}_{111} + (k_i^2 + k_e^2 - 2\pi^2)e^{2\tau_0} C^{c,s}_{112} = 2R\Omega_2\chi^{c,s}_0 \\ &\quad \times\left[C_1\left(\frac{k_e}{V_{Ae}} - \frac{k_i}{V_{Ai}}\right) - C_2 e^{2\tau_0}\left(\frac{k_e}{V_{Ae}} + \frac{k_i}{V_{Ai}}\right)\right], \end{aligned} \tag{90}$$

$$\begin{aligned} &(k_i^2 + k_e^2 - 2\pi^2)e^{2\tau_0} C^{c,s}_{111} + (k_i^2 - k_e^2)C^{c,s}_{112} = -2R\Omega_2\chi^{c,s}_0 \\ &\quad \times\left[C_1 e^{2\tau_0}\left(\frac{k_e}{V_{Ae}} + \frac{k_i}{V_{Ai}}\right) + C_2\left(\frac{k_i}{V_{Ai}} - \frac{k_e}{V_{Ae}}\right)\right], \end{aligned} \tag{91}$$

where $\chi^c_0 = \cos\sigma_0$ and $\chi^s_0 = \sin\sigma_0$. When deriving these equations we used equation (84) to eliminate $C^{c,s}_{11-}$ and $C^{c,s}_{11+}$. This is the system of two linear inhomogeneous algebraic equations for $C^{c,s}_{111}$ and $C^{c,s}_{112}$. Using the expression for Ω_1 it is straightforward to show that the determinant of this system is zero. We obtain this compatibility condition by calculating the determinant of matrix obtained by substituting the right-hand side of equations (90) and (91) for one of the two columns of the determinant of this system of equations, and then taking this determinant equal to zero. After a long but straightforward calculation we obtain that this determinant is equal to Ω_2 times a non-zero quantity. Hence, eventually the compatibility condition reduces to $\Omega_2 = 0$. This is a very important result because it implies that the correction to the oscillation frequency is at least of the order of q^2. However this is an expected result because it looks like a viable assumption that the oscillation frequency is independent of the sign of q. In that case the expansion of Ω with respect to q contains only even powers of q.

Next we take $m = 1$ and $n > 1$. Collecting terms proportional to $\sin(\pi ns)\cos\sigma$ and $\sin(\pi ns)\sin\sigma$ in equation (76) yields

$$\begin{aligned} &(k_i^2 - k_e^2)C^{c,s}_{n11} + (k_i^2 + k_e^2 - 2\pi^2 n^2)e^{2\tau_0} C^{c,s}_{n12} = 0, \\ &(k_i^2 + k_e^2 - 2\pi^2 n^2)e^{2\tau_0} C^{c,s}_{n11} + (k_i^2 - k_e^2)C^{c,s}_{n12} = 0. \end{aligned} \tag{92}$$

It is easy to show that the determinant of this system of equations for $C^{c,s}_{n11}$ and $C^{c,s}_{n12}$ is non-zero. Hence, we conclude that $C^{c,s}_{n11} = C^{c,s}_{n12} = 0$ and the Fourier expansion for ψ_2 does not contain the terms proportional to $\sin(\pi ns)\cos\sigma$ and $\sin(\pi ns)\sin\sigma$ with $n > 1$.

Finally we calculate the coefficients at $\sin(\pi ns)\cos 2\sigma$ and $\sin(\pi ns)\sin 2\sigma$. These coefficients are equal to w^c_{n2} and w^s_{n2}, respectively. Using equation (57) we obtain

$$2\frac{\partial}{\partial\sigma}\left(\frac{\partial\psi_1}{\partial\tau}\sin\sigma\cosh\tau\right) = 2\frac{d\Phi}{d\tau}(\cos\sigma_0\cos 2\sigma + \sin\sigma_0\sin 2\sigma)\cosh\tau\sin(\pi s). \tag{93}$$

Using equations (60), (85), (86), and (93) we obtain from equation (76)

$$
\begin{aligned}
&(k_i^2-k_e^2)C_{n21}^{c,s}+(k_i^2+k_e^2-2\pi^2n^2)e^{4\tau_0}C_{n22}^{c,s}\\
&\quad=-\delta_{1n}C_2e^{2\tau_0}(e^{2\tau_0}+1)\chi_0^{c,s}\mp(k_i^2-\pi^2n^2)\\
&\qquad\times[C_1(g_{ni}-g_{ne})+C_2e^{2\tau_0}(g_{ni}+e^{2\tau_0}g_{ne})]\chi_0^{s,c}, \quad (94)\\
&(k_i^2+k_e^2-2\pi^2n^2)e^{4\tau_0}C_{n21}^{c,s}+(k_i^2-k_e^2)C_{n22}^{c,s}\\
&\quad=-\delta_{1n}C_1e^{2\tau_0}(e^{2\tau_0}+1)\chi_0^{c,s}\pm(k_i^2-\pi^2n^2)\\
&\qquad\times[C_1e^{2\tau_0}(g_{ni}+e^{2\tau_0}g_{ne})+C_2(g_{ni}-g_{ne})]\chi_0^{s,c}, \quad (95)
\end{aligned}
$$

where δ_{1n} is the Kronecker delta, and the upper and lower signs in these equations correspond to the first and second superscripts, respectively. Equations (94) and (95) with the first superscript constitute the system of equations for C_{n21}^c and C_{n22}^c, while these equations with the second superscript constitute the system of equations for C_{n21}^s and C_{n22}^s. The determinants of the two systems are equal. When $n=1$ the condition that these determinants are equal to zero reduces to $\rho_e=\rho_i$ that contradicts to the assumption that $\rho_e<\rho_i$. For $n>1$ these determinants can be equal to zero for some discrete values of parameters, however this is a structurally unstable case. In the generic case the determinants are non-zero implying that there are unique solutions to equations (94) and (95). Using equations (67), (68), and (82), and equation (63) rewritten as

$$k_i^2-k_e^2=\pm e^{2\tau_0}(k_i^2+k_e^2-2\pi^2) \quad (96)$$

we obtain that the solutions to equations (94) and (95) are given by

$$C_{122}^c=Re^{2\tau_0}\cos\sigma_0\coth\tau_0,\quad C_{121}^c=\mp C_{122}^c, \quad (97)$$
$$C_{122}^s=Re^{2\tau_0}\sin\sigma_0\coth\tau_0,\quad C_{121}^s=\mp C_{122}^s, \quad (98)$$

for $n=1$ and by

$$
\begin{aligned}
C_{n21}^{c,s}&=-\frac{e^{-2\tau_0}(k_i^2-\pi^2)\chi_0^{s,c}}{e^{2\tau_0}(k_i^2+k_e^2-2\pi^2n^2)+(k_i^2+k_e^2-2\pi^2)}\\
&\quad\times[C_1(g_{ni}-g_{ne})+e^{2\tau_0}C_2(g_{ni}+e^{2\tau_0}g_{ne})], \quad (99)\\
C_{n22}^{c,s}&=\frac{e^{-2\tau_0}(k_i^2-\pi^2)\chi_0^{s,c}}{e^{2\tau_0}(k_i^2+k_e^2-2\pi^2n^2)+(k_i^2+k_e^2-2\pi^2)}\\
&\quad\times[C_2(g_{ni}-g_{ne})+e^{2\tau_0}C_1(g_{ni}+e^{2\tau_0}g_{ne})], \quad (100)
\end{aligned}
$$

for $n>1$.

