On the determination of additive source terms in the hyperbolic bio-heat equation

N. Kinash and D. Lesnic*
Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK
E-mails: N.Kinash@leeds.ac.uk, D.Lesnic@leeds.ac.uk (*corresponding author)

Abstract. We study the simultaneous recovery of two additive source components in the two-dimensional hyperbolic bio-heat equation from temperature measurements at two spatial cross-sections. The inverse problem is reduced to a source-free initial–boundary value problem for the mixed spatial derivative of the temperature. Using energy estimates, we prove the unique solvability of the inverse source problem and establish convergence to the parabolic limit at a rate proportional to the square root of the time-lag relaxation parameter.

Keywords: bio-heat transfer; inverse source problem; thermal biology; hyperbolic equation with small parameter.

1 Introduction

The Pennes bio-heat equation [18] is the most widely used model to describe heat transfer in living tissues. In its classical form, the temperature u in perfused tissue satisfies the parabolic partial differential equation

$$C_{\text{tissue}}\frac{\partial u}{\partial t} = \kappa\Delta u + C_{\text{blood}}W_{\text{blood}}(u_a - u) + Q_{\text{met}} + Q_{\text{ext}}, \tag{1}$$

where C_{tissue} and κ are the heat capacity and thermal conductivity of the tissue, respectively, (assumed, for simplicity, constant), $C_{\text{blood}}W_{\text{blood}}$ characterises the blood perfusion heat exchange (assumed, for simplicity, constant), u_a is the arterial blood temperature, Q_{met} is the metabolic heat generation and Q_{ext} represents an externally applied heat source. Equation (1) has been considered as modelling several biomedical engineering applications in hyperthermia, cryosurgery, laser ablation and thermal diagnostics [5, 24, 25]. Despite certain limitations, for example, the assumption that thermal equilibration occurs at the capillary level and the omission of directional blood flow effects [3, 10, 23], the Pennes equation has demonstrated some agreement with experimental data [23].

A well-known idealization of equation (1) is its parabolic nature, implying infinite speed of thermal signal propagation. Experimental evidence that this is not always adequate was provided by Mitra *et al.* [16], who observed hyperbolic heat conduction in processed meat, and by Kaminski [12], who showed that materials with non-homogeneous internal structures exhibit measurable thermal relaxation effects. These findings motivate the replacement of the classical Fourier law expressing the heat flux $\mathbf{q} = -\kappa\nabla u$ by the Maxwell–Cattaneo constitutive relation

$$\varepsilon\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\kappa\nabla u, \tag{2}$$

where $\varepsilon > 0$ is a thermal relaxation time-lag. Combining (2) with the energy balance and the Pennes perfusion term yields the hyperbolic bio-heat equation [1]

$$\varepsilon u^\varepsilon_{tt} + (1 + \varepsilon A)u^\varepsilon_t = \alpha\Delta u^\varepsilon - Au^\varepsilon + F^\varepsilon(t, \mathbf{x}), \tag{3}$$

where $\alpha = \kappa/C_{\text{tissue}} > 0$ is the thermal diffusivity of the tissue, $A = C_{\text{blood}}\,W_{\text{blood}}/C_{\text{tissue}} \geq 0$ is the blood perfusion coefficient and F is a source term. Equation (3) is a perturbed telegraph-type equation that reduces to the classical parabolic model (1) as $\varepsilon \searrow 0$.

In the setting of hyperthermia cancer treatment, where tissue is heated at 39–45°C in order to sensitise tumours subjected to radiotherapy and/or chemotherapy [4, 24], the distribution of internal heat sources is essential. However, these sources are typically inaccessible to direct measurement. Instead, temperature data can be acquired non-invasively via magnetic resonance thermometry [21] or infrared thermography [14] at specific measurement surfaces or interior cross-sections, which mathematically leads to the problem of determination of an unknown source function from such temperature observations [2, 20].

Inverse source problems for parabolic equations with various overdetermination conditions were studied extensively in [7, 13, 22], with stability results established by means of Carleman estimates [26] or integral methods [8]. Considerably less attention has been devoted to inverse problems for the hyperbolic bio-heat equation (3). Alosaimi and Lesnic [1] were concerned with the determination of a space-dependent source in the thermal-wave model of bio-heat transfer. Mochnacki and Paruch [17] considered the identification of external heat flux and relaxation time for the equation (3). Denisov [6] studied asymptotic expansions of solutions to inverse problems for hyperbolic equations with a small parameter multiplying the highest derivative.

In the present paper, we consider the simultaneous recovery of two additive source components $f^\varepsilon(t, x)$ and $g^\varepsilon(t, y)$ in the two-dimensional hyperbolic equation (3), with $F^\varepsilon(t, x, y) = f^\varepsilon(t, x) + g^\varepsilon(t, y)$, from temperature measurements at prescribed space cross-sections $x = x_*$ and $y = y_*$. A similar inverse source problem for the parabolic heat equation was considered in [9, 11]. The analysis for establishing the existence and uniqueness of solution of the inverse problem is based on reducing it to a well-posed initial–boundary value problem for the mixed derivative $v^\varepsilon = u^\varepsilon_{xy}$, followed by *a priori* energy estimates and the Galerkin method, see [2, Section 7.3]. We also prove a convergence estimate of order $O(\sqrt{\varepsilon})$ as $\varepsilon \searrow 0$, showing that the solution of the hyperbolic inverse source problem converges to that of the corresponding parabolic one.

