



Towards a classification of charge-3 monopoles with symmetry

H. W. Braden¹ · Linden Disney-Hogg¹

Received: 15 May 2023 / Revised: 7 July 2023 / Accepted: 17 July 2023 /
Published online: 28 July 2023
© The Author(s) 2023

Abstract

We classify all possible charge-3 monopole spectral curves with non-trivial automorphism group and within these identify those with elliptic quotients. By focussing on elliptic quotients the transcendental constraints for a monopole spectral curve become ones regarding periods of elliptic functions. We construct the Nahm data and new monopole spectral curves with D_6 and V_4 symmetry, the latter based on an integrable complexification of Euler's equations, and for which energy density isosurfaces are plotted. Extensions of our approach to higher charge and hyperbolic monopoles are discussed.

Keywords Monopoles · Automorphisms · Integrable systems · Toda · Computation

Mathematics Subject Classification 14H70 · 14H81 · 70S15

1 Introduction

In this letter, we shall classify and construct new charge-3 $SU(2)$ monopoles; throughout we will work just with this gauge group. To describe this work further we first recall some necessary background. Following the success of the ADHM construction of instanton solutions to the self-dual Yang–Mills equations Nahm introduced his eponymous equations to solving the reduction of the self-dual Yang–Mills equations $F = \star D\phi$ which describes monopoles; alternately these reduced equations arise as

Dedicated to Nigel Hitchin: for the 40th anniversary of his construction of monopoles—and much more.

✉ H. W. Braden
hwb@ed.ac.uk
Linden Disney-Hogg
A.L.Disney-hogg@sms.ed.ac.uk

¹ School of Mathematics and Maxwell Institute for Mathematical Sciences, The University of Edinburgh, Edinburgh EH9 3FD, Scotland, UK

the BPS limit of a Yang–Mills–Higgs system. Here a (Euclidean BPS) monopole is the data (A, ϕ) where A is a connection of a $SU(2)$ principal bundle over \mathbb{R}^3 with associated curvature F , and ϕ (the Higgs field) is a section of the adjoint bundle. These satisfy the equation $F = \star D\phi$ as well as the boundary conditions that as $r \rightarrow \infty$ (i) $|\phi| = 1 - \frac{k}{2r} + \mathcal{O}(r^{-2})$, where $|\phi|^2 = -\frac{1}{2} \text{Tr}(\phi^2)$ is the norm coming from the Killing form on $\mathfrak{su}(2)$ and r is the Euclidean distance from origin; (ii) $\frac{\partial|\phi|}{\partial\Omega} = \mathcal{O}(r^{-2})$ and Ω is a solid angle; (iii) $|D\phi| = \mathcal{O}(r^{-2})$. The (positive) integer k fixes the topological class of the data and is called the charge. Nahm’s modification of the ADHM construction then sought a triplet of $k \times k$ matrices of a real parameter $s \in [0, 2]$ such that these satisfied (i) the Nahm equations

$$\frac{dT_i}{ds} = \frac{1}{2} \sum_{j,k=1}^3 \epsilon_{ijk}[T_j, T_k], \tag{1}$$

together with (ii) the $T_i(s)$ are regular for all $s \in (0, 2)$ and with simple poles at $s = 0, 2$ if $k > 1$, whose residues form an irreducible k -dimensional representation of $SU(2)$; and (iii) $T_i(s) = -T_i^\dagger(s)$, $T_i(s) = T_i^T(2 - s)$. Hitchin in his seminal work [21] introduced yet a third description of monopoles, the *spectral curve*, and proved the equivalence of all three descriptions. Here the spectral curve $C \subset T\mathbb{P}^1 \xrightarrow{\pi} \mathbb{P}^1$ is a compact algebraic curve with no multiple components of genus $(k - 1)^2$ such that (i) C is real with respect to a (to be given) anti-holomorphic involution τ ; (ii) there is a family of line bundles \mathcal{L}^s on C such that \mathcal{L}^2 is trivial and $\mathcal{L}(k - 1) := \mathcal{L} \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(k - 1)$ is real; and (iii) $H^0(C, \mathcal{L}^s(k - 2)) = 0$ for all $s \in (0, 2)$. Introducing coordinates ζ, η on $T\mathbb{P}^1$ corresponding to the base \mathbb{P}^1 and fibre, respectively, then $\mathcal{L}^s \rightarrow T\mathbb{P}^1$ (and by restriction, to C) is the (holomorphic) line bundle defined by $\exp(s\eta/\zeta)$ with $\eta/\zeta \in H^1(T\mathbb{P}^1, \mathcal{O})$. Note that $T\mathbb{P}^1$ arises here because it is the mini-twistor space of oriented geodesics in Euclidean 3-space [20]. The action of τ is $(\zeta, \eta) \mapsto (-1/\bar{\zeta}, -\bar{\eta}/\zeta^2)$, and so a generic spectral curve satisfying the reality constraints may be written as the zero set of a polynomial

$$0 = P(\zeta, \eta) := \eta^k + \sum_{r=1}^k p_{2r}(\zeta)\eta^{k-r}, \quad p_{2r}(\zeta) = (-1)^r \zeta^{2r} \overline{p_{2r}(-1/\bar{\zeta})},$$

where p_{2r} is a polynomial of degree $2r$ in ζ . In what follows, we shall refer to *Nahm data* as matrices $\{T_i\}$ satisfying Nahm’s three constraints and a *monopole spectral curve* as a curve C satisfying Hitchin’s three constraints.