4.2.3 Third-order approximation

In this section we aim to obtain the corrections to the frequencies of the kink modes. Substituting equation (69) into equation (45), taking into account that $\Omega_2=0$, and collecting terms of the order of q^2 we obtain

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial\sigma^2}+\frac{\partial^2}{\partial\tau^2}\right)\bigg\{k^2\psi_3+\frac{\partial^2\psi_3}{\partial s^2}+\frac{2kR}{V_A}\Omega_3\psi_1\\
&\quad-\frac{\partial}{\partial s}\bigg[\frac{\partial\psi_2}{\partial\sigma}\cos\sigma\sinh\tau-\frac{\partial\psi_2}{\partial\tau}\sin\sigma\cosh\tau\\
&\quad+s\left(\frac{\partial\psi_1}{\partial\tau}\cos\sigma\sinh\tau+\frac{\partial\psi_1}{\partial\sigma}\sin\sigma\cosh\tau\right)\bigg]\bigg\}=0. \quad (101)
\end{aligned}
$$

The function ψ_3 must be continuous at $\tau=\pm\tau_0$. Substituting equation (69) into equation (51), collecting terms of the order of q^2, and taking into account that $\Omega_2=0$ we obtain that the following boundary condition is satisfied at $\tau=\pm\tau_0$:

$$
\begin{aligned}
&\bigg[\!\!\bigg[k^2\frac{\partial\psi_3}{\partial\tau}+\frac{\partial^3\psi_3}{\partial\tau\partial s^2}+\frac{2kR}{V_A}\Omega_3\frac{\partial\psi_1}{\partial\tau}+\\
&\quad+2\frac{\partial}{\partial\sigma}\left(\frac{\partial\psi_2}{\partial\tau}\sin\sigma\cosh\tau-s\frac{\partial\psi_1}{\partial\tau}\cos\sigma\sinh\tau\right)\\
&\quad-\frac{\partial^2}{\partial\tau\partial s}\bigg[\frac{\partial\psi_2}{\partial\sigma}\cos\sigma\sinh\tau-\frac{\partial\psi_2}{\partial\tau}\sin\sigma\cosh\tau\\
&\quad+s\left(\frac{\partial\psi_1}{\partial\tau}\cos\sigma\sinh\tau+\frac{\partial\psi_1}{\partial\sigma}\sin\sigma\cosh\tau\right)\bigg]\bigg]\!\!\bigg]=0. \quad (102)
\end{aligned}
$$

Similar to the previous section we look for the solution to this equation in the form $\psi_3=\psi_{3c}+\psi_{3p}$, where ψ_{3c} and ψ_{3p} are defined by

$$\left(\frac{\partial^2}{\partial\sigma^2}+\frac{\partial^2}{\partial\tau^2}\right)\left(k^2\psi_{3c}+\frac{\partial^2\psi_{3c}}{\partial s^2}\right)=0, \quad (103)$$

$$
\begin{aligned}
k^2\psi_{3p}+\frac{\partial^2\psi_{3p}}{\partial s^2}&=\frac{\partial}{\partial s}\bigg(\frac{\partial\psi_2}{\partial\sigma}\cos\sigma\sinh\tau-\frac{\partial\psi_2}{\partial\tau}\sin\sigma\cosh\tau\\
&\quad+s\frac{\partial\psi_1}{\partial\tau}\cos\sigma\sinh\tau+s\frac{\partial\psi_1}{\partial\sigma}\sin\sigma\cosh\tau\bigg). \quad (104)
\end{aligned}
$$

Both ψ_{3p} and ψ_{3c} must satisfy the boundary conditions

$$\psi_{3p}=\psi_{3c}=0\quad\text{at}\quad s=0,1. \quad (105)$$

Using equation (104) we transform equation (100) to

$$
\begin{aligned}
&\bigg[\!\!\bigg[k^2\frac{\partial\psi_{3c}}{\partial\tau}+\frac{\partial^3\psi_{3c}}{\partial\tau\partial s^2}+\frac{2kR}{V_A}\Omega_3\frac{\partial\psi_1}{\partial\tau}+2\frac{\partial}{\partial\sigma}\bigg(\frac{\partial\psi_2}{\partial\tau}\sin\sigma\cosh\tau\\
&\quad-s\frac{\partial\psi_1}{\partial\tau}\cos\sigma\sinh\tau\bigg)\bigg]\!\!\bigg]=0\quad\text{at}\quad\tau=\pm\tau_0. \quad (106)
\end{aligned}
$$

To calculate Ω_3 we only need to collect terms proportional to $\sin\sigma\sin(\pi s)$ and $\cos\sigma\sin(\pi s)$. In accordance with this, hereafter we only calculate terms proportional to $\sin\sigma\sin(\pi s)$ and $\cos\sigma\sin(\pi s)$. We look for the solution to equation (103) in the form of expansion similar to equation (78). Then we obtain that the solution regular as $\tau\to\pm\infty$ is

$$\psi_{3c}=\sin(\pi s)[H^c(\tau)\cos\sigma+H^s(\tau)\sin\sigma]+\dots \quad (107)$$

In this equation $H^c(\tau)$ and $H^s(\tau)$ are given by

$$
H^{c,s}(\tau)=\begin{cases}C_{3-}^{c,s}e^{\tau}, & \tau<-\tau_0,\\ C_{31}^{c,s}e^{\tau}+C_{32}^{c,s}e^{-\tau}, & |\tau|<\tau_0,\\ C_{3+}^{c,s}e^{-\tau}, & \tau>\tau_0,\end{cases} \quad (108)
$$

where $C_{3-}^{c,s}$, $C_{3+}^{c,s}$, $C_{31}^{c,s}$, and $C_{32}^{c,s}$ are constants, and the dots in equation (107) indicate terms not proportional to $\sin\sigma\sin(\pi s)$ or $\cos\sigma\sin(\pi s)$.

Next we proceed to equation (104). First of all, we note that the last two terms in the brackets on the right-hand side of this equation do not give terms proportional to $\sin\sigma$ or $\cos\sigma$. Only the terms in the Fourier expansion of ψ_2 proportional to $\sin 2\sigma$ or $\cos 2\sigma$ provide the terms proportional to $\sin\sigma$ or $\cos\sigma$ on the right-hand side of equation (104). Recalling that the expansion of ψ_2 in the double Fourier series is given by the equation similar to equation (78) however with w_{nm}^c and w_{nm}^s substituted for h_{nm}^c and h_{nm}^s, respectively, we obtain

$$\psi_2=\sum_{n=1}^{\infty}\sin(\pi ns)[w_{n2}^c(\tau)\cos 2\sigma+w_{n2}^s(\tau)\sin 2\sigma]+\dots, \quad (109)$$

where w_{n2}^c and w_{n2}^s are defined by equation (83), and the dots stand for terms not proportional to $\sin 2\sigma$ or $\cos 2\sigma$. Substituting equation (109) into equation (104) yields

$$k^2\psi_{3p} + \frac{\partial^2\psi_{3p}}{\partial s^2} = \pi\sum_{n=1}^{\infty} n\cos(\pi ns)\bigg[\bigg(w_{n2}^s\sinh\tau - \frac{1}{2}\frac{\partial w_{n2}^s}{\partial\tau}\cosh\tau\bigg)\cos\sigma - \bigg(w_{n2}^c\sinh\tau - \frac{1}{2}\frac{\partial w_{n2}^c}{\partial\tau}\cosh\tau\bigg)\sin\sigma\bigg] + \dots, \quad (110)$$

