Alternating algorithms applied to Cauchy problems for a fractional elliptic equation with fractional-order boundary conditions

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Abstract. The main objective of this work is to investigate the possibility of developing alternating algorithms that preserve the governing fractional differential equation when applied to solving associated Cauchy problems. To this end, we consider an ill-posed Cauchy problem associated to a fractional partial differential equation with fractional-order boundary conditions. We suggest an alternating algorithm inspired by the one of [14] developed for the classical Cauchy problem for the elliptic Laplace's equation. Such algorithm involves two types of problems whose stabilities are proved rigorously by two methods. The first method consists of the establishment of weak formulations for these problems and subsequently the stability can be obtained through the coercivity and continuity of the associated bilinear and linear forms, respectively. The second method is based on the use of the method of separation of variables. Subsequently, we develop several new inequalities based, essentially, on the asymptotic behaviour of Mittag-Leffler functions, to control the solutions in $L^2(H_0^1)$. Some numerical simulations are presented to support the main key inequalities involved in our analysis. The convergence of the sequence of solutions of the alternating algorithm towards the solution of the ill-posed Cauchy problem is proved to hold in the $L^2(H_0^1)$-seminorm.

Keywords: fractional differential equations; alternating iterative algorithm; Cauchy problems; convergence; stability; asymptotic behaviour of Mittag-Leffler functions.

1. Introduction to the problem, related literature and description of main results

The non-local character of fractional partial differential equations (PDEs) offers additional mathematical modelling options to many applications in viscoelasticity [12], geo-hydrology [2] and viral epidemics [5], to mention only a few. The objective of this paper is to devise and analyse iterative alternating algorithms for solving the fractional PDE: $\mathcal{P}^\alpha u + \Delta u = 0$ (posed in $\Omega_T := \Omega \times (0,T) \subset \mathbb{R}^{d+1}$), where $\mathcal{P}^\alpha$ is the conservative Caputo operator $D\left(\partial_t^{1-\beta}\right)$ of order $2-\beta \in (1,2)$ and $D = \partial_t$ denotes the usual first-order derivative with respect to the t variable [24], with given Dirichlet and Neumann data (Cauchy data) at $t = 0$ and a homogenous Dirichlet boundary condition on $\partial\Omega \times (0,T)$. It is worth mentioning that the word alternating comes from alternating the Dirichlet and Neumann data at each iteration.

Let $\beta \in (0,1)$ be given and let us now consider the Cauchy problem for the fractional PDE

$$-D\left(\partial_t^{1-\beta} u\right)(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \Omega_T := \Omega \times (0,T), \tag{1}$$

where Ω is an open bounded connected subset of $\mathbb{R}^d$ (with $d \in \mathbb{N}^* := \mathbb{N}\backslash\{0\}$) with a Lipschitz continuous boundary $\partial\Omega$ and $\partial_t^{1-\beta}$ is the Caputo derivative of order $1-\beta \in (0,1)$ given by (see, for instance, [21, Page 426])

$$\partial_t^{1-\beta}\varphi(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\varphi'(\tau)d\tau, \tag{2}$$

where Γ is the Gamma function, subjected to the following boundary conditions:

$$u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \partial\Omega \times (0,T), \tag{3}$$

$$\partial_t^{1-\beta} u(\boldsymbol{x},0) = \mathrm{G}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{4}$$

and

$$u(\boldsymbol{x},0) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{5}$$

where we assume that $f \in H_0^1(\Omega)$ and $\mathrm{G} \in L^2(\Omega)$. The conservative Caputo fractional-derivative operator $D\left(\partial_t^{1-\beta}\right)$ of fractional order $2-\beta \in (1,2)$ was considered, for instance, in [23, equation (1.1), Page 4331] with mixed Dirichlet-Neumann boundary conditions (4) and (5), in [25, equation (2.1), Page 1295] with inhomogeneous Dirichlet boundary conditions, and in [24, equation (2.6), Page 1359] with Neumann boundary conditions as in (4).

We note that, as $\beta \searrow 0$, the variable t represents a space coordinate and equation (1) becomes the elliptic Laplace's equation in $(d+1)$-dimensions modelling steady-state diffusion processes, whilst as $\beta \nearrow 1$, the variable t represents the usual time coordinate and equation (1) becomes a non-classical parabolic heat equation (with reversed time) in d-dimensions modelling transient diffusion processes given by $u_t(\boldsymbol{x},t) + \Delta u(\boldsymbol{x},t) = u_t(\boldsymbol{x},0)$ for $(\boldsymbol{x},t) \in \Omega_T$.

1.1. Ill-posedness of the Cauchy problem (1)–(5).

In this subsection, we show that the problem (1)–(5) is ill-posed. Indeed, using the method of separation of variables we seek its solution in the series form

$$u(\boldsymbol{x},t) = \sum_{m=1}^{\infty} \psi_m(t)\varphi_m(\boldsymbol{x}), \quad (\boldsymbol{x},t) \in \Omega_T, \tag{6}$$

where for all $m \in \mathbb{N}^*$, ψ_m satisfies

$$-D\left(\partial_t^{1-\beta}\psi_m\right)(t) + \mu_m\psi_m(t) = 0, \quad t \in (0,T), \tag{7}$$

$$\partial_t^{1-\beta}\psi_m(0) = \mathrm{G}_m := (\mathrm{G},\varphi_m)_{L^2(\Omega)} \quad \text{and} \quad \psi_m(0) = f_m := (f,\varphi_m)_{L^2(\Omega)}, \tag{8}$$

where $(\varphi_m,\mu_m)_{m\in\mathbb{N}^\star}$ is the solution of the eigenvalue problem (see, for instance, [1, Page 773])

$$-\Delta\varphi(\boldsymbol{x}) = \mu\varphi(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \quad \text{and} \quad \varphi(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega. \tag{9}$$

It is well-known that

$$0 < \mu_1 < \mu_2 < \ldots < \mu_m \to +\infty, \quad \text{as } m \to +\infty \tag{10}$$

and that the eigenfunctions set $(\varphi_m)_{m\in\mathbb{N}^*}$ may be taken to constitute an orthonormal basis of $L^2(\Omega)$.

To derive an explicit expression for the solution of (7)–(8), we apply the Laplace transform to both sides of (7) and use the first condition in (8) to get (see, for instance, [20, equation (2.239), Page 104])

$$-s\mathcal{L}\left(\partial_t^{1-\beta} v\right)(s) + \mathrm{G}_m + \mu_m \mathcal{L}\left(\psi_m\right)(s) = 0. \tag{11}$$

Using [20, equation (2.253), Page 106], equation (11) and the second condition in (8) imply that

$$-s\left(s^{1-\beta}\mathcal{L}\left(\psi_m\right)(s) - s^{-\beta} f_m\right) + \mathrm{G}_m + \mu_m \mathcal{L}\left(\psi_m\right)(s) = 0, \tag{12}$$

which leads to

$$\mathcal{L}\left(\psi_m\right)(s) = -\frac{\mathrm{G}_m + s^{1-\beta} f_m}{\mu_m - s^{2-\beta}}. \tag{13}$$

Let us consider the particular case of problem (1) with (3)–(5) when $\mathrm{G} \equiv 0$ and show its instability. In this case, equation (13) yields $\mathcal{L}\left(\psi_m\right)(s) = -\dfrac{s^{1-\beta} f_m}{\mu_m - s^{2-\beta}}$. Using [27, Page 3], this gives

$$\psi_m(t) = f_m E_{2-\beta,1}\left(\mu_m t^{2-\beta}\right). \tag{14}$$

and, from (6),

$$u(\boldsymbol{x}, t) = \sum_{m=1}^{\infty} \left(f, \varphi_m\right)_{L^2(\Omega)} E_{2-\beta,1}\left(\mu_m t^{2-\beta}\right) \varphi_m(\boldsymbol{x}), \tag{15}$$

where E_{α_1,α_2} is the two-parameter Mittag–Leffler function defined by

$$E_{\alpha_1,\alpha_2}(z) := \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha_1 m + \alpha_2)}, \quad z \in \mathbb{C}. \tag{16}$$

It is useful to notice that the Mittag-Leffler functions E_{α_1,α_2} are entire functions for all $\alpha_1 > 0$ and consequently, they are real analytic and the series (16) is termwise differentiable in $\mathbb{R}$, see [21, Page 432] and [8, Page 62]. The expression (15) shows that the problem is severely ill-posed because the terms $(f, \varphi_m)_{L^2(\Omega)}$ are amplified by the exponentially growing factor

$$E_{2-\beta,1}\left(\mu_m t^{2-\beta}\right) \geq \cosh\left(\sqrt{\mu_m t^{2-\beta}}\right) \geq \frac{\exp\left(\sqrt{\mu_m t^{2-\beta}}\right)}{2}, \tag{17}$$

which is large for large m.

The main aim of this paper is to restore the stability of the problem (1)–(5) using an alternating algorithm inspired by the one developed for the Cauchy problem of the elliptic Laplace equation in [14] along with a full convergence analysis. The ill-posed Cauchy problem associated to a similar problem for (1)–(5) (in which the conservative derivative operator $D\left(\partial_t^{1-\beta}\right)$ and the boundary condition (4) are replaced, respectively, by the Caputo fractional-derivative ∂_t^{δ} with

$\delta \in (1,2)$ and $\partial_t u(\cdot,0) = \mathrm{G}(\cdot)$ was considered, for instance, in [3, 4, 19]. In [3, 19] the stability was restored using a boundary-value problem regularization method and a Fourier truncation method, respectively. However, the former requires extending the solution domain and considering a non-local boundary condition, whilst the latter one is truncating the Fourier series expansion to achieve stable approximate solutions. In [4], the regularization is performed using an iterative algorithm, which is not alternating, but it preserves the original solution domain and the governing PDE, based on solving a sequence of associated well-posed problems. The algorithm considered in [4] is based on a formulation of a sequence of differential problems regularizing a similar problem to (7)–(8). The limit case when $\beta = 0$ in (1) and (4) and $\|u(\cdot,0) - f\|_{L^2(\Omega)} \leq \varepsilon$ was investigated in [6] (resp. [22]) where a non-local boundary value problem method (resp. a truncation method based on the fundamental solution to the elliptic Laplace's equation) were presented for stabilizing the resulting Cauchy ill-posed problem. Recently in [11], the limit case $\beta = 0$ in (1)–(5) (which generates the $(d+1)$-dimensional Laplace's equation) was considered. The approach followed in [11] was to approximate the integer derivatives in the elliptic Laplace's equation by fractional ones.

1.2. **Outline of the paper.** We first investigate in Section 2 the limit case when $\beta = 0$ in (1)–(5), which yields the classical ill-posed Cauchy problem (18)–(21) associated to the Laplace equation in $(d+1)$ dimensions. This problem was addressed recently in [11] (as stated above). However, we follow another approach to restore the stability, different from the one of [11]. Inspired by the algorithm developed in [14] for the Cauchy problem associated with the Laplace's equation $-\Delta u = 0$, we establish the alternating regularizing algorithm (23)–(25), which involves two types of problems. Under some reasonable assumptions on the data f and G of (18)–(21), we prove in Lemma 2.1 the well-posedness of these problems. To analyse the convergence of the algorithm (23)–(25), we first write its solutions under the form of eigenfunction expansions. We subsequently derive explicit formulas (see (45)–(46)) for the error between the coefficients ψ_m and ψ_m^n involved, respectively, in the eigenvalue expansion (6) of the solution u of the ill-posed problem (18)–(21) and the eigenfunction expansion (34) of the solutions $\Psi_m^{(n)}$ of the algorithm (23)–(25). Thanks to this explicit and concise error, we prove rigorously in Theorem 2.1 the convergence of solutions of the sequence (23)–(25) towards the solution of the ill-posed Cauchy problem (18)–(21).

In Section 3, we focus on the main target of this work: we establish an alternating algorithm for the problem (1)–(5) along with a novel and a full convergence analysis. One of the main difficulties in the analysis of stability and the convergence of the sequence of solutions of the problems involved is that we will deal with expansions involving the Mittag-Leffler functions (an area of fractional calculus still in progress) in which we need to perform several operations (some of them are not straightforward and others require very careful and specific treatments) like differentiation, integration and bounds. In addition to this, the arguments of the Mittag-Leffler functions involved in the stated expansions we will deal with are non-negative and consequently some known and popular techniques cannot be applied, see [11, Page 6] and [21, Lemma 3.1, Page 432] (which requires that the arguments should be negative). Using the algorithm developed in [14], we suggest in subsection 3.1 the alternating algorithm (116)–(118) for solving the ill-posed Cauchy problem (1)–(5). This algorithm includes two types of problems, like (74)–(77), whose well-posedness is proved rigorously using two methods. The first method consists of weak formulations which satisfy the hypotheses of Lax-Milgram lemma, see the weak formulation (104)–(107) of the problem (78)–(81), which is equivalent to (74)–(77). The well-posedness can then be deduced, as usual, from the coercivity of the bilinear forms and the continuity of the linear forms involved in the weak formulations. In the second method, we write the solution of (74)–(77) under the form of the eigenfunction expansion (111), which yields the differential problem (112)–(113), implying in turn the explicit form (115) for the coefficients ψ_m involved in (111). We then have to control the $L^2(H_0^1)$-norm of the expansion (111) with ψ_m given by (115). This second method is enriched by several new tools and bounds, and it is explained in Remark 3.1 and Appendix A. The stability result (212) (which confirms the $L^2(H_0^1)$-stability (109) derived by the first method) is based on bounding the norms of expansions involving Mittag-Leffler functions, which are not straightforward to compute and instead we employ some recent inequalities and asymptotic behaviour results of Mittag-Leffler functions, combined with some appropriate technical tools to control them. In fact, to evaluate the norm of the stated expansions, we have to control functions whose arguments tend to infinity (because of the presence of the sequence of eigenvalues (10)) and this is why we use the asymptotic behaviour of Mittag-Leffler functions at infinity. In subsection 3.2, we present a convergence analysis for the solutions of the sequence of the alternating algorithm (116)–(118) towards the solution of the ill-posed problem (1)–(5). The main ideas of this analysis along with highlights of the main contributions are sketched as follows:

1. We first compute the error e_m^n between the coefficients ψ_m (the coefficients of the solution u in the expression (6) of the ill-posed problem (1)–(5)) and ψ_m^n (the coefficients of the solutions $\Psi_m^{(n)}$ in the eigenfunctions basis of the algorithm (116)–(118)) in terms of some fractional order derivatives of expansions involving Mittag Leffler functions, (135)–(139).
2. We carefully compute the fractional derivatives of the expansions stated in previous item.
3. We derive an accurate recurrent relation (see (148)) from which we derive the explicit concise expansions (151) and (152) for the errors e_m^{2n+2} and e_m^{2n+1}, which yield in turn the expansions (153) and (154) for the errors e^{2n+2} and e^{2n+1}. The $L^2(H_0^1)$-norm of the error is then given by (156) and (157).
4. To prove the convergence in the $L^2(H_0^1)$-norm, i.e., $\|\nabla e^n\|_{L^2(0,T;H_0^1(\Omega))} \to 0$ in Theorem 3.2, we employ several new tools and techniques: Lemma 3.1 in which we prove some new bounds, Theorem 3.1 in which we prove the new inequality (181) and Appendix B in which we prove one of the main inequalities used in the proof of Theorem 3.1. To prove Lemma 3.1 and Theorem

3.1 we carefully use the asymptotic behaviour along with some known inequalities for Mittag-Leffler functions and some appropriate techniques in analysis. Numerical illustrations for some key inequalities are presented. Furthermore, in the case of one dimension the convergence of the developed algorithm is evidenced numerically in Section 4. Finally, Section 5 includes a summary of the main results obtained and some perspectives for possible future work.

2. Alternating algorithm for the Cauchy problem in the limit case when $\beta = 0$ in (1)–(5) (the elliptic Laplace equation)

In this section, we investigate the ill-posed problem considered in [11] (the limit case $\beta = 0$ in (1)–(5)), that is, the elliptic Laplace equation in $(d+1)$ dimensions:

$$u_{tt}(\boldsymbol{x}, t) + \Delta u(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \Omega_T := \Omega \times (0, T), \tag{18}$$

subjected to

$$u(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \Gamma_T := \partial\Omega \times (0, T), \tag{19}$$

$$u_t(\boldsymbol{x}, 0) = \mathrm{G}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{20}$$

and

$$u(\boldsymbol{x}, 0) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{21}$$

The solution of problem (18)–(21) can be computed using the method of separation of variables; however, this solution does not depend continuously on the data, see [11, Page 4]. To restore the stability, we will present an approach, different from the ones of [11], based on the use of the alternating algorithm developed in [14].

