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The Ace Paradox: the dynamics of expert pilot replacement and the demise of the Luftwaffe 1940–45

Niall J. MacKay[1], Paul E. Strong[2] and A. Jamie Wood[1,3]

Abstract

Paradoxically, the Second World War air forces with by far the highest-scoring "aces" lost the attritional war. The Luftwaffe had a slowly-declining group of aces replenished by poorly trained novices who rarely survived to become experts themselves. Allied aces, in contrast, had much lower kill tallies, but were routinely rotated into training roles, thereby improving the quality of newly deployed novices. We begin with the historical background, and then assess whether aces are "born" or "made". Having established the latter, we resolve the paradox with a dynamical model in which the development of novices into experts is facilitated by the latter's transfer into training roles. The model has a single globally stable equilibrium, with two distinct regimes separated by a bifurcation. In the "ace" regime, with low transfer of experts into training, the equilibrium has no experts and a rapid turnover of novices. In the "learning" regime, a higher transfer rate leads to a stable mixed cohort consisting mostly of experts, with vastly greater fighting strength. We discuss the implications, including for the training of drone pilots. The general conclusion is that combat experts should be transferred into training at a substantial fraction of their loss rate.

Keywords

combat modelling; air warfare; training; drones

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1. Introduction

Germany's *Luftwaffe* started the Second World War as perhaps the best air force in the world.[1–3] The Spanish Civil War had given it valuable combat experience, and it began the war with a cadre of expert pilots, many of whom went on to become enormously high-scoring "aces" with over 100 "kills". The Allies, by contrast, had few or no such high scorers. Their pilots typically were not allowed to remain in front-line combat for long enough to become so, and it was common for expert combat pilots to be transferred to training units, which rarely occurred in the *Luftwaffe*. Yet, in this attritional battle in which the goal was to shoot down as many enemy aircraft as possible, the Allies won: the *Luftwaffe* was destroyed by the United States Army Air Forces [USAAF] over Germany in 1944-45 in a manner which the *Luftwaffe* had notably failed to achieve in its battle with Britain's Royal Air Force [RAF] in 1940. Quantity and quality of aircraft offer a partial explanation, yet there remains a historical paradox to be resolved. Is it not self-evident that removing experts from combat, transferring them into training units, and then even letting them depart completely, is wasteful of expertise, and will reduce overall attritional success? Such expertise is hard-won, appears useful principally for its intended purpose of shooting down enemy aircraft – and yet was discarded.

The first question which must be answered is whether aces are "born" or "made". If they are "born" – what has sometimes been termed "survival of the fittest"[4] – then the purpose of training is to discover and select latent talent. If, in contrast, aces are at least to some extent "made" then even the most talented pilot has to ascend a learning curve in combat, and combat is a skill which a much wider group are capable of learning, if they can only survive their early combat encounters. In this case training and mentoring are crucial, and require close involvement from those who are already expert. It has been generally accepted at least since the Vietnam War that aces are at least to some extent "made", but a classic early study assumed that they are all "born" and proceeded on that basis. We challenge this thesis using its own data, as part of a broader review of the published literature on this question.

The centrepiece of our article is a model which offers a resolution of the "ace paradox". The point of using experts for training is of course to enable the creation of more experts. Yet no trainees, fresh from an advanced training unit, are immediately combat experts, even if they have been trained by experts. They have to become so. On doing so *quickly* both their lives and the combat success of their units depend. If having been trained by up-to-date combat experts enables combat novices to become experts just a little more quickly, then, tensioned against the lower combat life expectancy of novices than experts, this can be a crucial advantage. We construct a dynamical system which captures and quantifies this advantage using well established techniques.[5] The intention is not to produce a complex model precisely fitted to data. Rather it is, in the language of Epstein's classic essay, to "illuminate [the] core dynamics" of the process.[6]

But the results that emerge from the model are much stronger, and do much more than merely embody a mechanistic observation. The model has a transcritical bifurcation, a kind of "tipping point", between two regions of parameter space with significantly different outcomes (and which we refer to as "regimes"), which occurs at a particular value of the rate of transfer of experts into training roles. Such bifurcations are a feature of nonlinear systems, where they manifest themselves as an abrupt change of behaviour which is not intuitively predictable by simple arguments. Thus, rather than attempt to predict *quantitative* outcomes and generate a potentially misleading numerical fit, this simple model captures the crucial *qualitative* feature of the problem: the contrasting behaviour between the two distinct regimes separated by the bifurcation. Below the critical value in our model, in what we call the "ace" regime, the cadre of experts slowly dies out, and all that remains is constant in- and out-flow of novices – exactly as occurred in Germany and Japan in the later years of the war. Above the critical value, in the "learning" regime, there is a stable equilibrium mix of experts and novices which admits an optimal value at which most pilots are expert. Since (both intuitively and in an appropriate combat model) the fighting strength is a linear combination of experts and novices, in which experts are much more valuable than novices, "learning" is revealed to be vastly more effective than "ace". Thus the paradox is resolved: enabling more novices to survive and develop into combat experts, albeit with modest "kill" tallies, creates a fighting force which is more effective than one built around preserving and privileging very high-scoring aces. The practical lesson is a simple one: there should be planned, systematic re-deployment of combat experts into training roles at a higher level than one might intuitively expect.

The article is structured as follows. In Section 2 we give some historical background, setting out the context of the well-known paradox. In Section 3 we perform some simple statistical modelling to address the question of whether aces are "born" or "made", reviewing the published literature and testing and challenging the assumption that aces are simply "born". Having established that combat pilots must be "made" by their training and early combat experience, and that there is a historical paradox to be explained, we then do so by linking the two. In Section 4 we present the main dynamical model and its rationale. Section 5 conducts an analysis of the model and presents results, including a sequence of phase portraits of the model for increasing values of the transfer rate, followed by an analysis of natural variations of the model and then an overall summary. We conclude in Section 6 with comments on the historical and modern context, followed by a statement of the central practical lesson from our model, that hard-won combat expertise should be redeployed to training at a substantial fraction of its loss rate.

2. Historical Background

With its combat experience and extensive pre-war preparation, the *Luftwaffe* began the Second World War as one of the best-*trained* air forces in the world.[3] In particular, its focus on offensive counter-air operations and close air support led to its being very good at gaining temporary air supremacy over the battlefield and supporting ground forces. However, its defensive counter-air skills were initially limited and there was never a suitable longer-term plan to develop them. Above all "the *Luftwaffe* was slow to recognize that it was in an attritional air war and to implement the measures needed to wage one successfully"; indeed, "up to the summer of 1942, the training programme had run on a peacetime leisurely basis".[7] Thereafter, the programme was rapidly expanded, but "the rise from 1,662 new fighter pilots in 1942 to 3,276 in 1943 was barely enough ... to cover wastage at the front (2,870). ... Thus, there was virtually no opportunity to build up a pilot reserve".[2] The result, and one of the most important failures of the *Luftwaffe*, was a decline in pilot quality through the war. By 1944 the majority of *Luftwaffe* fighter pilots had less than a month's combat experience. Finally "it was the German pilot deficiencies much more than the aircraft technical deficiencies which gave the Allies such complete air domination".[8]

Worse, the quality of training became very poor. "German pilots had half the [total] training hours of their Allied counterparts. More costly was the fact that a German pilot received 60 to 80 hours of training in operational aircraft, while his opponent in the RAF or [US] Army Air Forces averaged 225".[2] Indeed many pilots could barely fly their aircraft, still less land them safely, with very high levels of non-combat losses.[9] Finally, even that limited training would often be compromised. Hans Jeschonnek, *Luftwaffe* Chief of Staff, would often commit training units in contingencies,[7] and was not a believer in training:

“Even with a plea from subordinates to increase training, Jeschonnek’s response was, ‘Let’s beat the Russians first, then we can start training’”.[10] Urbanke[11] gives a good sense of how far the *Luftwaffe*’s pilot training program had fallen by the last year of the war, and this was immediately understood by its opponents: “Instead of training a few pilots thoroughly, the Germans chose to produce a greater number of poorer pilots, [who would] gain experience in battle under the tutelage of veterans”[12] – but rarely survived to do so.

The *Luftwaffe*’s training problem existed within a broader “ace” culture.[13–17] The German military since the 18th century had developed a concept of pervasive individualism that venerated the ace in the hope that these individuals might set a heroic example to others. At its best, this culture empowered young soldiers to seek opportunities that might enable battlefield success, and underlay the German concept of *Auftragstaktik* (mission command).[18,19] At its worst, it could cause military personnel to put their own glory before the success of the wider group. This pattern was repeated in both the *Panzerwaffe* (armored force) and the *Ubootwaffe* (submarine force) where the few aces that survived attrition added to their medal collection while the rest were predated by adversaries who were eventually better trained and collaborated more effectively. In the *Ubootwaffe*, aces even used their less experienced peers to lure away escort vessels so they could conduct their own attacks.[20] There was a similar issue with a pseudo-heroic culture, closely linked to qualitative decline, in Japan’s military, although they were far less generous with medals.[21]

One resulting problem was that officers preferred to stay in the frontline instead of sharing their skills with others. As a result, the *Luftwaffe*, *Panzerwaffe* and *Ubootwaffe* all ended the war with a tiny number of remarkable individuals with very high scores while many of the rest were barely competent. The *Luftwaffe* had over 100 pilots with over 100 air victories – and no other air force had even one ace at this level, with the highest-scorers from the USA and UK almost all in the 20s. But, by 1945, many of the *Luftwaffe* aces’ colleagues “faced great difficulty in landing their aircraft, much less surviving combat”.[2]

Even if the *Luftwaffe* wanted pilots to share the latest innovations in tactics, a culture that encouraged pernicious individualism meant that the officers themselves actively avoided such pedestrian assignments – instructors were known as *Etappenschweine*, “rear-line pigs”, pilots not fit for the front line or whose motivation was questionable. There is also evidence to suggest that privileging status over sharing key skills can undermine the survival of those who are driven to aspire to equal the aces whom the propaganda machine highlights as exemplars.[22] The basic training courses covered the core flying skills, although according to Günther Rall this barely covered instrumentation,[23] and operational squadrons were assumed to complete the training process.