The paper is organised as follows. In Section 2, we formulate the inverse problem, state the existence and uniqueness theorem, and give its proof. In Section 3, we establish the convergence to the parabolic equation solution as $\varepsilon \searrow 0$ with an explicit error bound. Finally, Section 4 presents conclusions and possible future work.

2 Mathematical formulation

In the rectangular domain $\Omega = (0, l_1) \times (0, L_2)$ over the time interval $(0, T)$, we consider the inverse problem of finding the functions $u^\varepsilon(t, x, y)$, $f^\varepsilon(t, x)$ and $g^\varepsilon(t, y)$ satisfying the equation

$$\varepsilon u^\varepsilon_{tt} + (1+\varepsilon A)u^\varepsilon_t = \alpha(u^\varepsilon_{xx} + u^\varepsilon_{yy}) - Au^\varepsilon + f^\varepsilon(t, x) + g^\varepsilon(t, y), \quad (t, x, y) \in Q_T := (0, T) \times \Omega, \tag{4}$$

subjected to the initial and Neumann adiabatic boundary conditions

$$u^\varepsilon(0,x,y) = u_0(x,y), \quad u^\varepsilon_t(0,x,y) = u_1(x,y), \quad (x,y) \in \overline{\Omega}, \tag{5}$$

$$u^\varepsilon_x(t,0,y) = u^\varepsilon_x(t,l_1,y) = 0, \quad (t,y) \in [0,T] \times [0,l_2], \tag{6}$$

$$u^\varepsilon_y(t,x,0) = u^\varepsilon_y(t,x,l_2) = 0, \quad (t,x) \in [0,T] \times [0,l_1] \tag{7}$$

from the measurements of u^ε at $x_* \in [0,l_1]$ and $y_* \in [0,L_2]$, namely,

$$u^\varepsilon(t,x_*,y) = \beta(t,y), \quad (t,y) \in [0,T] \times [0,l_2], \tag{8}$$

$$u^\varepsilon(t,x,y_*) = \gamma(t,x), \quad (t,x) \in [0,T] \times [0,l_1], \tag{9}$$

In the particular case $A = 0$, we obtain the inverse problem formulated and analysed in [2, Section 7.3]. We assume consistency conditions ensuring that the initial data, boundary conditions, and over-determination conditions are mutually consistent at $t = 0$ and on the boundaries $x \in \{0,l_1\}$, $y \in \{0,l_2\}$, as follows:

- The first initial data in (5) consistent with the Neumann boundary data (6) and (7):

$$u_{0_x}(0,y) = u_{0_x}(l_1,y) = 0, \quad y \in [0,l_2], \quad u_{0_y}(x,0) = u_{0_y}(x,l_2) = 0, \quad x \in [0,l_1]. \tag{10}$$

- Initial data (5) consistent with the over-determination data (8) and (9) at $t = 0$:

$$u_0(x_*,y) = \beta(0,y), \quad u_1(x_*,y) = \beta_t(0,y), \quad y \in [0,l_2], \tag{11}$$

$$u_0(x,y_*) = \gamma(0,x), \quad u_1(x,y_*) = \gamma_t(0,x), \quad x \in [0,l_1]. \tag{12}$$

- Consistency of the two over-determination data (8) and (9) at the intersection point (x_*,y_*):

$$\beta(t,y_*) = \gamma(t,x_*), \quad t \in [0,T]. \tag{13}$$

  In particular, at $t = 0$: $u_0(x_*,y_*) = \beta(0,y_*) = \gamma(0,x_*)$.

- Consistency of the Neumann boundary data (6) and (7) with the over-determination data (8) and (9):

$$\beta_y(t,0) = \beta_y(t,l_2) = 0 = \gamma_x(t,0) = \gamma_x(t,l_1), \quad t \in [0,T]. \tag{14}$$

Theorem 2.1. *Let $u_0 \in H^5(\Omega)$, $u_1 \in H^4(\Omega)$, $\beta \in C^2([0,T] \times [0,L_2])$, $\gamma \in C^2([0,T] \times [0,l_1])$, and suppose that the compatibility conditions (10)–(14) are satisfied. Then, there is a unique solution $(u^\varepsilon, F^\varepsilon) \in H^2(Q_T) \times L^2(Q_T)$ of the problem (4)–(9), where $F^\varepsilon(t,x,y) = f^\varepsilon(t,x) + g^\varepsilon(t,y)$. The individual components $f^\varepsilon(t,x)$ and $g^\varepsilon(t,y)$ can be uniquely determined under an additional condition like $g^\varepsilon(t,\zeta_2) = 0$ for some $\zeta_2 \in (0,l_2)$ or $\int_0^{l_2} g^\varepsilon(t,y)\,dy = 0$.*

Proof. We follow the proof of Theorem 7.3 in [2]. Let us denote $v^\varepsilon(t,x,y) = u^\varepsilon_{xy}(t,x,y)$. Since f^ε depends only on (t,x) and g^ε depends only on (t,y), we have $\partial_y f^\varepsilon = 0$ and $\partial_x g^\varepsilon = 0$, so $\partial_{xy}(f^\varepsilon + g^\varepsilon) = 0$. The Neumann boundary conditions (6) and (7) on u^ε imply homogeneous Dirichlet conditions on $v^\varepsilon = u^\varepsilon_{xy}$ at the boundary (since $u^\varepsilon_x = 0$ on $x \in \{0,l_1\}$ implies