Let us denote $S_0 := \Omega \times \{0\}$ and $S_T := \Omega \times \{T\}$, and analogously to [14, Page 46], let us define $\mathcal{R}^{(n)}$ as the linear operator that assigns (see Lemma 2.1 below)

$$(f, \mathrm{G}, \eta) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \mapsto \mathcal{R}^{(n)}(f, \mathrm{G}, \eta) = \Psi^{(n)} \in H^1(\Omega_T), \tag{22}$$

where $\Psi^{(n)}$ is defined as follows:

$$\begin{cases} \Psi_{tt}^{(0)}(\boldsymbol{x}, t) + \Delta\Psi^{(0)}(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \Omega_T, \\ \Psi^{(0)}(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \Gamma_T, \\ \Psi^{(0)}(\boldsymbol{x}, 0) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \Psi_t^{(0)}(\boldsymbol{x}, T) = \eta(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{23}$$

$$\begin{cases} \Psi_{tt}^{(2n+1)}(\boldsymbol{x}, t) + \Delta\Psi^{(2n+1)}(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \Omega_T, \\ \Psi^{(2n+1)}(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \Gamma_T, \\ \Psi_t^{(2n+1)}(\boldsymbol{x}, 0) = \mathrm{G}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \Psi^{(2n+1)}(\boldsymbol{x}, T) = \Psi^{(2n)}(\boldsymbol{x}, T), & \boldsymbol{x} \in \Omega, \end{cases} \tag{24}$$

and

$$\begin{cases} \Psi^{(2n+2)}_{tt}(\boldsymbol{x},t) + \Delta\Psi^{(2n+2)}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Omega_T, \\ \Psi^{(2n+2)}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Gamma_T, \\ \Psi^{(2n+2)}(\boldsymbol{x},0) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \Psi^{(2n+2)}_{t}(\boldsymbol{x},T) = \Psi^{(2n+1)}_{t}(\boldsymbol{x},T), & \boldsymbol{x} \in \Omega. \end{cases} \tag{25}$$

We first state the following lemma.

Lemma 2.1 (Well-posedness of the problems involved in (23)–(25))**.** *Consider the mixed problems of the following forms:*

$$\begin{cases} \Psi_{tt}(\boldsymbol{x},t) + \Delta\Psi(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Omega_T, \\ \Psi(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Gamma_T, \\ \Psi(\boldsymbol{x},0) = \zeta(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \Psi_{t}(\boldsymbol{x},T) = \theta(\boldsymbol{x}), & \boldsymbol{x} \in \Omega \end{cases} \tag{26}$$

and

$$\begin{cases} \overline{\Psi}_{tt}(\boldsymbol{x},t) + \Delta\overline{\Psi}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Omega_T, \\ \overline{\Psi}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Gamma_T, \\ \overline{\Psi}_t(\boldsymbol{x},0) = \overline{\zeta}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \overline{\Psi}(\boldsymbol{x},T) = \overline{\theta}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{27}$$

The following well-posedness results hold for the problems (26) and (27):

1. **Problems of the form (26).** *Assume that $\zeta \in H^1_0(\Omega)$ and $\theta \in L^2(\Omega)$ and let us consider the subspace of $H^1(\Omega_T)$,*

$$\mathcal{V} := \left\{ v \in H^1(\Omega_T) : \tilde{\gamma}(v)\big|_{\Gamma_T} = 0 \quad \text{and} \quad \tilde{\gamma}(v)\big|_{S_0} = \zeta \right\}, \tag{28}$$

where $\tilde{\gamma}$ is the classical trace operator from $H^1(\Omega_T)$ to $L^2(\partial\Omega_T)$. There exists a unique (weak) solution $\Psi \in \mathcal{V}$ of the problem (26) and the following estimate holds:

$$\|\Psi_t\|_{L^2(\Omega_T)} + \|\nabla\Psi\|_{L^2(\Omega_T)} \le C_1 \left(\|\theta\|_{L^2(\Omega)} + \|\nabla\zeta\|_{(L^2(\Omega))^d} \right). \tag{29}$$

2. **Problems of the form (27).** *Assume that $\overline{\theta} \in H^1_0(\Omega)$ and $\overline{\zeta} \in L^2(\Omega)$ and let us consider the subspace of $H^1(\Omega_T)$,*

$$\overline{\mathcal{V}} := \left\{ v \in H^1(\Omega_T) : \tilde{\gamma}(v)\big|_{\Gamma_T} = 0 \quad \text{and} \quad \tilde{\gamma}(v)\big|_{S_T} = \overline{\zeta} \right\}. \tag{30}$$

There exists a unique (weak) solution $\overline{\Psi} \in \overline{\mathcal{V}}$ of the problem (27) and the following estimate holds:

$$\|\overline{\Psi}_t\|_{L^2(\Omega_T)} + \|\nabla\overline{\Psi}\|_{L^2(\Omega_T)} \le C_2 \left(\|\overline{\zeta}\|_{L^2(\Omega)} + \|\nabla\overline{\theta}\|_{(L^2(\Omega))^d} \right). \tag{31}$$

Proof. The proof is classical and can be performed using the weak formulations of the problems (26) and (27), where the Lax-Milgram can be applied (see, for instance, the case of Dirichlet-Neumann mixed boundary conditions in the problem [7, equations (3.71)–(3.73), Pages 95–96]). The well-posedness can be then justified using the coercivity of the bilinear forms and the continuity of the linear forms involved in the stated weak formulations. □

2.1. **Convergence of solutions of the algorithm (23)–(25) towards the solution of the ill-posed problem (18)–(21).** The proof of the convergence of the sequence of solutions of the well-posed problems (23)–(25) towards the solution of the ill-posed problem (18)–(21) can performed using some ideas from [14] along with some appropriate techniques (see Remark 2.1 below). However, we will use a different approach based on the use of eigenfunction expansions along with an adaptation of the techniques of [13, Pages 1210–1211]. This approach will serve later when we deal with the original ill-posed fractional problem (1)–(5). Writing the solution of (18)–(21) as in (6), this yields

$$\psi_m''(t) - \mu_m \psi_m(t) = 0, \quad t \in (0,T), \tag{32}$$

$$\psi_m'(0) = \mathrm{G}_m \quad \text{and} \quad \psi_m(0) = f_m. \tag{33}$$

Let us now express the sequence $\Psi^{(n)}$ defined by (23)–(25) as

$$\Psi^{(n)}(\boldsymbol{x},t) = \sum_{m=1}^{\infty} \psi_m^n(t)\varphi_m(\boldsymbol{x}). \tag{34}$$

The sequence ψ_m^n is then defined by

$$\begin{cases} \left(\psi_m^0\right)''(t) - \mu_m \psi_m^0(t) = 0, \quad t \in (0,T), \\ \psi_m^0(0) = f_m \quad \text{and} \quad \left(\psi_m^0\right)'(T) = \eta_m := (\eta, \varphi_m)_{L^2(\Omega)}, \end{cases} \tag{35}$$

$$\begin{cases} \left(\psi_m^{2n+1}\right)''(t) - \mu_m \psi_m^{2n+1}(t) = 0, \quad t \in (0,T), \\ \left(\psi_m^{2n+1}\right)'(0) = \mathrm{G}_m \quad \text{and} \quad \psi_m^{2n+1}(T) = \psi_m^{2n}(T), \end{cases} \tag{36}$$

and

$$\begin{cases} \left(\psi_m^{2n+2}\right)''(t) - \mu_m \psi_m^{2n+2}(t) = 0, \quad t \in (0,T), \\ \psi_m^{2n+2}(0) = f_m \quad \text{and} \quad \left(\psi_m^{2n+2}\right)'(T) = \left(\psi_m^{2n+1}\right)'(T). \end{cases} \tag{37}$$

We remark that the sequence ψ_m^n given by (35)–(37) is the sequence of the alternating algorithm of [13, Section 1.2, Page 1210] corresponding to the initial value problem (32)–(33). The convergence proof is based on the use of some ideas provided in [13, Section 1.2, Pages 1210–1211]. Let us consider the error

$$e_m^n(t) := \psi_m(t) - \psi_m^n(t).$$

Using (32), (33) and (35)–(37) we obtain

$$\begin{cases} \left(e_m^0\right)''(t) - \mu_m e_m^0(t) = 0, \quad t \in (0,T), \\ \psi_m^0(0) = 0 \quad \text{and} \quad \left(e_m^0\right)'(T) = \Lambda_m := \psi_m'(T) - \eta_m, \end{cases} \tag{38}$$

$$\begin{cases} \left(e_m^{2n+1}\right)''(t) - \mu_m e_m^{2n+1}(t) = 0, \quad t \in (0,T), \\ \left(e_m^{2n+1}\right)'(0) = 0 \quad \text{and} \quad e_m^{2n+1}(T) = e_m^{2n}(T), \end{cases} \tag{39}$$

and

$$\begin{cases} \left(e_m^{2n+2}\right)''(t) - \mu_m e_m^{2n+2}(t) = 0, \quad t \in (0,T), \\ e_m^{2n+2}(0) = 0 \quad \text{and} \quad \left(e_m^{2n+2}\right)'(T) = \left(e_m^{2n+1}\right)'(T). \end{cases} \tag{40}$$

Resolving the differential problems (38)–(40) gives

$$e_m^0(t) = \frac{\Lambda_m}{\sqrt{\mu_m}\cosh(T\sqrt{\mu_m})}\sinh(t\sqrt{\mu_m}), \tag{41}$$

$$e_m^{2n+1}(t) = \frac{e_m^{2n}(T)}{\cosh(T\sqrt{\mu_m})}\cosh(t\sqrt{\mu_m}), \tag{42}$$

and

$$e_m^{2n+2}(t) = \frac{\left(e_m^{2n+1}\right)'(T)}{\sqrt{\mu_m}\cosh(T\sqrt{\mu_m})}\sinh(t\sqrt{\mu_m}). \tag{43}$$

The expansions (42)–(43) yield

$$e_m^{2n+2}(T) = \frac{\left(e_m^{2n+1}\right)'(T)\tanh(T\sqrt{\mu_m})}{\sqrt{\mu_m}} = \tanh^2(T\sqrt{\mu_m})e_m^{2n}(T). \tag{44}$$

This with (41) yield $e_m^{2n+2}(T) = \dfrac{\Lambda_m \tanh^{2n+3}(T\sqrt{\mu_m})}{\sqrt{\mu_m}}$. Inserting this value in (42) yields

$$e_m^{2n+1}(t) = \frac{\Lambda_m \cosh(t\sqrt{\mu_m})}{\sqrt{\mu_m}\cosh(T\sqrt{\mu_m})}\tanh^{2n+1}(T\sqrt{\mu_m}). \tag{45}$$

In the same way, we get

$$e_m^{2n+2}(t) = \frac{\Lambda_m \sinh(t\sqrt{\mu_m})}{\sqrt{\mu_m}\cosh(T\sqrt{\mu_m})}\tanh^{2n+2}(T\sqrt{\mu_m}). \tag{46}$$

Let us now prove the following theorem of convergence whose hypotheses are similar to the ones of [13, Theorem 2, Page 1212].

Theorem 2.1 (The convergence of the alternating algorithm given by (34) and (23)–(25))**.** *Let $\Psi^{(n)}$ be the sequence given by (34) and (23)–(25). Assume that the solution of the Cauchy problem (18)–(21) with $f \in H_0^1(\Omega)$ and $\mathrm{G} \in L^2(\Omega)$ is satisfying $u \in L^2\left(0,T;H_0^1(\Omega)\right)$. Then, for any initial guess $\eta \in L^2(\Omega)$ provided in the last branch of (23), while assuming that $u_t(\cdot,T) \in L^2(\Omega)$, the following convergence result holds:*

$$\|\partial_t e^n\|_{L^2(\Omega_T)} + \|\nabla e^n\|_{L^2(\Omega_T)} \to 0, \quad \text{as} \quad n \to +\infty, \tag{47}$$

where

$$e^n(\boldsymbol{x},t) := u(\boldsymbol{x},t) - \Psi^{(n)}(\boldsymbol{x},t). \tag{48}$$

Proof. Using (45) and (46), the error (48) is satisfying

$$e^{2n+2}(\boldsymbol{x},t)=\sum_{m=1}^{\infty}\frac{\Lambda_m\sinh(t\sqrt{\mu_m})}{\sqrt{\mu_m}\cosh(T\sqrt{\mu_m})}\tanh^{2n+2}(T\sqrt{\mu_m})\varphi_m(\boldsymbol{x}) \tag{49}$$

and

$$e^{2n+1}(t)=\sum_{m=1}^{\infty}\frac{\Lambda_m\cosh(t\sqrt{\mu_m})}{\sqrt{\mu_m}\cosh(T\sqrt{\mu_m})}\tanh^{2n+1}(T\sqrt{\mu_m})\varphi_m(\boldsymbol{x}). \tag{50}$$

To differentiate with respect to t the series in (49), we remark that, since $0\le\tanh(s)<1$ for all $s\in\mathbb{R}$ and $u_t(\cdot,T)-\eta\in L^2(\Omega)$ (and consequently $\sum_{m=1}^{\infty}\Lambda_m^2$ is convergent), we have

$$\sum_{m=1}^{\infty}\left|\left(\frac{\Lambda_m\sinh(t\sqrt{\mu_m})}{\sqrt{\mu_m}\cosh(T\sqrt{\mu_m})}\tanh^{2n+2}(T\sqrt{\mu_m})\right)_t\right|^2\le\sum_{m=1}^{\infty}\Lambda_m^2<+\infty. \tag{51}$$

This allows us to write

$$\partial_t e^{2n+2}(\boldsymbol{x},t)=\sum_{m=1}^{\infty}\frac{\Lambda_m\cosh(t\sqrt{\mu_m})}{\cosh(T\sqrt{\mu_m})}\tanh^{2n+2}(T\sqrt{\mu_m})\varphi_m(\boldsymbol{x}). \tag{52}$$

Similarly, we differentiate (50) with respect to t to get

$$\partial_t e^{2n+1}(\boldsymbol{x},t)=\sum_{m=1}^{\infty}\frac{\Lambda_m\sinh(t\sqrt{\mu_m})}{\cosh(T\sqrt{\mu_m})}\tanh^{2n+1}(T\sqrt{\mu_m})\varphi_m(\boldsymbol{x}). \tag{53}$$

These relations yield

$$\|\partial_t e^{2n+2}\|^2_{L^2(\Omega_T)}=\sum_{m=1}^{\infty}\frac{\Lambda_m^2\tanh^{4n+4}(T\sqrt{\mu_m})}{\cosh^2(T\sqrt{\mu_m})}\int_0^T\cosh^2(t\sqrt{\mu_m})dt \tag{54}$$

and

$$\|\partial_t e^{2n+1}\|^2_{L^2(\Omega_T)}=\sum_{m=1}^{\infty}\frac{\Lambda_m^2\tanh^{4n+2}(T\sqrt{\mu_m})}{\cosh^2(T\sqrt{\mu_m})}\int_0^T\sinh^2(t\sqrt{\mu_m})dt. \tag{55}$$

In addition to this, using [10, Page 141], we have

$$\|\nabla e^{2n+2}\|^2_{L^2(\Omega_T)}=\sum_{m=1}^{\infty}\frac{\Lambda_m^2\tanh^{4n+4}(T\sqrt{\mu_m})}{\cosh^2(T\sqrt{\mu_m})}\int_0^T\sinh^2(t\sqrt{\mu_m})dt \tag{56}$$

and

$$\|\nabla e^{2n+1}\|^2_{L^2(\Omega_T)}=\sum_{m=1}^{\infty}\frac{\Lambda_m^2\tanh^{4n+2}(T\sqrt{\mu_m})}{\cosh^2(T\sqrt{\mu_m})}\int_0^T\cosh^2(t\sqrt{\mu_m})dt. \tag{57}$$

We have to prove that $\|\partial_t e^n\|^2_{L^2(\Omega_T)}\to 0$ and $\|\nabla e^n\|^2_{L^2(\Omega_T)}\to 0$ (using formulas (54)–(57)), as $n\to\infty$. Let us justify, for instance, that $\|\partial_t e^n\|^2_{L^2(\Omega_T)}\to 0$, as $n\to\infty$. We have that

$$\int_0^T\cosh^2(t\sqrt{\mu_m})dt=\frac{1}{2}\int_0^T\left(1+\cosh(2\sqrt{\mu_m}t)\right)dt=\frac{T}{2}+\frac{1}{4\sqrt{\mu_m}}\sinh(2T\sqrt{\mu_m})$$

and

$$\sum_{m=1}^{\infty}\frac{\Lambda_m^2\tanh^{4n+4}(T\sqrt{\mu_m})}{\cosh^2(T\sqrt{\mu_m})}\int_0^T\cosh^2(t\sqrt{\mu_m})dt=A_n+B_n,$$

where

$$A_n:=\frac{T}{2}\sum_{m=1}^{\infty}\frac{\Lambda_m^2\tanh^{4n+4}(T\sqrt{\mu_m})}{\cosh^2(T\sqrt{\mu_m})}$$

and

$$B_n := \frac{1}{4}\sum_{m=1}^{\infty}\frac{\Lambda_m^2 \sinh(2T\sqrt{\mu_m})\tanh^{4n+4}(T\sqrt{\mu_m})}{\sqrt{\mu_m}\cosh^2(T\sqrt{\mu_m})} = \frac{1}{2}\sum_{m=1}^{\infty}\frac{\Lambda_m^2 \tanh^{4n+5}(T\sqrt{\mu_m})}{\sqrt{\mu_m}}.$$

We prove that both terms A_n and B_n tend to zero, as $n \to \infty$.