where the dots indicate terms not proportional to $\cos\sigma$ or $\sin\sigma$. The solution to this equation satisfying equation (105) is

$$\psi_{3p} = \pi\sum_{n=1}^{\infty} n\bigg(\frac{\sin[k(1-s)] + (-1)^n\sin(ks)}{\sin k(\pi^2n^2-k^2)} - \frac{\cos(\pi ns)}{\pi^2n^2-k^2}\bigg)\bigg[\bigg(w_{n2}^s\sinh\tau - \frac{1}{2}\frac{dw_{n2}^s}{d\tau}\cosh\tau\bigg)\cos\sigma - \bigg(w_{n2}^c\sinh\tau - \frac{1}{2}\frac{dw_{n2}^c}{d\tau}\cosh\tau\bigg)\sin\sigma\bigg] + \dots \quad (111)$$

Calculating the Fourier series with respect to s for this expression yields

$$\psi_{3p} = \frac{8\sin(\pi s)}{\pi^2-k^2}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\bigg[\bigg(w_{2n,2}^s\sinh\tau - \frac{1}{2}\frac{dw_{2n,2}^s}{d\tau}\cosh\tau\bigg)\cos\sigma - \bigg(w_{2n,2}^c\sinh\tau - \frac{1}{2}\frac{dw_{2n,2}^c}{d\tau}\cosh\tau\bigg)\sin\sigma\bigg] + \dots, \quad (112)$$

where the dots indicate terms proportional to $\sin(2\pi s)$, $\sin(3\pi s)$, and so on. Using equation (83) we transform this equation to

$$\psi_{3p} = \frac{8\sin(\pi s)}{\pi^2-k^2}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\bigg[\bigg(h_{2n,2}^s\sinh\tau - \Phi g_{2n}\cos\sigma_0 - \frac{1}{2}\frac{dh_{2n,2}^s}{d\tau}\cosh\tau\bigg)\cos\sigma - \bigg(h_{2n,2}^c\sinh\tau + \Phi g_{2n}\sin\sigma_0 - \frac{1}{2}\frac{dh_{2n,2}^c}{d\tau}\cosh\tau\bigg)\sin\sigma\bigg] + \dots \quad (113)$$

Using the condition that $\psi_3 = \psi_{3c} + \psi_{3p}$ is continuous at $\tau = \pm\tau_0$ we obtain with the aid of equations (59), (60), (80), (85), (107), (108), and (113)

$$C_{3-}^c = C_{31}^c + e^{2\tau_0}C_{32}^c + \sum_{n=1}^{\infty}\frac{8n}{4n^2-1}\bigg\{\frac{1}{\pi^2-k_i^2}\big[C_{2n,21}^s + e^{4\tau_0}C_{2n,22}^s + (C_1 - C_2e^{4\tau_0})g_{2n,e}\cos\sigma_0\big] - \frac{1}{\pi^2-k_e^2}\times\big[C_{2n,21}^s - e^{2\tau_0}C_{2n,22}^s + (C_1 + C_2e^{2\tau_0})g_{2n,e}\cos\sigma_0\big]\bigg\}, \quad (114)$$

$$C_{3-}^s = C_{31}^s + e^{2\tau_0}C_{32}^s - \sum_{n=1}^{\infty}\frac{8n}{4n^2-1}\bigg\{\frac{1}{\pi^2-k_i^2}\big[C_{2n,21}^c + e^{4\tau_0}C_{2n,22}^c - (C_1 - C_2e^{4\tau_0})g_{2n,e}\sin\sigma_0\big] - \frac{1}{\pi^2-k_e^2}\times\big[C_{2n,21}^c - e^{2\tau_0}C_{2n,22}^c - (C_1 + C_2e^{2\tau_0})g_{2n,e}\sin\sigma_0\big]\bigg\}, \quad (115)$$

$$C_{3+}^c = e^{2\tau_0}C_{31}^c + C_{32}^c - \sum_{n=1}^{\infty}\frac{8n}{4n^2-1}\bigg\{\frac{1}{\pi^2-k_i^2}\big[e^{4\tau_0}C_{2n,21}^s + C_{2n,22}^s + (C_1e^{4\tau_0} - C_2)g_{2n,e}\cos\sigma_0\big] + \frac{1}{\pi^2-k_e^2}\times\big[e^{2\tau_0}C_{2n,21}^s - C_{2n,22}^s + (C_1e^{2\tau_0} + C_2)g_{2n,e}\cos\sigma_0\big]\bigg\}, \quad (116)$$

$$C_{3+}^s = e^{2\tau_0}C_{31}^s + C_{32}^s + \sum_{n=1}^{\infty}\frac{8n}{4n^2-1}\bigg\{\frac{1}{\pi^2-k_i^2}\big[e^{4\tau_0}C_{2n,21}^c + C_{2n,22}^c - (C_1e^{4\tau_0} - C_2)g_{2n,e}\sin\sigma_0\big] + \frac{1}{\pi^2-k_e^2}\times\big[e^{2\tau_0}C_{2n,21}^c - C_{2n,22}^c - (C_1e^{2\tau_0} + C_2)g_{2n,e}\sin\sigma_0\big]\bigg\}. \quad (117)$$

Finally we use the boundary condition equation (106). Using equations (66) and (109) we obtain

$$2\frac{\partial}{\partial\sigma}\bigg(\frac{\partial\psi_2}{\partial\tau}\sin\sigma\cosh\tau - s\frac{\partial\psi_1}{\partial\tau}\cos\sigma\sinh\tau\bigg) = -\sin(\pi s)\cosh\tau\bigg(\frac{dw_{12}^c}{d\tau}\cos\sigma + \frac{dw_{12}^s}{d\tau}\sin\sigma\bigg) + \dots, \quad (118)$$

where dots indicate terms not proportional to $\sin(\pi s)\cos\sigma$ or $\sin(\pi s)\sin\sigma$. It follows that the coefficients at $\sin(\pi s)\cos\sigma$ and $\sin(\pi s)\sin\sigma$ in the expression inside the double brackets in equation (106) must be continuous at $\tau = \pm\tau_0$. Using equations (66), (107), and (118) we obtain from this condition that at $\tau = \pm\tau_0$ the two boundary conditions must be satisfied:

$$\bigg[\bigg[(k^2-\pi^2)\frac{dH^{c,s}}{d\tau} + \frac{2kR\chi_0^{c,s}}{V_A}\Omega_3\frac{d\Phi}{d\tau} - \frac{dw_{12}^{c,s}}{d\tau}\cosh\tau\bigg]\bigg] = 0. \quad (119)$$