Our understanding of monopoles, the self-duality equations and Nahm’s equation have developed greatly in the 40 years since [21]. The moduli space of monopoles of a given charge has attracted much attention, and a rational map description [14] allows different insights and facilitates numerical solutions; integrable systems techniques have also been brought to the fore [11, 17]. Yet despite this progress very few monopole spectral curves have been found in the intervening period owing to the transcendental nature of the Hitchin conditions (see [8]). While monopoles of charge 1 and 2 are well-understood (for a review, see [12]) little progress has been made for higher

charges. In all cases known the use of symmetry to simplify the conditions has been required; in nearly all of these we may quotient by the group of symmetries to an elliptic curve. Motivated by this history our first result here is to classify all possible charge-3 monopole spectral curves by their automorphism groups and within these identify those with elliptic quotients. From these we will then construct new monopole spectral curves with D_6 and V_4 symmetry. In Sect. 2 we prove:

Theorem 1.1 *Let $\mathcal{C} \subset T\mathbb{P}^1$ be a charge-3 monopole spectral curve with $H \leq \text{Aut}(\mathcal{C})$ such that the quotient genus $g(\mathcal{C}/H) = 1$. Then, up to an automorphism of $T\mathbb{P}^1$, the curve is given by the vanishing of one of the following 5 forms:*

1. $\eta^3 + \eta[(a+ib)\zeta^4 + c\zeta^2 + (a-ib)] + [(d+ie)\zeta^6 + (f+ig)\zeta^4 - (f-ig)\zeta^2 - (d-ie)],$
2. $\eta^3 + \eta[a(\zeta^4 + 1) + b\zeta^2] + ic\zeta(\zeta^4 - 1),$
3. $\eta^3 + a\eta\zeta^2 + ib\zeta(\zeta^4 - 1),$
4. $\eta^3 + a\eta\zeta^2 + b(\zeta^6 - 1),$
5. $\eta^3 + ia\zeta(\zeta^4 - 1),$

where $a, b, c, d, e, f, g \in \mathbb{R}$.

Our result does not itself guarantee the existence of monopole spectral curves in these families, and the classes intersect (for example, 5 is a special case of 3). In previous works monopole spectral curves of the form 3 and 5 have been understood in [26] and [22] as corresponding to charge-3 twisted line scattering and the tetrahedrally-symmetric monopole, respectively, while one special case of the form 2 was understood in [25] as the class of inversion-symmetric monopoles, with another in [21] as the axially-symmetric 3-monopole. Curves of the form 2 had been observed in [23, (3.71)], but the Hitchin constraints were only imposed for a restricted subset. In Theorem 3.2 we determine the general monopole spectral curve and Nahm data in class 4 and in Theorem 4.3 the same for class 2. This provides the necessary data in order to plot energy density isosurfaces following [24]¹; Fig. 1 gives for example a previously unknown V_4 configuration. Along with the class of charge-3 monopoles described via an implicit condition in [9], these form all the charge-3 monopole spectral curves currently known, which fit together as shown in Fig. 2 for some parameter values. Figure 3 shows the relations between the symmetry groups of the curves.

Our approach is as follows. In Sect. 2 of the paper we will prove Theorem 1.1. Once we have the automorphism groups of interest we will take the procedure introduced in [22] and developed in [24–26] and apply this to the relevant symmetry. This procedure is recalled in Appendix B where Nahm's equations for case 4 are reduced to the Toda equations before further reduction is described in the text. Similarly in Appendix C Nahm's equations for the V_4 symmetric case are determined. This yields a complex extension of the Euler equations. We then show how these equations are solved in terms of elliptic functions on the quotient elliptic curve, first in Sect. 3 for the D_6 -symmetric monopole and then in Sect. 4 for the V_4 -symmetric monopole. Here the rationale for focussing on elliptic quotients is most evident: the transcendental constraints implicit in the works of Hitchin and Ercolani-Sinha become ones regarding periods of elliptic

¹ We are grateful to Paul Sutcliffe for providing us with the initial code which we modified to make our plots.

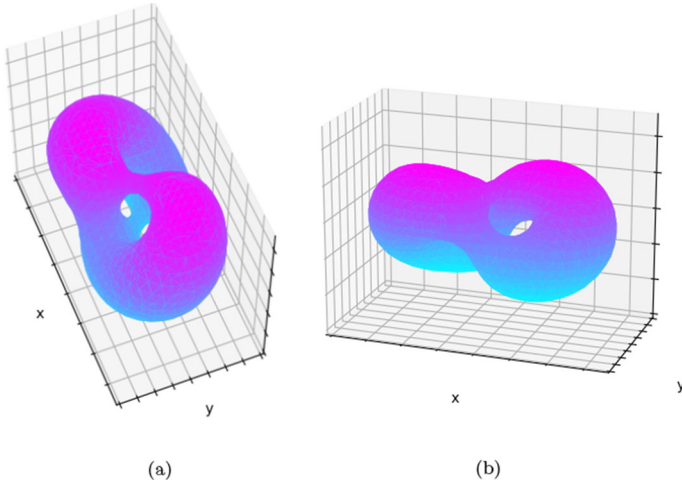


Fig. 1 Surface of constant energy density $\mathcal{E} = 0.18$ for the V_4 monopole given by the parameters (see Theorem 4.3) $m = 0.6, \alpha = -2.0, \text{sgn} = 1$