(i) We prove that $\lim\limits_{n\to\infty} A_n = 0$. Since $\cosh^2(T\sqrt{\mu_m}) > 1$ and $0 < \tanh(T\sqrt{\mu_m}) < 1$,

$$0 < \frac{\Lambda_m^2 \tanh^{4n+4}(T\sqrt{\mu_m})}{\cosh^2(T\sqrt{\mu_m})} < \Lambda_m^2.$$

This implies that

$$\sum_{m=1}^{\infty}\frac{\Lambda_m^2 \tanh^{4n+4}(T\sqrt{\mu_m})}{\cosh^2(T\sqrt{\mu_m})} \le \sum_{m=1}^{\infty}\Lambda_m^2 < \infty.$$

In addition to this, $\lim_{n\to+\infty}\frac{\Lambda_m^2 \tanh^{4n+4}(T\sqrt{\mu_m})}{\cosh^2(T\sqrt{\mu_m})} = 0$, for all $m \in \mathbb{N}^\star$. This yields, using the Dominated Convergence Theorem, $\lim_{n\to\infty} A_n = 0$.

(ii) We prove that $\lim_{n\to\infty} B_n = 0$. The above reasoning together with the fact that for all $m \ge 1$ we have $\mu_m \ge \mu_1$ can also be applied to prove that $\lim_{n\to\infty} B_n = 0$. This completes the proof of Theorem 2.1. $\square$

Remark 2.1 (Another proof for the convergence of (23)–(25)). *Another new possible proof for the convergence (47), using the ideas of* [14, Pages 45–48] *along with some appropriate techniques, can be performed. Let us sketch the main steps of this convergence proof.*

First step. Using the formulations (22)–(27), we have that

$$u - \Psi^{(n)} = \mathcal{R}^{(n)}\left(0, 0, u_t|_{t=T} - \eta\right). \tag{58}$$

Our aim is to prove that $\mathcal{R}^{(n)}(0,0,\Lambda) \to 0$ (in some sense given below), as $n \to \infty$, for any "smooth" function Λ. This in turn implies the convergence of the sequence $\Psi^{(n)}$ (defined by (23)–(25)) towards u (satisfying (18)–(21)).

Second step. Let Λ be a "smooth" function defined on Ω. We prove that the following convergence holds:

$$\mathcal{S}^n := \int_{\Omega_T}\{|\partial_t\rho_n|^2 + |\nabla\rho_n|^2\}(\boldsymbol{x},t)dtd\boldsymbol{x} \to 0, \quad \text{as } n \to \infty, \tag{59}$$

where, for the sake of simplicity of notations, we have set

$$\rho_n := \mathcal{R}^{(n)}(0,0,\Lambda). \tag{60}$$

Using Green's formula in space Ω and integration by parts in $(0,T)$ together with the fact that $\rho_{2n}(\cdot,0)\partial_t\rho_{2n-1}(\cdot,0) = 0$, we are able to show that

$$\int_{\Omega_T}\left[(\partial_t\rho_{2n} - \partial_t\rho_{2n-1})^2 + |\nabla(\rho_{2n} - \rho_{2n-1})|^2\right](\boldsymbol{x},t)dtd\boldsymbol{x}$$

$$= \int_{\Omega_T}\left[(\partial_t\rho_{2n-1})^2 + |\nabla\rho_{2n-1}|^2 - \left((\partial_t\rho_{2n})^2 + |\nabla\rho_{2n}|^2\right)\right](\boldsymbol{x},t)dtd\boldsymbol{x} \tag{61}$$

and

$$\int_{\Omega_T}\left[\left(\partial_t\rho^{2n+1}-\partial_t\rho^{2n}\right)^2+\left|\nabla\left(\rho_{2n+1}-\rho_{2n}\right)\right|^2\right](\boldsymbol{x},t)dtd\boldsymbol{x}$$
$$=\int_{\Omega_T}\left[\left(\partial_t\rho_{2n}\right)^2+|\nabla\rho_{2n}|^2-\left(\left(\partial_t\rho_{2n+1}\right)^2+|\nabla\rho_{2n+1}|^2\right)\right](\boldsymbol{x},t)dtd\boldsymbol{x}. \tag{62}$$

For $n=2m$, thanks to (62),

$$\begin{aligned}\mathcal{S}^n-\mathcal{S}^{n+1}=\mathcal{S}^{2m}-\mathcal{S}^{2m+1}&=\int_{\Omega_T}\left[\left(\partial_t\rho_{2m}\right)^2+|\nabla\rho_{2m}|^2\right](\boldsymbol{x},t)dtd\boldsymbol{x}\\&\quad-\int_{\Omega_T}\left[\left(\partial_t\rho_{2m+1}\right)^2+|\nabla\rho_{2m+1}|^2\right](\boldsymbol{x},t)dtd\boldsymbol{x}\\&=\int_{\Omega_T}\left[\left(\partial_t\rho_{2m+1}-\partial_t\rho_{2m}\right)^2+\left|\nabla\left(\rho_{2m+1}-\rho_{2m}\right)\right|^2\right](\boldsymbol{x},t)dtd\boldsymbol{x}\geq 0,\end{aligned} \tag{63}$$

whilst for $n=2m-1$, thanks to (61),

$$\begin{aligned}\mathcal{S}^n-\mathcal{S}^{n+1}=\mathcal{S}^{2m-1}-\mathcal{S}^{2m}&=\int_{\Omega_T}\left[\left(\partial_t\rho_{2m-1}\right)^2+|\nabla\rho_{2m-1}|^2\right](\boldsymbol{x},t)dtd\boldsymbol{x}\\&\quad-\int_{\Omega_T}\left[\left(\partial_t\rho_{2m}\right)^2+|\nabla\rho_{2m}|^2\right](\boldsymbol{x},t)dtd\boldsymbol{x}\\&=\int_{\Omega_T}\left[\left(\partial_t\rho_{2m}-\partial_t\rho_{2m-1}\right)^2+\left|\nabla\left(\rho_{2m}-\rho_{2m-1}\right)\right|^2\right](\boldsymbol{x},t)dtd\boldsymbol{x}\geq 0.\end{aligned} \tag{64}$$

Therefore, the sequence $\left(\mathcal{S}^n\right)_n$ is decreasing. In addition, $\left(\mathcal{S}^n\right)_n$ is bounded from below by 0. Therefore, $\left(\mathcal{S}^n\right)_n$ is convergent to a limit value $l\geq 0$. To conclude the convergence result (59), we follow two stages.

(i) **Convergence for a set of functions** Λ. For a given function $\xi\in L^2(\Omega)$, we consider the iteration:

$$\Phi^{(n)}=\mathcal{R}^{(n)}\left(0,0,\xi\right). \tag{65}$$

Let us now take Λ given by

$$\Lambda(\boldsymbol{x})=\Phi_t^{(2)}(\boldsymbol{x},T)-\xi(\boldsymbol{x}). \tag{66}$$

According to [14, Page 45], the mapping $\mathcal{R}^{(n)}$ is linear. This implies that

$$\begin{aligned}\rho_n(\boldsymbol{x})=\mathcal{R}^{(n)}\left(0,0,\Lambda(\boldsymbol{x})\right)=\mathcal{R}^{(n)}\left(0,0,\Phi_t^{(2)}(\boldsymbol{x},T)-\xi(\boldsymbol{x})\right)&=\mathcal{R}^{(n)}\left(0,0,\Phi_t^{(2)}(\boldsymbol{x},T)\right)\\&\quad-\mathcal{R}^{(n)}\left(0,0,\xi(\boldsymbol{x})\right).\end{aligned} \tag{67}$$

In particular, we have

$$\rho_{2n}(\boldsymbol{x})=\mathcal{R}^{(2n)}\left(0,0,\Phi_t^{(2)}(\boldsymbol{x},T)\right)-\mathcal{R}^{(2n)}\left(0,0,\xi(\boldsymbol{x})\right). \tag{68}$$

The alternating nature of the iterative algorithm (23)–(25) implies that

$$\mathcal{R}^{(2n)}\left(0,0,\Phi_t^{(2)}(\boldsymbol{x},T)\right)=\mathcal{R}^{(2n+2)}\left(0,0,\xi(\boldsymbol{x})\right). \tag{69}$$

Gathering (68) and (69) implies that

$$\rho_{2n}(\boldsymbol{x}) = \mathcal{R}^{(2n+2)}\left(0,0,\xi(\boldsymbol{x})\right) - \mathcal{R}^{(2n)}\left(0,0,\xi(\boldsymbol{x})\right)$$
$$= \mathcal{R}^{(2n+2)}\left(0,0,\xi(\boldsymbol{x})\right) - \mathcal{R}^{(2n+1)}\left(0,0,\xi(\boldsymbol{x})\right) + \mathcal{R}^{(2n+1)}\left(0,0,\xi(\boldsymbol{x})\right) - \mathcal{R}^{(2n)}\left(0,0,\xi(\boldsymbol{x})\right). \quad (70)$$

Using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ together with (70) and (59) imply that

$$\mathcal{S}^{2n} \leq 2\int_{\Omega_T} \left\{ \left|\left(\mathcal{R}^{(2n+2)} - \mathcal{R}^{(2n+1)}\right)_t\right|^2 + \left|\nabla\left(\mathcal{R}^{(2n+2)} - \mathcal{R}^{(2n+1)}\right)\right|^2 \right\}(\boldsymbol{x},t)dtd\boldsymbol{x}$$
$$+2\int_{\Omega_T} \left\{ \left|\left(\mathcal{R}^{(2n+1)} - \mathcal{R}^{(2n)}\right)_t\right|^2 + \left|\nabla\left(\mathcal{R}^{(2n+1)} - \mathcal{R}^{(2n)}\right)\right|^2 \right\}(\boldsymbol{x},t)dtd\boldsymbol{x}, \quad (71)$$

where, for the sake of simplicity of notations, we have written $\mathcal{R}^{(m)}$ instead of $\mathcal{R}^{(m)}\left(0,0,\xi(\boldsymbol{x})\right)$ (for $m = 2n+2, 2n+1, 2n$). Using the same reasoning as that used to get (61) and (62), estimate (71) yields

$$\mathcal{S}^{2n} \leq 2\int_{\Omega_T} \left\{ |\mathcal{R}_t^{(2n+1)}|^2 + |\nabla\mathcal{R}^{(2n+1)}|^2 - \left(|\mathcal{R}_t^{(2n+2)}|^2 + |\nabla\mathcal{R}^{(2n+2)}|^2\right) \right\}(\boldsymbol{x},t)dtd\boldsymbol{x}$$
$$+2\int_{\Omega_T} \left\{ |\mathcal{R}_t^{(2n)}|^2 + |\nabla\mathcal{R}^{(2n)}|^2 - \left(|\mathcal{R}_t^{(2n+1)}|^2 + |\nabla\mathcal{R}^{(2n+1)}|^2\right) \right\}(\boldsymbol{x},t)dtd\boldsymbol{x}$$
$$= 2\int_{\Omega_T} \left\{ |\mathcal{R}_t^{(2n)}|^2 + |\nabla\mathcal{R}^{(2n)}|^2 - \left(|\mathcal{R}_t^{(2n+2)}|^2 + |\nabla\mathcal{R}^{(2n+2)}|^2\right) \right\}(\boldsymbol{x},t)dtd\boldsymbol{x}. \quad (72)$$

The sequence $\int_{\Omega_T} \left\{ |\mathcal{R}_t^{(2n)}|^2 + |\nabla\mathcal{R}^{(2n)}|^2 \right\} dtd\boldsymbol{x}$ is then decreasing. Since it is bounded from below, it is convergent. Letting $n \to \infty$ in (72) and using the fact that the sequence $\mathcal{S}^n$ is convergent (see above) yield (59), i.e.,

$$\lim_{n\to\infty} \mathcal{S}^n = \lim_{n\to\infty} \mathcal{S}^{2n} = 0. \quad (73)$$

(ii) **Density argument.** Using a density argument similar to the one used in [14, Pages 47–48] we can prove that the set of functions $\Lambda \in L^2(\Omega)$, which can be written in the form (66) where the sequence $\Phi^{(n)}$ is defined by (65) with $\xi \in L^2(\Omega)$, is dense in $L^2(\Omega)$.

Remark 2.2. *The algorithm of this section has a regularizing character* [13, 14] *and, in case the input data (20) and (21) are noisy, stable solutions may be achieved by stopping the iterations according to various criteria* [16, 18].

3. A formulation of an alternating algorithm for the ill-posed fractional problem (1)–(5) and its convergence

As in Section 2, we establish an alternating algorithm for the ill-posed fractional problem (1)–(5) along with a full convergence analysis. Such algorithm is a sequence of well-posed problems and some of them are of the form:

$$-D\left(\partial_t^{1-\beta} u\right)(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \Omega_T, \quad (74)$$

subjected to

$$u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \Gamma_T, \tag{75}$$

$$\partial_t^{1-\beta} u(\boldsymbol{x},T) = Q(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{76}$$

and

$$u(\boldsymbol{x},0) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \tag{77}$$

satisfying $f(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \partial\Omega$. Without loss of generality, we can write the problem (74)–(77) as, (using the new function $\overline{u}(\boldsymbol{x},t) := u(\boldsymbol{x},t) - f(\boldsymbol{x})$),

$$-D\left(\partial_t^{1-\beta}\overline{u}\right)(\boldsymbol{x},t) - \Delta\overline{u}(\boldsymbol{x},t) = \Delta f(\boldsymbol{x}) =: F(\boldsymbol{x}), \quad (\boldsymbol{x},t) \in \Omega_T, \tag{78}$$

$$\overline{u}(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \Gamma_T, \tag{79}$$

$$\partial_t^{1-\beta}\overline{u}(\boldsymbol{x},T) = Q(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{80}$$

and

$$\overline{u}(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \Omega. \tag{81}$$

To establish a convenient weak formulation for the problem (78)–(81), let us first consider the following fractional differential version (in which we only have the t variable) of (78)–(81):

$$-D\left(\partial_t^{1-\beta}u\right)(t) = 0, \quad t \in (0,T), \tag{82}$$

equipped with the following mixed boundary conditions:

$$u|_{t=0} = a_0 \tag{83}$$

and

$$\partial_t^{1-\beta}u|_{t=T} = a_1. \tag{84}$$

The equation (82) was previously considered in [24, Page 1359] but with fractional Neumann boundary conditions at both ends $t = 0$ and $t = T$. The problem (82)–(84) was considered in [23], where a least-squares mixed weak formulation (based on the introduction of intermediate variables) was derived, see [23, Theorem 3.2], from which it is possible to derive a well-posedness result for (82)–(84). However, we will not use this path of mixed formulations (which uses the splitting of the fractional differential equation into a system of two equations), but instead we will use the results of [24] (which dealt with Neumann boundary conditions) to derive a direct weak formulation for the problem (82)–(84). This latter path will also serve to derive a convenient weak formulation for the full problem (78)–(81).