Using equations (59), (80), (83), and (108) we obtain from equation (119)

$$(k_i^2-\pi^2)C_{3-}^{c,s} + \frac{2k_iR\chi_0^{c,s}}{V_{Ai}}\Omega_3C_- - (1+e^{-2\tau_0})C_{12-}^{c,s} = (k_e^2-\pi^2)(C_{31}^{c,s} - e^{2\tau_0}C_{32}^{c,s}) + \frac{2k_eR\chi_0^{c,s}}{V_{Ae}}\Omega_3(C_1 - e^{2\tau_0}C_2) - (1+e^{-2\tau_0})(C_{121}^{c,s} - e^{4\tau_0}C_{122}^{c,s}), \quad (120)$$

$$(k_i^2-\pi^2)C_{3+}^{c,s} + \frac{2k_iR\chi_0^{c,s}}{V_{Ai}}\Omega_3C_+ - (1+e^{-2\tau_0})C_{12+}^{c,s} = (\pi^2-k_e^2)(e^{2\tau_0}C_{31}^{c,s} - C_{32}^{c,s}) - \frac{2k_eR\chi_0^{c,s}}{V_{Ae}}\Omega_3(e^{2\tau_0}C_1 - C_2) + (1+e^{-2\tau_0})(e^{4\tau_0}C_{121}^{c,s} - C_{122}^{c,s}). \quad (121)$$

When deriving equations (120) and (121) we took into account that $g_1 = 0$. Using equations (60), (85), (86), and (114)–(117) we transform equations (120) and (121) to

$$(k_i^2-k_e^2)C_{31}^{c,s} + e^{2\tau_0}(k_i^2+k_e^2-2\pi^2)C_{32}^{c,s} = \Pi_-^{c,s}, \quad (122)$$
$$e^{2\tau_0}(k_i^2+k_e^2-2\pi^2)C_{31}^{c,s} + (k_i^2-k_e^2)C_{32}^{c,s} = \Pi_+^{c,s}, \quad (123)$$

where the expressions for $\Pi_-^{c,s}$ and $\Pi_+^{c,s}$ are given in Appendix C. Equations (122) and (123) for quantities with the superscript 'c' constitute the system of two linear inhomogeneous equations for C_{31}^c and C_{32}^c, while equations (122) and (123) for quantities with the superscript 's' constitute the system of two linear inhomogeneous equations for C_{31}^s and C_{32}^s. The determinants of these systems are equal to zero. This implies that the two systems have solutions only when the compatibility conditions are satisfied. The simplest way to obtain these conditions is to multiply equation (122) by $(k_i^2-k_e^2)$ and subtract from the obtained result equation (123) multiplied by $e^{2\tau_0}(k_i^2+k_e^2-2\pi^2)$. Then using equation (96) we write the two compatibility conditions in a unified form as

$$\Pi_-^{c,s} = \pm\Pi_+^{c,s}, \quad (124)$$

where as before the signs plus and minus correspond to $\Omega_1 = \Omega_+$ and $\Omega_1 = \Omega_-$, respectively. It is shown in Appendix D that equa-

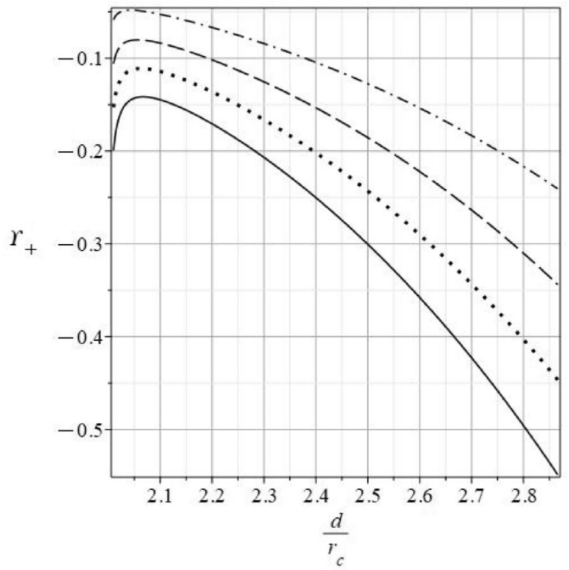

Figure 3. Dependence of Υ_+ on the ratio of the distance between the tube axes d and the tube cross-section radius r_c. The solid, dotted, dashed, and dash–dotted lines correspond to $\rho_i/\rho_e = 2$, 3, 5, and 10, respectively.

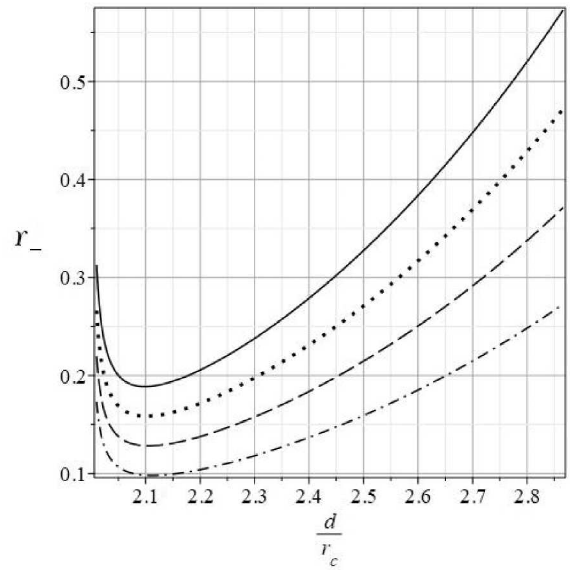

Figure 4. Dependence of Υ_- on the ratio of the distance between the tube axes d and the tube cross-section radius r_c. The solid, dotted, dashed, and dash–dotted lines correspond to $\rho_i/\rho_e = 2$, 3, 5, and 10, respectively.

tion (124) reduces to

$$\Omega_3 = \Omega_\pm \Upsilon_\pm, \tag{125}$$

where $\Upsilon_\pm$ are given by equation (D6). Analysing the expressions for $\Upsilon_\pm$ we can notice two problems. The first one is that $|\Upsilon_\pm| \to \infty$ as $\rho_i - \rho_e \to 0$. It follows from equation (9) that the distance between the centres of the two tubes is $d = 2x_c = 2a \coth \tau_0$. The ratio of this distance and the tube radius is $d/r_c = 2\cosh \tau_0$. Hence, taking $\tau_0 \to \infty$, we obtain the equilibrium with the highly separated tubes. The second problem is that $|\Upsilon_\pm| \to \infty$ as $\tau_0 \to \infty$. However, both problems are only apparent. If we take $\rho_i - \rho_e \to 0$ or $\tau_0 \to \infty$ then we obtain $\Omega_+ - \Omega_- \to 0$, so we obtain one double eigenvalue of the problem rather than two different eigenvalues. In this case the expansion given by equation (65) is not valid anymore. Rather we must use the expansion with respect to $|q|^{1/2}$. Hence, our analysis is only valid under the restrictions $\rho_i/\rho_e > 1$ and $\tau_0 \lesssim 1$.

We numerically calculated the dependences of Υ_+ and Υ_- on the distance between the tube axes d. These dependences are shown in Figs 3 and 4. We can see that braiding results in the reduction of Ω_+ and the enchantment of Ω_-. Hence it reduces the difference between Ω_+ and Ω_-.

5 SUMMARY AND CONCLUSIONS

In this article we studied kink oscillations of braided coronal magnetic loops. We considered the simplest model of a braided coronal loop that consists of just two identical magnetic tubes. We assumed that the tube cross-sections are circular and do not change along the tubes. The equilibrium plasma density is constant inside the tubes, and it is also constant in the surrounding plasma.