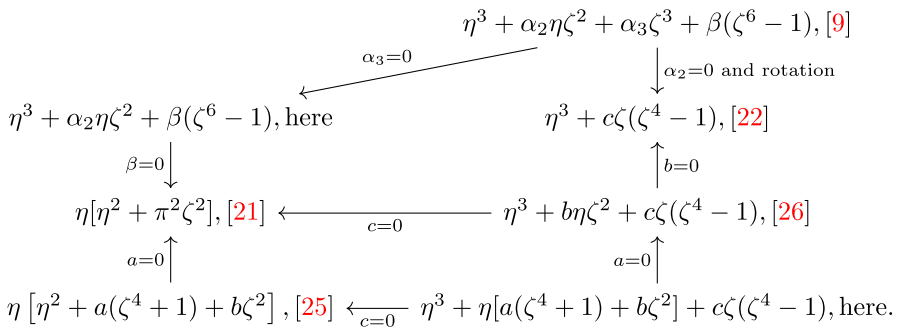
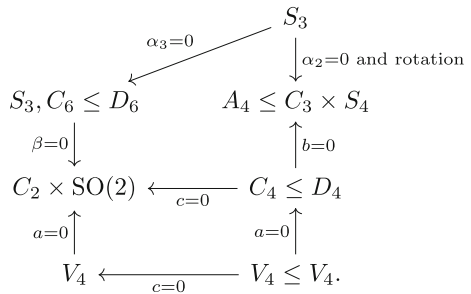


Fig. 2 Known charge-3 spectral curves and their relations. We do not specify the constraints on the parameters

Fig. 3 Automorphism groups of known charge-3 spectral curves and their relations, presented as G or $H \leq G$ where G is the full automorphism group and H is the subgroup quotienting to an elliptic curve when it exists



functions. We relegate to Appendix A a number of properties of elliptic and related functions used in the text and proofs of some statements requiring these. We will not deal with the remaining C_2 -symmetric case here discussing this further in Sect. 5 which is a conclusion.

2 Classifying curves by automorphism group

In this section, we determine the charge-3 monopole spectral curves we shall focus on, beginning with minimal restrictions and gradually imposing these.

A monopole spectral curve is a compact algebraic curve \mathcal{C} lying in Euclidean mini-twistor space \mathbb{MT} , the space of oriented lines in Euclidean 3-space. If the direction of the oriented line is given by ζ , an affine coordinate of $[\zeta_0 : \zeta_1] \in \mathbb{P}^1$, and $\eta \in \mathbb{C}$ describes the point in the plane perpendicular to this through which the line passes then we have $\eta\partial_\zeta \in T\mathbb{P}^1 \cong \mathbb{MT}$. A generic charge- k monopole spectral curve may then be written as the zero set of a polynomial $0 = \eta^k + \sum_{r=1}^k p_{2r}(\zeta_0, \zeta_1)\eta^{k-r}$, where p_{2r} is a homogeneous polynomial of degree $2r$ in ζ_0, ζ_1 ; equivalently a polynomial of degree $2r$ in ζ . Now $T\mathbb{P}^1$ is non-compact and two compactifications of this are common, either by inclusion in the (singular) weighted projective space $\mathbb{P}^{1,1,2} = \{(\zeta_0, \zeta_1, \eta) \in \mathbb{C}^3 \setminus \{0\} \mid (\zeta_0, \zeta_1, \eta) \sim (\lambda\zeta_0, \lambda\zeta_1, \lambda^2\eta), \lambda \in \mathbb{C}^*\}$, or (as by Hitchin) in the Hirzebruch surface \mathbb{F}_2 . We adopt the former view and note that the singular point $[0 : 0 : 1]$ does not lie in $T\mathbb{P}^1$ and hence on \mathcal{C} . Next, via the Veronese embedding, we have $\iota : \mathbb{P}^{1,1,2} \hookrightarrow \mathbb{P}^3$, $\iota([\zeta_0 : \zeta_1 : \eta]) = [\zeta_0^2 : \zeta_0\zeta_1 : \zeta_1^2 : \eta]$. Under this a homogeneous polynomial of degree $2r$ becomes a homogeneous polynomial of degree r in the new coordinates and $\mathbb{P}^{1,1,2}$ becomes a cone, a quadric, over the cone point $\iota([0 : 0 : 1])$ (see, for example, [37, §8.2.11]). Thus a monopole spectral curve may be viewed as the complete intersection of a quadric cone and a degree- k hypersurface in \mathbb{P}^3 . This is known to be a curve of genus $(k-1)^2$ (see for example [18, Exercise V.2.9]) which is non-hyperelliptic for $k \geq 3$ ([18, Exercise IV.5.1]). For $k = 3$ we then have that \mathcal{C} is a non-hyperelliptic genus-4 curve.