To derive a weak formulation for problem (82)–(84), we multiply equation (82) by a smooth test function v satisfying $v|_{t=0} = 0$, integrate by parts together with the boundary condition (84) to get

$$\left(\partial_t^{1-\beta}u, Dv\right)_{L^2(0,T)} - a_1 v(T) = 0. \tag{85}$$

This can be written, using the left fractional integration, as

$$\left({}_0I_t^{\beta}Du, Dv\right)_{L^2(0,T)} - a_1 v(T) = 0, \tag{86}$$

where

$$ {}_0I_t^{\beta}\omega(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1}\omega(\tau)d\tau. \tag{87} $$

The right fractional integration is given by

$$ {}_tI_T^{\beta}\omega(x) = \frac{1}{\Gamma(\beta)} \int_t^T (\tau-t)^{\beta-1}\omega(\tau)d\tau. \tag{88} $$

Using [24, Lemma 2.1, Page 1360], equation (86) implies that

$$ \left({}_0I_t^{\beta/2}Du, {}_tI_T^{\beta/2}Dv\right)_{L^2(0,T)} - a_1v(T) = 0. \tag{89} $$

This yields, according to the definitions of the left and right Caputo derivatives (see [24, Page 1359] and [15, Page 2117]),

$$ -\left(\partial_t^{1-\beta/2}u, {}_R\partial_t^{1-\beta/2}v\right)_{L^2(0,T)} = a_1v(T), \tag{90} $$

where

$$ {}_R\partial_t^{\beta}u(t) = \frac{1}{\Gamma(1-\beta)} \int_t^T (\tau-t)^{-\beta}Du(\tau)d\tau. \tag{91} $$

The weak formulation of the problem (82)–(84) is then given by (similar to the one of [24, equation (5.1)]): find $u \in H_{C,l}^{1-\beta/2,a_0}(0,T)$ such that

$$ a(u,v) = b(v),\ \forall v \in H_{C,l}^{1-\beta/2,0}(0,T), \tag{92} $$

where (see [24, Pages 1360–1361]),

$$ H_{C,l}^{\mu}(0,T) = \left\{v \in L^2(0,T):\ \partial_t^{\mu}v \in L^2(0,T)\right\},\ \forall \mu \in (0,1), $$

the subscript l stands for 'left',

$$ \begin{aligned} H_{C,l}^{1-\beta/2,a_0}(0,T) &= \left\{v \in H_{C,l}^{1-\beta/2}(0,T):\ v(0) = a_0\right\}, \\ H_{C,l}^{1-\beta/2,0}(I) &= \left\{v \in H_{C,l}^{1-\beta/2}(0,T):\ v(0) = 0\right\}, \end{aligned} \tag{93} $$

$$ a(u,v) = -\left(\partial_t^{1-\beta/2}u, {}_R\partial_t^{1-\beta/2}v\right)_{L^2(0,T)}, \quad b(v) = a_1v(T). $$

It is useful to notice that the spaces $H_{C,l}^{1-\beta/2}(0,T)$ and $H_{C,r}^{1-\beta/2}(0,T)$, where the subscript r stands for 'right', coincide [25, Lemmata 2.1-2.2]. Recall that $H_{C,l}^{1-\beta/2}(0,T) \hookrightarrow \mathcal{C}(0,T)$, see [24, Page 1372] and [24, Theorem 3.2, Page 1366], and therefore the values of $v(0)$ involved in the definition (93) make sense.

To prove the well-posedness of the problem (92), we use the Lax-Milgram lemma. To this end, we introduce the auxiliary function $\widetilde{u} = u - a_0$. This yields that $\widetilde{u} \in H_{C,l}^{1-\beta/2,0}(0,T)$ and

$$ a(\widetilde{u},v) = b(v) - a(a_0,v) = b(v), \quad \forall v \in H_{C,l}^{1-\beta/2,0}(0,T). \tag{94} $$

We check now the hypotheses of the Lax-Milgram lemma, using the techniques of [24, Page 1372]:

- *The coercivity of the bilinear form $a(\cdot,\cdot)$.* Using the techniques of [24, Page 1372], for all $v \in H_{C,l}^{1-\beta/2,0}(0,T)$ we have
$$a(v,v) = -\left(\partial_t^{1-\beta/2} v, {}_R\partial_t^{1-\beta/2} v\right)_{L^2(0,T)} = \cos\left(\frac{\beta\pi}{2}\right) \|\partial_t^{1-\beta/2} v\|^2_{L^2(0,T)}. \tag{95}$$
Thanks to the Poincaré inequality (see, for instance, [3] and [9, Proposition 3.2]), $\|\partial_t^{1-\beta/2} v\|_{L^2(0,T)}$ is a norm on the space $H_{C,l}^{1-\beta/2,0}(0,T)$.
- *The continuity of the bilinear form $a(\cdot,\cdot)$* follows as in [24, Page 1372].
- *The continuity of the linear form $b(\cdot,\cdot)$.* Using [24, Theorem 3.2], we have
$$|b(v)| \le |a_1|\ |v(1)| \le |a_1|\ \|D^{1-\beta/2} v\|_{L^2(0,T)}. \tag{96}$$

Consequently, there exists a unique $\widetilde{u} \in H_{C,l}^{1-\beta/2,0}(0,T)$ solving (94) satisfying
$$\|\partial_t^{1-\beta/2}\widetilde{u}\|_{L^2(0,T)} \le C_3(\beta)|a_1|. \tag{97}$$
This implies that, using the Poincaré inequality and the fact that $\widetilde{u} = u - a_0$,
$$\|u\|_{L^2(0,T)} + \|\partial_t^{1-\beta/2} u\|_{L^2(0,T)} \le C_4\left(|a_0| + |a_1|\right). \tag{98}$$
The problem (82)–(84) is therefore well-posed.

Let us now consider the following problem:
$$-D\left(\partial_t^{1-\beta} u\right)(t) = 0, \quad t \in (0,T), \tag{99}$$
with the boundary conditions
$$u|_{t=0} = a_0 \tag{100}$$
and
$$\partial_t^{1-\beta} u|_{t=0} = b_0. \tag{101}$$
This problem is also well-posed. Indeed, the solution of (99)–(101) is
$$u(t) = a_0 + \frac{b_0}{\Gamma(2-\beta)} t^{1-\beta}. \tag{102}$$
We therefore have that
$$|u(t)| \le |a_0| + \frac{|a_1|}{\Gamma(2-\beta)} T^{1-\beta} \le \max\left\{1, \frac{|T^{1-\beta}|}{\Gamma(2-\beta)}\right\}\left(|a_0| + |b_0|\right). \tag{103}$$

Using the weak formulation (92) of the problem (82)–(84) combined with the ideas of [15] and the Green's formula in space, we obtain the following weak formulation for the problem (78)–(81): Find $\overline{u} \in Z^{1-\beta/2}$ such that, for all $v \in Z^{1-\beta/2}$ (recall that $\Delta f = F$ and $f\big|_{\partial\Omega} = 0$):
$$\begin{aligned} &-\left(\partial_t^{1-\beta/2}\overline{u}, {}_R\partial_t^{1-\beta/2} v\right)_{L^2(\Omega_T)} + \int_0^T \left(\nabla\overline{u}(\cdot,t), \nabla v(\cdot,t)\right)_{L^2(\Omega)} dt \\ &= (F,v)_{L^2(\Omega_T)} + \left(Q, v(\cdot,T)\right)_{L^2(\Omega)} = -\left(\nabla f, \nabla v\right)_{L^2(\Omega_T)} + \left(Q, v(\cdot,T)\right)_{L^2(\Omega)}, \end{aligned} \tag{104}$$
where, as in [15, Pages 2112–2113],
$$Z^{1-\beta/2} := H_{C,l}^{1-\beta/2,0}\left(0,T; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right) \tag{105}$$

with

$$H_{C,l}^{1-\beta/2}\left(0,T;L^2(\Omega)\right)=\left\{v\in L^2(0,T;L^2(\Omega));\ \partial_t^{1-\beta/2}v\in L^2(0,T;L^2(\Omega))\right\} \tag{106}$$

and

$$H_{C,l}^{1-\beta/2,0}\left(0,T;L^2(\Omega)\right)=\left\{v\in H_{C,l}^{1-\beta/2}\left(0,T;L^2(\Omega)\right);\ v(\cdot,0)=0\right\}. \tag{107}$$

The well-posedness of the problem (78)–(81) can be justified through the weak formulation (104) which gives, by taking $v=\overline{u}$ and using coercivity,

$$\|\partial_t^{1-\beta/2}\overline{u}\|_{L^2(0,T;L^2(\Omega))}+\|\nabla\overline{u}\|_{L^2(0,T;L^2(\Omega))}\le C_5\left(\|\nabla f\|_{L^2(\Omega)}+\|Q\|_{L^2(\Omega)}\right). \tag{108}$$

This implies, since $\overline{u}(\boldsymbol{x},t)=u(\boldsymbol{x},t)-f(\boldsymbol{x})$ and under the assumption that $f\in H_0^1(\Omega)$, that the solution u of (74)–(77) satisfies

$$\|\partial_t^{1-\beta/2}u\|_{L^2(0,T;L^2(\Omega))}+\|\nabla u\|_{L^2(0,T;L^2(\Omega))}\le C_6\left(\|\nabla f\|_{L^2(\Omega)}+\|Q\|_{L^2(\Omega)}\right). \tag{109}$$

It is worth mentioning that the case when $\beta=0$ in the estimate (109) leads to the one of (29) (in Lemma 2.1) corresponding to the problem (26), which is the limit case when $\beta=0$ in the problem (74)–(77) with $(\zeta,\theta):=(f,Q)$.

Using similar techniques to those employed to prove [4, Corollary 3.2, Page 32] together with (109) lead to the following $L^\infty(L^2)$–estimate.

Corollary 3.1 (A new $L^\infty(L^2)$–estimate for the solution of (74)–(77))**.** *Assume that $f\in H_0^1(\Omega)$ and $Q\in L^2(\Omega)$. Then there exists a unique (weak) solution $u=\overline{u}+f$ to the problem (74)–(77) in the sense that $\overline{u}\in Z^{1-\beta/2}$ is satisfying (104) for all $v\in Z^{1-\beta/2}$. In addition to this, the following $L^\infty(L^2)$–estimate holds:*

$$\|u\|_{L^\infty(0,T;L^2(\Omega))}\le C_7\left(\|\nabla f\|_{L^2(\Omega)}+\|Q\|_{L^2(\Omega)}\right). \tag{110}$$

Remark 3.1 (Another new proof for the $L^2(H_0^1)$–estimate involved in (109)). *Previously we have proved the stability result (109) for problem (74)–(77) through the weak formulation (104). Another possible way to justify, for instance, the $L^2(H_0^1)$–estimate involved in (109) is to use the eigenfunction expansions. To this end, we write the solution of (74)–(77) as in (6), that is,*

$$u(\boldsymbol{x},t)=\sum_{m=1}^{\infty}\psi_m(t)\varphi_m(\boldsymbol{x}),\quad (\boldsymbol{x},t)\in\Omega_T, \tag{111}$$

where

$$-D\left(\partial_t^{1-\beta}\psi_m\right)(t)+\mu_m\psi_m(t)=0,\quad t\in(0,T), \tag{112}$$

$$\partial_t^{1-\beta}\psi_m(T)=Q_m:=(Q,\varphi_m)_{L^2(\Omega)}\quad\text{and}\quad\psi_m(0)=f_m. \tag{113}$$

The solution of (112) is given by, for $\alpha:=2-\beta\in(1,2)$ (see (13) and [27, Pages 3 and 5]*):*

$$\psi_m(t)=E_{\alpha,1}(\mu_m t^\alpha)f_m+\partial_t^{\alpha-1}\psi_m(0)t^{\alpha-1}E_{\alpha,\alpha}(\mu_m t^\alpha)Q_m. \tag{114}$$

Taking into account the boundary conditions (113), the expression (114) becomes as:

$$\psi_m(t) = \left(E_{\alpha,1}(\mu_m t^\alpha) - \frac{t^{\alpha-1} E_{\alpha,\alpha}(\mu_m t^\alpha) \mu_m T E_{\alpha,2}(\mu_m T^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} \right) f_m + \frac{t^{\alpha-1} E_{\alpha,\alpha}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} Q_m. \quad (115)$$

Using the asymptotic behaviour results of [17, inequality (2.1) and equation (2.7)] *and* [26, Lemma 1.1]*, along with some additional appropriate techniques, we are able to prove the $L^2(H_0^1)$–estimate involved in (109), see Appendix A.*

3.1. **Formulation of an alternating algorithm for the ill-posed fractional problem (1)–(5).** Inspired by the algorithm developed in [14] for the Cauchy problem of the elliptic Laplace equation in $\mathbb{R}^{d+1}$ and also by the ideas given in the previous Section 2 dealing with the limit case $\beta = 0$ in (1)–(5), we suggest the following alternating algorithm for the problem (1)–(5):

$$\begin{cases} -D\left(\partial_t^{1-\beta}\Psi^{(0)}\right)(\boldsymbol{x},t) - \Delta\Psi^{(0)}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Omega_T, \\ \Psi^{(0)}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Gamma_T, \\ \Psi^{(0)}(\boldsymbol{x},0) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \partial_t^{1-\beta}\Psi^{(0)}(\boldsymbol{x},T) = \eta(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \quad (116)$$

$$\begin{cases} -D\left(\partial_t^{1-\beta}\Psi^{(2n+1)}\right)(\boldsymbol{x},t) - \Delta\Psi^{(2n+1)}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Omega_T, \\ \Psi^{(2n+1)}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Gamma_T, \\ \partial_t^{1-\beta}\Psi^{(2n+1)}(\boldsymbol{x},0) = \mathrm{G}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \Psi^{(2n+1)}(\boldsymbol{x},T) = \Psi^{(2n)}(\boldsymbol{x},T), & \boldsymbol{x} \in \Omega, \end{cases} \quad (117)$$

and

$$\begin{cases} -D\left(\partial_t^{1-\beta}\Psi^{(2n+2)}\right)(\boldsymbol{x},t) - \Delta\Psi^{(2n+2)}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Omega_T, \\ \Psi^{(2n+2)}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Gamma_T, \\ \Psi^{(2n+2)}(\boldsymbol{x},0) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \partial_t^{1-\beta}\Psi^{(2n+2)}(\boldsymbol{x},T) = \partial_t^{1-\beta}\Psi^{(2n+1)}(\boldsymbol{x},T), & \boldsymbol{x} \in \Omega. \end{cases} \quad (118)$$

This sequence is completely defined by

$$f(\boldsymbol{x}) = \Psi^{(0)}(\boldsymbol{x},0), \quad \partial_t^{1-\beta}\Psi^0(\boldsymbol{x},T) = \eta(\boldsymbol{x}) \quad \text{and} \quad \partial_t^{1-\beta}\Psi^{(2n+1)}(\boldsymbol{x},0) = \mathrm{G}(\boldsymbol{x}).$$

The stability of the problems of type (116) and (118) was proved before. The stability of the problems of type (117) can be justified in a similar way with slightly changes of spaces.

3.2. **Convergence of the alternating algorithm.** To prove the convergence of the alternating algorithm (116)–(118), we first write the error (119), which satisfies (120)–(122), in the eigenfunction expanssion (123). This yields the sequence of fractional order differential problems (124)–(126), whose solutions are the coefficients e_m^n involved in (123). Subsequently:

1. We derive the relations (136)–(139) and compute the fractional derivatives involved.

2. We derive the explicit and concise formulas (151)–(152) for e_m^n.

3. The $L^2(H_0^1)$-norm of the errors are given in (156) and (157). To prove that these errors tend to zero as $n \to \infty$ (stated in Theorem 3.2), we employ new techniques developed in the proofs of Lemma 3.1, Theorem 3.1 and Appendix B.