The western allies did things differently.[24–26] Pilots in both the USAAF and RAF were rotated out of the front line after fixed tours of duty. These were originally calibrated in Britain during the First World War to offer pilots an even chance of survival, and had settled at 200 hours’ operational flying in RAF Fighter Command in 1940.[50] As a result, in the Second World War, Allied aces had a much higher survival rate than those of the *Luftwaffe*.[28] Sometimes they moved into training, sometimes to “fly a desk”, and sometimes then volunteering for a return to combat (for a classic career see Wellum[29]). The RAF rapidly learned the importance of training for combat,[30] and created “Operational Training Units” precisely to bridge the gap from training to fighting. In the USAAF, at the beginning of the war, of course, almost no instructors had combat experience, and the problem persisted into 1943. But determined efforts were made to change this, and by 1945 “over 90% of the instructors were combat veterans”.[25]

Of course the Allies enjoyed multiple advantages, especially of resources. But there remains the paradox that in allowing their aces to shoot down enemies in their hundreds Germany and Japan somehow "squandered their most experienced pilots".[9] The same volume laments unsuccessful air forces' "failing to do the math beforehand".

3. Are aces born or made?

We now move on to some simple statistical modelling of the title question of this section, which is best seen as two opposed, simplistically extreme conceptions: that aces are

born: every pilot has a fixed, constant (over all encounters) probability of being killed in any decisive encounter, but each pilot's probability is different,
or
made: all pilots have the same probability of being killed in the nth decisive encounter, which declines with n.

As we shall see, the "**born**" hypothesis is not supported by the data, which instead point to expert pilots needing to be "**made**", whatever their innate talent. However, the two simple hypotheses are clearly straw men, and most opinion would agree that the truth is a mixture of the two. The most comprehensive study is that of Youngling et al.,[31] prominent within which is a survey of 373 experienced combat pilots themselves. The pilots were told "Evidence ... indicates that a combat fighter pilot's chances of survival are dramatically improved if he survives his first five air-to-air encounters", and asked "Is this a reflection of A: The pilot's learning through experience, B: A 'Survival of the Fittest' phenomenon; the poorer pilots are shot down". For A the proportion "Strongly agree[ing]" or "Agree[ing]" was 86%. For B it was 53%, with 32% "Disagree[ing]". Thus about half of pilot opinion was that both effects are important, with a large minority favouring learning exclusively. That aces are simply **born** then appears to be untenable, with its implication that a born ace, fresh from training, has no further learning to do in combat, and never improves further.

The only technical analysis in the published literature originates with Weiss[4] and its influence lingers on.[32,33] Later studies[34–36] note that Weiss begged the question: he assumed, on the basis merely that **made** "seemed less likely ... than ... survival of the fittest", that the latter was correct, and proceeded on that basis. We can test his hypothesis using his data, which was derived from a mix of First and Second World War sources. To do so we had to infer numerical values (Table 1) from Weiss's aggregate data as represented (without detail or error bars) in Figure 1 of Bilikam & Frick,[34] where they are given as probability of pilot loss p_n in the nth decisive encounter, conditional on having survived the $n-1$th (so that, in the terminology of survival analysis, this is a hazard function rather than an event density). A "decisive encounter" means that the pilot either made a kill or was himself shot down.

[Table 1]

Weiss implements **born** by assigning to each pilot an innate probability q of surviving each encounter, with no learning taking place. Pilots differ, and their loss probability $p = 1 - q$ is distributed according to a density function $f(p)$. Weiss uses the moments of the distribution to infer the shape of $f(p)$, but his results appear inconsistent with the data above. He claims that "at best fewer than 15% of pilots had a better-than-even chance of surviving [*i.e.* not being shot down in] their first combat", which gives a maximum first-encounter survival probability of 57.5% (as $0.15 \times 1 + 0.85 \times 0.5 = 0.575$, if 15% are certain to survive their first combat, while 85% have an even chance), which is still less than the 62% of the data (as $0.62 = 1 - 0.38$, with $\langle p \rangle_1 = 0.38$ from Table 1).

To investigate this further, write as $\langle p \rangle_n$ the average probability *over surviving pilots* of being killed in the nth encounter, given survival at the $n-1$th. Even though each individual **born** pilot's probability is constant, $\langle p \rangle_n$ declines with n, as weaker pilots – those with higher p – are successively removed from the distribution.

We model **born** pilots' abilities as a Beta distribution

$$f(p) = B(p; \alpha, \beta) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha, \beta)} \tag{1}$$

(where $B(\alpha, \beta)$ is the Beta function, and is simply the integral from 0 to 1 of the numerator). Beta acts as the repository of accumulating information (in Bayesian updating, the "conjugate") for binomial probability distributions, and through its two parameters α, β covers a large class of smoothly-varying possibilities for $f(p)$, including the uniform distribution when $\alpha = \beta = 1$ and Jeffreys (uninformative) prior when $\alpha = \beta = \frac{1}{2}$.

It then follows that

$$\begin{aligned} \langle p \rangle_n &= \frac{\int_0^1 p(1-p)^{n-1} p^{\alpha-1}(1-p)^{\beta-1}\, dp}{\int_0^1 (1-p)^{n-1} p^{\alpha-1}(1-p)^{\beta-1}\, dp} \\ &= \frac{B(\alpha+1, \beta+n-1)}{B(\alpha, \beta+n-1)} \\ &= \frac{\Gamma(\alpha+1)\Gamma(\beta+n-1)}{\Gamma(\alpha+\beta+n)} \frac{\Gamma(\alpha+\beta+n-1)}{\Gamma(a)\Gamma(\beta+n-1)} \\ &= \frac{\alpha}{\alpha+\beta+n-1}, \end{aligned} \tag{2}$$

where Γ is the gamma function and B is the beta function. The optimal fit is shown in Figure 1, and is evidently poor. Since Beta is the distribution family of choice for probabilistic data of this type, this is strong evidence against the **born** model.

[Figure 1]

In fact the data points decay approximately exponentially, but with a slightly accelerating rate of decline. The best two-parameter fit we have found is a logistic curve, shown in Figure 2,

$$\langle p \rangle_n = \frac{1}{1 + \frac{1-p}{p} a^{n-1}}, \tag{3}$$

parametrized by p and a. In this model, and writing $\langle q \rangle_n = 1 - \langle p \rangle_n$, the survival odds ratio $\frac{\langle q \rangle_n}{\langle p \rangle_n}$ increases by a constant factor of approximately 1.4 with each successive encounter. If n is replaced by continuous time t, then $q(t) := \langle q \rangle_t$ obeys the Verhulst (logistic) equation $\frac{dq}{dt} = rq(1-q)$.

This a natural model for **made**, in which all pilots have the same increasing survival probability q_n at the nth encounter (so that $q_n = \langle q \rangle_n$, and its complement $p_n = 1 - q_n = \langle p \rangle_n$, with parameter $p = p_1$). The rate of proportionate improvement in q_n is then proportional to the loss probability,

$$\frac{q_{n+1} - q_n}{q_n} \propto p_n. \tag{4}$$

Learning thus takes place only in encounters with potentially deadly foes. Just as gamers and sportsmen learn from playing accomplished adversaries, so pilots continue to learn principally from meeting enemies of comparable skill.

[Figure 2]

Of course **born** versus **made** may be a false dichotomy. Frick,[37] drawing on Weiss's data, constructs a model which is a hybrid of constant-hazard **born** and exponentially-declining-hazard **made**, although this is conceived differently: the **born** pilots in his model are weaker; they "have a higher initial hazard rate and do not learn from experience", in contrast to **made** pilots who "have a lower initial hazard rate and learn from experience".

In our approach we can easily create a **made/born** hybrid. However, when one fits a three-parameter hybrid combination

$$\langle p \rangle_n = \frac{\alpha + \beta}{\alpha + \beta a^{n-1}} \frac{\alpha}{\alpha + \beta + n - 1}, \tag{5}$$

which combines the logistic **made** model with a Beta-based **born** model, it is striking that the best fit is achieved when α and β are of order 10^7, effectively a point estimate of the second fraction in the product. That is, almost all the fit is achieved by the first fraction, and almost none by the second – the model is almost entirely of **made**, not **born**. The data thus point to the logistic learning model of experts **made**, with no element of the simplistic model in which pilots arrive in combat **born** (as combat pilots) knowing everything they will ever know.

Certainly one of the greats of air fighting was quite content to come down firmly on one side: "Great pilots are made not born ... the end product is only fashioned by steady coaching, much practice, and experience".[38] Combat experts require advanced, expert-led, challenging training: "a strong consensus does exist ... that air combat training is closely linked to success or failure".[35] In the jet era[39] such training is exemplified by the USN's "Top Gun" school and USAF "Red Flag" exercises,[40] and its essence is the "deliberate practice of difficult and challenging matters".[32]

4. A Dynamical Model for the Ace Paradox

With all of the above in hand, we now move on to the core conceptual development of this article, which is to create a deterministic dynamical model for the development of combat experts. There appears to be no comparable attempt in the literature to capture the dynamics of pilot replacement in this way.

Our model divides pilots in front-line squadrons (only) into two groups, of combat "novices" and combat "experts" (see Figure 3, explained below). This is of course a simplification of what in reality would be dynamics on a continuous spectrum of ability, which would require a much more sophisticated model (a partial differential equation, with time and expertise as the independent variables). Our model's essence is that exposure of novices to experts *in training* is the primary means of development of novices into experts *in combat*. Thus all pilots arrive in combat as novices, and then have to ascend a steep learning curve, with the rapidity (and possibility) of ascent determined solely by the extent of involvement of experts in their prior training. It is thus a model of **made**, in which learning is the only source of new experts and is solely due to experts removed from combat, but it is robust to a range of variations which relax these assumptions and are described in Section 5.3 – specifically, that

– some experts (after a tour in training) may then return to combat,
– some of the effect of experts happens as mentoring in combat rather than in training,
– there is some spontaneous emergence of experts (without expert help),
– some pilots arrive in combat as (**born**) experts, or
– expertise decays with time.