$\partial_y(u_x^\varepsilon) = 0$, i.e. $v^\varepsilon = 0$, on those faces; similarly for $y \in \{0, l_2\}$). After differentiating in (4)–(7) with respect to both x and y we obtain the following system for $v^\varepsilon(t,x,y)$:

$$\varepsilon v_{tt}^\varepsilon(t,x,y) + (1+\varepsilon A) v_t^\varepsilon(t,x,y) = \alpha \Delta v^\varepsilon(t,x,y) - A v^\varepsilon(t,x,y), \quad (t,x,y) \in Q_T, \tag{15}$$

$$v^\varepsilon(0,x,y) = u_{0_{xy}}(x,y) =: v_0(x,y), \quad v_t^\varepsilon(0,x,y) = u_{1_{xy}}(x,y) =: v_1(x,y), \quad (x,y) \in \overline{\Omega}, \tag{16}$$

$$v^\varepsilon(t,0,y) = v^\varepsilon(t,l_1,y) = 0, \quad (t,y) \in [0,T] \times [0,l_2], \tag{17}$$

$$v^\varepsilon(t,x,0) = v^\varepsilon(t,x,l_2) = 0. \quad (t,x) \in [0,T] \times [0,l_1]. \tag{18}$$

This is a standard linear hyperbolic IBVP with homogeneous Dirichlet data and no forcing term. Assuming that v^ε is smooth enough we can derive *a priori* estimates needed for establishing the existence of solution.

Multiplying (15) by v_t^ε in $L^2(\Omega)$ and integrating by parts (using Green's formula and conditions (17) and (18)) we obtain

$$\frac{\varepsilon}{2}\frac{d}{dt}\|v_t^\varepsilon(t,\cdot)\|^2 + (1+\varepsilon A)\|v_t^\varepsilon(t,\cdot)\|^2 + \frac{1}{2}\frac{d}{dt}\left(\alpha\|\nabla v^\varepsilon(t,\cdot)\|^2 + A\|v^\varepsilon(t,\cdot)\|^2\right) = 0, \tag{19}$$

where the norm $\|\cdot\|$ is in $L^2(\Omega)$. Integrating (19) with respect to t and using (16) we get

$$\begin{aligned}\varepsilon\|v_t^\varepsilon(t,\cdot)\|^2 + 2(1+\varepsilon A)\int_0^t \|v_t^\varepsilon(\tau,\cdot)\|^2 d\tau + \alpha\|\nabla v^\varepsilon(t,\cdot)\|^2 + A\|v^\varepsilon(t,\cdot)\|^2 \\ = \varepsilon\|v_1\|^2 + \alpha\|\nabla v_0\|^2 + A\|v_0\|^2 =: C_1, \quad t \in [0,T].\end{aligned} \tag{20}$$

The Poincaré–Friedrichs inequality states $\|w\| \le C_{\mathrm{PF}}\|\nabla w\|$ for any function $w \in H_0^1(\Omega)$. In case of the rectangle $\Omega = (0,l_1) \times (0,l_2)$, the best constant $C_{\mathrm{PF}} = \frac{\min\{l_1,l_2\}}{\pi}$. Since v^ε satisfies homogeneous Dirichlet boundary conditions (17) and (18), we may apply this inequality to obtain $\|v^\varepsilon(t,\cdot)\|^2 \le C_{\mathrm{PF}}^2\|\nabla v^\varepsilon(t,\cdot)\|^2$. Then from (20) we obtain

$$\|v^\varepsilon(t,\cdot)\| \le C_2, \qquad t \in [0,T], \tag{21}$$

with $C_2 = C_{\mathrm{PF}}\sqrt{C_1/\alpha}$. Taking the inner product of (15) with v_{txx}^ε in $L^2(\Omega)$ and integrating by parts in x (using the homogeneous Dirichlet conditions (17) to eliminate boundary terms), we obtain

$$\begin{aligned}\frac{\varepsilon}{2}\frac{d}{dt}\|v_{tx}^\varepsilon(t,\cdot)\|^2 + (1+\varepsilon A)\|v_{tx}^\varepsilon(t,\cdot)\|^2 \\ + \frac{1}{2}\frac{d}{dt}\left(\alpha\|v_{xx}^\varepsilon(t,\cdot)\|^2 + \alpha\|v_{yx}^\varepsilon(t,\cdot)\|^2 + A\|v_x^\varepsilon(t,\cdot)\|^2\right) = 0,\end{aligned} \tag{22}$$

where we have also used integration by parts with respect to y to express the term

$$\int_\Omega v_{txx}^\varepsilon(t,\cdot)v_{yy}^\varepsilon(t,\cdot)d\Omega = -\int_\Omega v_{tx}^\varepsilon(t,\cdot)v_{yyx}^\varepsilon(t,\cdot)d\Omega = \int_\Omega v_{txy}^\varepsilon(t,\cdot)v_{yx}^\varepsilon(t,\cdot)d\Omega.$$

Integrating (22) with respect to t we obtain

$$\begin{aligned}\varepsilon\|v_{tx}^\varepsilon(t,\cdot)\|^2 + 2(1+\varepsilon A)\int_0^t \|v_{tx}^\varepsilon(\tau,\cdot)\|^2 d\tau + \alpha\|v_{xx}^\varepsilon(t,\cdot)\|^2 + \alpha\|v_{xy}^\varepsilon(t,\cdot)\|^2 + A\|v_x^\varepsilon(t,\cdot)\|^2 \\ = \varepsilon\|v_{1_x}\|^2 + \alpha\|\nabla v_{0_x}\|^2 + A\|v_{0_x}\|^2 =: C_3, \quad t \in [0,T].\end{aligned} \tag{23}$$