In 1895 Wiman [39] classified all non-hyperelliptic genus-4 curves by their automorphism group and gave explicit defining equations for these. Wiman's classification had two families: curves arose either as the intersections of a cubic surface and non-singular quadric in \mathbb{P}^3 , or as the intersection of a cubic surface and quadric cone in \mathbb{P}^3 . Thus charge-3 monopole spectral curves with automorphism group must lie in Wiman's second family. (The two rulings of the non-singular quadric of Wiman's first family lead to projections from the curve to $\mathbb{P}^1 \times \mathbb{P}^1$, which is relevant for spectral curves of hyperbolic monopoles; this will be developed elsewhere.) We note that although $\text{Aut}(\mathbb{P}^3) = \text{PGL}(4, \mathbb{C})$ differs from $\text{Aut}(\mathbb{P}^{1,1,2}) \cong \mathbb{C}^3 \rtimes (\text{GL}(2, \mathbb{C})/\{\pm \text{Id}_2\})$ (with the natural action of $\text{GL}(2, \mathbb{C})$ on (ζ_0, ζ_1) inducing that on $(\zeta_0^2, \zeta_0\zeta_1, \zeta_1^2)$), Wiman, in determining his normal forms, considered only those transformations of $\text{Aut}(\mathbb{P}^3)$ that preserved the cone, and so his normal forms include all possible charge-3 monopole spectral curves. In Table 1, we give those curves in Wiman's classification which lie on a cone presenting² these in terms of a curve given by the vanishing of a

² We set $y = 1$ in Wiman's notation, so as to make clear the connection to monopole spectral curves.

Table 1 Potential charge-3 monopole spectral curves with nontrivial automorphism group and those (with subgroups) quotienting to genus 1

P	G	c_G	H	c_H
$z^3 + z(ax^4 + bx^2 + c) + (dx^6 + ex^4 + fx^2 + g)$	C_2	$(1; 2^6)$	C_2	$(1; 2^6)$
$z^3 + z(ax^4 + bx^2 + c) + dx(x^4 + ex^2 + f)$	C_2	$(2; 2^2)$		
$z^3 + z[a(x^4 + 1) + bx^2] + x[c(x^4 + 1) + dx^2]$	$C_2 \times C_2$	$(0; 2^7)$		
$z^3 + z[a(x^4 + 1) + bx^2] + x(x^4 - 1)$	$C_2 \times C_2$	$(1; 2^3)$	C_2^2	$(1; 2^3)$
$z^3 + azx^2 + x(x^4 + 1)$	D_4	$(0; 2^4, 4)$	C_4	$(1; 4^2)$
$z^3 + z(x^4 + a) + (bx^4 + c)$	C_4	$(0; 2, 4^4)$		
$z^3 + azx^2 + x^6 + bx^3 + 1$	S_3	$(0; 2^6)$		
$z^3 + azx^2 + x^6 + 1$	D_6	$(0; 2^5)$	S_3 and C_6	$(1; 2^2)$
$z^3 + z(ax^3 + b) + (x^6 + cx^3 + d)$	C_3	$(1; 3^3)$	C_3	$(1; 3^3)$
$z^3 + az(x^3 + 1) + (x^6 + 20x^3 - 8)$	A_4	$(0; 2, 3^3)$		
$z^3 + az + x^6 + b$	C_6	$(0; 2, 6^3)$		
$z^3 + z + x^6$	C_{12}	$(0; 4, 6, 12)$		
$z^3 + az + x^5 + b$	C_5	$(0; 5^4)$		
$z^3 + z + x^5$	C_{10}	$(0; 5, 10^2)$		
$z^3 - (x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f)$	C_3	$(0; 3^6)$		
$z^3 - (x^6 + ax^4 + bx^2 + 1)$	C_6	$(0; 2^2, 3^3)$		
$z^3 - x(x^4 + ax^2 + 1)$	$C_6 \times C_2$	$(0; 2^2, 3, 6)$		
$z^3 - (x^6 + ax^3 + 1)$	$C_3 \times S_3$	$(0; 2^2, 3^2)$		
$z^3 - (x^5 + 1)$	C_{15}	$(0; 3, 5, 15)$		
$z^3 - (x^6 + 1)$	$C_6 \times S_3$	$(0; 2, 6^2)$		
$z^3 - x(x^4 + 1)$	$C_3 \times S_4$	$(0; 2, 3, 12)$	A_4	$(1; 2)$

polynomial $P(x, z)$. We also write down their full automorphism group $G := \text{Aut}(\mathcal{C})$ and the corresponding signature $c := c_G = (g_0; c_1, \dots, c_r)$ giving the quotient genus $g_0 = g(\mathcal{C}/G)$ and the ramification indices c_i of the quotient map $\mathcal{C} \rightarrow \mathcal{C}/G$ (see [30]). These have been calculated with the help of the information available from [29]. We make some remarks about Table 1.