Taking into account (1)–(5) and (116)–(118), the error

$$e^n(\boldsymbol{x},t) := u(\boldsymbol{x},t) - \Psi^{(n)}(\boldsymbol{x},t). \tag{119}$$

satisfies

$$\begin{cases} -D\left(\partial_t^{1-\beta} e^{2n+2}\right)(\boldsymbol{x},t) - \Delta e^{2n+2}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Omega_T, \\ e^0(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Gamma_T, \\ e^0(\boldsymbol{x},0) = 0, & \boldsymbol{x} \in \Omega, \\ \partial_t^{1-\beta} e^0(\boldsymbol{x},T) = \partial_t^{1-\beta} u(\boldsymbol{x},T) - \eta(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{120}$$

$$\begin{cases} -D\left(\partial_t^{1-\beta} e^{2n+1}\right)(\boldsymbol{x},t) - \Delta e^{2n+1}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Omega_T, \\ e^{(2n+1)}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Gamma_T, \\ \partial_t^{1-\beta} e^{2n+1}(\boldsymbol{x},0) = 0, & \boldsymbol{x} \in \Omega, \\ e^{2n+1}(\boldsymbol{x},T) = e^{2n}(\boldsymbol{x},T), & \boldsymbol{x} \in \Omega \end{cases} \tag{121}$$

and

$$\begin{cases} -D\left(\partial_t^{1-\beta} e^{2n+2}\right)(\boldsymbol{x},t) - \Delta e^{2n+2}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Omega_T, \\ e^{2n+2}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Gamma_T, \\ e^{2n+2}(\boldsymbol{x},0) = 0, & \boldsymbol{x} \in \Omega, \\ \partial_t^{1-\beta} e^{2n+2}(\boldsymbol{x},T) = \partial_t^{1-\beta} e^{2n+1}(\boldsymbol{x},T), & \boldsymbol{x} \in \Omega. \end{cases} \tag{122}$$

As in (58), this sequence can be written as $\mathcal{R}^{(n)}(0,0,\partial_t^{1-\beta} u|_{t=T} - \eta) = e^n$. Further, writing e^n in the eigenfunction expansion

$$e^n(\boldsymbol{x},t) := \sum_{m=1}^{\infty} e_m^n(t)\varphi_m(\boldsymbol{x}), \tag{123}$$

using (120)–(122), yields

$$\begin{cases} -D\left(\partial_t^{1-\beta} e_m^0\right)(t) + \mu_m e_m^0(t) = 0, \quad t \in (0,T), \\ e_m^0(0) = 0 \quad \text{and} \quad \partial_t^{1-\beta} e_m^0(T) = \Lambda_m := \partial_t^{1-\beta}\psi_m(T) - \eta_m, \end{cases} \tag{124}$$

$$\begin{cases} -D\left(\partial_t^{1-\beta} e_m^{2n+1}\right)(t) + \mu_m e_m^{2n+1}(t) = 0, \quad t \in (0,T), \\ \partial_t^{1-\beta} e_m^{2n+1}(0) = 0 \quad \text{and} \quad e_m^{2n+1}(T) = e_m^{2n}(T), \end{cases} \tag{125}$$

and

$$\begin{cases} -D\left(\partial_t^{1-\beta} e_m^{2n+2}\right)(t) + \mu_m e_m^{2n+2}(t) = 0, \quad t \in (0,T), \\ e_m^{2n+2}(0) = 0 \quad \text{and} \quad \partial_t^{1-\beta} e_m^{2n+2}(T) = \partial_t^{1-\beta} e_m^{2n+1}(T). \end{cases} \tag{126}$$

The coefficients e_m^n involved in (123) also satisfy

$$e_m^n(t) = \psi_m(t) - \Psi_m^{(n)}(t), \tag{127}$$

where $\Psi_m^{(n)}$ are the coefficients of the sequence $\Psi^{(n)}$ defined by (116)–(118) in the eigenfunction basis, i.e.,

$$\Psi^{(n)}(\boldsymbol{x},t)=\sum_{m=1}^{\infty}\Psi_m^{(n)}(t)\varphi_m(\boldsymbol{x}). \tag{128}$$

The functions $\Psi_m^{(n)}(t)$ also satisfy, thanks to (116)–(118),

$$\begin{cases} -D\left(\partial_t^{1-\beta}\Psi_m^{(0)}\right)(t)+\mu_m\Psi_m^{(0)}(t)=0, \quad t\in(0,T),\\ \Psi_m^{(0)}(0)=f_m \quad \text{and} \quad \partial_t^{1-\beta}\Psi_m^{(0)}(T)=\eta_m, \end{cases} \tag{129}$$

$$\begin{cases} -D\left(\partial_t^{1-\beta}\Psi_m^{(2n+1)}\right)(t)+\mu_m\Psi_m^{(2n+1)}(t)=0, \quad t\in(0,T),\\ \partial_t^{1-\beta}\Psi_m^{(2n+1)}(0)=\mathrm{G}_m \quad \text{and} \quad \Psi_m^{(2n+1)}(T)=\psi_m^{(2n)}(T), \end{cases} \tag{130}$$

and

$$\begin{cases} -D\left(\partial_t^{1-\beta}\Psi_m^{(2n+2)}\right)(t)+\mu_m\Psi_m^{(2n+2)}(t)=0, \quad t\in(0,T),\\ \Psi_m^{(2n+2)}(0)=f_m \quad \text{and} \quad \partial_t^{1-\beta}\Psi_m^{(2n+2)}(T)=\partial_t^{1-\beta}\psi_m^{(2n+1)}(T). \end{cases} \tag{131}$$

Using (13) together with (124) imply that

$$\mathcal{L}\left(e_m^n\right)(s)=-\frac{\partial_t^{1-\beta}e_m^n(0)}{\mu_m-s^{2-\beta}}-\frac{s^{1-\beta}e_m^n(0)}{\mu_m-s^{2-\beta}}. \tag{132}$$

This gives, using [27, page 3] and [20, equation (4.11), page 140],

$$e_m^n(t)=e_m^n(0)E_{2-\beta,1}\left(\mu_m t^{2-\beta}\right)+\partial_t^{1-\beta}e_m^n(0)t^{1-\beta}E_{2-\beta,2-\beta}\left(\mu_m t^{2-\beta}\right). \tag{133}$$

Gathering this expression with (124)–(126) leads to

$$e_m^{2n}(t)=\partial_t^{1-\beta}e_m^{2n}(0)t^{1-\beta}E_{2-\beta,2-\beta}\left(\mu_m t^{2-\beta}\right) \text{ and } e_m^{2n+1}(t)=e_m^{2n+1}(0)E_{2-\beta,1}\left(\mu_m t^{2-\beta}\right). \tag{134}$$

Using again (124)–(126) and (134) leads to

$$\partial_t^{1-\beta}e_m^0(0)=\frac{\Lambda_m}{\partial_t^{1-\beta}\left(t^{1-\beta}E_{2-\beta,2-\beta}\left(\mu_m t^{2-\beta}\right)\right)|_{t=T}},$$

$$\partial_t^{1-\beta}e_m^{2n}(0)=\frac{\partial_t^{1-\beta}e_m^{2n-1}(T)}{\partial_t^{1-\beta}\left(t^{1-\beta}E_{2-\beta,2-\beta}\left(\mu_m t^{2-\beta}\right)\right)|_{t=T}} \text{ and } e_m^{2n+1}(0)=\frac{e_m^{2n}(T)}{E_{2-\beta,1}\left(\mu_m T^{2-\beta}\right)}. \tag{135}$$

Inserting these values into (134) yields

$$e_m^0(t)=\frac{\Lambda_m}{\partial_t^{1-\beta}\left(t^{1-\beta}E_{2-\beta,2-\beta}\left(\mu_m t^{2-\beta}\right)\right)|_{t=T}}t^{1-\beta}E_{2-\beta,2-\beta}\left(\mu_m t^{2-\beta}\right), \tag{136}$$

$$e_m^{2n}(t)=\frac{\partial_t^{1-\beta}e_m^{2n-1}(T)}{\partial_t^{1-\beta}\left(t^{1-\beta}E_{2-\beta,2-\beta}\left(\mu_m t^{2-\beta}\right)\right)|_{t=T}}t^{1-\beta}E_{2-\beta,2-\beta}\left(\mu_m t^{2-\beta}\right), \tag{137}$$

and

$$e_m^{2n+1}(t)=\frac{e_m^{2n}(T)}{E_{2-\beta,1}\left(\mu_m T^{2-\beta}\right)}E_{2-\beta,1}\left(\mu_m t^{2-\beta}\right). \tag{138}$$

Gathering (137) and (138) yields

$$e_m^{2n}(t)=\frac{t^{1-\beta}E_{2-\beta,2-\beta}\left(\mu_m t^{2-\beta}\right)\partial_t^{1-\beta}E_{2-\beta,1}\left(\mu_m t^{2-\beta}\right)|_{t=T}}{\partial_t^{1-\beta}\left(t^{1-\beta}E_{2-\beta,2-\beta}\left(\mu_m t^{2-\beta}\right)\right)|_{t=T}E_{2-\beta,1}\left(\mu_m T^{2-\beta}\right)}e_m^{2n-2}(T). \tag{139}$$

To compute the fractional derivatives involved in the right-hand side of (139), we use the following relation between the Caputo and left Riemann-Liouville derivatives, see, for instance, [15, equation (2.6), Page 2111]:

$$ {}_0\partial_t^\rho v(t) - \frac{v(0)}{\Gamma(1-\rho)} t^{-\rho} = \partial_t^\rho v(t), \quad \rho \in (0,1), \tag{140} $$

where

$$ {}_0\partial_t^\gamma v(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t (t-\tau)^{-\gamma} v(\tau) d\tau. \tag{141} $$

For the sake of simplicity of computations, we set $\alpha := 2-\beta \in (1,2)$. The coefficient of $e_m^{2n-2}(T)$ in (139) becomes:

$$ \frac{t^{\alpha-1} E_{\alpha,\alpha}(\mu t^\alpha)\, \partial_t^{\alpha-1} E_{\alpha,1}(\mu t^\alpha)\,|_{t=T}}{\partial_t^{\alpha-1}\left(t^{\alpha-1} E_{\alpha,\alpha}(\mu t^\alpha)\right)|_{t=T} E_{\alpha,1}(\mu T^\alpha)} \tag{142} $$

where, for the sake of simplicity of notations, we have removed the index m from μ_m. The expansion (142) contains fractional derivatives, which we will compute as follows.
(i) We compute $\partial_t^{\alpha-1} E_{\alpha,1}(\mu t^\alpha)\,|_{t=T}$. Applying [8, equation (4.10.14), Page 87] or [20, equation (1.82), Page 21]), we have

$$ {}_0\partial_t^{\alpha-1} E_{\alpha,1}(\mu t^\alpha) = t^{1-\alpha} E_{\alpha,2-\alpha}(\mu t^\alpha), \quad t > 0. \tag{143} $$

Using now (141), equation (143) implies that (recall that $E_{\alpha,1}(0) = 1$)

$$ \partial_t^{\alpha-1} E_{\alpha,1}(\mu t^\alpha) = t^{1-\alpha}\left(E_{\alpha,2-\alpha}(\mu t^\alpha) - \frac{1}{\Gamma(\alpha)}\right) = t^{1-\alpha}\left(\sum_{j\geq 0} \frac{(\mu t^\alpha)^j}{\Gamma(j\alpha+2-\alpha)} - \frac{1}{\Gamma(\alpha)}\right). \tag{144} $$

This gives

$$ \partial_t^{\alpha-1} E_{\alpha,1}(\mu t^\alpha) = t^{1-\alpha} \sum_{j\geq 1} \frac{(\mu t^\alpha)^j}{\Gamma(j\alpha+2-\alpha)} = \mu t \sum_{j\geq 0} \frac{(\mu t^\alpha)^j}{\Gamma(j\alpha+2)} = \mu t E_{\alpha,2}(\mu t^\alpha). \tag{145} $$

(ii) We compute $\partial_t^{\alpha-1}\left(t^{\alpha-1} E_{\alpha,\alpha}(\mu t^\alpha)\right)$. Using again relation (141) and [8, equation (4.10.14), Page 87] or [20, equation (1.82), Page 21] leads to, since $t^{\alpha-1} E_{\alpha,\alpha}(\mu t^\alpha)\,|_{t=0} = 0$,

$$ \partial_t^{\alpha-1}\left(t^{\alpha-1} E_{\alpha,\alpha}(\mu t^\alpha)\right) = {}_0\partial_t^{\alpha-1}\left(t^{\alpha-1} E_{\alpha,\alpha}(\mu t^\alpha)\right) = E_{\alpha,1}(\mu t^\alpha). \tag{146} $$

Using (142), (145) and (146), the coefficient of $e_m^{2n-2}(T)$ in (139) is reduced to:

$$ \frac{t^{\alpha-1} E_{\alpha,\alpha}(\mu t^\alpha)\, \mu T E_{\alpha,2}(\mu T^\alpha)}{\left(E_{\alpha,1}(\mu T^\alpha)\right)^2}. $$

This gives

$$ e_m^{2n+2}(t) = \frac{t^{\alpha-1} E_{\alpha,\alpha}(\mu_m t^\alpha)\, \mu_m T E_{\alpha,2}(\mu_m T^\alpha)}{\left(E_{\alpha,1}(\mu_m T^\alpha)\right)^2} e_m^{2n}(T), \tag{147} $$

and by substituting $t = T$ in (147) and taking $n-1$ in place of n,

$$ e_m^{2n}(T) = \frac{E_{\alpha,\alpha}(\mu_m T^\alpha)\, \mu_m T^\alpha E_{\alpha,2}(\mu_m T^\alpha)}{\left(E_{\alpha,1}(\mu_m T^\alpha)\right)^2} e_m^{2n-2}(T). \tag{148} $$

Using (136) and (146) we obtain

$$e_m^{2n}(T) = \mathbb{S}_m^n e_m^0(T) = \mathbb{S}_m^n \frac{\Lambda_m T^{\alpha-1} E_{\alpha,\alpha}\left(\mu_m T^\alpha\right)}{\partial_t^{\alpha-1}\left(t^{\alpha-1} E_{\alpha,\alpha}\left(\mu_m t^\alpha\right)\right)|_{t=T}} = \mathbb{S}_m^n \frac{\Lambda_m T^{\alpha-1} E_{\alpha,\alpha}\left(\mu_m T^\alpha\right)}{E_{\alpha,1}\left(\mu T^\alpha\right)}. \tag{149}$$

where

$$\mathbb{S}_m := \frac{E_{\alpha,\alpha}\left(\mu_m T^\alpha\right) \mu_m T^\alpha E_{\alpha,2}\left(\mu_m T^\alpha\right)}{\left(E_{\alpha,1}\left(\mu_m T^\alpha\right)\right)^2}. \tag{150}$$

Substituting $e_m^{2n}(T)$ by its value of (149) in (147) leads to

$$\begin{aligned} e_m^{2n+2}(t) &= \frac{t^{\alpha-1} E_{\alpha,\alpha}\left(\mu_m t^\alpha\right) \mu_m T E_{\alpha,2}\left(\mu_m T^\alpha\right)}{\left(E_{\alpha,1}\left(\mu_m T^\alpha\right)\right)^2} \frac{\Lambda_m T^{\alpha-1} E_{\alpha,\alpha}\left(\mu_m T^\alpha\right)}{E_{\alpha,1}\left(\mu T^\alpha\right)} \mathbb{S}_m^n \\ &= \frac{t^{\alpha-1} E_{\alpha,\alpha}\left(\mu_m t^\alpha\right)}{E_{\alpha,1}\left(\mu_m T^\alpha\right)} \mathbb{S}_m^{n+1} \Lambda_m. \end{aligned} \tag{151}$$

Substituting $e_m^{2n}(T)$ by its value of (149) in (138) we obtain

$$\begin{aligned} e_m^{2n+1}(t) = \frac{E_{\alpha,1}\left(\mu_m t^\alpha\right)}{E_{\alpha,1}\left(\mu_m T^\alpha\right)} e_m^{2n}(T) &= \frac{E_{\alpha,1}\left(\mu_m t^\alpha\right)}{E_{\alpha,1}\left(\mu_m T^\alpha\right)} \mathbb{S}_m^n \frac{\Lambda_m T^{\alpha-1} E_{\alpha,\alpha}\left(\mu_m T^\alpha\right)}{E_{\alpha,1}\left(\mu T^\alpha\right)} \\ &= \frac{E_{\alpha,1}\left(\mu_m t^\alpha\right) T^{\alpha-1} E_{\alpha,\alpha}\left(\mu_m T^\alpha\right)}{\left(E_{\alpha,1}\left(\mu_m T^\alpha\right)\right)^2} \mathbb{S}_m^n \Lambda_m. \end{aligned} \tag{152}$$

Substituting e_m^{2n+2} and e_m^{2n+1} by their values of (151) and (152) in (123) yields

$$e^{2n+2}(\boldsymbol{x}, t) = \sum_{m=1}^{\infty} \frac{t^{\alpha-1} E_{\alpha,\alpha}\left(\mu_m t^\alpha\right)}{E_{\alpha,1}\left(\mu_m T^\alpha\right)} \mathbb{S}_m^{n+1} \Lambda_m \varphi_m(\boldsymbol{x}) \tag{153}$$

and

$$e^{2n+1}(\boldsymbol{x}, t) = \sum_{m=1}^{\infty} \frac{E_{\alpha,1}\left(\mu_m t^\alpha\right) T^{\alpha-1} E_{\alpha,\alpha}\left(\mu_m T^\alpha\right)}{\left(E_{\alpha,1}\left(\mu_m T^\alpha\right)\right)^2} \mathbb{S}_m^n \Lambda_m \varphi_m(\boldsymbol{x}). \tag{154}$$

It is useful to examine the computations (153) and (154) in the limit case $\beta = 0$ (and consequently $\alpha = 2$) and compare the results with the ones of Section 2. To this end, we use the following formulas [8, equation (4.2.2), Page 57]:

$$E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}} \quad \text{and} \quad E_{2,1}(z) = \cosh(\sqrt{z}). \tag{155}$$

We remark that, thanks to the formulas (155), the limit case when $\beta = 0$ in the errors (153) and (154) yields (49) and (50) obtained for the Cauchy problem of the Laplace equation (18)–(21) (the limit case when $\beta = 0$ in (1)–(5)). Consider now

$$\|e^{2n+2}\|^2_{L^2(0,T;H_0^1(\Omega))} = \sum_{m=1}^{\infty} \mu_m \frac{\mathbb{S}_m^{2n+2} \Lambda_m^2}{E_{\alpha,1}^2\left(\mu_m T^\alpha\right)} \int_0^T \left(t^{\alpha-1} E_{\alpha,\alpha}\left(\mu_m t^\alpha\right)\right)^2 dt \tag{156}$$

and

$$\|e^{2n+1}\|^2_{L^2(0,T;H_0^1(\Omega))} = \sum_{m=1}^{\infty} \mu_m \frac{\mathbb{S}_m^{2n} \Lambda_m^2 T^{2\alpha-2} E_{\alpha,\alpha}^2\left(\mu_m T^\alpha\right)}{E_{\alpha,1}^4\left(\mu_m T^\alpha\right)} \int_0^T \left(E_{\alpha,1}\left(\mu_m t^\alpha\right)\right)^2 dt. \tag{157}$$

To prove that $\lim_{n\to\infty} \|e^{2n+2}\|_{L^2(0,T;H_0^1(\Omega))} = \lim_{n\to\infty} \|e^{2n+1}\|_{L^2(0,T;H_0^1(\Omega))} = 0$, we use the following Lemmata 3.1 and 3.2.