Similarly, taking the inner product of (15) with v^ε_{tyy} in $L^2(\Omega)$ and integrating by parts in y (using (18)), we obtain

$$\varepsilon\|v^\varepsilon_{ty}(t,\cdot)\|^2+2(1+\varepsilon A)\int_0^t\|v^\varepsilon_{ty}(\tau,\cdot)\|^2d\tau+\alpha\|v^\varepsilon_{xy}(t,\cdot)\|^2+\alpha\|v^\varepsilon_{yy}(t,\cdot)\|^2+A\|v^\varepsilon_{y}(t,\cdot)\|^2$$
$$=\varepsilon\|v_{1_y}\|^2+\alpha\|\nabla v_{0_y}\|^2+A\|v_{0_y}\|^2=:C_4,\quad t\in[0,T]. \tag{24}$$

Lemma 2.2. *The following inequalities hold:*

$$\|v^\varepsilon(t,\cdot)\|^2_{H^2(\Omega)}\le C_5\varepsilon+C_6,\ t\in[0,T], \tag{25}$$
$$\varepsilon\|v^\varepsilon_{tt}\|^2_{L^2(Q_T)}\le C_7\varepsilon+C_8, \tag{26}$$
$$\|v^\varepsilon_{t}\|^2_{L^2(Q_T)}\le C_9\varepsilon^2+C_{10}\varepsilon+C_{11}. \tag{27}$$

where the positive constants C_5 to C_{11} depend only on α, A, T, $\|v_0\|_{H^2(\Omega)}$ and $\|v_1\|_{H^1(\Omega)}$.

Proof. From the estimates (20), (21)–(24) we immediately obtain (25). To prove (26), multiply (15) by v^ε_{tt} in $L^2(\Omega)$ to get

$$\varepsilon\|v^\varepsilon_{tt}(t,\cdot)\|^2+\frac{1+\varepsilon A}{2}\frac{d}{dt}\|v^\varepsilon_{t}(t,\cdot)\|^2=\alpha(v^\varepsilon_{xx}(t,\cdot),v^\varepsilon_{tt}(t,\cdot))+\alpha(v^\varepsilon_{yy}(t,\cdot),v^\varepsilon_{tt}(t,\cdot))$$
$$-A(v^\varepsilon(t,\cdot),v^\varepsilon_{tt}(t,\cdot)),\quad t\in[0,T], \tag{28}$$

and integrate over $(0,T)$ to obtain

$$\varepsilon\|v^\varepsilon_{tt}\|^2_{L_2(Q_T)}+\frac{1+\varepsilon A}{2}\|v^\varepsilon_{t}(T,\cdot)\|^2=\frac{1+\varepsilon A}{2}\|v_1\|^2+\alpha\int_0^T(v^\varepsilon_{xx},v^\varepsilon_{tt})\,dt+\alpha\int_0^T(v^\varepsilon_{yy},v^\varepsilon_{tt})\,dt$$
$$-A\int_0^T(v^\varepsilon,v^\varepsilon_{tt})\,dt. \tag{29}$$

The key idea is to estimate each integral in the right-hand side of (29) using integration by parts in t rather than applying the Cauchy inequality directly to v^ε_{tt} (which would produce constants depending on ε^{-1}). First, we have

$$I_1:=\alpha\int_0^T(v^\varepsilon_{xx},v^\varepsilon_{tt})\,dt=\alpha\big[(v^\varepsilon_{xx},v^\varepsilon_{t})\big]_{t=0}^{t=T}-\alpha\int_0^T(v^\varepsilon_{xxt},v^\varepsilon_{t})\,dt. \tag{30}$$

For the integral on the right-hand side of (30), we integrate by parts in x, using the homogeneous Dirichlet conditions (17) to eliminate the boundary terms, and use (23) to get

$$-\alpha\int_0^T(v^\varepsilon_{xxt},v^\varepsilon_{t})\,dt=\alpha\int_0^T\|v^\varepsilon_{xt}(\tau,\cdot)\|^2\,d\tau\le\frac{\alpha C_3}{2(1+\varepsilon A)}. \tag{31}$$

The term at $t=0$ in (30) involves only the initial data (16). Namely, $(v^\varepsilon_{xx}(0,\cdot),v^\varepsilon_t(0,\cdot))=(v_{0_{xx}},v_1)$, so by the Cauchy–Schwarz inequality $|(v_{0_{xx}},v_1)|\le\|v_{0_{xx}}\|\,\|v_1\|$. For the term at $t=T$ in (30) we apply the Cauchy inequality with a free parameter $\delta>0$:

$$\alpha\big|(v^\varepsilon_{xx}(T,\cdot),v^\varepsilon_t(T,\cdot))\big|\le\frac{\alpha\delta}{2}\|v^\varepsilon_{xx}(T,\cdot)\|^2+\frac{\alpha}{2\delta}\|v^\varepsilon_t(T,\cdot)\|^2. \tag{32}$$

The term $\alpha\|v^\varepsilon_{xx}(T,\cdot)\|^2 \le C_3$ by (23). The second term $\|v^\varepsilon_t(T,\cdot)\|^2$ will be absorbed into the left-hand side of (29) at the end of the argument.