Other variations are also possible, such as time-varying rates, but these would only be relevant if they caused the model behaviour to straddle the bifurcation, and there is no historical evidence of the complex dynamics this would produce.

4.1. Model

We posit two groups of combat pilots, a cadre $E(t)$ of experienced "experts" (some of whom become aces but who are above all survivors), and a group of $N(t)$ "novices". These two groupings can be considered as a division of the total stock of available combat-ready pilots. The independent variable is "time", but this can also be read as "sorties" (number of combat missions flown) if intensity varies; the time variable can be nonlinearly rescaled. The evolution is then

$$\frac{dN}{dt} = -\nu N + r - \beta \frac{E}{E+K} N, \tag{6}$$

$$\frac{dE}{dt} = -\epsilon E - \gamma E + \beta \frac{E}{E+K} N. \tag{7}$$

The Greek parameters are pure rates, measured in units of inverse time, and correspond to constant probability of occurrence per unit time: ν of a novice being shot down, ϵ of an expert being shot down (with $\nu > \epsilon$), γ of experts being transferred out of combat and into training units (thus enabling experts to contribute to training), and β the maximal rate at which novices become experts. Also included in ν and ϵ (and contributing greatly to $\nu \gg \epsilon$ for the late-war Luftwaffe) are non-combat losses. The other parameters are r, the number of novice pilots leaving training per unit time and entering combat squadrons, and a saturation K with the dimensions of number of pilots, about which we say more below. The parameters are shown in Table 2.

[Table 2]

4.2. Rationale

The rationale for the model begins with the observation that the simplest way to distinguish experts from novices – before we begin to assess their offensive capability – is that the latter have a higher probability of being shot down. At its simplest this is the assumption that $\nu > \epsilon$. Thus, for example, $\nu = 1$ per month would be to say that a novice pilot has a life expectancy t distributed as $\nu e^{-\nu t}$ with mean ν^{-1} or one month, and a $1/e \simeq 37\%$ chance of surviving a month. In fact towards the end of the war "the average life expectancy of a JG 300 pilot for the last year of the war amounted to just 11 hours of flying – barely four sorties ... including check and ferry flights",[41] which translates to an astonishing $\nu \simeq 10$ per month. A more typical value of ν would be much lower.

If aces were **born** rather than **made** then this is where we would stop. The distribution $f(p)$ of innate ability would simply collapse into two bins, experts E and (perpetual) novices N, with no transfer between the bins (and thus no coupling of the differential equations), but rather simply inflow from training schools and constant loss rates $\nu > \epsilon$.

Notice that we do not make any assertion about an underlying combat model beyond specifying that the loss rates experienced by each group of pilots on one side are in proportion to the number in that group. In particular there is no intrinsic need to posit an underlying Lanchester-type model for two-sided attritional combat or its scaling (in the form of a "linear law" or "square law"). However, we note that if the other side experiences attrition in the same way then losses are proportional to *both* sides' numbers, and we have the Lanchester "unaimed fire" model – a sum of duels, each of which is the outcome of a hunt. The evidence is that, other than some asymmetry between the attacker and defender of surface targets, this is a good approximation to attritional air combat.[42–44] The correct model is then an unaimed-fire "Lanchester $(2, 2)$ model", a heterogeneous model with two types of unit (experts and novices) on each side.[45] Since we have not been able to find it in the literature, we give in an Appendix a derivation of the conserved quantity for this model. This is a Lanchester linear law, in which each side's fighting strength is a linear combination of expert and novice numbers, with the experts being weighted by their greater longevity and greater kill rate.

The core of our model is the transfer rate from N to E and the way in which this is enabled by E. Of course not all experts are good teachers, and not all survivors become aces. Since we are not looking here at the expert group's effectiveness against their opponents, the crucial feature is survival, and the capacity of some proportion of experts to pass on the ability to survive. Our decision parameter is therefore the rate gamma, γ, the proportion of experts per unit time who are redeployed from front-line squadrons into training units. We assume that after a fixed period of time acting as trainers, experts are then lost from the model, "flying a desk" or moving to a safer role, although some may choose to return to combat (see Sect.5.3), perhaps as commanders. This is consistent with the basic conception of dangerous deployments within liberal-democratic states, that after an appropriate tour (or number of tours) of duty, the combat expert is entitled to some reduced measure of danger (although it may also be that for many their combat expertise, and their mental health, eventually decline – again see Sect.5.3). It is also the essence of the paradox – surely it must be wasteful, considered on military grounds alone, to create expert combat pilots and then simply allow them to leave the system?

The crux is then the rate of conversion of combat novices into experts, which is dependent on the novices' prior training by experts. We assume that there is a maximum rate β at which well-trained novices, fresh from their training, become expert after their arrival in combat. Sub-optimal training then reduces this through a simple saturating form controlled by a saturation level K. In this approach, if the availability of experts in training units is high, additional experts bring very little additional value, and the rate approaches its maximum value β. However, if very few experts are deployed into training then the conversion rate increases linearly as their number increases. A crossover between these effects occurs around $E \simeq K$. An important sub-case, the spontaneous conversion of novices into experts without need of expert help, is the limit $K \to 0$. The crossover parameter K, which for small E varies as the reciprocal of the expert training effect, is taken to be proportional to the number of trainees per instructor, which is itself proportional to the output rate of trainees divided by the transfer rate of expert instructors. Thus $K = rc/\gamma$ for some dimensionless parameter c, whose meaning is explained below.

Whilst more complex functional forms are possible, this is the classic method in mathematical ecology of capturing the essential dynamics,[46] and is structurally stable, in that the same results follow from any functional form for the conversion which (i) is zero for $\gamma = 0$, (ii) increases linearly in E for small E, (iii) approaches a maximum value for large E. More complex forms would allow us to capture a steeper approach to the maximum, but we have no evidence to support this, and such forms merely complicate the algebra without providing further insight. Similarly, more complex forms are possible for the dependence of K on the other parameters but, in the absence of real-world justification for greater complexity, simple proportionality is both natural and tractable.

One could augment the model with further bins, but this is needless to capture the core dynamics. For example, in the same manner as in simple epidemiological models we could include the actual number $C(t)$ of "cadet" trainees, with a fixed probability of a cadet graduating per unit time, and the number of expert "instructors" $I(t)$, who exit the role with constant probability. This gives two further differential equations,

$$\frac{dC}{dt} = r - \alpha C, \qquad \frac{dI}{dt} = \gamma E - dI, \tag{8}$$

which are linear, while the original two equations become

$$\frac{dN}{dt} = -\nu N + \alpha C - \beta \frac{I}{I + kC} N, \qquad \frac{dE}{dt} = -\epsilon E - \gamma E + \beta \frac{I}{I + kC} N, \tag{9}$$

so that crossover happens when the ratio of instructors to cadets exceeds a certain level, k.

Assuming a separation of timescales between combat and training, $\frac{dC}{dt} = 0$, $\frac{dI}{dt} = 0$ is then established with $\alpha C = r$ and $\gamma E = dI$. The Holling term is then

$$\frac{I}{I + kC} = \frac{\frac{\gamma E}{d}}{\frac{\gamma E}{d} + k\frac{r}{\alpha}} = \frac{E}{E + \frac{krd}{\gamma\alpha}}, \tag{10}$$

which is equal to the original Holling term $\frac{E}{E+K}$ if $K = \frac{krd}{\gamma\alpha}$. This matches our earlier form $K = rc/\gamma$ if $c = kd/\alpha$. That is, c is the product of the crossover ratio k of instructors to cadets (at any one time) and the ratio of cadets' to instructors' residence times in training units. Most simply put, $1/c$ is the total number of cadets graduated per instructor during the latter's time as a trainer.

The equations then revert to their original form (6,7). Thus the additional C and I bins are unnecessary to capture the dominant dynamics: only the two groups of combat pilots, E and N, are essential provided there is a separation of timescales between training and combat. Even if the timescales were to be comparable, because the equations for cadets and instructors are linear their presence would at most cause delay dynamics (in the manner of the linear chain trick[47]). The dynamics of the conversion of novice to expert due to prior training is where the essence of the process is to be found. The model scheme is depicted in Figure 3.

[Figure 3]

5. Analysis and Results

The goal of the analysis is to work out the relationship between the negative and positive effects of expert-transfer γ on the cohort of experts: negative because of the transfer $-\gamma E$ out of combat, positive because of its effect on the rate of novice-to-expert conversion. More concisely stated, the aim is to determine how variation of the decision parameter γ affects the outcome of the model.

As we shall see, there are two distinct regimes, which we call "ace", for low values of γ, and "learning", for high values, according to the use made of expert pilots. These two regimes have distinctly different qualitative features, and are separated by a transcritical bifurcation, a common feature in the theory of dynamical systems.

5.1. Fixed-points and stability

First we observe that when $E = 0$ it remains so, for then $\frac{dE}{dt} = 0$, and $\frac{dN}{dt} = r - \nu N$. In contrast when $N = 0$ we have $\frac{dN}{dt} = r$, so N is increasing, and $\frac{dE}{dt} = -(\gamma + \epsilon)E$, so E is initially decreasing.