To describe the plasma motion we used the linear ideal MHD equations in the cold plasma approximation. We introduced parameter q characterizing the value of braiding. When q increases, the braiding becomes stronger. We note that $q = 0$ corresponds to the lack of braiding. When there is no braiding there are two kinds of kink modes, fast and slow. We used the regular perturbation method to calculate the correction to the frequency of the fast and slow oscillations for $|q| \ll 1$. We found that these corrections are proportional to q^2.

We obtained the analytical expressions for the coefficients of proportionality at q^2 for the corrections to the fast, Υ_+, and slow, Υ_-, kink modes. These expressions are very complex, so we calculate the dependences of Υ_+ and Υ_- on the relative distance between the tube axes, d/r_c, for various values of the densities inside and outside the tubes, ρ_i/ρ_e. We can see from the numerical results that braiding reduces the fast frequency and increases the slow frequency thus reducing the difference between the two frequencies. The increase in the distance between the tube axes enhances the effect. Also, the smaller the density ratio, the stronger the effect is.

We now discuss the implications of the obtained results for coronal seismology. The main ideas of coronal seismology have been formulated by H. Rosenberg (1970), Y. Uchida (1970), and B. Roberts, P. M. Edwin & A. O. Benz (1984). However the real flourishing of coronal seismology started after the waves and oscillations in the solar corona were observed by *Transition Region and Coronal Explorer* and reported by M. J. Aschwanden et al. (1999) and V. M. Nakariakov et al. (1999). The recent review of coronal seismology is given by V. M. Nakariakov et al. (2024). The method of coronal seismology has been used to estimate the magnetic field magnitude, the atmospheric scale height, and the transverse structure of coronal magnetic loops. All these studies are based on using the model of monolithic coronal loops. To our knowledge there have been no attempt of using coronal seismology to determine if loops are monolithic or composite. The existence of two eigenmodes, fast and slow, in the model of two parallel magnetic tubes provides the pathway for making the distinction between the two models of coronal loops. If the two frequencies of kink oscillations of a coronal loop are found then this would be a strong argument in favour of the composite model of this loop. It follows from equation (64) that the difference between frequencies of the fast and slow modes of oscillations in the system of two straight tubes is the largest when $\tau_0 = 0$, that is, when the two tubes touch each other. In this case $\Omega_+/\Omega_- = (\rho_i/\rho_e)^{1/2}$. Hence, the ratio of the two frequencies can be quite large and, in principle, we can hope to find two different frequencies of kink oscillations of composite coronal magnetic loops. We found that braiding reduces Ω_+/Ω_-. Since it is quite possible that composite loops are braided this makes it less probable to detect two different frequencies of kink oscillations of the same coronal loop. However our results have been found in the weak braiding approximation. First of all, it is not clear if Ω_+/Ω_- is a monotonic function of braiding parameter q. Even if it is monotonically decreasing then it is still possible that the correction to Ω_+/Ω_- is small for a finite value of q. We see that studying the effect of braiding on the frequencies of kink oscillations of the two-tube system is important for coronal seismology. To study this effect we plan to solve equations (45) and (46) with the boundary conditions equations (47), (50), and (51) numerically.

DATA AVAILABILITY

There are no new data associated with this article.

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APPENDIX A: CURVILINEAR COORDINATE SYSTEM

In this section we derive various expressions used in our analysis. Here, we use the auxiliary Cartesian coordinates x, y, z and introduce the position vector $\boldsymbol{r} = (x, y, z)$. We also introduce the covariant basis vectors

$$\boldsymbol{\epsilon}_\sigma = \frac{1}{a}\frac{\partial \boldsymbol{r}}{\partial \sigma}, \quad \boldsymbol{\epsilon}_\tau = \frac{1}{a}\frac{\partial \boldsymbol{r}}{\partial \tau}, \quad \boldsymbol{\epsilon}_s = \frac{1}{a}\frac{\partial \boldsymbol{r}}{\partial s},$$

where the coefficient a^{-1} is introduced to have the covariant basis vectors dimensionless. We start by obtaining expressions for Cartesian coordinates in terms of curvilinear coordinate. We use the relations

$$\begin{aligned} x &= r\cos\phi = r\cos\phi_0\cos(\phi-\phi_0) - r\sin\phi_0\sin(\phi-\phi_0), \\ y &= r\sin\phi = r\sin\phi_0\cos(\phi-\phi_0) + r\cos\phi_0\sin(\phi-\phi_0). \end{aligned} \tag{A1}$$

Using equation (5) yields

$$\begin{aligned} r\cos\phi_0 = x_0 &= \frac{a\sinh\tau}{\cosh\tau - \cos\sigma}, \\ r\sin\phi_0 = y_0 &= \frac{a\sin\sigma}{\cosh\tau - \cos\sigma}. \end{aligned} \tag{A2}$$

With the aid of these expressions and equations (4) and (7) we obtain

$$\begin{aligned} x &= a\frac{\sinh\tau\cos(\kappa s) - \sin\sigma\sin(\kappa s)}{\cosh\tau - \cos\sigma}, \\ y &= a\frac{\sinh\tau\sin(\kappa s) + \sin\sigma\cos(\kappa s)}{\cosh\tau - \cos\sigma}, \end{aligned} \tag{A3}$$

where

$$\kappa = \frac{AL}{\sqrt{B_0^2 + A^2(2R^2 - r^2)}}. \tag{A4}$$

Finally it follows from equation (7) that

$$z = \frac{sL}{\Lambda}, \quad \Lambda = \sqrt{\frac{B_0^2 + A^2(2R^2 - r^2)}{B_0^2 + 2A^2(R^2 - r^2)}}. \tag{A5}$$

We recall that the expression for r in terms of curvilinear coordinates is given by equation (6). Hence, equations (A3) and (A5) give the expression of Cartesian coordinates in terms of curvilinear coordinates.

The next step is to obtain the expressions for the vectors of covariant basis in terms of unit vectors of Cartesian coordinates. Using equations (A3)–(A5) we obtain

$$\begin{aligned} \boldsymbol{\epsilon}_\sigma = {} & \left(-X + \frac{s\chi Z\sin\sigma\cosh\tau}{(\cosh\tau - \cos\sigma)^2}\right)\boldsymbol{e}_x - \left(Y + \frac{s\chi W\sin\sigma\cosh\tau}{(\cosh\tau - \cos\sigma)^2}\right)\boldsymbol{e}_y \\ & + \frac{s\chi(B_0^2 + 2A^2R^2)\sin\sigma\cosh\tau}{aA\sqrt{B_0^2 + 2A^2(R^2 - r^2)}(\cosh\tau - \cos\sigma)^2}\boldsymbol{e}_z, \end{aligned} \tag{A6}$$

$$\begin{aligned} \boldsymbol{\epsilon}_\tau = {} & \left(Y + \frac{s\chi Z\cos\sigma\sinh\tau}{(\cosh\tau - \cos\sigma)^2}\right)\boldsymbol{e}_x - \left(X + \frac{s\chi W\cos\sigma\sinh\tau}{(\cosh\tau - \cos\sigma)^2}\right)\boldsymbol{e}_y \\ & + \frac{s\chi(B_0^2 + 2A^2R^2)\cos\sigma\sinh\tau}{aA\sqrt{B_0^2 + 2A^2(R^2 - r^2)}(\cosh\tau - \cos\sigma)^2}\boldsymbol{e}_z, \end{aligned} \tag{A7}$$