Collecting (30)–(32), we have

$$|I_1| = \alpha\left|\int_0^T (v^\varepsilon_{xx}, v^\varepsilon_{tt})\,dt\right| \le \frac{(\alpha+\delta)C_3}{2} + \alpha\|v_{0_{xx}}\|\,\|v_1\| + \frac{\alpha}{2\delta}\|v^\varepsilon_t(T,\cdot)\|^2. \tag{33}$$

Similarly, with y replacing x and the Dirichlet conditions (18) replacing (17), we have

$$|I_2| = \alpha\left|\int_0^T (v^\varepsilon_{yy}, v^\varepsilon_{tt})\,dt\right| \le \frac{(\alpha+\delta)C_4}{2} + \alpha\|v_{0_{yy}}\|\,\|v_1\| + \frac{\alpha}{2\delta}\|v^\varepsilon_t(T,\cdot)\|^2. \tag{34}$$

Finally,

$$I_3 := -A\int_0^T (v^\varepsilon, v^\varepsilon_{tt})\,dt = -A\big[(v^\varepsilon, v^\varepsilon_t)\big]_{t=0}^{t=T} + A\int_0^T \|v^\varepsilon_t(\tau,\cdot)\|^2\,d\tau. \tag{35}$$

By (20), $2(1+\varepsilon A)\int_0^T \|v^\varepsilon_t(\tau,\cdot)\|^2\,d\tau \le C_1$, so the integral in (35) is bounded by $\frac{AC_1}{2(1+\varepsilon A)}$. The term at $t=0$ in (35) is $A|(v_0,v_1)| \le \frac{A}{2}\|v_0\|^2 + \frac{A}{2}\|v_1\|^2$. The term at $t=T$ in (35) satisfies

$$A\big|(v^\varepsilon(T,\cdot), v^\varepsilon_t(T,\cdot))\big| \le \frac{A\delta}{2}\|v^\varepsilon(\cdot,T)\|^2 + \frac{A}{2\delta}\|v^\varepsilon_t(T,\cdot)\|^2,$$

where $\|v^\varepsilon(T,\cdot)\|^2 \le C_2^2$ by (21). To summarise,

$$|I_3| = A\left|\int_0^T (v^\varepsilon, v^\varepsilon_{tt})\,dt\right| \le \frac{AC_1}{2(1+\varepsilon A)} + \frac{A}{2}\left(\|v_0\|^2 + \|v_1\|^2\right) + \frac{A\delta C_2^2}{2} + \frac{A}{2\delta}\|v^\varepsilon_t(T,\cdot)\|^2. \tag{36}$$

Substituting the estimates (33), (34) and (36) into (29), the total coefficient of $\|v^\varepsilon_t(T,\cdot)\|^2$ on the right-hand side is $\frac{2\alpha+A}{2\delta}$. The left-hand side of (29) contains $\frac{1+\varepsilon A}{2}\|v^\varepsilon_t(T,\cdot)\|^2$. Moving all $\|v^\varepsilon_t(T,\cdot)\|^2$ terms to the left, its multiplying coefficient becomes non-negative provided we choose, say $\delta = 2\alpha + A$. We can then drop the resulting non-negative term $\big(\frac{1+\varepsilon A}{2} - \frac{2\alpha+A}{2\delta}\big)\|v^\varepsilon_t(T,\cdot)\|^2$ from the left-hand side of (29) to obtain (using also (20)–(24)) the estimate (26). Finally, from equations (15), (26) and the estimates (20), (21)–(24) we also obtain the inequality (27). $\square$

The constant C_8/ε in (26) grows as $\varepsilon \searrow 0$, but remains finite for each fixed $\varepsilon > 0$, which is sufficient for the existence theorem. The obtained estimates in (25) and (26) yield the existence of a solution $v^\varepsilon \in H^2(Q_T)$ to the linear direct problem (15)–(18) by means of the Galerkin method (by seeking approximate solutions v^ε_N as finite linear combinations of eigenfunctions of the Laplacian in Ω with homogeneous Dirichlet boundary data). The estimates (20), (21)–(25), (26) are derived first at the level of the Galerkin approximation, uniformly in N, and then passing to the limit $N \to \infty$ yields a solution in $H^2(Q_T)$.

The uniqueness of solution to the problem (15)–(18) follows from its linearity and homogeneous structure. If v^ε_1 and v^ε_2 are two solutions in $H^2(Q_T)$, their difference $w = v^\varepsilon_1 - v^\varepsilon_2$ satisfies the same partial differential equation with zero initial and boundary conditions. Multiplying the resulting equation by w_t and integrating exactly as in (19) and (20) gives

$\alpha\|\nabla w(t,\cdot)\|^2 + A\|w(t,\cdot)\|^2 = 0$ for all $t \in [0,T]$; which by the Poincaré–Friedrichs inequality then gives $\|w(t,\cdot)\| = 0$ for all $t \in [0,T]$, and hence $w \equiv 0$.

Once we have proved that the IBVP (15)–(18) has a unique solution $v^\varepsilon \in H^2(Q_T)$, we can now define the function

$$u^\varepsilon(t,x,y) := \int_{x_*}^{x}\int_{y_*}^{y} v^\varepsilon(t,\xi,\eta)\,d\eta\,d\xi + \beta(t,y) + \gamma(t,x) - \gamma(t,x_*), \tag{37}$$

by integrating $u^\varepsilon_{xy} = v^\varepsilon$ and

$$\begin{aligned} F^\varepsilon(t,x,y) := f^\varepsilon(t,x) + g^\varepsilon(t,y) &= \varepsilon u^\varepsilon_{tt} + (1+\varepsilon A)u^\varepsilon_t - \alpha(u^\varepsilon_{xx} + u^\varepsilon_{yy}) + Au^\varepsilon \\ &= \varepsilon\gamma_{tt}(t,x) + (1+\varepsilon A)\gamma_t(t,x) - \alpha\gamma_{xx}(t,x) - \alpha\int_{x_*}^{x} v^\varepsilon_y(t,\xi,y_*)\,d\xi \\ +A\gamma(t,x) - \varepsilon\gamma_{tt}(t,x_*) - (1+\varepsilon A)\gamma_t(t,x_*) &- \alpha\int_{y_*}^{y} v^\varepsilon_x(t,x_*,\eta)\,d\eta - A\gamma(t,x_*) \\ &+\varepsilon\beta_{tt}(t,y) + (1+\varepsilon A)\beta_t(t,y) - \alpha\beta_{yy}(t,y) + A\beta(t,y). \end{aligned} \tag{38}$$