- The label D_n refers to the dihedral group of order $2n$, following the convention [29].
- Wiman's parameters are to be understood as generic: there may exist specific values of the parameters for which the automorphism group is larger than that indicated.
- Wiman provides a form where the z^2 term is always zero, equivalent to centring the monopole.
- All the curves given are irreducible, so we can only find reducible spectral curves as limiting members of the above families.
- The completeness of the above data on signatures and elliptic quotients is reliant on the completeness of the data of the LMFDB.
- We recognise the curve with $C_3 \times S_4$ symmetry as corresponding to the tetrahedrally-symmetric monopole.

Not all curves on the list will yield monopoles spectral curves; for example by the following result.

Proposition 2.1 ([10]) *There are only two curves in the family $\eta^3 + \chi(\zeta^6 + b\zeta^3 + 1) = 0$, $\chi, b \in \mathbb{R}$, that correspond to BPS monopoles; these are tetrahedrally-symmetric monopole spectral curves.*

2.1 Moduli space dimension

We now also briefly discuss another aspect of this group theoretic approach that can be used to identify particularly tractable monopole curves.

There is a moduli space N_k of charge- k monopoles up to gauge transform, which one typically enlarges by a phase to M_k which has a natural action of the Euclidean group $E(3)$ and circle group $U(1)$. Associated with this is the submanifold of (strongly-) centred charge- k monopoles $M_k^0 \subset M_k$ with action of the orthogonal group $O(3)$ which parametrises monopoles up to gauge transform with fixed centre [4, 31]. The spectral curves corresponding to such monopoles have $a_1(\zeta) = 0$, while the corresponding Nahm data have $\text{Tr}(T_i) = 0$. M_k^0 is a totally geodesic manifold of real dimension $4(k-1)$, and given $G \leq O(3)$ we may consider the submanifold of the moduli space of G -invariant strongly-centred monopoles $(M_k^0)^G$, which is also totally geodesic [6]. This we shall distinguish from $(M_k^0)^{[G]}$ which is the moduli space of monopoles invariant under an element of the conjugacy class of G inside $SO(3)$; clearly $(M_k^0)^G \subset (M_k^0)^{[G]}$ and when G is discrete we have $\dim_{\mathbb{R}}(M_k^0)^G = \dim_{\mathbb{R}}(M_k^0)^{[G]} - \dim_{\mathbb{R}} SO(3)$. As scattering of magnetic monopoles is approximated by geodesic motion in the moduli space [34], if one can find $G \leq O(3)$ for which $\dim(M_k^0)^G = 1$, it is known that the corresponding 1-parameter family of monopoles corresponds to a scattering

process. We may use group theory to help identify such families via the following result.

Proposition 2.2 *For discrete G , $\dim_{\mathbb{R}}(M_3^0)^G \leq 3g_0 - 3 + r$.*

Proof [30, Lemma 3.1] gives that the complex dimension of each component of the locus of equivalence classes of genus $g \geq 2$ curves admitting an action of a group isomorphic to G with signature c is (provided it is non-empty) $\delta(g, G, c) := 3(g_0 - 1) + r = \dim_{\mathbb{C}} \mathcal{M}_{g_0, r}$ the moduli space of genus g_0 curves with r marked points. The $SO(3)$ action on $(M_3^0)^{[G]}$ is trivial on the moduli space of curves because it induces a birational isomorphism. The result then follows as the $SO(3)$ orbits of the moduli space of monopoles will form a component of this locus, hence³ $\dim_{\mathbb{R}}(M_3^0)^G = \dim_{\mathbb{R}}(M_3^0)^{[G]} - \dim_{\mathbb{R}} SO(3) \leq \dim_{\mathbb{R}}(\mathcal{M}_{g_0, r}^{\tau})$, and using the fact from Teichmüller theory that $\dim_{\mathbb{R}}(\mathcal{M}_{g_0, r}^{\tau}) = \dim_{\mathbb{C}} \mathcal{M}_{g_0, r}$ [16, §3.1]. □

Remark 2.3 Note in the above we could have used $H \leq G$ and its corresponding signature, but this would have given a weaker bound as $\delta(g, G, c_G) \leq \delta(g, H, c_H)$ [30].

The remarkable fact about this is that the bound depends on the signature only. The curves of [22], with automorphism group H of order $2k(k - 1)$ for $k = 3, 4, 6$, have signature $c_H = (1; k - 1)$ [22, Proposition 4]. For these curves and all known monopole spectral curves of charge 3 we have $\dim(M_k^0)^H = \delta(g, H, c_H) - 1$, and hence one might conjecture that this is always true in the case $g_0 = 1$. A calculation for the case of inversion-symmetric monopoles considered in [6] for which there is a C_2 action $\eta \mapsto -\eta$ shows that such a conjecture would certainly not be true for all g_0 .

2.2 Genus-1 reductions

Table 1 gives us a list of putative spectral curves with symmetry before we have imposed the further constraints of Hitchin. We know from [21] that Nahm’s equations correspond to a linear flow in the Jacobian of the corresponding spectral curve \mathcal{C} ; the direction of this linear flow is given by the Ercolani-Sinha vector U [17]. Braden [7] has shown that when we have a symmetry group G we may be able to reduce to the quotient curve $\mathcal{C} \xrightarrow{\pi} \mathcal{C}' := \mathcal{C}/G$ and reduced Ercolani-Sinha vector U' when $U = \pi^*U'$. For example charge- k monopoles with C_k symmetry reduce to questions about a genus- $(k - 1)$ hyperelliptic curve. The $k = 3$ case was studied in [9].