Lemma 3.1 (A new technical lemma). *Let $\mathbb{L}_m$ and $\mathbb{M}_m$ be the sequences given by, for $m \in \mathbb{N}^*$,*

$$\mathbb{L}_m = \frac{\mu_m}{E^2_{\alpha,1}\left(\mu_m T^\alpha\right)} \int_0^T \left(t^{\alpha-1} E_{\alpha,\alpha}\left(\mu_m t^\alpha\right)\right)^2 dt \tag{158}$$

and

$$\mathbb{M}_m = \frac{\mu_m T^{2\alpha-2} E^2_{\alpha,\alpha}\left(\mu_m T^\alpha\right)}{E^4_{\alpha,1}\left(\mu_m T^\alpha\right)} \int_0^T \left(E_{\alpha,1}\left(\mu_m t^\alpha\right)\right)^2 dt. \tag{159}$$

Then $\mathbb{L}_m$ and $\mathbb{M}_m$ are bounded.

Proof. We have, using the change of variable $z^\alpha := \mu_m t^\alpha$ with $z \geq 0$,

$$\begin{aligned} \mathbb{L}_m &= \frac{\mu_m \mu_m^{-\frac{1}{\alpha}}}{E^2_{\alpha,1}(\mu_m T^\alpha)} \int_0^{T\mu_m^{\frac{1}{\alpha}}} \left(\mu_m^{-\frac{1}{\alpha}} z\right)^{2\alpha-2} E^2_{\alpha,\alpha}(z^\alpha) dz \\ &= \frac{\mu_m^{\frac{1}{\alpha}-1}}{E^2_{\alpha,1}(\mu_m T^\alpha)} \int_0^{T\mu_m^{\frac{1}{\alpha}}} z^{2\alpha-2} E^2_{\alpha,\alpha}(z^\alpha) dz. \end{aligned} \tag{160}$$

Using [17, equation (2.7)] yields

$$E_{\alpha,\alpha}(z^\alpha) \sim (z^\alpha)^{\frac{1-\alpha}{\alpha}} e^{z^{\alpha}{}^{\frac{1}{\alpha}}}/\alpha = z^{1-\alpha} e^z/\alpha, \quad \text{as} \quad z \to \infty. \tag{161}$$

This implies that

$$z^{2\alpha-2} E^2_{\alpha,\alpha}(z^\alpha) \sim \frac{e^{2z}}{\alpha^2}, \quad \text{as} \quad z \to \infty. \tag{162}$$

This together with the fact that (recall that $\alpha > 1$) the function $z \mapsto \frac{z^{2\alpha-2} E^2_{\alpha,\alpha}(z^\alpha)}{\frac{e^{2z}}{\alpha^2}}$ is continuous (and consequently it is bounded on any compact subset in which the maximum and minimum values are attained at some points) and non-negative imply that there exists a positive constant C_8 (which depends only on α) such that for all $z \geq 0$,

$$0 \leq \frac{z^{2\alpha-2} E^2_{\alpha,\alpha}(z^\alpha)}{\frac{e^{2z}}{\alpha^2}} \leq C_8. \tag{163}$$

Here, for brevity, we have omitted including the lengthy standard arguments of the separate handling of compact intervals and mentioned only the asymptotic behaviour (162) to conclude the boundness in (163).

Using again [17, equation (2.7)] (recall that $\lim\limits_{m\to\infty} \mu_m = +\infty$), we get

$$E_{\alpha,1}(\mu_m T^\alpha) \quad \sim \quad e^{(\mu_m T^\alpha)^{\frac{1}{\alpha}}}/\alpha = e^{\mu_m^{\frac{1}{\alpha}} T}/\alpha, \quad \text{as} \quad m \to \infty.$$

This gives

$$E^2_{\alpha,1}(\mu_m T^\alpha) \sim e^{2\mu_m^{\frac{1}{\alpha}} T}/\alpha^2, \quad \text{as} \quad m \to \infty. \tag{164}$$

This together with the fact that the sequence $\frac{E^2_{\alpha,1}(\mu_m T^\alpha)}{\frac{e^{2\mu_m^{\frac{1}{\alpha}} T}}{\alpha^2}}$ is positive imply that for some two positive constants C_9 and C_{10} (depending on α and T) we have, for all $m \geq \mathbb{N}^*$,

$$C_9 \leq \frac{E^2_{\alpha,1}(\mu_m T^\alpha)}{\frac{e^{2\mu_m^{\frac{1}{\alpha}} T}}{\alpha^2}} \leq C_{10}. \tag{165}$$

Gathering (160), (163) and (165), and using (10) together with the fact that $\frac{1}{\alpha}-1<0$ yield

$$0<\mathbb{L}_m \leq \frac{C_8\mu_m^{\frac{1}{\alpha}-1}}{C_9 e^{2\mu_m^{\frac{1}{\alpha}}T}}\int_0^{T\mu_m^{\frac{1}{\alpha}}} e^{2z}dz = \frac{C_8\mu_m^{\frac{1}{\alpha}-1}}{2C_9 e^{2\mu_m^{\frac{1}{\alpha}}T}}\left(e^{2T\mu_m^{\frac{1}{\alpha}}}-1\right) \leq \frac{C_8}{2C_9}\mu_1^{\frac{1}{\alpha}-1}. \tag{166}$$

This means that $\mathbb{L}_m$ is bounded. Let us now prove that the sequence $\mathbb{M}_m$ is bounded. We have, using the change of variables $z^\alpha := \mu_m t^\alpha$ with $z \geq 0$,

$$\left(\frac{1}{E_{\alpha,1}(\mu_m T^\alpha)}\right)^2 \int_0^T E_{\alpha,1}^2(\mu_m t^\alpha)dt = \frac{\mu_m^{-\frac{1}{\alpha}}}{E_{\alpha,1}^2(\mu_m T^\alpha)}\int_0^{T\mu_m^{\frac{1}{\alpha}}} E_{\alpha,1}^2(z^\alpha)dz. \tag{167}$$

Using [17, equation (2.7)] yields

$$E_{\alpha,1}(z^\alpha) \sim e^{z^{\alpha\frac{1}{\alpha}}}/\alpha = e^z/\alpha, \quad \text{as} \quad z\to\infty. \tag{168}$$

This implies that

$$E_{\alpha,1}^2(z^\alpha) \sim \frac{e^{2z}}{\alpha^2}, \quad \text{as} \quad z\to\infty. \tag{169}$$

This together with the fact that $z \mapsto \frac{E_{\alpha,1}^2(z^\alpha)}{\frac{e^{2z}}{\alpha^2}}$ is continuous and positive imply that there exists two positive constants C_{11} and C_{12} (which are depending only on α) such that for all $z \geq 0$,

$$C_{11} \leq \frac{E_{\alpha,1}^2(z^\alpha)}{\frac{e^{2z}}{\alpha^2}} \leq C_{12}. \tag{170}$$

Gathering (165), (167) and (170) yields

$$\begin{aligned}\left(\frac{1}{E_{\alpha,1}(\mu_m T^\alpha)}\right)^2 \int_0^T E_{\alpha,1}^2(\mu_m t^\alpha)dt &\leq \frac{\alpha^2\mu_m^{-\frac{1}{\alpha}}}{C_9 e^{2\mu_m^{\frac{1}{\alpha}}T}}\frac{C_{12}}{\alpha^2}\int_0^{T\mu_m^{\frac{1}{\alpha}}} e^{2z}dz \\ &= \frac{C_{12}\mu_m^{-\frac{1}{\alpha}}}{2C_9 e^{2\mu_m^{\frac{1}{\alpha}}T}}\left(e^{2T\mu_m^{\frac{1}{\alpha}}}-1\right) \leq \frac{C_{12}\mu_m^{-\frac{1}{\alpha}}}{2C_9}.\end{aligned} \tag{171}$$

On the other hand, taking $z^\alpha := \mu_m T^\alpha$ in (163) leads to

$$\mu_m T^{2\alpha-2}E_{\alpha,\alpha}^2(\mu_m T^\alpha) \leq \mu_m \mu_m^{\frac{2-2\alpha}{\alpha}}\frac{e^{2\mu_m^{\frac{1}{\alpha}}T}}{\alpha^2}C_8 = \mu_m^{\frac{2}{\alpha}-1}\frac{e^{2\mu_m^{\frac{1}{\alpha}}T}}{\alpha^2}C_8. \tag{172}$$

Gathering this with (165) implies that

$$\frac{\mu_m T^{2\alpha-2}E_{\alpha,\alpha}^2(\mu_m T^\alpha)}{E_{\alpha,1}^2(\mu_m T^\alpha)} \leq \mu_m^{\frac{2}{\alpha}-1}\frac{e^{2\mu_m^{\frac{1}{\alpha}}T}}{\alpha^2}C_8\frac{\alpha^2}{C_9}e^{-2T\mu_m^{\frac{1}{\alpha}}} = \frac{C_8}{C_9}\mu_m^{\frac{2}{\alpha}-1}. \tag{173}$$

Gathering now the definition (159) of $\mathbb{M}_m$, (171), (173), the fact that $\alpha \in (1,2)$ and (10) yield

$$0<\mathbb{M}_m \leq \frac{C_{12}\mu_m^{-\frac{1}{\alpha}}}{2C_9}\frac{C_8}{C_9}\mu_m^{\frac{2}{\alpha}-1} = C_{13}\mu_m^{\frac{1}{\alpha}-1} \leq C_{13}\mu_1^{\frac{1}{\alpha}-1}. \tag{174}$$

This implies that $\mathbb{M}_m$ is bounded. This completes the proof of Lemma 3.1. □

Using (16) and that $\lim_{\gamma\searrow 0}\frac{1}{\Gamma(\gamma)}=0$, we can define $E_{\alpha,0}(z)=\sum_{m=1}^\infty z^m/\Gamma(\alpha m)$.

Lemma 3.2. *The following relations hold:*

$$E_{\alpha,1}^2(z) \geq E_{\alpha,0}(z)E_{\alpha,2}(z), \quad z \in (0,\infty), \tag{175}$$

$$\frac{d}{dz}\left(zE_{\alpha,2}(z^\alpha)\right) = E_{\alpha,1}\left(z^\alpha\right), \quad z \in (0,\infty), \tag{176}$$

$$\frac{d}{dz}\left(E_{\alpha,1}(z^\alpha)\right) = z^{\alpha-1}E_{\alpha,\alpha}\left(z^\alpha\right), \quad z \in (0,\infty). \tag{177}$$

Proof. In [17, equation (2.8)] it is remarked that

$$E_{\alpha,\gamma+1}^2(z) \geq E_{\alpha,\gamma}(z)E_{\alpha,\gamma+2}(z), \quad z \in (0,\infty), \tag{178}$$

given $\alpha > 0$ and $\gamma > 0$. Below we wish to show that this inequality also holds as $\gamma \searrow 0$ in the form of (175). Although (175) can be proved for any $\alpha > 0$, for the sake of simplicity of its proof, let us assume that $\alpha \in (1,2)$, which is also the case we are dealing with throughout the paper. According to the definition (16), we have

$$E_{\alpha,\gamma+1}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \gamma + 1)} = \sum_{m=0}^{\infty} \theta_m(\gamma), \tag{179}$$

where, for all $m \geq 1$,

$$\theta_m(\gamma) = \frac{z^m}{\Gamma(\alpha m + \gamma + 1)} \leq \frac{z^m}{\Gamma(\alpha m + 1)}.$$

Since the series $\sum_{m=0}^{\infty} \theta_m(0) = 1 + \sum_{m=1}^{\infty} \frac{z^m}{\Gamma(\alpha m+1)}$ is convergent, it follows that the series in (179) is uniformly convergent with respect to γ and $\lim_{\gamma\searrow 0} E_{\alpha,\gamma+1}(z) = E_{\alpha,1}(z)$. Similarly, we can obtain that $\lim_{\gamma\searrow 0} E_{\alpha,\gamma+2}(z) = E_{\alpha,2}(z)$. Again, according to (16), we have

$$E_{\alpha,\gamma}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \gamma)} = \frac{1}{\Gamma(\gamma)} + \sum_{m=1}^{\infty} \frac{z^m}{\Gamma(\alpha m + \gamma)}. \tag{180}$$

For $m \geq 1$, we have that $0 \leq \frac{z^m}{\Gamma(\alpha m+\gamma)} \leq \frac{z^m}{\Gamma(\alpha m)}$ and since the series $\sum_{m=1}^{\infty} \frac{z^m}{\Gamma(\alpha m)}$ is convergent it follows that the series $\sum_{m=1}^{\infty} \frac{z^m}{\Gamma(\alpha m+\gamma)}$ is uniformly convergent with respect to γ. Finally, using that $\lim_{\gamma\searrow 0} \frac{1}{\Gamma(\gamma)} = 0$ we obtain that $\lim_{\gamma\searrow 0} E_{\alpha,\gamma}(z) = E_{\alpha,0}(z)$. This concludes the proof of (175).

Formula (176) can be deduced from [8, equation (4.3.1)]. Using that $z^\alpha E_{\alpha,\alpha}(z^\alpha) = E_{\alpha,0}(z^\alpha)$, formula (177) can also be deduced from [8, equation (4.3.1)]. □

Theorem 3.1 (A new bound result)**.** *For all $m \in \mathbb{N}^*$,*

$$0 < \mathbb{S}_m < 1. \tag{181}$$

Proof. Let us set $z_T^\alpha := \mu_m T^\alpha$ such that $z \geq 0$ (and consequently $T = z_T \mu_m^{-\frac{1}{\alpha}}$) in (150) to get

$$\mathbb{S}_m = \frac{z_T^\alpha E_{\alpha,\alpha}\left(z_T^\alpha\right) E_{\alpha,2}\left(z_T^\alpha\right)}{\left(E_{\alpha,1}\left(z_T^\alpha\right)\right)^2} = \kappa_1(z_T)\kappa_2(z_T), \tag{182}$$

where $z \mapsto \kappa_1(z)$ and $z \mapsto \kappa_2(z)$ are two functions defined for any $z \geq 0$ by

$$\kappa_1(z) := \frac{zE_{\alpha,2}(z^\alpha)}{E_{\alpha,1}(z^\alpha)} \quad \text{and} \quad \kappa_2(z) := \frac{z^{\alpha-1}E_{\alpha,\alpha}(z^\alpha)}{E_{\alpha,1}(z^\alpha)}. \tag{183}$$

To justify that (181) holds, it suffices to prove that $0 \leq \kappa_1(z) < 1$ and $0 \leq \kappa_2(z) \leq 1$, for all $z \geq 0$ (see also Remark 3.2 which provides another path to justify (181) using only $0 \leq \kappa_1(z) \leq 1$ and $0 \leq \kappa_2(z) \leq 1$ for all $z \geq 0$ together with a property of Mittag-Leffler functions). The property that κ_1 and κ_2 are non-negative stems from the facts that the numerators and denominators involved in their definitions are non-negative for all $z \geq 0$ (according to the definition (16) of Mittag-Leffler functions $E_{\alpha,2}(z^\alpha)$, $E_{\alpha,1}(z^\alpha)$ and $E_{\alpha,\alpha}(z^\alpha) > 0$ for all $z \geq 0$).
(i) We prove that $\kappa_1(z) < 1$. Using [17, equation (2.9)] we have that $\kappa_1(z) \leq 1$. The strict inequality $\kappa_1(z) < 1$ is established in Appendix B.
(ii) We prove that $\kappa_2(z) \leq 1$. Let us consider the function (it is the difference between the numerator and denominator of $\kappa_2(z)$)

$$\chi(z) := z^{\alpha-1}E_{\alpha,\alpha}(z^\alpha) - E_{\alpha,1}(z^\alpha). \tag{184}$$

We show that $\chi(z) \leq 0$. To this end, we compute the derivative $\chi'(z)$ using [8, equations (4.2.3) and (4.3.1)] to obtain that for all $z > 0$,

$$\chi'(z) = z^{\alpha-2}E_{\alpha.\alpha-1}(z^\alpha) - z^{\alpha-1}E_{\alpha,\alpha}(z^\alpha) = z^{\alpha-2}E_{\alpha.\alpha-1}(z^\alpha)\left(1 - \frac{zE_{\alpha,\alpha}(z^\alpha)}{E_{\alpha.\alpha-1}(z^\alpha)}\right). \tag{185}$$