$$\boldsymbol{\epsilon}_s = \kappa(-Z\boldsymbol{e}_x + W\boldsymbol{e}_y) + \frac{L}{a\Lambda}\boldsymbol{e}_z, \tag{A8}$$

where

$$X = \frac{\sin\sigma\sinh\tau\cos(\kappa s) + (\cos\sigma\cosh\tau - 1)\sin(\kappa s)}{(\cosh\tau - \cos\sigma)^2}, \tag{A9}$$

$$Y = \frac{\sin\sigma\sinh\tau\sin(\kappa s) - (\cos\sigma\cosh\tau - 1)\cos(\kappa s)}{(\cosh\tau - \cos\sigma)^2}, \tag{A10}$$

$$Z = \frac{\sin\sigma\cos(\kappa s) + \sinh\tau\sin(\kappa s)}{\cosh\tau - \cos\sigma}, \tag{A11}$$

$$W = \frac{\sinh\tau\cos(\kappa s) - \sin\sigma\sin(\kappa s)}{\cosh\tau - \cos\sigma}, \tag{A12}$$

$$\chi = \frac{a^2A^3L}{[B_0^2 + A^2(2R^2 - r^2)]^{3/2}}. \tag{A13}$$

The lengths of vectors of the covariant basis are defined by

$$|\boldsymbol{\epsilon}_\sigma|^2 = \frac{1}{(\cosh\tau - \cos\sigma)^2} - \frac{s\chi\sin 2\sigma\cosh 2\tau}{2(\cosh\tau-\cos\sigma)^4} + \frac{s^2\chi^2\sin^2\sigma\cosh^2\tau}{a^2(\cosh\tau-\cos\sigma)^4}\left[r^2 + \frac{(B_0^2+2A^2R^2)^2}{A^2[B_0^2+2A^2(R^2-r^2)]}\right], \tag{A14}$$

$$|\boldsymbol{\epsilon}_\tau|^2 = \frac{1}{(\cosh\tau - \cos\sigma)^2} + \frac{s\chi\sin 2\sigma\cosh 2\tau}{2(\cosh\tau-\cos\sigma)^4} + \frac{s^2\chi^2\cos^2\sigma\sinh^2\tau}{a^2(\cosh\tau-\cos\sigma)^4}\left[r^2 + \frac{(B_0^2+2A^2R^2)^2}{A^2[B_0^2+2A^2(R^2-r^2)]}\right], \tag{A15}$$

$$|\boldsymbol{\epsilon}_s| = \frac{L}{a}. \tag{A16}$$

We also calculate the scalar products of vectors of the covariant basis,

$$\boldsymbol{\epsilon}_\sigma\cdot\boldsymbol{\epsilon}_\tau = \frac{s\chi(1-\cos 2\sigma\cosh 2\tau)}{2(\cosh\tau-\cos\sigma)^4} + \frac{s^2\chi^2r^2\sin 2\sigma\sinh 2\tau}{4a^2(\cosh\tau-\cos\sigma)^4} + \frac{s^2\chi^2(B_0^2+2A^2R^2)^2\sin 2\sigma\sinh 2\tau}{4a^2A^2(\cosh\tau-\cos\sigma)^4[B_0^2+2A^2(R^2-r^2)]}, \tag{A17}$$

$$\boldsymbol{\epsilon}_\sigma\cdot\boldsymbol{\epsilon}_s = \frac{\kappa[a^2\cos\sigma\sinh\tau - s\chi r^2\sin\sigma\cosh\tau]}{a^2(\cosh\tau-\cos\sigma)^2} + \frac{s\chi L(B_0^2+2A^2R^2)\sin\sigma\cosh\tau}{a^2A(\cosh\tau-\cos\sigma)^2\sqrt{B_0^2+A^2(2R^2-r^2)}}, \tag{A18}$$

$$\boldsymbol{\epsilon}_\tau\cdot\boldsymbol{\epsilon}_s = -\frac{\kappa[a^2\sin\sigma\cosh\tau + s\chi r^2\cos\sigma\sinh\tau]}{a^2(\cosh\tau-\cos\sigma)^2} + \frac{s\chi L(B_0^2+2A^2R^2)\cos\sigma\sinh\tau}{a^2A(\cosh\tau-\cos\sigma)^2\sqrt{B_0^2+A^2(2R^2-r^2)}}. \tag{A19}$$

Using equations (A14)–(A19) and the bi-linearity of scalar product we now calculate the scalar product of any two vectors defined by their components in the covariant basis.

Here, we use the triple product of the basis vectors of covariant basis. Using equations (A6)–(A12) we obtain

$$(\boldsymbol{\epsilon}_\sigma, \boldsymbol{\epsilon}_\tau, \boldsymbol{\epsilon}_s) = \frac{L\Delta}{a}, \quad \Delta = \frac{1}{\Lambda(\cosh\tau-\cos\sigma)^2}, \tag{A20}$$

where $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$. The divergence of vector $\boldsymbol{b}$ is defined by

$$\nabla\cdot\boldsymbol{b} = \frac{1}{L\Delta}\left[\frac{\partial(\boldsymbol{b},\boldsymbol{\epsilon}_\tau,\boldsymbol{\epsilon}_s)}{\partial\sigma} + \frac{\partial(\boldsymbol{b},\boldsymbol{\epsilon}_s,\boldsymbol{\epsilon}_\sigma)}{\partial\tau} + \frac{\partial(\boldsymbol{b},\boldsymbol{\epsilon}_\sigma,\boldsymbol{\epsilon}_\tau)}{\partial s}\right]. \tag{A21}$$

Using the properties of the triple product we transform this expression to

$$\nabla\cdot\boldsymbol{b} = \frac{1}{a\Delta}\left[\frac{\partial(\Delta b_\sigma)}{\partial\sigma} + \frac{\partial(\Delta b_\tau)}{\partial\tau} + \frac{\partial(\Delta b_s)}{\partial s}\right]. \tag{A22}$$

We can invert the relations between $\boldsymbol{\epsilon}_\sigma$, $\boldsymbol{\epsilon}_\tau$, and $\boldsymbol{\epsilon}_s$, and $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_x$ given by equations (A6)–(A8) to obtain

$$\boldsymbol{e}_x = -\big[X(\cosh\tau-\cos\sigma)^2 + 2s\chi W\Lambda^2\cos\sigma\sinh\tau\big]\boldsymbol{\epsilon}_\sigma + \big[Y(\cosh\tau-\cos\sigma)^2 + 2s\chi W\Lambda^2\sin\sigma\cosh\tau\big]\boldsymbol{\epsilon}_\tau + \frac{s\chi\Lambda W(B_0^2+2A^2R^2)}{AL\sqrt{B_0^2+2A^2(R^2-r^2)}}\boldsymbol{\epsilon}_s, \tag{A23}$$

$$\boldsymbol{e}_y = -\big[Y(\cosh\tau-\cos\sigma)^2 + 2s\chi Z\Lambda^2\cos\sigma\sinh\tau\big]\boldsymbol{\epsilon}_\sigma - \big[X(\cosh\tau-\cos\sigma)^2 - 2s\chi Z\Lambda^2\sin\sigma\cosh\tau\big]\boldsymbol{\epsilon}_\tau + \frac{s\chi\Lambda Z(B_0^2+2A^2R^2)}{AL\sqrt{B_0^2+2A^2(R^2-r^2)}}\boldsymbol{\epsilon}_s, \tag{A24}$$

$$L\boldsymbol{e}_z = a\Lambda[\kappa(-\boldsymbol{\epsilon}_\sigma\cos\sigma\sinh\tau + \boldsymbol{\epsilon}_\tau\sin\sigma\cosh\tau) + \boldsymbol{\epsilon}_s]. \tag{A25}$$