Now we calculate the nullclines, the curves on which precisely one of E or N does not change. These are, respectively, that $\frac{dN}{dt} = 0$ on

$$N = \frac{r}{\nu}\left(1 - \frac{\beta E}{(\beta + \nu)E + \nu rc/\gamma}\right) \tag{11}$$

and $\frac{dE}{dt} = 0$ on $E = 0$ and on

$$N = \frac{\epsilon + \gamma}{\beta}\left(E + \frac{rc}{\gamma}\right). \tag{12}$$

When both $\frac{dN}{dt} = 0$ and $\frac{dE}{dt} = 0$ we have a fixed point, a constant solution or equilibrium, and there are two of these:

FP1:

$$E = 0, \quad N = \frac{r}{\nu},$$

FP2:

$$E = \frac{r}{\beta + \nu}\left(\frac{\beta}{\epsilon + \gamma} - \frac{c\nu}{\gamma}\right), \quad N = \frac{r}{\beta + \nu}\left(1 + c + \frac{\epsilon c}{\gamma}\right).$$

There are then two distinct regimes, for values of γ either side of a critical value

$$\gamma_c = \frac{\nu\epsilon c}{\beta - \nu c}. \tag{13}$$

When $\gamma < \gamma_c$, FP1 is stable and FP2, which has $E < 0$, is unstable. When $\gamma > \gamma_c$, stability is reversed: FP1 is unstable and FP2, which now has $E > 0$, is stable. The stability of the fixed points is determined by the eigenvalues of the local stability (Jacobian) matrix. Linear stability analysis (using Mathematica[48]) confirms that for realistic values of $\gamma > \gamma_c$, FP2 is a stable node. (There is a range of larger γ in which it becomes a stable focus, and beyond which it reverts to a stable node, but as this does not affect model outcome or conclusions we do not give details.) Thus the fixed points pass through each other and exchange stability as γ passes through the critical value γ_c, in a transcritical bifurcation (see Figure 4).

[Figure 4]

To restate, we then have either the

Ace regime ($\gamma < \gamma_c$): FP2 is inaccessible (since $E < 0$ there) and unstable, FP1 is stable, and all trajectories approach the final values $E = 0$ and $N = r/\nu$. Expert pilots are all eventually shot down, and all that remains is the constant replenishment r of novices from training schools, who are then shot down at rate ν, giving a constant number $N = r/\nu$ of combat pilots who do not survive to become experts;

or the

Learning regime ($\gamma > \gamma_c$): FP1 still exists, but is unstable. Rather the stable fixed point is FP2, at which there are positive numbers of both experts and novices, and all trajectories eventually approach this point. Further, within the **learning** regime there is an optimal value of γ, a value

$$\gamma^* = \frac{\epsilon}{\sqrt{\frac{\beta}{\nu c}} - 1} = \left(1 + \sqrt{\frac{\beta}{\nu c}}\right)\gamma_c \tag{14}$$

at which the value of E at FP2 is maximized. At this $\gamma = \gamma^*$ we write

$$E^* = \frac{r\nu}{\epsilon(\beta + \nu)}\left(\sqrt{\frac{\beta}{\nu}} - \sqrt{c}\right)^2, \tag{15}$$

$$N^* = \frac{r}{\beta + \nu}\left(1 + \sqrt{\frac{\beta c}{\nu}}\right). \tag{16}$$

Notice that the existence of the **learning** regime requires that $\beta > \nu c$, so that it is possible for well-trained novices to become experts faster than they are lost.

5.2. Sample phase diagrams

The model and its analysis are best illustrated by some sample plots. These are “phase portraits”, plots of E against N, in which the direction of flow is shown by a vector field. To aid interpretation we also plot sample trajectories (thick lines, as detailed below), and the nullclines (thinner solid lines), the curves on which the flow is, as can be seen in the plots, either horizontal or vertical.

As a concrete illustrative example we take numbers loosely corresponding to the Royal Air Force in the battle of Britain, with $r = 300$ novices per month. Recall that any exponential decay rate d gives rise to an exponential distribution with expected (average) time $1/d$. Thus $\nu = 1$ per month corresponds to an expected novice survival time of one month, and a $1/e \simeq 37\%$ chance of surviving this long while still inexpert. Next, $\epsilon = 0.25$ per month corresponds to an expected expert survival time of four months, and a $e^{-0.25} \simeq 78\%$ chance of surviving one month. Finally, the maximum rate at which combat novices become expert is $\beta = 2$ per month, so that that around $1 - e^{-0.5}$ $\simeq$40% of novices become expert within a week, while only $e^{-2} \simeq 14\%$ of surviving novices remain so after a month. We take $c = 0.1$, so that the utility of experts in training falls off when each expert trainer graduates a total of ten trainees, – for example, through each expert training four cohorts at a trainee-to-instructor ratio of 2.5. For this model $\gamma_c \simeq 0.013$, and $\gamma^* \simeq 0.07$. Numbers are plotted on axes with $0 \leq E \leq 800, 0 \leq N \leq 800$, so that E and N are shown to approximately the same scale. We provide plots for five increasing values of γ, each with qualitative differences. To aid visualization of results, we plot two sample trajectories, beginning at $E = 600$ and $N = 0$ (in blue) and $E = 600, N = 600$ (in green). In each case nullclines are shown as thin red lines, while the field of red arrows indicates the direction of the flow $(\dot{N}, \dot{E})$. In all of Figures 5–9 the global stability of the equilibrium is evident: the flow from *any* point in the positive quadrant eventually approaches the sole stable equilibrium.

Figure 5 shows the model in the **ace** regime at $\gamma = 0.003$, with almost no use made of experts in developing novices, as may have been typical of the late-WW2 Luftwaffe. The dynamics of experts and of novices are almost independent: experts die out, and the system rapidly reaches the equilibrium in which incoming novices do not survive long enough to become experts.

[Figure 5]

Figure 6 shows the model still in the **ace** regime at $\gamma = 0.01$, a modest, but non-zero, rate of novice-to-expert conversion such as might occur in the combat squadrons themselves with minimal active support of the process. Here $\gamma < \gamma_c$, and the stable fixed point is at $E = 0$ and $N = 300$. The same starting points now see expert numbers initially varying slowly until novice numbers stabilise at the nullcline, whose non-equilibrium N is not very different from the final $N = 300$. After this, however, E enters a gradual decline until, eventually, E approaches zero. The two-stage, two-speed process which is becoming evident here occurs because $\nu/\epsilon \gg 1$. Notice how the starting values of $E = N = 600$ have created a deceptive initial stage (before the inevitable decline) in which E is stable, even increasing slightly.

[Figure 6]

Figure 7 moves from this poor outcome into the **learning** regime with $\gamma = 0.02$. The trajectories initially behave in a way broadly similar to the ace-regime Figure 6. While they are evolving it would not be clear that there is going to be a crucially different outcome: instead of approaching $E = 0$, the trajectories will approach a stable, positive-E fixed point, here of $E = 241, N = 235$.

In particular, in both Figures 6 and 7 we see the same initial increase in E from the starting values $E = N = 600$. In such circumstances it might appear better to avoid redeploying experts into training entirely, minimizing γ and $-\frac{dE}{dt} \simeq (\gamma + \epsilon)E$ and thereby preserving combat expertise. But such an argument could easily shift γ from above to below the critical γ_c, with catastrophic consequences.

[Figure 7]

Figure 8 shows the **learning** regime with γ not only greater than the critical value but approximately optimal, with $\gamma = 0.07$. The stable fixed point is around $E = 482, N = 145$.

[Figure 8]

Figure 9 remains in the **learning** regime but with needlessly high γ. The fixed point is sub-optimal, and we can now see that the number of experts in the trajectory from $E = 600, N = 0$ declines needlessly. Note that in all of Figures 3–7 the *global* stability of the equilibrium is evident: the flow from *any* point in the positive quadrant eventually approaches the single equilibrium.

[Figure 9]

Ours is what physicists call an "effective" model: its purpose is to ignore irrelevant detail, and to use the remaining relevant variables to discriminate between qualitatively distinct outcomes, quantifying only approximately. In particular it would be inappropriate to try to fit a single trajectory to Luftwaffe data,[49] which is strongly seasonal (see for example Ager,[22] Supp.Mats, Figure A.2), and subject to changing underlying conditions (such as the 1941-42 successes against an inferior opponent in the Russian campaign[2]). Nevertheless it is clear that the *Luftwaffe*'s 1943-45 story is typical of Figures 5 and 6: their numbers of expert pilots never stabilised, and they were undoubtedly in the "ace" regime.

5.3. Model variations

The model is sufficiently general that a number of variations of the assumptions are possible without altering the dynamics. For example, some proportion of experts may later choose to return to combat. It is also possible that some part of the positive effect of experts on novices takes place as mentoring in combat, rather than through redeployment into training. Both of these important effects are equivalent to variations in ϵ, γ and K without any change in the model and its analysis.

5.3.1. Spontaneous emergence of experts. A first variation on the model itself might be to allow, alongside the benefits of expert tuition, some additional spontaneous conversion of novices into experts without expert help. This can be effected by adding a small constant term ηN to $\frac{dE}{dt}$ (and subtracting it from $\frac{dN}{dt}$). The effect on the analysis is that the trivial $\frac{dE}{dt} = 0$ nullcline on $E = 0$ is lost, while the non-trivial $\frac{dE}{dt} = 0$ nullcline is no longer a line but becomes slightly curved, passing through the origin. The transcritical bifurcation is lost and there is always a positive, stable fixed point. However, it falls far short of optimality. What has happened is that even with $\gamma = 0$ there is now still some small number of experts due to the endogenous conversion of novices. For example, if we take the model with the parameter values of Section 5.2 at $\gamma = 0$ and add a autonomous conversion rate $\eta = 0.1$, the fixed point is $E = 109, N = 273$.

5.3.2. Some aces are born. A further variation would be to include some **born** experts, splitting the output from training schools into $r(1-x)$ novices and rx experts, for some small $x \in (0,1)$. In the absence of any **made** effect (that is, with $\gamma = \beta = 0$) this leads to a stable equilibrium

$$E = \frac{rx}{\epsilon}, \quad N = \frac{r(1-x)}{\nu}. \tag{17}$$

Reinstating γ, β results in a model whose analysis is algebraically complex but which essentially reproduces that of the previous paragraph: the $\frac{dE}{dt} = 0$ nullcline is curved, passing through the E-axis at $N = 0$, $E = rx/\epsilon$; the transcritical bifurcation is lost; and there is always a stable, positive fixed point, with a value E greater than the $E = \frac{rx}{\epsilon}$ which is the inevitable result of there existing **born** experts. Again, however, this value is greatly enhanced by the **made** effect. In the numerical example of section 5.2, if we allow 10% of pilots to be **born** experts so that $x = 0.1$, then there will always be at least $rx/\epsilon = 120$ experts. At the approximately optimal $\gamma = 0.07$, the maximal number of experts (which is $E = 482$ at $x = 0$, see Figure 8) is enhanced to around $E \simeq 530$.