Notwithstanding the attendant simplifications, the list of Table 1 is too long for the purposes of this letter and we require a further criterion to reduce this. Here we adopt the following: does the genus-4 spectral curve (assumed with real structure) quotient

³ As described in Earle, points in the moduli space of genus $g \geq 2$ Riemann surfaces correspond to equivalence classes of complex structures on a fixed smooth surface of the corresponding genus. There is a natural conjugate involution on this moduli space of complex structures, such that any complex structure fixed under an orientation-reversing involution of the surface corresponds to a fixed point of the conjugate involution of the moduli space. Hence we may consider the moduli space $\mathcal{M}_{g_0, r}^{\tau} \subset \mathcal{M}_{g_0, r}$ of curves invariant under τ by this procedure.

(either by $\text{Aut}(\mathcal{C})$ or a subgroup) to an elliptic curve? The rationale for this is that the remaining of Hitchin’s conditions are most straightforwardly answered for elliptic curves; equivalently the Ercolani-Sinha constraint becomes one on the real period of an elliptic curve. There are also a number of curves known with this property [22, 24–26].

Thus we seek curves \mathcal{C} with real structure from Wiman’s list for which there exists $H \leq \text{Aut}(\mathcal{C})$ such that $g(\mathcal{C}/H) = 1$. Here we may use the database of [29] which has enumerated all the possible H and the corresponding signatures for genus-4 curves. We may then use our knowledge of the explicit forms of the curves to match up these cases, which leaves us with the reduced list in the final two columns of Table 1. As previously noted, the $H = A_4$ case corresponds to the tetrahedrally-symmetric monopole [22], and the $H = C_4$ case has already been solved in [26]. We also see that the cases $H = S_3$ and $H = C_6$ arise from the same curve, indicating that the curve has two distinct quotients to an elliptic curve.

In the following sections we will investigate in more detail the two new cases $H = C_6$ (or equivalently $H = S_3$) with full automorphism group $G = D_6$, and $H = V_4$ (with full automorphism group $G = V_4$); we do not treat the C_2 case here. We will begin with the D_6 case which is both illustrative and simpler, though ultimately the new solutions and their scattering family are less interesting.

Before turning to these however we may complete the proof of Theorem 1.1. With the exception of the $H = C_3$ curve, imposing reality on the curves with groups H listed in Table 1 yields the curves of Theorem 1.1 (and in the same order). Note not all $M \in \text{GL}_2(\mathbb{C})/\langle -1 \rangle \leq \text{Aut}(\mathbb{P}^{1,1,2})$ will commute with the action of τ , but $S \in \text{SU}(2)$ will. One may use Schur decomposition to write $M = STS^{-1}$ for some $S \in \text{SU}(2)$, T an upper-triangular matrix, and so when imposing reality on Wiman’s normal forms one should consider the orbits under upper-triangular matrices. The only real forms present in the orbit of the $G = H = C_3$ family have $a = b = 0$ in the corresponding defining equation P ; the resulting curve then lies in the family described by Proposition 2.1. Only the tetrahedrally-symmetric monopole within this family quotients to an elliptic curve and by a rotation this may written as $\eta^3 + ia\zeta(\zeta^4 - 1) = 0$, the final entry of the Theorem. We have thus established Theorem 1.1.

3 D_6 monopoles

To understand the spectral curves of this section it helps to first begin with the general centred C_k invariant spectral curve \mathcal{C} (with reality imposed),

$$\eta^k + \alpha_2 \eta^{k-2} \zeta^2 + \alpha_3 \eta^{k-3} \zeta^3 + \dots + \alpha_{k-1} \eta \zeta^{k-1} + \alpha_k \zeta^k + \beta[\zeta^{2k} + (-1)^k] = 0, \tag{2}$$

where $\alpha_k, \beta \in \mathbb{R}$. This is invariant under $s : (\eta, \zeta) \rightarrow (\omega\eta, \omega\zeta)$, $\omega = \exp(2\pi i/k)$ with $C_k = \langle s \rangle$. The work of [7] shows (2) is the unbranched cover of the hyperelliptic curve

$$y^2 = (x^k + \alpha_2 x^{k-2} + \alpha_3 x^{k-3} + \dots + \alpha_k)^2 - (-1)^k 4\beta^2, \tag{3}$$

where $x = \eta/\zeta$ and $y = \beta[\zeta^k - (-1)^k \zeta^{-k}]$. The curve (2) also has the symmetry $t : (\zeta, \eta) \mapsto (-1/\zeta, -\eta/\zeta^2)$ and $G = \langle s, t \rangle = D_k$ is the full automorphism group. The transformation t corresponds to a reflection $\underline{v} \rightarrow \text{diag}(1, -1, 1)\underline{v}$ in $O(3, \mathbb{R})$ and it becomes the hyperelliptic involution $t : (x, y) \rightarrow (x, -y)$ on the quotient curve. For $k = 3$ we are describing the curve in Table 1 with full automorphism group $G = S_3$ ($= D_3$ in [29] notation). Further, the work of [7] shows that Nahm’s equations for every charge- k monopole with C_k rotational symmetry are equivalent to the $A_{k-1}^{(1)}$ affine Toda equations (in real Flaschka variables)

$$a'_i = \frac{1}{2}a_i(b_i - b_{i+1}), \quad b'_i = a_i^2 - a_{i-1}^2, \tag{4}$$

where i is taken mod k , and we use' to denote $\frac{d}{ds}$. These equations may also be found in other ways. In Appendix B we describe how taking the C_k invariant polynomials $Q_i = \zeta_0^i \zeta_1^i, i = 1, \dots, k$ and $Q_{k+1} = \zeta_0^k - \zeta_1^k$ as the inputs to the procedure of [22] we obtain equations (4).