Next we calculate the vector products of basis vectors of the covariant coordinates. First we use equations (A6)–(A8) to express these products in terms of unit vectors of Cartesian coordinates. Then we use equations (A23)–(A25) to finally obtain the expressions in terms of basis vectors of the covariant coordinates. As a result we obtain

$$\begin{aligned}\boldsymbol{\epsilon}_\sigma\times\boldsymbol{\epsilon}_s = {}& \boldsymbol{\epsilon}_\sigma\left[\frac{s\chi L\Lambda}{a(\cosh\tau-\cos\sigma)^2}\left(1-\cos 2\sigma\cosh 2\tau - \frac{r^2s\chi}{a^2}\Lambda^2\sin 2\sigma\sinh 2\tau\right) + \frac{a\kappa^2\Lambda\sin 2\sigma\sinh 2\tau}{4L(\cosh\tau-\cos\sigma)^2}\right] \\ & - \frac{L\boldsymbol{\epsilon}_\tau}{a}\left\{\frac{1}{\Lambda} - \frac{s\chi\Lambda}{(\cosh\tau-\cos\sigma)^2}\big[\sin 2\sigma\sinh 2\tau - \left(4a^{-2}r^2s\chi\Lambda^2 + a^2\kappa^2L^{-2}\right)\sin^2\sigma\cosh^2\tau\big]\right\} \\ & + \frac{\boldsymbol{\epsilon}_s}{(\cosh\tau-\cos\sigma)^2}\left[\frac{s\chi(B_0^2+2A^2R^2)\cos\sigma\sinh\tau}{aA\sqrt{B_0^2+2A^2(R^2-r^2)}}\times\left(1-\frac{2r^2s\chi\Lambda^2\sin\sigma\cosh\tau}{a^2}\right) - \frac{a^2\kappa\Lambda\sin\sigma\cosh\tau}{L}\right],\end{aligned} \tag{A26}$$

$$\begin{aligned}\boldsymbol{\epsilon}_\tau\times\boldsymbol{\epsilon}_s = {}& \frac{L\boldsymbol{\epsilon}_\sigma}{a}\left\{\frac{1}{\Lambda} + \frac{s\chi\Lambda}{(\cosh\tau-\cos\sigma)^2}\big[\sin 2\sigma\sinh 2\tau + \left(4a^{-2}r^2s\chi\Lambda^2 + a^2\kappa^2L^{-2}\right)\cos^2\sigma\sinh^2\tau\big]\right\} \\ & - \frac{L\Lambda\boldsymbol{\epsilon}_\tau}{(\cosh\tau-\cos\sigma)^2}\left[s\chi(1-\cos 2\sigma\cosh 2\tau) + \left(\frac{s^2\chi^2r^2\Lambda^2}{a^3} + \frac{a\kappa^2}{4L^2}\right)\sin 2\sigma\sinh 2\tau\right] \\ & + \frac{\boldsymbol{\epsilon}_s}{(\cosh\tau-\cos\sigma)^2}\left[\frac{a\kappa\Lambda\cos\sigma\sinh\tau}{L} + \frac{B_0^2+2A^2R^2}{a^3A}\times\frac{s\chi(a^2\sin\sigma\cosh\tau + 2r^2\Lambda^2\cos\sigma\sinh\tau)}{\sqrt{B_0^2+2A^2(R^2-r^2)}}\right].\end{aligned} \tag{A27}$$

Finally we give the expression for the curl of a vector. It reads

$$\nabla\times\boldsymbol{b} = \frac{1}{L\Delta}\begin{vmatrix}\boldsymbol{\epsilon}_\sigma & \boldsymbol{\epsilon}_\tau & \boldsymbol{\epsilon}_s \\ \dfrac{\partial}{\partial\sigma} & \dfrac{\partial}{\partial\tau} & \dfrac{\partial}{\partial s} \\ \boldsymbol{b}\cdot\boldsymbol{\epsilon}_\sigma & \boldsymbol{b}\cdot\boldsymbol{\epsilon}_\tau & \boldsymbol{b}\cdot\boldsymbol{\epsilon}_s\end{vmatrix}. \tag{A28}$$

APPENDIX B: ASYMPTOTIC FORMULAE

In this section we use the results presented in the previous section to obtain asymptotic formulae valid for $\epsilon \ll 1$. Using equations (15) and (20) we obtain from equations (A4), (A5), and (A13)

$$\kappa = q + \mathcal{O}(\epsilon^2), \quad \Lambda = 1 + \mathcal{O}(\epsilon^2), \quad \chi = \epsilon^2\alpha^2q^3 + \mathcal{O}(\epsilon^4), \quad \alpha = a/R = \mathcal{O}(1). \tag{B1}$$

Using these formulae we obtain from equations (A14), (A15), and (A17)–(A19)

$$|\boldsymbol{\epsilon}_\sigma|^2 = \frac{1+\mathcal{O}(\epsilon^2)}{(\cosh\tau-\cos\sigma)^2}, \quad |\boldsymbol{\epsilon}_\tau|^2 = \frac{1+\mathcal{O}(\epsilon^2)}{(\cosh\tau-\cos\sigma)^2}, \tag{B2}$$

$$\boldsymbol{\epsilon}_\sigma\cdot\boldsymbol{\epsilon}_\tau = \mathcal{O}(\epsilon^2), \tag{B3}$$

$$\boldsymbol{\epsilon}_\sigma\cdot\boldsymbol{\epsilon}_s = qu\big[1+\mathcal{O}(\epsilon^2)\big], \tag{B4}$$

$$\boldsymbol{\epsilon}_\tau\cdot\boldsymbol{\epsilon}_s = -qv\big[1+\mathcal{O}(\epsilon^2)\big], \tag{B5}$$

where u and v are defined by equations (25) and (26). Finally, we give the approximation expressions for the vector products:

$$\boldsymbol{\epsilon}_\sigma\times\boldsymbol{\epsilon}_s = -\frac{L}{a}\big[\boldsymbol{\epsilon}_\tau + \mathcal{O}(\epsilon^2)\big], \quad \boldsymbol{\epsilon}_\tau\times\boldsymbol{\epsilon}_s = \frac{L}{a}\big[\boldsymbol{\epsilon}_\sigma + \mathcal{O}(\epsilon^2)\big]. \tag{B6}$$