Notice that the experience of the Luftwaffe as described in Section 2 does not support the hypotheses that experts are to any great extent **born** or emerge spontaneously. Since no force has been sufficiently foolish to assume this subsequently, we are limited to this early evidence.

5.3.3. Decay of expertise. Another potentially important missing feature is the decay of expertise. This might occur because tactics are evolving and equipment improving, or due to "burn-out", the limited capacity of experts to continue to endure and survive combat while maintaining effectiveness (as was observed by the RAF in the battle of Britain[50]). This applies as much among pilots of unmanned as of manned vehicles.[51]

At its simplest this effect would manifest as a higher rate ϵ of expert attrition. It is unsurprising that as ϵ increases so does γ_c, (13), and the system is in the **ace** regime for a wider range of parameter values. But it is striking that the optimal γ^*, (14), also increases with ϵ, although the resulting equilibrium value $E = E^*$ (15) at γ^* declines. Thus, paradoxically, the higher the rate of expert loss, the more experts should be diverted into training. Even though we are seemingly reducing the pool of combat experts through this choice, it maximizes the stable number of experts. Failure to do so runs the risk of losing the expert cadre altogether via a transition to the **ace** regime.

5.4. Summary

The optimal deployment of hard-won combat expertise is a subtle problem for which verbal arguments are insufficient but which has not hitherto been properly tackled using the mathematical techniques.

We began with a well known historical paradox: that, during World War Two, Germany had more and higher-scoring aces by far than the Allied powers – and yet lost the air campaign. This paradox can be explored by looking at the progression of *Luftwaffe* performance through the war and by examining the testimony of their adversaries. Most German aces emerged early in the war and swiftly dominated the scoring rate of their units. *Jagdgeschwader* 26 was one of the highest scoring wings in the battle of Britain; according to their war diary, four of the 150 pilots that took part in the battle claimed 31% of the unit's kills, Adolf Galland alone claiming 14%[52]. The focus on the aces' kill rates meant that other flying skills, even those of aces, were not prioritized.[53] Analyses and memoirs of the campaign deplore the weakness of *Luftwaffe* training, which resulted in novices having a very short combat life expectancy (including unsustainable losses from accidents). This is the "ace paradox", in which an air force has a cadre of high-scoring "aces" yet has much lower sustainable fighting strength, because the expert cohort does not regenerate itself.

But any argument that this training should have been more than minimal fails if fighting ability is an innate skill, if aces are "born" rather than "made". This claim, however, is straightforwardly falsified. If we then accept the consensus view that aces are to a large extent "made", and that involvement from combat experts in training, mentoring and development is valuable, how can this be modelled, and what are the implications?

We introduced a robust dynamical-systems model of the ace paradox in which redeployment of experts into training increases the rate at which combat novices, after deployment in the front line, themselves become experts. In our model there is an important emergent feature: a transcritical bifurcation separates two regimes, each with a distinctive globally-stable equilibrium. The "ace" regime, which appears to best match the historical experience of the *Luftwaffe*, lays bare the essence of the paradox. It has no experts at its equilibrium, even though it may still deploy a small proportion of available experts into training. Rather there is a rapid turnover of novices, and a slower decline of the number of remaining experts until they are eliminated. In the "learning" regime, by contrast, above-threshold redeployment of experts into training, who are then lost from the front line, paradoxically enables a stable equilibrium mix of experts and novices. Further, there is an optimal redeployment rate, much higher than the bifurcation threshold, which produces an equilibrium at which there are vastly more experts than novices, thus creating a mixed cohort of hugely greater fighting strength than one based on "aces". If we depart from the base model, allowing variations in which some experts are "born" or develop spontaneously, this forces the existence of a small stable number of experts, and the bifurcation necessarily disappears. However, the optimum remains much higher when experts are used for training than when not.

6. Conclusions

The challenges of the ace paradox are relevant to all operations that require expert skills which combine a high degree of technical mastery and appreciation of a complex environment – the "situational awareness" that can only be developed with training and experience. For most pilots who are to some extent natural potential aces, their skills are not magical: as in music or sport, alongside the few geniuses there are many with moderate talent which cannot develop without hard work and, crucially, expert tuition. These pilots' skills must be learned and internalized rapidly, to prevent a fresh combat pilot from soon being added to an enemy ace's tally.

6.1. Historical warfare

The contrast between "ace" and "learning" approaches was seen in other branches during World War Two. Submarine warfare is a particular example, where the idea extends to progress in tactics. We have argued elsewhere[17] that the Germans were hindered in the 1939–45 battle of the Atlantic by their "ace" tactics, while on the Allied side the development and inculcation of tactical skills was the central purpose of the Western Approaches Tactical Unit, which "enabled the Royal Navy to gain a better understanding of the threat they were facing, facilitated the development of counters to German tactics and technology, revealed weaknesses in their adversary's approach to the campaign, and disseminated what we would now call 'best practice' to every Allied commander involved in the battle".[54]

In technological warfare of all kinds, it is usually both cheaper and (crucially, in war) quicker and more feasible to replace the weapon than to replace the expert who wields it. A classic example is the 1940 battle of Britain,[55] in which the Royal Air Force's saturated constraint was the availability of pilots, not of airframes – it is a cliché that for any pilot shot down and recovered there was immediately a new warplane available. Thus the basic position of the democratic powers is supported by the model: that the lives of those who serve are of great value, far greater even than that of the expensive weapons they control; and that everyone who places their lives in danger is entitled to expect that this has a time limit, a horizon beyond which they can expect to return to a life relatively free from danger. Far from being in tension with the demands of war, such principles are, we have shown, consistent with the aim of developing the stable, regenerating body of expertise necessary to employ such weapons.

6.2. Modern warfare

In the present day, many weapons systems are expensive and slow to produce, and the same is true of the expertise to use them. Among the western powers, numbers have tended to be small, with an emphasis on quality – and for many years this worked. With recent wars indicating a return to numbers and the primacy of attrition, planning now needs to include the capacity to scale-up not only equipment production but also replenishment of the reservoir of expertise. Combat expertise should always be considered a valuable resource, to be exploited and disseminated as widely as possible, and then carefully retained and husbanded.

This starts at the bottom, with the infantryman, and has always done so: recall mediaeval requirements for training with the English warbow, a weapon which was revolutionary only when combined with a large body of intensively trained experts, created by widespread, long-term tuition and practice.[56] In modern war there may well be more such weapons, cheap and quick to produce but where the comparative advantage is to be found in a force's capacity to use them expertly.[44] If such weapons are slower and more expensive to produce, their improved retention through the expertise of their wielders becomes even more vital.

Unmanned Combat Aerial Vehicle (UCAV, "drone") pilots, in particular, have to master skills which include situational awareness in a constantly evolving battle environment. Lighter UCAV variants must operate from forward positions, and their pilots are then vulnerable as ground targets (even though the UCAV itself can be shot down without losing a pilot). Although prior expert training may not greatly enhance combat novices' survival, if it speeds their fighting development then (with $\epsilon \simeq \nu$, since experts and novices are likely to be equally vulnerable on the ground) our model describes their circumstances, and the two distinct regimes still occur. Of course it may be that experts can offer training online while remaining in combat – but they can only do so if they survive, and preserving a body of expertise by judiciously rotating experts out of the front line will typically be worthwhile.

Ukraine is an important case. "Sky Guards" units, using Wild Hornets FPV Interceptors, are facing drones equipped with UCAV pursuit detectors and evasion algorithms.[57] To stay ahead of their adversaries, Ukrainian drone pilots must constantly adapt and evolve – future requirements may include the capacity to work effectively with Artificial Intelligence co-pilots, itself likely to be an expert skill. Training is a crucial part of Ukraine's drone warfare operations,[58] and our model provides support for more active and systematized alternation of combat tours and training deployments than appears to exist at present.[59]

The 2026 Iran war amplifies such lessons.[60] As NATO powers build their drone capacity, it is worth quoting one of these lessons at length: "Another key factor is personnel. Ukraine's counter-drone effectiveness depends on large numbers of trained operators organized into specialized units responsible for detecting, tracking, and engaging aerial threats. .... This requires dedicated training pipelines, standardized tactics, and constant exercises that simulate large-scale drone attacks. For the United States, building a counter-drone force will require not just technology procurement but also training programs and unit structures designed specifically for drone defense missions." We would add that the cycle of rotation into training of experts will need to be at the core of such structures, not only because of the lessons of our model but also to enable immediate response of the training programme to rapidly changing technology and conditions, just as the RAF and USAAF achieved during the Second World War.

6.3. Optimal implementation

Experts' combat knowledge should be used for two things: in combat, and in training and mentoring. The goal is to achieve maximum fighting power through the optimal balance of these, replenishing capacity by enabling novices to survive and develop into experts themselves. This balance is given by the optimal value γ^* (14) of the decision parameter γ, the proportion of experts redeployed into training per unit time. Notice first that γ^* *increases* with ϵ: an increasing loss rate of experts requires their conservation and *increased* use in training, both for optimality and to avoid the danger of falling into the **ace** regime (since γ_c, (13), also increases with ϵ). Next, one has to believe that a **learning** regime exists (*i.e.* that **ace** is not inevitable, a rather nihilistic position), so that $\beta > \nu c$. This being so, notice that the optimal training transfer-rate is at least double the critical value: that is, $\gamma^* > 2\gamma_c$, from (14).