Applying the compatibility conditions (10)-(14) we can verify that the functions $u^\varepsilon(t,x,y)$ and $F^\varepsilon(t,x,y) = f^\varepsilon(t,x) + g^\varepsilon(t,y)$ defined by (37) and (38) satisfy the inverse problem (4)–(9). This concludes the proof of Theorem 2.1.

□

Remark 2.3. The elimination of the unknown additive source, as explained at the beginning of the proof of Theorem 2.1, required high regularity initial data assumptions, which were technically necessary. We also note that Theorem 7.3 in [2] stated that $u_0 \in H^6(\Omega)$ and $u_1 \in H^6(\Omega)$, which is even more stringent than our assumptions that $u_0 \in H^5(\Omega)$ and $u_1 \in H^4(\Omega)$. Further research would be necessary to relax these assumptions, possibly by using other techniques [7, 8] for solving inverse source problems.

3 Convergence to the parabolic limit

The following theorem gives the order of convergence of the solution $(u^\varepsilon, F^\varepsilon)$ of the hyperbolic inverse source problem to the solution (u, F) of the corresponding parabolic inverse source problem, as $\varepsilon \searrow 0$.

Theorem 3.1. *Let the conditions of Theorem 2.1 be fulfilled. Then*

$$\|u^\varepsilon - u\|_{H^2(Q_T)} + \|F^\varepsilon - F\|_{L_2(Q_T)} \le \sqrt{\varepsilon(C_{12}\varepsilon^3 + C_{13}\varepsilon^2 + C_{14}\varepsilon + C_{15})}, \tag{39}$$

where the positive constants C_{12} to C_{15} are independent of ε, and $u(t,x,y)$ and $F(t,x,y) = f(t,x) + g(t,y)$ is the solution of the parabolic inverse problem

$$u_t = \alpha(u_{xx} + u_{yy}) - Au + f(t,x) + g(t,y), \quad (t,x,y) \in Q_T, \tag{40}$$

$$u(0,x,y) = u_0(x,y), \quad (x,y) \in \overline{\Omega}, \tag{41}$$

$$u_x(t,0,y) = u_x(t,l_1,y) = 0, \quad (t,y) \in [0,T] \times [0,l_2], \tag{42}$$

$$u_y(t,x,0) = u_y(t,x,l_2) = 0, \quad (t,x) \in [0,T] \times [0,l_1], \tag{43}$$

$$u(t,x_*,y) = \beta(t,y), \quad (t,y) \in [0,T] \times [0,l_2], \tag{44}$$

$$u(t,x,y_*) = \gamma(t,x), \quad (t,x) \in [0,T] \times [0,l_1]. \tag{45}$$

Proof. The unique solution of the inverse source problem (40)–(45) is given by

$$u(t,x,y) := \int_{x_*}^{x}\int_{y_*}^{y} v(t,\xi,\eta)\,d\xi\,d\eta + \beta(t,y) + \gamma(t,x) - \gamma(t,x_*), \tag{46}$$

and

$$\begin{aligned} F(t,x,y) := f(t,x) + g(t,y) = u_t - \alpha(u_{xx} + u_{yy}) + Au = \gamma_t(t,x) - \alpha\gamma_{xx}(t,x) - \\ \alpha\int_{x_*}^{x} v_y(t,\xi,y_*)\,d\xi + A\gamma(t,x) - \gamma_t(t,x_*) - A\gamma(t,x_*) + \beta_t(t,y) - \alpha\beta_{yy}(t,y) \\ -\alpha\int_{y_*}^{y} v_x(t,x_*,\eta)\,d\eta + A\beta(t,y), \end{aligned} \tag{47}$$

where $v = u_{xy}$ is the solution of the parabolic problem

$$v_t(t,x,y) = \alpha\big(v_{xx}(t,x,y) + v_{yy}(t,x,y)\big) - Av(t,x,y), \quad (t,x,y) \in Q_T, \tag{48}$$

$$v(0,x,y) = u_{0_{xy}}(x,y) =: v_0(x,y), \quad (x,y) \in \overline{\Omega}, \tag{49}$$

$$v(t,0,y) = v(t,l_1,y) = 0, \qquad (t,y) \in [0,T] \times [0,l_2], \tag{50}$$

$$v(t,x,0) = v(t,x,l_2) = 0, \quad (t,x) \in [0,T] \times [0,l_1]. \tag{51}$$

Following the scheme of proof of Theorem 2.1, if $u_0 \in H^5(\Omega)$, then problem (48)–(51) has a unique solution $v \in W_2^{1,2}(Q_T)$. The existence and uniqueness of v follows from standard parabolic theory (see e.g [22]): multiplying (48) by v_t in $L^2(\Omega)$ and integrating by parts yields the energy estimate $\|\nabla v(t,\cdot)\|^2 \le C\|\nabla v_0\|^2$ uniformly in t, and the Galerkin approximation gives existence of a solution.