In general the solutions to the Toda system (4) linearise on the genus- $(k - 1)$ Jacobian of the curve (3). For $k = 3$ this was the approach taken in [9] where a family of monopoles including the tetrahedrally-symmetric monopole was investigated. Are simplifications possible? In the remainder of this section, we shall show that for $k = 3$ a one-parameter family of elliptic Nahm data exists with C_3 symmetry. As this is contrary to results in the literature we begin with four results that lead to such an elliptic reduction before determining the parameters that yield Nahm data. Three particular points in the family will be identified before concluding with a description of the scattering described by the family.

3.1 Four lessons

For $k = 3$ equations (4) take the form (with $a_0 \equiv a_3, b_0 \equiv b_3$)

$$\begin{aligned} a'_0 &= \frac{1}{2}a_0(b_3 - b_1), & a'_1 &= \frac{1}{2}a_1(b_1 - b_2), & a'_2 &= \frac{1}{2}a_2(b_2 - b_3), \\ b'_1 &= a_1^2 - a_0^2, & b'_2 &= a_2^2 - a_1^2, & b'_3 &= a_0^2 - a_2^2, \end{aligned} \tag{5}$$

coming from Nahm matrices derived in Appendix B given by

$$\begin{aligned} T_1 &= \frac{1}{2} \begin{pmatrix} 0 & a_1 & -a_0 \\ -a_1 & 0 & a_2 \\ a_0 & -a_2 & 0 \end{pmatrix}, & T_2 &= \frac{1}{2i} \begin{pmatrix} 0 & a_1 & a_0 \\ a_1 & 0 & a_2 \\ a_0 & a_2 & 0 \end{pmatrix}, \\ T_3 &= \frac{-i}{2} \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}. \end{aligned} \tag{6}$$

⁴ This is the notation of [19].

Here, we find the constants

$$\alpha_2 = b_1 b_2 + b_1 b_3 + b_2 b_3 + a_0^2 + a_1^2 + a_2^2, \alpha_3 = b_1 b_2 b_3 + b_1 a_2^2 + b_2 a_0^2 + b_3 a_1^2, \quad \beta = a_0 a_1 a_2,$$

for the (centred) spectral curve

$$\eta^3 + \alpha_2 \eta \zeta^2 + \alpha_3 \zeta^3 + \beta(\zeta^6 - 1) = 0$$

which covers

$$y^2 = (x^3 + \alpha_2 x + \alpha_3)^2 + 4\beta^2. \tag{7}$$

We have 6 differential equations, 6 variables and three conserved quantities.

3.1.1 Direct simplification

Appendix B shows that we may use the constants $\alpha_2, \alpha_3, 0 = \sum b_i$ to eliminate the b_i , resulting in the equations

$$0 = \sum_{i=0}^2 a_i^2 - \alpha_2 - \frac{1}{3}(d_1^2 + d_1 d_2 + d_2^2), \quad 0 = a_1^2 d_2 - a_2^2 d_1 + \alpha_3 + \frac{1}{3}\alpha_2(d_1 - d_2) + \frac{1}{27}(d_1 - d_2)^3,$$

where we have introduced $d_i = 2a'_i/a_i$. Using $\beta = a_0 a_1 a_2$ to eliminate a_0 we then have two nonlinear ODE's in two variables, the maximal reduction one can achieve with generic α_i and β . The appendix shows further that if $a_1^2 = a_2^2$ additional simplification is possible; and that we may consistently set $a_1^2 - a_2^2 = 0$ provided $b_2 a_1^2 = 0$. As $b'_2 = a_2^2 - a_1^2$, this means we can consistently set $a_1^2 = a_2^2$ and $b_2 = 0$. Making these restrictions we find $\alpha_3 = 0$ and that we reduce to one equation

$$a_1^2 \left(2 \frac{da_1}{ds} \right)^2 = \beta^2 + 2a_1^6 - \alpha_2 a_1^4.$$

Upon setting $u = a_1^2$ this becomes

$$\left(\frac{du}{ds} \right)^2 = \beta^2 + 2u^3 - \alpha_2 u^2, \tag{8}$$

to which we shall return. We record that the j -invariant of the associated elliptic curve $y^2 = \beta^2 + 2u^3 - \alpha_2 u^2$ is $16\alpha_2^6/(\beta^2[\alpha_2^3 - 27\beta^2])$.