APPENDIX C: EXPRESSIONS FOR $\Pi_\pm^{c,s}$

$$\begin{aligned}
\Pi_-^c &= -2R\Omega_3\bigg[C_1\left(\frac{k_i}{V_{Ai}} - \frac{k_e}{V_{Ae}}\right) \\
&\quad + C_2 e^{2\tau_0}\left(\frac{k_i}{V_{Ai}} + \frac{k_e}{V_{Ae}}\right)\bigg]\cos\sigma_0 + 2e^{2\tau_0}(e^{2\tau_0}+1)C^c_{122} \\
&\quad + \sum_{n=1}^{\infty}\frac{8n}{4n^2-1}\Big\{(C_1 - C_2 e^{4\tau_0})g_{2n,e}\cos\sigma_0 + C^s_{2n,21} \\
&\quad + e^{4\tau_0}C^s_{2n,22} - \frac{\pi^2-k_i^2}{\pi^2-k_e^2}\big[(C_1 + C_2 e^{2\tau_0})g_{2n,e}\cos\sigma_0 \\
&\quad + C^s_{2n,21} - e^{2\tau_0}C^s_{2n,22}\big]\Big\},
\end{aligned} \tag{C1}$$

$$\begin{aligned}
\Pi_-^s &= -2R\Omega_3\bigg[C_1\left(\frac{k_i}{V_{Ai}} - \frac{k_e}{V_{Ae}}\right) \\
&\quad + C_2 e^{2\tau_0}\left(\frac{k_i}{V_{Ai}} + \frac{k_e}{V_{Ae}}\right)\bigg]\sin\sigma_0 + 2e^{2\tau_0}(e^{2\tau_0}+1)C^s_{122} \\
&\quad + \sum_{n=1}^{\infty}\frac{8n}{4n^2-1}\Big\{(C_1 - C_2 e^{4\tau_0})g_{2n,e}\sin\sigma_0 - C^c_{2n,21} \\
&\quad - e^{4\tau_0}C^c_{2n,22} + \frac{\pi^2-k_i^2}{\pi^2-k_e^2}\big[C^c_{2n,21} - e^{2\tau_0}C^c_{2n,22} \\
&\quad - (C_1 + C_2 e^{2\tau_0})g_{2n,e}\sin\sigma_0\big]\Big\},
\end{aligned} \tag{C2}$$

$$\begin{aligned}
\Pi_+^c &= -2R\Omega_3\bigg[C_1 e^{2\tau_0}\left(\frac{k_i}{V_{Ai}} + \frac{k_e}{V_{Ae}}\right) \\
&\quad + C_2\left(\frac{k_i}{V_{Ai}} - \frac{k_e}{V_{Ae}}\right)\bigg]\cos\sigma_0 + 2e^{2\tau_0}(e^{2\tau_0}+1)C^c_{121} \\
&\quad - \sum_{n=1}^{\infty}\frac{8n}{4n^2-1}\Big\{(C_1 e^{4\tau_0} - C_2)g_{2n,e}\cos\sigma_0 + e^{4\tau_0}C^s_{2n,21} \\
&\quad + C^s_{2n,22} + \frac{\pi^2-k_i^2}{\pi^2-k_e^2}\big[(C_1 e^{2\tau_0} + C_2)g_{2n,e}\cos\sigma_0 \\
&\quad + e^{2\tau_0}C^s_{2n,21} - C^s_{2n,22}\big]\Big\},
\end{aligned} \tag{C3}$$

$$\begin{aligned}
\Pi_+^s &= -2R\Omega_3\bigg[C_1 e^{2\tau_0}\left(\frac{k_i}{V_{Ai}} + \frac{k_e}{V_{Ae}}\right) \\
&\quad + C_2\left(\frac{k_i}{V_{Ai}} + \frac{k_e}{V_{Ae}}\right)\bigg]\sin\sigma_0 + 2e^{2\tau_0}(e^{2\tau_0}+1)C^s_{121} \\
&\quad - \sum_{n=1}^{\infty}\frac{8n}{4n^2-1}\Big\{(C_1 e^{4\tau_0} - C_2)g_{2n,e}\sin\sigma_0 - e^{4\tau_0}C^c_{2n,21} \\
&\quad - C^c_{2n,22} - \frac{\pi^2-k_i^2}{\pi^2-k_e^2}\big[e^{2\tau_0}C^c_{2n,21} - C^c_{2n,22} \\
&\quad - (C_1 e^{2\tau_0} + C_2)g_{2n,e}\sin\sigma_0\big]\Big\}.
\end{aligned} \tag{C4}$$

APPENDIX D: EVALUATION OF COMPATIBILITY CONDITIONS

In this section we evaluate the compatibility conditions given by equation (124). First we consider equation (124) with the superscript 'c'. Using equations (68), (C1), and (C3) we obtain

$$\begin{aligned}
&4RC_1\Omega_3\bigg[\frac{k_i}{V_{Ai}} - \frac{k_e}{V_{Ae}} \mp e^{2\tau_0}\left(\frac{k_i}{V_{Ai}} + \frac{k_e}{V_{Ae}}\right)\bigg]\cos\sigma_0 \\
&\quad = 2e^{2\tau_0}(e^{2\tau_0}+1)(C^c_{122} \mp C^c_{121}) - 8\Big[(1-e^{2\tau_0})\frac{\pi^2-k_i^2}{\pi^2-k_e^2} \\
&\qquad - 1 - e^{4\tau_0}\Big]\sum_{n=1}^{\infty}\frac{n(C^s_{2n,21} \pm C^s_{2n,22} + 2C_1 g_{2n,e}\cos\sigma_0)}{4n^2-1}.
\end{aligned} \tag{D1}$$

Now we transform the left-hand side of equation (D1). We use the relations

$$\frac{k_{i,e}}{V_{Ai,e}} = \frac{R\Omega_1\rho_{i,e}}{\rho V_A^2}, \qquad k_{i,e}^2 = \frac{R^2\Omega_1^2\rho_{i,e}}{\rho V_A^2}. \tag{D2}$$

It follows from these relations that

$$\frac{k_{i,e}}{V_{Ai,e}} = \frac{k_{i,e}^2}{R\Omega_1}. \tag{D3}$$

Using equations (96) and (D3) we obtain

$$\frac{k_i}{V_{Ai}} - \frac{k_e}{V_{Ae}} \mp e^{2\tau_0}\left(\frac{k_i}{V_{Ai}} + \frac{k_e}{V_{Ae}}\right) = \mp\frac{2\pi^2 e^{2\tau_0}}{R\Omega_1}. \tag{D4}$$

Next we use equations (67), (97), (99), (100), and (D4) to transform equation (D1) to

$$(\Omega_3 - \Omega_1\Upsilon_\pm)\cos\sigma_0 = 0, \tag{D5}$$

where

$$\begin{aligned}
\Upsilon_\pm &= \mp\frac{\rho V_A^2 e^{2\tau_0}(e^{2\tau_0}+1)\coth\tau_0}{2\pi^2R^2\Omega_1^2(\rho_i-\rho_e)} \pm \frac{2}{\pi^2}\bigg[1 \pm e^{4\tau_0} \\
&\quad - (1 \mp e^{2\tau_0})\frac{\pi^2-k_i^2}{\pi^2-k_e^2}\bigg]\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\bigg[g_{2n,e}e^{-2\tau_0} \\
&\quad \mp \frac{e^{-4\tau_0}(k_i^2-\pi^2)[g_{2n,i}(1\mp e^{2\tau_0}) - g_{2n,e}(1 \pm e^{4\tau_0})]}{e^{2\tau_0}(k_i^2+k_e^2-2\pi^2n^2) + k_i^2 + k_e^2 - 2\pi^2}\bigg].
\end{aligned} \tag{D6}$$

Evaluation of equation (124) with the superscript 's' is the complete repetition of calculation bys that results in equation (D5). This gives

$$(\Omega_3 - \Omega_1\Upsilon_\pm)\sin\sigma_0 = 0. \tag{D7}$$

Equation (125) follows from equations (D5) and (D7).

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