Denote $\zeta = v - v^\varepsilon$, where v solves (48)–(51) and v^ε solves (15)–(18). Subtracting the equations, ζ satisfies

$$\zeta_t = \alpha(\zeta_{xx} + \zeta_{yy}) - A\zeta + \varepsilon(v_{tt}^\varepsilon + Av_t^\varepsilon), \quad (t,x,y) \in Q_T, \tag{52}$$

$$\zeta(0,x,y) = 0, \quad (x,y) \in \overline{\Omega}, \tag{53}$$

$$\zeta(t,0,y) = \zeta(t,l_1,y) = 0, \quad (t,y) \in [0,T] \times [0,L_2], \tag{54}$$

$$\zeta(t,x,0) = \zeta(t,x,L_2) = 0, \quad (t,x) \in [0,T] \times [0,l_1]. \tag{55}$$

Multiplying (52) by ζ in $L^2(\Omega)$ and integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|\zeta(t,\cdot)\|^2 + \alpha\|\nabla\zeta(t,\cdot)\|^2 + A\|\zeta(t,\cdot)\|^2 = \varepsilon(v_{tt}^\varepsilon(t,\cdot) + Av_t^\varepsilon(t,\cdot), \zeta(t,\cdot)).$$

Apply the Cauchy inequality with a free parameter $\delta > 0$ to the right-hand side:

$$\varepsilon|\,(v_{tt}^\varepsilon(t,\cdot) + Av_t^\varepsilon(t,\cdot), \zeta(t,\cdot))\,| \le \frac{\varepsilon^2}{\delta}\left(\|v_{tt}^\varepsilon(t,\cdot)\|^2 + A^2\|v_t^\varepsilon(t,\cdot)\|^2\right) + \frac{\delta}{2}\|\zeta(t,\cdot)\|^2. \tag{56}$$

When $A > 0$, choosing $\delta = A$ gives $\frac{\delta}{2}\|\zeta(t,\cdot)\|^2 = \frac{A}{2}\|\zeta(t,\cdot)\|^2$, which is absorbed by $A\|\zeta(t,\cdot)\|^2$ on the left-hand side of (57). When $A = 0$, the Poincaré–Friedrichs inequality $\|\zeta(t,\cdot)\|^2 \le$

$C_{\mathrm{PF}}^2\|\nabla\zeta(t,\cdot)\|^2$ gives $\frac{\delta}{2}\|\zeta(t,\cdot)\|^2 \leq \frac{\delta C_{\mathrm{PF}}^2}{2}\|\nabla\zeta(t,\cdot)\|^2$; choosing $\delta = \alpha/C_{\mathrm{PF}}^2$ ensures this is absorbed by $\alpha\|\nabla\zeta(t,\cdot)\|^2$ on the left-hand side of (57). In either case, after absorbing and dropping the positive remainder on the left, one obtains (using Lemma 2.2)

$$\begin{aligned}\frac{d}{dt}\|\zeta(t,\cdot)\|^2 + \alpha'\|\nabla\zeta(t,\cdot)\|^2 + A'\|\zeta(t,\cdot)\|^2 &\leq \frac{\varepsilon^2}{\delta}\left(\|v_{tt}^\varepsilon(t,\cdot)\|^2 + A^2\|v_t^\varepsilon(t,\cdot)\|^2\right)\\ &\leq \varepsilon\left(A_1\varepsilon^3 + A_2\varepsilon^2 + A_3\varepsilon + A_4\right). \qquad (57)\end{aligned}$$

Integrating (57) over $(0,t)$ and using (53) we get

$$\|\zeta(t,\cdot)\|^2 + \int_0^t \left(\alpha'\|\nabla\zeta(\cdot,\tau)\|^2 + A'\|\zeta(\cdot,\tau)\|^2\right)d\tau \leq \varepsilon\left(A_1\varepsilon^3 + A_2\varepsilon^2 + A_3\varepsilon + A_4\right)T, \quad t \in [0,T]. \tag{58}$$

Multiply (52) by ζ_{xx} in $L^2(\Omega)$ and integrate over Ω to obtain

$$(\zeta_t, \zeta_{xx}) = \alpha(\zeta_{xx}, \zeta_{xx}) + \alpha(\zeta_{yy}, \zeta_{xx}) - A(\zeta, \zeta_{xx}) + \varepsilon(v_{tt}^\varepsilon + Av_t^\varepsilon, \zeta_{xx}). \tag{59}$$

Since ζ satisfies homogeneous Dirichlet conditions (54) and (55), integration by parts in x and y gives

$$\begin{aligned}(\zeta_t, \zeta_{xx}) = -\tfrac{1}{2}\tfrac{d}{dt}\|\zeta_x(t,\cdot)\|^2, \quad \alpha(\zeta_{xx}, \zeta_{xx}) = \alpha\|\zeta_{xx}(t,\cdot)\|^2, \quad \alpha(\zeta_{yy}, \zeta_{xx}) = \alpha\|\zeta_{xy}(t,\cdot)\|^2,\\ -A(\zeta, \zeta_{xx}) = A\|\zeta_x(t,\cdot)\|^2. \qquad (60)\end{aligned}$$