Using [17, Page 396], the function $z \mapsto \frac{zE_{\alpha,\alpha}(z^\alpha)}{E_{\alpha.\alpha-1}(z^\alpha)}$ is increasing and $\lim_{z\to\infty} \frac{zE_{\alpha,\alpha}(z^\alpha)}{E_{\alpha.\alpha-1}(z^\alpha)} = 1$. Consequently, $\chi' \geq 0$ and thus χ is increasing, and

$$\chi(z) \leq \lim_{z\to\infty}\left(z^{\alpha-1}E_{\alpha,\alpha}(z^\alpha) - E_{\alpha,1}(z^\alpha)\right). \tag{186}$$

Applying the asymptotic behaviour result of [26, Lemma 1.1] yields

$$\begin{aligned}\chi(z) = z^{\alpha-1}\left(\frac{1}{\alpha}(z^\alpha)^{\frac{1-\alpha}{\alpha}} e^{(z^\alpha)^{\frac{1}{\alpha}}} + O\left((z^\alpha)^{-1-1}\right)\right) - \frac{1}{\alpha}(z^\alpha)^{\frac{1-1}{\alpha}} e^{(z^\alpha)^{\frac{1}{\alpha}}} + O\left(z^{-\alpha}\right) \\ = O\left(z^{-\alpha-1} + O\left(z^{-\alpha}\right)\right), \quad \text{as} \quad z \to \infty.\end{aligned} \tag{187}$$

This implies that $\lim_{z\to\infty}\chi(z) = 0$. Gathering this with (186) leads to the desired inequality $\kappa_2(z) \leq 1$ (recall that $E_{\alpha,1}(z^\alpha) \geq 1 > 0$ for all $z \geq 0$). This completes the proof of Lemma 3.1. □

Remark 3.2 (Another possible proof for (181)). *In the proof of Lemma 3.1, we have used the inequalities* $0 \leq \kappa_1(z) < 1$ *(the strict inequality* $\kappa_1(z) < 1$ *is justified in Appendix B) and* $0 \leq \kappa_2(z) \leq 1$ *to justify that (181) holds through (182). However, it is possible to justify that (181) holds using only inequalities* $0 \leq \kappa_1(z) \leq 1$ *and* $0 \leq \kappa_2(z) \leq 1$ *(which have been shown in the proof of Theorem 3.1), i.e., without needing the strict inequality* $\kappa_1(z) < 1$ *for all* $z \geq 0$*. Indeed, the two inequalities* $0 \leq \kappa_1(z) \leq 1$ *and* $0 \leq \kappa_2(z) \leq 1$ *for all* $z \geq 0$ *imply that* $0 \leq \kappa_1(z)\kappa_2(z) \leq 1$*. Assume, by contradiction, that* $\kappa_1(z)\kappa_2(z) < 1$ *does not hold, i.e., for* *some* $z_0 > 0$ *(when* $z_0 = 0$ *we have* $\kappa_1(z_0)\kappa_2(z_0) = 0$*), we have* $\kappa_1(z_0)\kappa_2(z_0) = 1$*. This implies, since* $0 \leq \kappa_1(z_0) \leq 1$ *and* $0 \leq \kappa_2(z_0) \leq 1$*,*

$$\kappa_1(z_0) = \frac{z_0 E_{\alpha,2}\left(z_0^\alpha\right)}{E_{\alpha,1}\left(z_0^\alpha\right)} = 1 \quad \textit{and} \quad \kappa_2(z_0) = \frac{z_0^{\alpha-1} E_{\alpha,\alpha}\left(z_0^\alpha\right)}{E_{\alpha,1}\left(z_0^\alpha\right)} = 1. \tag{188}$$

The first identity in (188) imply that, using the fact that the function $\kappa_1(z)$ *is increasing (see (175) and (214)),*

$$\kappa_1(z_0) \leq \kappa_1(z) \leq 1, \quad \forall z \in [z_0, +\infty).$$

This implies that

$$\kappa_1(z) = 1, \quad \forall z \in [z_0, +\infty).$$

On the other hand, using the second inequality in (188) implies $\chi(z_0) = 0$*. As stated before,* χ *is increasing and consequently,*

$$0 = \chi(z_0) \leq \chi(z) \leq \lim_{z\to\infty} \chi(z) = 0, \quad \forall z \in [z_0, +\infty), \tag{189}$$

and hence

$$\chi(z) = 0, \quad \forall z \in [z_0, +\infty). \tag{190}$$

This gives, using (175), that

$$1 = \frac{z^{\alpha-1} E_{\alpha,\alpha}\left(z^\alpha\right)}{E_{\alpha,1}\left(z^\alpha\right)} = \frac{\left(E_{\alpha,1}\left(z^\alpha\right)\right)'}{E_{\alpha,1}\left(z^\alpha\right)}, \quad \forall z \in [z_0, +\infty) \tag{191}$$

Integrating (191) over $z \in (z_0, t)$ *for any* $t \geq z_0$ *leads to*

$$\ln\left(E_{\alpha,1}\left(t^\alpha\right)\right) = t + \ln\left(E_{\alpha,1}\left(z_0^\alpha\right)\right) - z_0, \quad \forall t \in [z_0, +\infty), \tag{192}$$

This gives the contradiction that the functions $E_{\alpha,1}\left(t^\alpha\right)$ *and* e^t *are identical up to a multiplicative constant. Therefore, we have that for all* $z \geq 0$*,* $\kappa_1(z)\kappa_2(z) < 1$*. This implies in particular by taking* $z := z_T$ *and using (182), that* $\mathbb{S}_m = \kappa_1(z_T)\kappa_2(z_T) < 1$*.*

Using the Mathematica software `https://www.wolfram.com/mathematica/`, the stated inequalities $0 \leq \kappa_1(z) < 1$ and $0 \leq \kappa_2(z) \leq 1$ in the previous proof of Theorem 3.1 are illustrated in Figure 1 for various values of α.

We are now able to state and prove the convergence of the alternating algorithm (116)–(118) towards the solution of the Cauchy problem (1)–(5).

Theorem 3.2 (The convergence of the alternating algorithm given by (116)–(118))**.** *Let* $\Psi^{(n)}$ *be the sequence given by (116)–(118). Assume that the solution of the Cauchy problem (1)–(5) with* $f \in H_0^1(\Omega)$ *and* $\mathrm{G} \in L^2(\Omega)$ *is satisfying* $u \in L^2\left(0, T; H_0^1(\Omega)\right)$*. Then, for any initial guess* $\eta \in L^2(\Omega)$ *provided in the last branch of (116), while assuming that* $\partial_t^{1-\beta} u(\cdot, T) \in L^2(\Omega)$*, the following convergence result holds:*

$$\|\nabla e^n\|_{L^2(0,T;H_0^1(\Omega))} \to 0, \quad \text{as} \quad n \to \infty, \tag{193}$$

where e^n is the accuracy error given by (119).

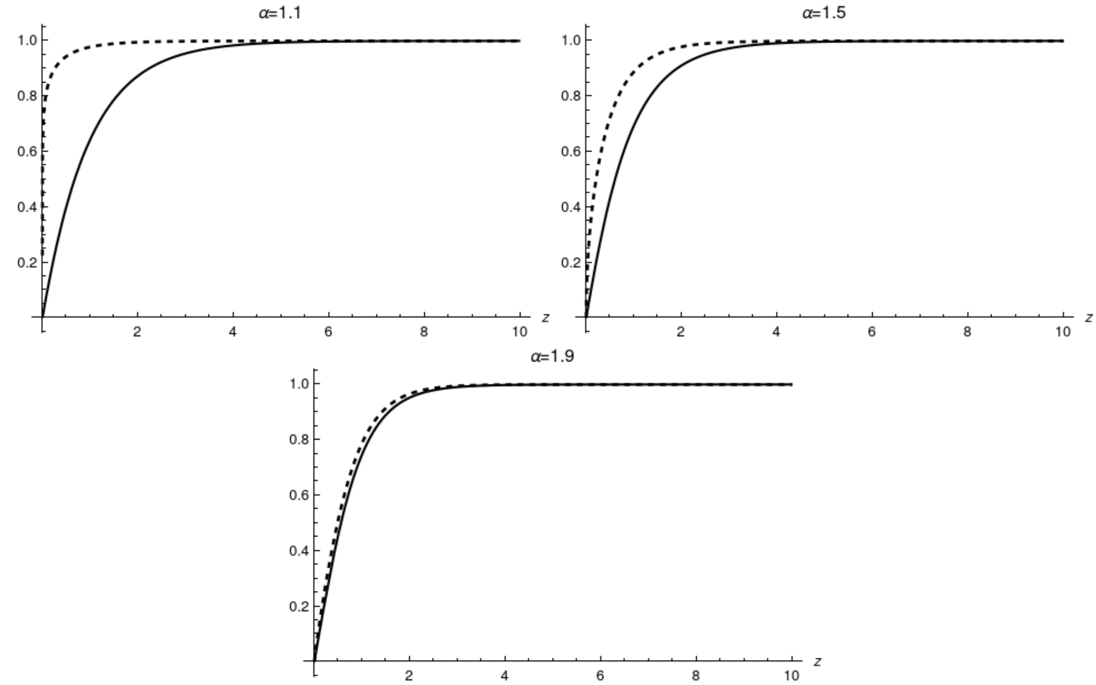

FIGURE 1. The functions $\kappa_1(z)$ shown by continuous line (——) and $\kappa_2(z)$ shown by dashed line $(- - -)$, as functions of $z \in [0,10]$, for various values of (a) $\alpha = 1.1$, (b) $\alpha = 1.5$ and (c) $\alpha = 1.9$.

Proof. We remark that the series (156) and (157) are uniformly convergent (since they satisfy the Weierstrass M-test); indeed, using Lemma 3.1 and Theorem 3.1,

$$\mu_m \frac{\mathbb{S}_m^{2n+2}\Lambda_m^2}{E_{\alpha,1}^2\left(\mu_m T^\alpha\right)} \int_0^T \left(t^{\alpha-1}E_{\alpha,\alpha}\left(\mu_m t^\alpha\right)\right)^2 dt = \mathbb{S}_m^{2n+2}\Lambda_m^2\mathbb{L}_m \leq \Lambda_m^2 \sup_{m\in\mathbb{N}^*} \mathbb{L}_m \tag{194}$$

and

$$\mu_m \frac{\mathbb{S}_m^{2n+2}\Lambda_m^2 T^{2\alpha-2}E_{\alpha,\alpha}^2\left(\mu_m T^\alpha\right)}{E_{\alpha,1}^4\left(\mu_m T^\alpha\right)} \int_0^T \left(E_{\alpha,1}\left(\mu_m t^\alpha\right)\right)^2 dt = \mathbb{S}_m^{2n+2}\Lambda_m^2\mathbb{M}_m \leq \Lambda_m^2 \sup_{m\in\mathbb{N}^*} \mathbb{M}_m, \tag{195}$$

where the series

$$\sum_{m\in\mathbb{N}^*} \Lambda_m^2 = \|\partial_t^{1-\beta}u(\cdot,T) - \eta\|_{L^2(\Omega)}^2 < \infty.$$

In addition to this, thanks to Theorem 3.1,

$$\lim_{n\to\infty} \mu_m \frac{\mathbb{S}_m^{2n+2}\Lambda_m^2}{E_{\alpha,1}^2\left(\mu_m T^\alpha\right)} \int_0^T \left(t^{\alpha-1}E_{\alpha,\alpha}\left(\mu_m t^\alpha\right)\right)^2 dt = 0$$

and

$$\lim_{n\to\infty} \mu_m \frac{\mathbb{S}_m^{2n+2}\Lambda_m^2 T^{2\alpha-2}E_{\alpha,\alpha}^2\left(\mu_m T^\alpha\right)}{E_{\alpha,1}^4\left(\mu_m T^\alpha\right)} \int_0^T \left(E_{\alpha,1}\left(\mu_m t^\alpha\right)\right)^2 dt = 0.$$

Taking into account the above facts, equations (156) and (157), and the Dominated Convergence Theorem, we deduce that

$$\lim_{n\to\infty} \|e^{2n+2}\|_{L^2(0,T;H_0^1(\Omega))}^2 = \sum_{m=1}^{\infty} \lim_{n\to\infty} \mu_m \frac{\mathbb{S}_m^{2n+2}\Lambda_m^2}{E_{\alpha,1}^2\left(\mu_m T^\alpha\right)} \int_0^T \left(t^{\alpha-1}E_{\alpha,\alpha}\left(\mu_m t^\alpha\right)\right)^2 dt = 0. \tag{196}$$

and similarly, we have

$$\lim_{n\to\infty} \|e^{2n+1}\|^2_{L^2(0,T;H_0^1(\Omega))} = 0. \tag{197}$$

This completes the proof of the theorem. $\square$

Remark 3.3 (Extension to the non-homogeneous case of equation (1)). *The alternating algorithm (116)–(118) for solving the problem (1)–(5) and its convergence analysis can be extended to the non-homogeneous case when there is a function $P(\boldsymbol{x}, t)$ in the r.h.s. (right-hand-side) of (1). In this case, the equations (116)–(118) are changed slightly only by adding the function P to the r.h.s. of their first branches and consequently the error (119) is satisfying (120)–(122) (the function P disappears when subtracting the first branches of (116)–(118) from (1) with r.h.s. P). The convergence analysis of Subsection 3.2 remains then valid. The proof of stability of the problems involved in the alternating algorithm (116)–(118) with r.h.s. P in the first branches can be justified using the techniques at the beginning of Section 3 and Remark 3.1.*

4. An illustration for the convergence of the algorithm in one dimension

Let us consider the one-dimensional ($d = 1$) domain $\Omega = (0, 1)$. Consider also the inhomogeneous form of equation (1) with right-hand side $P(x, t)$, with the boundary conditions (3)–(5) having the exact solution given by $u(x, t) = (1 + t^{2+\alpha})x(1 - x)$ for $(x, t) \in \Omega_T = \Omega \times (0, T) := (0, 1) \times (0, 1)$. The functions f and G are given, respectively, by $x(1 - x)$ and 0.

We calculate the errors (153) and (154) in the $L^2(L^2)$ and $L^2(H_0^1)$-norms (the $L^2(H_0^1)$-norms are obtained by taking the square-roots in (156) and (157), whereas the errors in the $L^2(L^2)$-norm are defined by the same expressions but omitting the term μ_m in the sums) by truncating the sums after a sufficiently large number of terms, such as at $m = 20$. The eigenfunctions φ_m and the eigenvalues μ_m are given, respectively, by $\varphi_m(x) = \sqrt{2}\sin(m\pi x)$ and $\mu_m = \pi^2 m^2$. We consider the alternating algorithm (as described in Remark 3.3) with $\eta = 0$. The term Λ_m is then given by $\partial_t^{1-\beta}\psi_m(t)|_{t=1} = \frac{\sqrt{2}\Gamma(3+\alpha)}{24}\int_0^1 x(1 - x)\sin(m\pi x)dx$. The results obtained using the Mathematica symbolic computation software for $\alpha = 1.5$ are presented in Table 1. From this table it can be seen that the norms of the errors are decreasing to zero, as n increases.

Table 1. The norms of truncated errors e_{20}^{2n+2} and e_{20}^{2n+1} for $\alpha = 1.5$.

n	The norms of the error e_{20}^{2n+2}		The norms of the error e_{20}^{2n+1}	
	$\|e_{20}^{2n+2}\|_{L^2(L^2)}$	$\|e_{20}^{2n+2}\|_{L^2(H_0^1)}$	$\|e_{20}^{2n+1}\|_{L^2(L^2)}$	$\|e_{20}^{2n+1}\|_{L^2(H_0^1)}$
100	3.3355×10^{-2}	1.0487×10^{-1}	3.3395×10^{-2}	1.0499×10^{-1}
500	4.8607×10^{-3}	1.5842×10^{-2}	4.8665×10^{-3}	1.5859×10^{-2}
1000	6.3783×10^{-4}	4.6695×10^{-3}	6.3819×10^{-4}	4.6700×10^{-3}

The above computational simulation has illustrated the convergence of the alternating algorithm in the simplest case of $d = 1$. Nevertheless, in case Ω is a bounded irregular domain in $\mathbb{R}^d$ with $d > 1$, the direct mixed problems (116)–(118) would have to be discretised and solved numerically.