The main lesson, of how many experts to deploy to training, is best considered in terms of the dimensionless numbers of the model. Notice that the dimensionless ratios of rates of expert training-transfer to expert losses, both the optimal γ^*/ϵ and the critical γ_c/ϵ, are determined (13,14) by $\frac{\beta}{\nu c}$, which is the ratio of two dimensionless numbers, $\frac{\beta}{\nu}$ and c. Recall that the crossover value c is the ratio of *rates* of throughput of instructors to trainees at which the benefits of increased expert training begin to level off. If this happens at (say) two trainees for every instructor but each instructor is in post for five cycles of trainees (graduating a total of $1/c = 10$ trainees), then $c = 0.5 \times 0.2 = 0.1$. The second dimensionless number, the ratio β/ν, is the average life expectancy of a novice entering combat divided by the minimum time it takes for an expertly-trained novice to become an expert, which we took in our sample numerical case to be 2. With these rough numbers the outcome is that $\frac{\beta}{\nu c} = 2 \times 10 = 20$, and $\gamma^*/\epsilon = 0.29$, so that for optimal **learning** experts should be redeployed to training at around 30% of their loss rate, while the transition into **ace** occurs if this ratio drops below around 5%. This optimal rate is moderately robust to variations. For example, imagine that, in an air force under extreme duress, combat novices have equal chances of surviving to become expert or of being shot down, so that $\beta/\nu \simeq 1$. Then $\frac{\beta}{\nu c} = 10$, and percentage redeployment should be nearly 50%. More generally, doubling or halving $\frac{\beta}{\nu c}$ gives a range of around 20%–50% for γ^*/ϵ, and 3%–10% for γ_c/ϵ. In words, experts should optimally be re-deployed into training at a substantial fraction of their loss rate, probably between 20% and 50%. Over an attritional campaign anything less is sub-optimal, and as this fraction drops towards 10% there is a severe danger of the extinction of expertise. The ultimate conclusion is that, if combat experts are indeed **made**, then a substantial proportion of them should be diverted to the creation of more experts.

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A. Appendix: The Lanchester $(2, 2)$ unaimed fire model

A.1. Derivation of conserved quantity

The $(2, 2)$ heterogeneous Lanchester unaimed-fire model is

$$\frac{dR_1}{dt} = -b_{11}R_1B_1 - b_{12}R_1B_2, \tag{18}$$

$$\frac{dR_2}{dt} = -b_{21}R_2B_1 - b_{22}R_2B_2, \tag{19}$$

$$\frac{dB_1}{dt} = -r_{11}B_1R_1 - r_{12}B_1R_2, \tag{20}$$

$$\frac{dB_2}{dt} = -r_{21}B_2R_1 - r_{22}B_2R_2. \tag{21}$$

For general values of r_{ij}, b_{ij} there is no conserved quantity and we can make no headway. But

$$\frac{dR_1/R_1}{dR_2/R_2} = \frac{d(\log R_1)}{d(\log R_2)} = \frac{b_{11}B_1 + b_{12}B_2}{b_{21}B_1 + b_{22}B_2} \tag{22}$$

is constant if $b_{11}/b_{21} = b_{12}/b_{22}$; call this quantity b. Similarly for Blue losses, writing $r := r_{11}/r_{21} = r_{12}/r_{22}$.

This is equivalent to the requirement that the matrices b_{ij} and r_{ij} have rank equal to one. In words, and supposing that $r_{ij} = b_{ij}$, this says that an expert (type 2) makes n times as many kills (of each type of opponent, expert or novice) as does a novice (type 1), and any pilot kills m times as many novices as experts;

$$r_{ij} = b_{ij} = \begin{pmatrix} 1 & n \\ \frac{1}{m} & \frac{n}{m} \end{pmatrix}. \tag{23}$$

At its simplest, experts survive m times as long as novices, during which they achieve n times the kill rate.

We then have $d(\log R_1) = b\, d(\log R_2)$ and $d(\log B_1) = r\, d(\log R_2)$, so that

$$\frac{R_2^b}{R_1} = \frac{R_2(0)^b}{R_1(0)} \quad \text{and} \quad \frac{B_2^r}{B_1} = \frac{B_2(0)^r}{B_1(0)}. \tag{24}$$

Now looking at the casualty exchange ratio we have

$$\frac{dR_2}{dB_2} = \frac{b_{21}B_1 + b_{22}B_1}{r_{21}R_1 + r_{22}R_2}\frac{R_2}{B_2}, \tag{25}$$

and thus

$$\left(r_{21}\frac{R_1}{R_2} + r_{22}\right) dR_2 = \left(b_{21}\frac{B_1}{B_2} + b_{22}\right) dB_2 \tag{26}$$

or

$$\left(r_{21}R_2^{b-1}\frac{R_1(0)}{R_2(0)^b} + r_{22}\right) dR_2 = \left(b_{21}B_2^{r-1}\frac{B_1(0)}{B_2(0)^r} + b_{22}\right) dB_2, \tag{27}$$

which integrates to give

$$\frac{1}{b}r_{21}R_2^b\frac{R_1(0)}{R_2(0)^b} + r_{22}R_2 - \frac{1}{r}b_{21}B_2^r\frac{B_1(0)}{B_2(0)^r} - b_{22}B_2 = \text{ constant} \tag{28}$$

or finally

$$\frac{1}{b}r_{21}R_1 + r_{22}R_2 - \frac{1}{r}b_{21}B_1 - b_{22}B_2 = \text{ constant}. \tag{29}$$

Thus, in the separable case, the $(2, 2)$ Lanchester unaimed fire model obeys a linear law. With $b = r = \frac{1}{m}$, this law is that

$$R_2 + \frac{1}{nm}R_1 - B_2 - \frac{1}{nm}B_1 \tag{30}$$

is conserved. Thus each side's fighting strength, the quantity to be optimized, is the sum of its experts and d times its novices, $E + dN$ where we write $d := \frac{1}{nm}$.

A.2. Implications for the Ace Paradox model

In the main text the objective was to maximize E, the number of experts at the stable fixed point in the learning regime. In the linear law (30), by contrast, the objective is to maximize $E + dN$. In this subsection we determine how this modifies the earlier results. A shift from maximizing E to maximizing $E + dN$ of course has no effect on the transition between regimes at $\gamma = \gamma_c$.

We see that d is the product of $\frac{1}{m} = \frac{\epsilon}{\nu}$ and the novice-to-expert kill ratio $\frac{1}{n}$, so that $d \ll 1$. At the fixed point the fighting strength is, for general $\gamma > \gamma_c$,

$$F := E + dN = \frac{r}{\beta + \nu}\left\{\frac{\beta}{\epsilon + \gamma} - \frac{c\nu}{\gamma} + d\left(1 + c + \frac{\epsilon c}{\gamma}\right)\right\}, \tag{31}$$

and at the optimal $\gamma = \gamma^*$ is

$$F^* := E^* + dN^* = \frac{r}{\beta + \nu}\left\{\frac{\nu}{\epsilon}\left(\sqrt{\frac{\beta}{\nu}} - \sqrt{c}\right)^2 + d\left(1 + \sqrt{\frac{\beta c}{\nu}}\right)\right\}. \tag{32}$$

These expressions are not very illustrative. The essential point is that the admixture of novices has very little effect, because, first, d is small; second, at optimal $\gamma = \gamma^*$ the equilibrium E is much larger than N; and finally, at its maximum E is flat and changes only at second order in $\gamma - \gamma^*$. To make this precise we work out the maximal $F = F_{\max}$, which occurs at a value of γ slightly different (by approximately 0.001) from γ^*, and subtract from it the value $F = F^*$ at $\gamma = \gamma^*$. With the values of the parameters in Section 5.2, and supposing $d = \left(\frac{\epsilon}{\nu}\right)^2 = \frac{1}{16}$, the fighting strengths $F_{\max}$ and F^* are around 491 (composed of 482 experts, as in Figure 8, with an admixture of novices worth around nine experts), and the difference $F_{\max} - F^*$ is approximately 0.02 of an expert pilot.

Thus, using the linear law fighting strength $E + dN$ rather than the number of experts E as an objective makes no significant difference either to the optimal expert re-deployment rate γ^* or to the resulting maximum fighting strength.

Finally, it is worth emphasizing that the mixed-force unaimed-fire model of A.1 above is precisely that: unaimed, with each expert or novice choosing targets uniformly at random. If, for example, experts were directed to seek out and attack enemy experts, the model would be altered.

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n	1	2	3	4	5	6	7	8	9	10	11	12
$\langle p \rangle_n$	0.380	0.312	0.255	0.200	0.159	0.120	0.085	0.057	0.035	0.023	0.014	0.011

Table 1: **Average pilot loss probability $\langle p \rangle_n$ as a function of encounter number** n. Data inferred from Figure 1, Bilikam & Frick, 1970.

Parameter	Dimensions	Meaning	Value in Figs 5-9
ν	T^{-1}	rate of combat novice loss	1 per month
ϵ	T^{-1}	rate of combat expert loss	0.25 per month
β	T^{-1}	maximal rate of novice-to-expert conversion	2 per month
γ	T^{-1}	rate of transfer of experts to training roles; decision parameter	varies
r	PT^{-1}	rate of arrival in front line of novices from training	300 pilots per month
K	P	crossover parameter: value of E at which utility of experts in training roles levels off	(varies with γ)
c	none	alternative crossover parameter	0.1
		equal to reciprocal of total number of trainees graduated per instructor	10

Table 2: **Model parameters**. Dimensions are P (pilots) and T (time). Dependent variables $E(t)$ and $N(t)$ have dimensions of PT^{-1}.