From the Cauchy inequality we have

$$\varepsilon|\left(v_{tt}^\varepsilon(t,\cdot) + Av_t^\varepsilon(t,\cdot), \zeta_{xx}(t,\cdot)\right)| \leq \frac{\varepsilon^2}{\alpha}\left(\|v_{tt}^\varepsilon(t,\cdot)\|^2 + A^2\|v_t^\varepsilon(t,\cdot)\|^2\right) + \frac{\alpha}{2}\|\zeta_{xx}(t,\cdot)\|^2. \tag{61}$$

Integrating with respect to $t \in (0,T)$ the first identity in (60), and using (59)–(61) and (53), we obtain

$$\begin{aligned}0 \geq -\frac{1}{2}\|\zeta_x(T,\cdot)\|^2 = \int_0^T (\zeta_t, \zeta_{xx})dt &= \alpha\|\zeta_{xx}\|_{L^2(Q_T)}^2 + \alpha\|\zeta_{xy}\|_{L^2(Q_T)}^2 + A\|\zeta_x\|_{L^2(Q_T)}^2\\ &\qquad + \varepsilon\int_0^T \left(v_{tt}^\varepsilon(t,\cdot) + Av_t^\varepsilon(t,\cdot), \zeta_{xx}(t,\cdot)\right)dt\\ \Longrightarrow\ \alpha\|\zeta_{xx}\|_{L^2(Q_T)}^2 &\leq \frac{\varepsilon^2}{\alpha}\left(\|v_{tt}^\varepsilon\|_{L^2(Q_T)}^2 + A^2\|v_t^\varepsilon\|_{L^2(Q_T)}^2\right) + \frac{\alpha}{2}\|\zeta_{xx}\|_{L_2(Q_T)}^2.\end{aligned}$$

Gathering this with (57) we obtain

$$\|\zeta_{xx}\|_{L^2(Q_T)}^2 \leq \varepsilon\left(A_1'\varepsilon^3 + A_2'\varepsilon^2 + A_3'\varepsilon + A_4'\right). \tag{62}$$

A similar estimate can be obtained for

$$\|\zeta_{yy}\|_{L^2(Q_T)}^2 \leq \varepsilon\left(A_1'\varepsilon^3 + A_2'\varepsilon^2 + A_3'\varepsilon + A_4'\right). \tag{63}$$

Now, from $v = u_{xy}$, $v^\varepsilon = u_{xy}^\varepsilon$ and $\zeta = v - v^\varepsilon$, we obtain that $u - u^\varepsilon = \int_{x_*}^x \int_{y_*}^y \zeta(t,\xi,\eta)\,d\xi\,d\eta$ by (46). Differentiating twice, the derivatives $(u-u^\varepsilon)_{xx}$, $(u-u^\varepsilon)_{yy}$ and $(u-u^\varepsilon)_{xy}$ are all expressible in terms of ζ_{xx}, ζ_{yy}, ζ_{xy}, ζ_x and ζ_y. Since $\|\zeta_{xy}(t,\cdot)\|^2 = (\zeta_{xx}(t,\cdot),\zeta_{yy}(t,\cdot)$ we have that $\|\zeta_{xy}\|^2_{L^2(Q_T)} \leq \frac{1}{2}(\|\zeta_{xx}\|^2_{L^2(Q_T)} + \|\zeta_{yy}\|^2_{L^2(Q_T)})$. Therefore, the estimates (62) and (63) yield the bound for $\zeta_{xy} = v_{xy} - v^{\varepsilon}_{xy}$ in $L_2(Q_T)$. Combining this with (57), (62) and (63) give $\|u - u^{\varepsilon}\|_{H^2(Q_T)} \leq \frac{1}{2}\sqrt{\varepsilon(C_{12}\varepsilon^3 + C_{13}\varepsilon^2 + C_{14}\varepsilon + C_{15})}$, and, using (57), a similar estimate holds for

$$F^{\varepsilon}(t,x,y) - F(t,x,y) = \alpha \int_{x_*}^{x} \left(v_y(t,\xi,y_*) - v^{\varepsilon}_y(t,\xi,y_*)\right) d\xi$$

$$+\alpha \int_{y_*}^{y} (v_x(t,x_*,\eta) - v^{\varepsilon}_x(t,x_*,\eta))\, d\eta = \alpha \left(\int_{x_*}^{x} \zeta_y(t,\xi,y_*)\, d\xi + \int_{y_*}^{y} \zeta_x(t,x_*,\eta)\, d\eta \right).$$

Theorem 3.1 is proved. □

4 Conclusions

We have established the well-posedness of the inverse problem of simultaneously recovering two additive source components $f^{\varepsilon}(t,x)$ and $g^{\varepsilon}(t,y)$ in the two-dimensional hyperbolic bio-heat equation from temperature measurements at two prescribed cross-sections. For each fixed value of the relaxation parameter $\varepsilon > 0$, existence and uniqueness of the solution were proved by reducing the inverse problem to a source-free initial–boundary value problem for the mixed derivative $v^{\varepsilon} = u^{\varepsilon}_{xy}$ and applying energy estimates together with the Galerkin method (Theorem 2.1). The convergence of the hyperbolic PDE solution $(u^{\varepsilon}, F^{\varepsilon})$ to the corresponding parabolic PDE solution (u, F) was established with the rate $O(\sqrt{\varepsilon})$ in the $H^2(Q_T) \times L^2(Q_T)$ norm (Theorem 3.1), providing a rigorous justification for using the classical Pennes model as the zero-relaxation limit of the Maxwell–Cattaneo model in inverse source reconstruction. The numerical implementation of the inverse source problem, including the effect of noisy data and the behaviour of the Tikhonov regularisation, as $\varepsilon \searrow 0$, would complement the theoretical results and will be the subject of future work.

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