5. Conclusions and perspectives

In this paper, an ill-posed Cauchy problem associated to a fractional elliptic equation with fractional order derivative boundary conditions has been investigated. In the classical integer case of $\alpha = 2$ resulting in the Cauchy problem for the Laplace equation $u_{tt} + \Delta u = 0$, we suggested an alternating regularising algorithm using the ideas of of [14], which was developed for the Cauchy problem associated to the elliptic Laplace's equation. This algorithm contains two types of problems whose weak formulations have been developed under some conditions on the data. These weak formulations allowed to derive stability of the solutions of the algorithm in $L^2(H_0^1)$. The convergence of the sequence of solutions of the algorithm towards to the solution of the ill-posed Cauchy problem was rigorously proved. Furthermore, we have extended the alternating algorithm to the general fractional case $\alpha \in (1, 2)$ (our main problem). Such algorithm involves two types of problems whose stabilities were established using two different methods. The first method relies on weak formulations of the problems involved in the iterative algorithm and subsequently the stability was obtained through the coercivity of the bilinear forms and the continuity of the linear forms involved. The second method was based on the use of method of variation of variables. Subsequently, we developed several new inequalities to control the solutions in some convenient spaces. These inequalities are based essentially on the asymptotic behaviour of Mittag-Leffler functions. To the best of our knowledge, this is the first time to use this approach which involves the existing asymptotic behaviours of Mittag-Leffler functions together with the method of separation of variables in order to control (to bound) solutions of fractional partial differential equations. The stability results proved for these two types of problems generalize the ones obtained for the limit case of $\alpha = 2$; when we replace $\alpha = 2$ in these stability results we get the ones obtained in our analysis in the case of the Cauchy problem associated to the Laplace equation in $\mathbb{R}^{d+1}$.

The results obtained in this article can be extended to the problem (1)–(5) when the Laplace operator Δ involved in (1) is replaced by the symmetric uniformly elliptic operator

$$\mathcal{L}u(\boldsymbol{x}, t) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(\boldsymbol{x})\frac{\partial u}{\partial x_j}(\boldsymbol{x}, t)\right), \quad \boldsymbol{x} = (x_1, x_2, \ldots, x_d) \in \Omega, \;\; t \in (0, T),$$

with time-independent coefficients $a_{ij} \in \mathcal{C}^1(\overline{\Omega})$, $a_{ij} = a_{ji}$ and, for some positive constant ν,

$$\nu \sum_{i=1}^{d} \xi_i^2 \leq \sum_{i,j=1}^{d} a_{ij}(\boldsymbol{x})\xi_i\xi_j, \quad \forall \boldsymbol{x} \in \overline{\Omega}, \;\; \forall \xi := (\xi_1, \xi_2 \ldots, \xi_d) \in \mathbb{R}^d.$$

One of the main perspectives of this work is to derive a convergence rate (believed to be of logarithmic type) and to extend these results to the general fractional partial differential equation $-D\left(\partial_t^{1-\beta}\right)u + Au = 0$ with A a self-adjoint positive-definite unbounded operator. Another future work would be to follow the approach of [21] to first define a weak formulation for the problem (74)–(77), as in [21, Definition 2.1, Page 428], and subsequently, derive regularity results of the solution u under some assumptions on the data f and Q, as in [21, Theorems 2.3 and 2.4].

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Appendix A

Using the expansion (6) of u together with (115), [10, Page 141] and inequality $(a+b)^2 \leq 2(a^2+b^2)$, we get

$$\begin{aligned} \|\nabla u\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \|\nabla u(\cdot,t)\|^2_{L^2(\Omega)} dt = \int_0^T \left(\sum_{m=1}^{\infty} \mu_m |\psi_m(t)|^2 \right) dt \\ &\leq 2 \sum_{m=1}^{\infty} \mu_m \left(\int_0^T \mathbb{T}_1(t) dt \right) |f_m|^2 + 2 \sum_{m=1}^{\infty} \mathbb{L}_m |Q_m|^2, \end{aligned} \tag{198}$$

where

$$\mathbb{T}_1(t) := \left(E_{\alpha,1}(\mu_m t^\alpha) - \frac{t^{\alpha-1} E_{\alpha,\alpha}(\mu_m t^\alpha) \mu_m T E_{\alpha,2}(\mu_m T^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} \right)^2 \tag{199}$$

and $\mathbb{L}_m$ is given by (158). Using Lemma 3.1, $\mathbb{L}_m$ is bounded and consequently the second term on the right hand side of (198) can be bounded as, thanks to (166),

$$2\sum_{m=1}^{\infty}\mathbb{L}_m|Q_m|^2 \leq \frac{C_8}{C_9}\mu_1^{\frac{1}{\alpha}-1}\|Q\|^2_{L^2(\Omega)}. \tag{200}$$

We now estimate the first term on the right hand side of (198). To this, we bound $\mathbb{T}_1$ as, using inequality $(a+b)^2 \leq 2(a^2+b^2)$,

$$\mathbb{T}_1 \leq 2\left(\mathbb{T}_2 + \mathbb{T}_3\right), \tag{201}$$

where, for $z^\alpha := \mu_m t^\alpha$,

$$\mathbb{T}_2 = \left(E_{\alpha,1}(z^\alpha) - z^{\alpha-1}E_{\alpha,\alpha}(z^\alpha)\right)^2 \tag{202}$$

and

$$\mathbb{T}_3 = \left(\left(\mu_m^{-\frac{1}{\alpha}}E_{\alpha,1}(\mu_m T^\alpha) - TE_{\alpha,2}(\mu_m T^\alpha)\right)\frac{\mu_m t^{\alpha-1}E_{\alpha,\alpha}(z)}{E_{\alpha,1}(\mu_m T^\alpha)}\right)^2. \tag{203}$$

This implies that

$$\sum_{m=1}^{\infty}\mu_m\left(\int_0^T \mathbb{T}_1(t)dt\right)|f_m|^2 \leq 2\sum_{m=1}^{\infty}\mu_m\left(\int_0^T \mathbb{T}_2 dt\right)|f_m|^2 + 2\sum_{m=1}^{\infty}\mu_m\left(\int_0^T \mathbb{T}_3 dt\right)|f_m|^2. \tag{204}$$

We begin to estimate the second term on the right-hand side of (204). Applying twice the asymptotic behaviour result of [26, Lemma 1.1] we get

$$\mu_m^{-\frac{1}{\alpha}}E_{\alpha,1}(\mu_m T^\alpha) - TE_{\alpha,2}(\mu_m T^\alpha) = \mathcal{O}\left(\mu_m^{-\frac{1}{\alpha}-1}\right).$$

This gives

$$\mu_m\left(\mu_m^{-\frac{1}{\alpha}}E_{\alpha,1}(\mu_m T^\alpha) - TE_{\alpha,2}(\mu_m T^\alpha)\right) = \mathcal{O}\left(\mu_m^{-\frac{1}{\alpha}}\right). \tag{205}$$

Gathering this with (203), the definition (158) of $\mathbb{L}_m$, and Lemma 3.1 imply

$$\begin{aligned}\sum_{m=1}^{\infty}\mu_m\left(\int_0^T \mathbb{T}_3 dt\right)|f_m|^2 &= \sum_{m=1}^{\infty}\left(\mu_m\left(\mu_m^{-\frac{1}{\alpha}}E_{\alpha,1}(\mu_m T^\alpha) - TE_{\alpha,2}(\mu_m T^\alpha)\right)\right)^2\mathbb{L}_m|f_m|^2 \\ &\leq C_{14}\|f\|^2_{L^2(\Omega)}. \end{aligned} \tag{206}$$

Let us now estimate the first term on the right-hand side of (204). To this end, we use again the asymptotic behaviour result of [26, Lemma 1.1] to get

$$\begin{aligned}E_{\alpha,1}(z^\alpha) - z^{\alpha-1}E_{\alpha,\alpha}(z^\alpha) &\sim \frac{1}{\alpha}e^{(z^\alpha)^{\frac{1}{\alpha}}} + \mathcal{O}\left(z^{-\alpha}\right) - z^{\alpha-1}\left(\frac{1}{\alpha}\left(z^\alpha\right)^{\frac{1-\alpha}{\alpha}}e^{(z^\alpha)^{\frac{1}{\alpha}}} + \mathcal{O}\left(z^{-2\alpha}\right)\right) \\ &= \mathcal{O}\left(z^{-\alpha}\right), \quad \text{as} \quad z \to +\infty. \end{aligned} \tag{207}$$

This implies in particular that, since $\left(E_{\alpha,1}(z^\alpha) - z^{\alpha-1}E_{\alpha,\alpha}(z^\alpha)\right)/z^{-\alpha}$ is continuous away from 0 (and consequently is bounded on any interval $\left[\mu_1^{\frac{1}{\alpha}}T, a\right]$ with $a > \mu_1^{\frac{1}{\alpha}}T$), there exists a positive constant C_{15} such that, for all $z \geq \mu_1^{\frac{1}{\alpha}}T$,

$$\left|\frac{E_{\alpha,1}(z^\alpha) - z^{\alpha-1}E_{\alpha,\alpha}(z^\alpha)}{z^{-\alpha}}\right| \leq C_{15}. \tag{208}$$

Let us write, using the change of variable $z^\alpha := \mu_m t^\alpha$ with $z \geq 0$ and (208) (recall that $1 - 2\alpha < 0$),

$$\begin{aligned}\int_0^T \mathbb{T}_2(z)dt = \mu_m^{-\frac{1}{\alpha}} \int_0^{\mu_m^{\frac{1}{\alpha}}T} \mathbb{T}_2(z)dz &= \mu_m^{-\frac{1}{\alpha}} \left(\int_0^{\mu_1^{\frac{1}{\alpha}}T} \mathbb{T}_2(z)dz + \int_{\mu_1^{\frac{1}{\alpha}}T}^{\mu_m^{\frac{1}{\alpha}}T} \mathbb{T}_2(z)dz \right) \\ &\leq \mu_m^{-\frac{1}{\alpha}} \left(\int_0^{\mu_1^{\frac{1}{\alpha}}T} \mathbb{T}_2(z)dz + C_{15}^2 \int_{\mu_1^{\frac{1}{\alpha}}T}^{\mu_m^{\frac{1}{\alpha}}T} z^{-2\alpha}dz \right) \\ &\leq C_{16}\mu_m^{-\frac{1}{\alpha}} \left(\int_0^{\mu_1^{\frac{1}{\alpha}}T} \mathbb{T}_2(z)dz + \left(\mu_1^{\frac{1}{\alpha}}T\right)^{1-2\alpha} - \left(\mu_m^{\frac{1}{\alpha}}T\right)^{1-2\alpha} \right) \\ &\leq C_{16}\mu_m^{-\frac{1}{\alpha}} \left(\int_0^{\mu_1^{\frac{1}{\alpha}}T} \mathbb{T}_2(z)dz + \left(\mu_1^{\frac{1}{\alpha}}T\right)^{1-2\alpha} \right).\end{aligned}$$

This implies that, using the fact $\mathbb{T}_2(z)$ is continuous (and consequently, $\int_0^{\mu_1^{\frac{1}{\alpha}}T} \mathbb{T}_2(z)dz$ is bounded above by a constant which does not depend on m),

$$\int_0^T \mathbb{T}_2(z)dt \leq C_{17}\mu_m^{-\frac{1}{\alpha}}. \tag{209}$$

This yields the following bound on the first term of the right-hand side of (204), since $1-\frac{1}{\alpha} \leq \frac{1}{2}$:

$$\sum_{m=1}^{\infty} \mu_m \left(\int_0^T \mathbb{T}_2(t)dt \right) |f_m|^2 \leq C_{17} \sum_{m=1}^{\infty} \mu_m^{1-\frac{1}{\alpha}} |f_m|^2 \leq C_{17} \sum_{m=1}^{\infty} \mu_m^{\frac{1}{2}} |f_m|^2 \leq C_{17} \|f\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)}, \tag{210}$$

where, the space $\dot{H}^s(\Omega)$ (see [10, Page 141] and [21, 428]) is defined by, for $s > 0$,

$$\dot{H}^s(\Omega) = \{v \in L^2(\Omega) : \quad \sum_{m=1}^{\infty} \mu_m^s |\left(v, \varphi_m\right)_{L^2(\Omega)}|^2 < +\infty\}$$

equipped with the norm

$$\|v\|^2_{\dot{H}^s(\Omega)} = \sum_{m=1}^{\infty} \mu_m^s |\left(v, \varphi_m\right)_{L^2(\Omega)}|^2.$$

In the particular case when $s = 1$, we have $\dot{H}^1(\Omega) = H_0^1(\Omega)$ (see [10, Page 141]). It is useful to state that $\dot{H}^1(\Omega) = H_0^1(\Omega) \subset \dot{H}^{\frac{1}{2}}(\Omega)$ since, for $v \in H_0^1(\Omega)$,

$$\sum_{m=1}^{\infty} \mu_m^{\frac{1}{2}} |\left(v, \varphi_m\right)_{L^2(\Omega)}|^2 = \sum_{m=1}^{\infty} \frac{\mu_m}{\mu_m^{\frac{1}{2}}} |\left(v, \varphi_m\right)_{L^2(\Omega)}|^2 \leq \frac{1}{\sqrt{\mu_1}} \sum_{m=1}^{\infty} \mu_m |\left(v, \varphi_m\right)_{L^2(\Omega)}|^2 < +\infty. \tag{211}$$

Gathering now (198), (200), (204), (206) and (210) yields

$$\|\nabla u\|_{L^2(0,T;L^2(\Omega))} \leq C \left(\|f\|_{L^2(\Omega)} + \|f\|_{\dot{H}^{\frac{1}{2}}(\Omega)} + \|Q\|_{L^2(\Omega)} \right). \tag{212}$$

A first interesting remark is that the regularity assumption on f, which is needed in (212), is reduced to $f \in \dot{H}^{\frac{1}{2}}(\Omega)$ (instead of the assumption $f \in H_0^1(\Omega)$ needed in (109)). A second useful remark is that using (211) and under the assumption that $f \in H_0^1(\Omega)$, the stability result (212) together with the Poincaré inequality (that is, $\|v\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)}$ for any

$v \in H_0^1(\Omega)$) yield the $L^2(H_0^1)$–estimate involved in (109), which was previously proved in Section 3 using the weak formulation (104).

Appendix B

From [17, equation (2.9)] we only have that

$$\kappa_1(z) := \frac{zE_{\alpha,2}\left(z^\alpha\right)}{E_{\alpha,1}\left(z^\alpha\right)} \leq 1, \quad z \in [0,\infty). \tag{213}$$

The aim of this appendix is to explain that the inequality (213) is strict, that is, $\kappa_1(z) < 1$ for all $z \geq 0$ (which was used in the proof of Theorem 3.1). Indeed, using (175)–(177), the numerator of the derivative $\kappa_1'(z)$ (the denominator of $\kappa_1'(z)$ is positive) satisfies

$$\xi(z^\alpha) = E_{\alpha,1}^2\left(z^\alpha\right) - z^\alpha E_{\alpha,\alpha}\left(z^\alpha\right)E_{\alpha,2}\left(z^\alpha\right) = E_{\alpha,1}^2\left(z^\alpha\right) - E_{\alpha,0}\left(z^\alpha\right)E_{\alpha,2}\left(z^\alpha\right) \geq 0. \tag{214}$$

The function $z \mapsto \xi(z)$ is strictly positive on $(0,+\infty)$ since, thanks (214) and [17, equation (2.1)] with $\beta_2 := \alpha$ and $\beta_1 := 2-\alpha$ (note that the hypothesis $\beta_2 > \beta_1 > 0$ of [17, Theorem 1] is satisfied since $\alpha > 1$), we have

$$\begin{aligned}
\xi(z) = E_{\alpha,1}^2\left(z\right) - E_{\alpha,\alpha}\left(z\right) z \sum_{j=0}^{\infty} \frac{z^j}{\Gamma\left(\alpha j+2\right)} &= E_{\alpha,1}^2\left(z\right) - E_{\alpha,\beta_2}\left(z\right) \sum_{j=0}^{\infty} \frac{z^{j+1}}{\Gamma\left(\alpha j+2\right)} \\
&= E_{\alpha,(\beta_1+\beta_2)/2}^2\left(z\right) - E_{\alpha,\beta_2}\left(z\right) \sum_{j=1}^{\infty} \frac{z^j}{\Gamma\left(\alpha j+2-\alpha\right)} \\
&= E_{\alpha,(\beta_1+\beta_2)/2}^2\left(z\right) - E_{\alpha,\beta_2}\left(z\right) \left(E_{\alpha,\beta_1}(z) - \frac{1}{\Gamma\left(2-\alpha\right)}\right) \\
&= E_{\alpha,(\beta_1+\beta_2)/2}^2\left(z\right) - E_{\alpha,\beta_2}\left(z\right) E_{\alpha,\beta_1}(z) + \frac{E_{\alpha,\beta_2}(z)}{\Gamma\left(2-\alpha\right)} > 0.
\end{aligned}$$

Consequently, $z \mapsto \kappa_1(z)$ is strictly increasing on $(0,+\infty)$. In addition to this, using, for instance, the asymptotic behaviour results [17, equation (2.7)] or [26, Lemma 1.1], we have that $\lim_{z\to+\infty} \kappa_1(z) = 1$. Therefore, $\kappa_1(z) < 1$ for all $z > 0$. This together with the fact that $\kappa_1(0) = 0$ yield $\kappa_1(z) < 1$ for all $z \geq 0$.