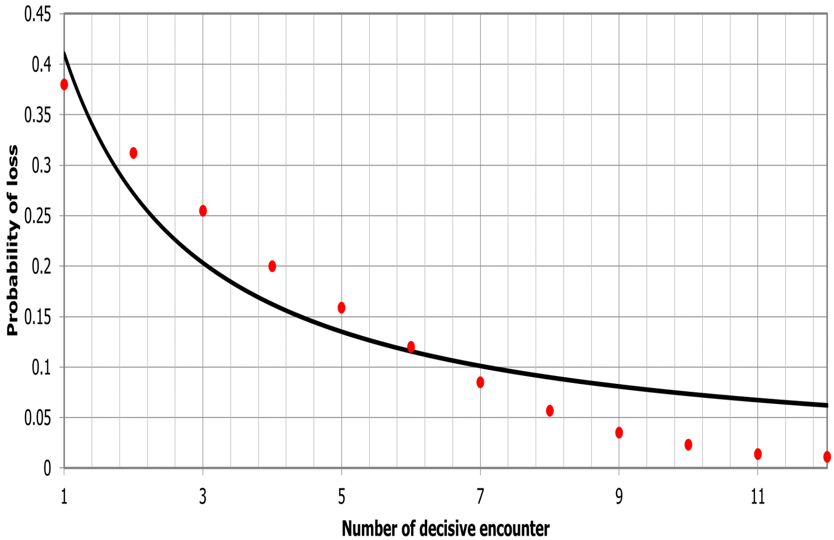

Figure 1: **Optimal Beta fit to probability of loss**, $p_n = \alpha/(\alpha + \beta + n - 1)$, with $\alpha = 0.805, \beta = 1.155$. Figure produced in *MyCurveFit*.

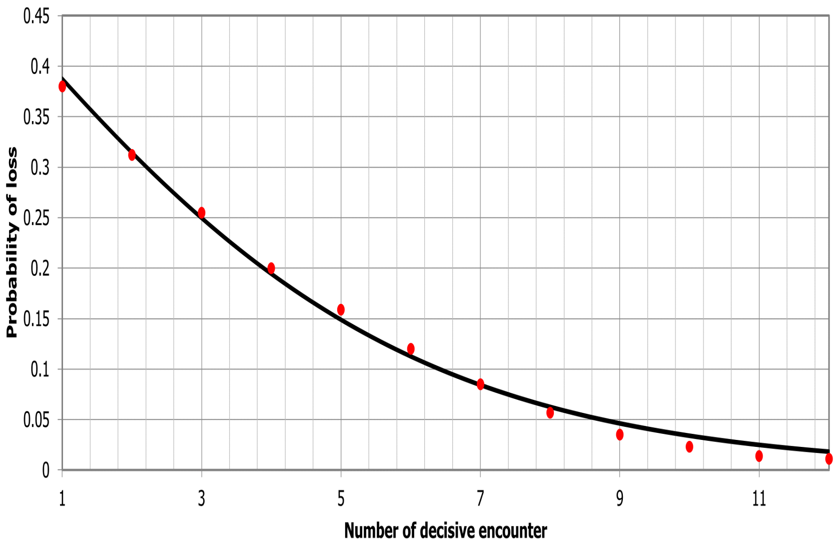

Figure 2: **Optimal Logistic fit to probability of loss**, $p_n = (1 + \frac{1-p}{p} e^{r(x-1)})^{-1}$, with $p = 0.388, a = 1.379$, with $\sum R^2 = 0.996$. Figure produced in *MyCurveFit*.

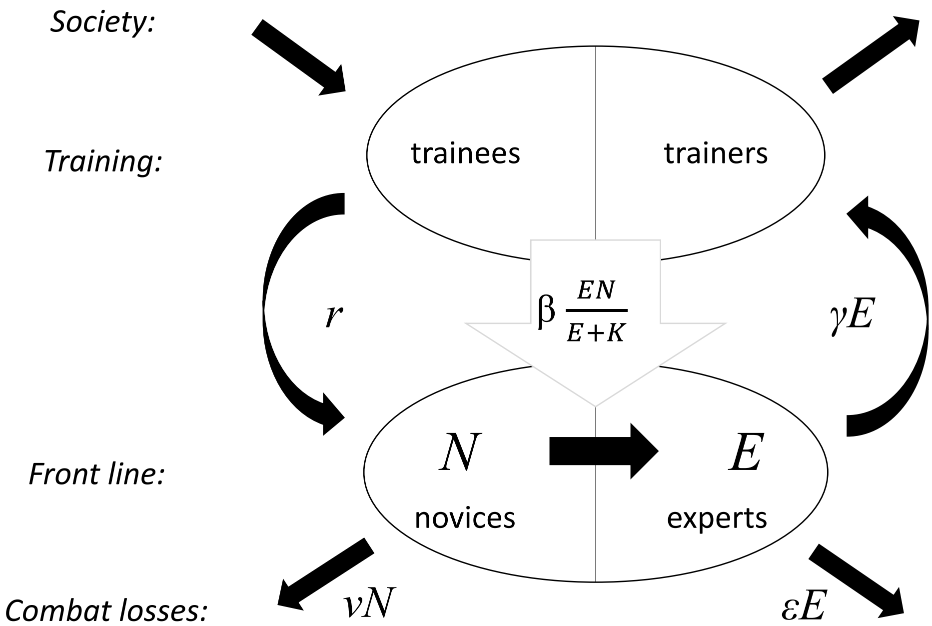

Figure 3: **Model structure for the ace paradox.** There are two dynamical variables, combat novices N and combat experts E. Solid black arrows denote transfers. The central blank arrow denotes influence: contact between trainees and expert trainers causes more rapid conversion of combat novices to experts via a Hollings type-II response. Figure produced in *MS PowerPoint*.

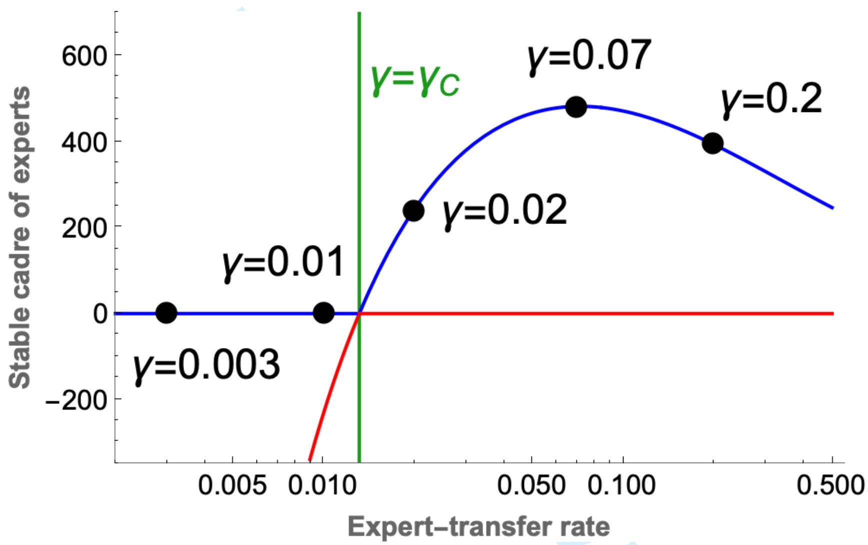

Figure 4: **The transcritical bifurcation in equilibrium numbers of experts plotted against expert-transfer rate, for** $r = 300, \nu = 1, \epsilon = 0.25, \beta = 2$**.** The stable equilibrium is in blue (with sample values of γ, displayed as large solid dots, corresponding to the phase diagrams of Figures 5-9). The unstable equilibrium is in red, with the two values exchanging stability at the critical value γ_c (in green) at which the non-trivial equilibrium curve passes through the $E = 0$ line. Figure produced in *Mathematica*.

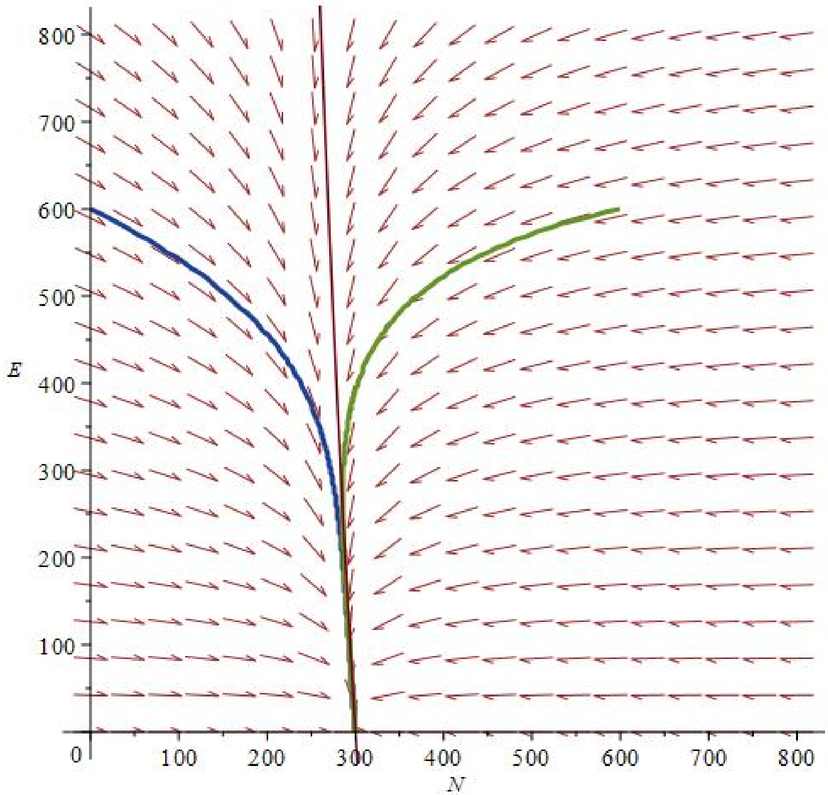

Figure 5: **Ace regime; experts die out.** An almost nil rate of expert re-deployment, $\gamma = 0.003$, results in global stability of $(N, E) = (300, 0)$, constantly-replenishing novices with no experts alongside them. Flow directions and $dN/dt = 0$ nullcline (thin line) in red; sample trajectories (thick lines) in blue and green. Figure produced in *Maple*.

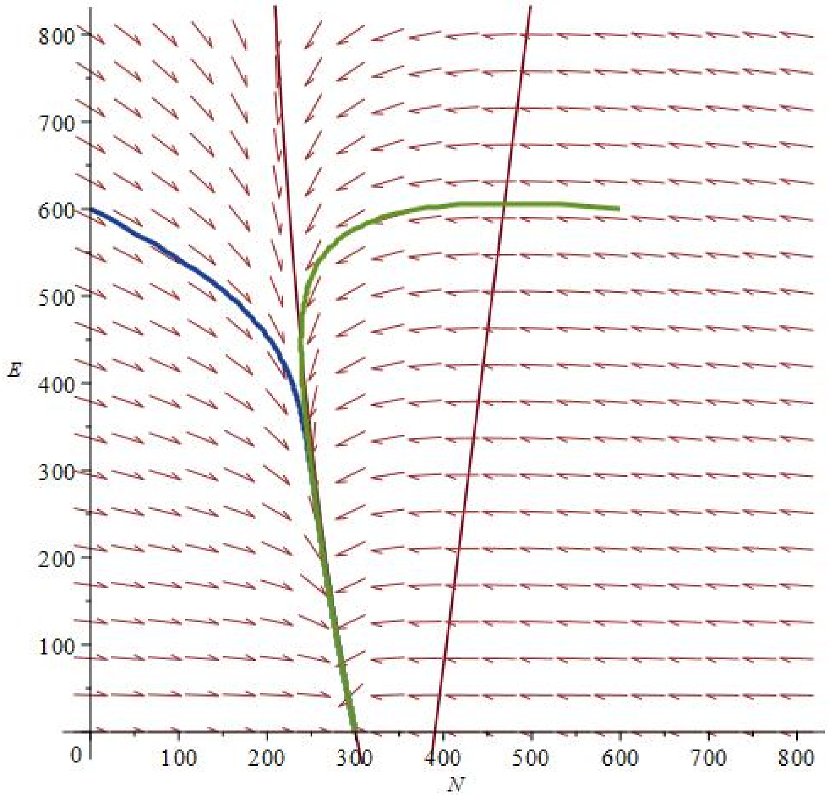

Figure 6: **Ace regime, close to criticality; numbers appear to stabilise, but then experts slowly die out.** Expert re-deployment $\gamma = 0.01$ is slightly below the critical value; the fixed point remains $(N, E) = (300, 0)$. Flow directions and nullclines (thin lines: $dN/dt = 0$ to the left, $dE/dt = 0$ to the right) in red; sample trajectories (thick lines) in blue and green. Figure produced in *Maple*.

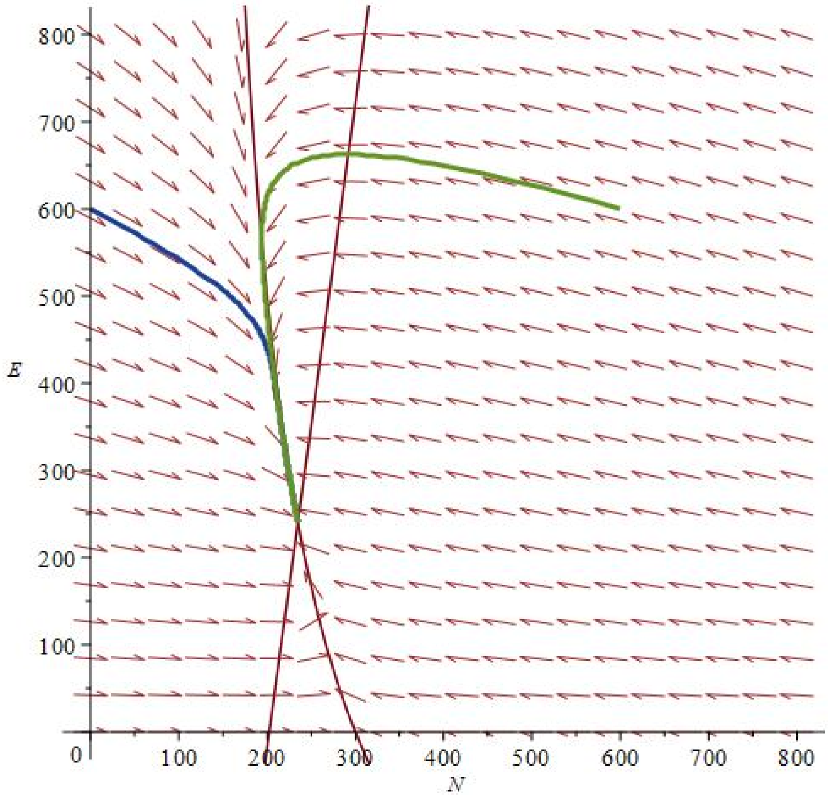

Figure 7: **Learning regime, close to criticality; numbers appear to stabilise, then experts are slowly lost but approach a non-zero stable value.** Expert re-deployment $\gamma = 0.02$ is just above the critical value, with resulting fixed point at approximately $(N, E) = (235, 241)$, a roughly equal mix of novices and experts. Flow directions and nullclines (thin lines: downwards curve $dN/dt = 0$, upwards line $dE/dt = 0$, intersecting at the fixed point) in red; sample trajectories (thick lines) in blue and green. Figure produced in *Maple*.

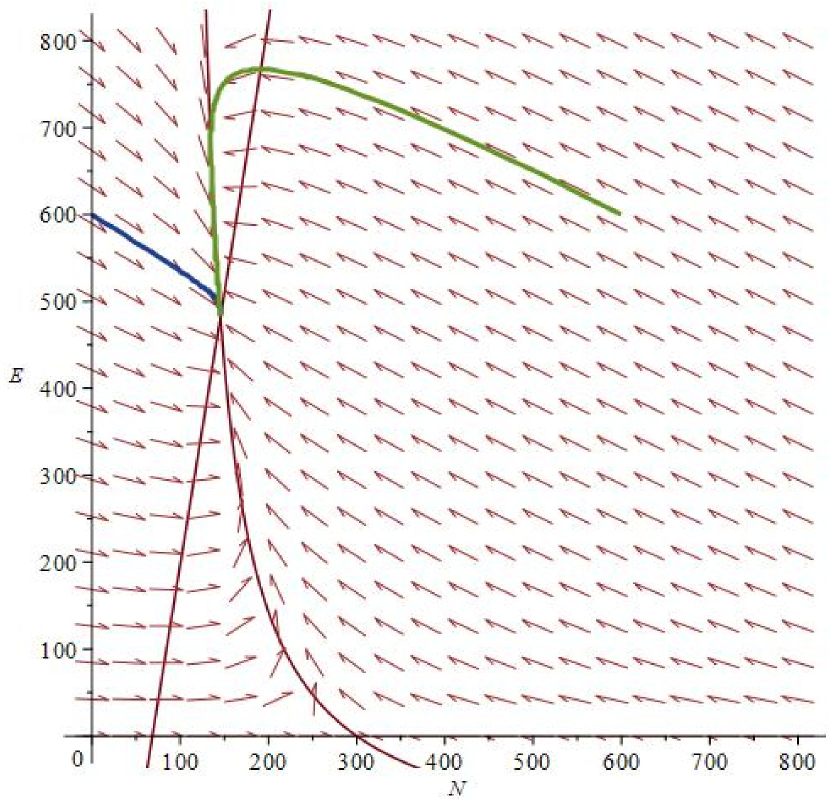

Figure 8: **Learning regime, maximizing stable number of experts.** Expert re-deployment $\gamma = 0.07$ results in fixed point at approximately $(N, E) = (145, 482)$, a predominantly expert group of pilots together with around a quarter novices. Flow directions and nullclines (thin lines: downwards curve $dN/dt = 0$, upwards line $dE/dt = 0$, intersecting at the fixed point) in red; sample trajectories (thick lines) in blue and green. Figure produced in *Maple*.

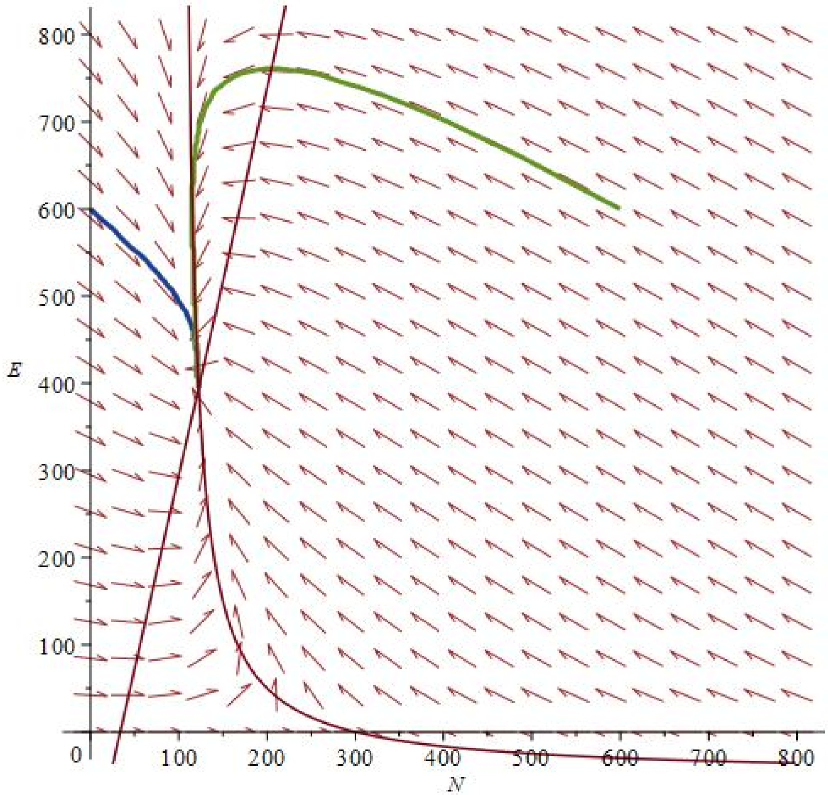

Figure 9: **Learning regime, with needlessly rapid initial exodus of experts into training roles.** High expert re-deployment $\gamma = 0.2$ leads to fixed point at $(N, E) = (123, 394)$, still a predominantly expert group of pilots. Flow directions and nullclines (thin lines: downward curve $dN/dt = 0$, upwards line $dE/dt = 0$, intersecting at the fixed point) in red; sample trajectories (thick lines) in blue and green. Figure produced in *Maple*.