3.1.2 Sutcliffe's ansatz

Some time ago, in the context of Seiberg-Witten theory, Sutcliffe [35] gave an ansatz for charge- k cyclically symmetric monopoles in terms of affine Toda theory. With $a_j =$

$\gamma e^{(q_j - q_{j+1})/2}$, $b_j = q'_j$ equations (4) follow from a Hamiltonian⁵ $H = \frac{1}{2} \sum_{j=1}^k b_j^2 - \sum_{j=0}^{k-1} a_j^2 = \frac{1}{2} \sum_{j=1}^k p_j^2 - \gamma^2 \sum_{j=0}^{k-1} e^{q_j - q_{j+1}}$. (The constant γ here is to account for the constant $\prod_{i=0}^{k-1} a_i = \gamma^k = (-1)^{k-1} \beta$.) Sutcliffe showed that for $k = 2$ Nahm data could be constructed, but for $k = 3$ although he could solve the equations he couldn't find solutions with the correct pole behaviour. The solution was obtained from the infinite chain solution as follows. We have from

$$\left(\ln a_j^2\right)'' = -a_{j-1}^2 + 2a_j^2 - a_{j+1}^2$$

and the standard elliptic function identity for the Weierstrass \wp -function

$$\frac{d^2}{du^2} \ln[\wp(u) - \wp(v)] = -\wp(u+v) + 2\wp(u) - \wp(u-v)$$

that with $u = ju_0 + t + t_0$ and $v = u_0$ then

$$\begin{aligned} \frac{d^2}{dt^2} \ln[\wp(ju_0 + t + t_0) - \wp(u_0)] &= -\wp([j + 1]u_0 + t + t_0) \\ &+ 2\wp(ju_0 + t + t_0) - \wp([j - 1]u_0 + t + t_0) \end{aligned}$$

and we may identify $a_j = \wp(ju_0 + t + t_0) - \wp(u_0)$. This yields the solution for the infinite chain and we must still impose periodicity to obtain a solution. Imposing periodicity yields (for $k = 3$) that $a_j = \wp(2jK/3 + t) - \wp(2K/3)$ which is equivalent to the solution of [35] which is given in Jacobi elliptic functions.⁶ Now the ansatz employed here forces only one of the a_j to be singular at any point, and this means the pole condition on the Nahm matrices cannot be satisfied. If we are to find an alternative solution that does indeed yield a monopole then this would suggest that one appropriate route would be to pick a simplification which forces multiple variables to have poles simultaneously. Such is the case when $a_1^2 = a_2^2$ found previously.

⁵ This Hamiltonian is unbounded from below: such is necessary as the monopole boundary conditions require a pole at $s = 0, 2$ hence the momenta and correspondingly the potential must also be unbounded below. Thus while the dynamical system being described is integrable, a corresponding interpretation in terms of a mechanical system is less helpful. Further, while the a_i will always be real throughout we have freedom to choose their sign and we will make sign choices for the a_i where we cannot take $\log a_i$ and obtain real values for the associated Toda position variables.

⁶ To make connection with [35] we use Lawden's notation [28, §6.3.1, §6.9] Thus for $k = 3$ we take $u_0 = 2K/3$. Now

$$\begin{aligned} dc^2(u) &= \frac{\wp(u) - e_2}{\wp(u) - e_1} = 1 + \frac{e_1 - e_2}{\wp(u) - e_1} = 1 + \frac{1}{e_1 - e_3} [\wp(u + \omega_1) - e_1] = 1 + [\wp(u + \omega_1) - e_1], \\ cs^2(2K/3) &= \wp(2K/3) - e_1. \end{aligned}$$

Note Sutcliffe's ' q_j^2 ' is our a_j . Then [35, 3.41] is $dc^2(u) - 1 - cs^2(2K/3) = \wp(u + \omega_1) - \wp(2K/3) = \wp(u + K) - \wp(2K/3)$ and so with $u = 2jK/3 + t + K$ (his choice of δ) we get $a_j = \wp(2jK/3 + t) - \wp(2K/3)$ and the corresponding asymptotics given in [35, 3.42-45].

3.1.3 Imposing symmetry on Nahm matrices

We next show that $a_1^2 = a_2^2$ follows from the symmetry⁷ $\underline{v} \rightarrow A\underline{v} := \text{diag}(1, -1, -1)\underline{v}$ in $\text{SO}(3, \mathbb{R})$. From [31, Equation (8.169)] we know that the conditions for the Nahm matrices to be symmetric under A are that

$$T_1 = CT_1C^{-1}, \quad -T_2 = CT_2C^{-1}, \quad -T_3 = CT_3C^{-1},$$

for some constant invertible matrix C . Recalling the form of the T_i from (6) we see that as T_3 is diagonal and traceless the only way to achieve the invariance under A is if at least one of the b_i is 0 and C permutes the other two. By conjugating with a permutation matrix we can without loss of generality pick $b_2 = 0$ so $b_1 = -b_3$ which

gives that the generic C is $C = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}$. Picking a generic a, b, c we get

$$a_0(a + c) = a_1a - a_2b = a_1c + a_2b = a_1b + a_2c = a_2a - a_1b = 0.$$

To avoid having an $a_i = 0$ we required $a = -c$, and so these reduce to

$$a_1a - a_2b = 0 = a_1b - a_2a,$$

and consequently $(a/b)^2 = 1$ and $a_1 = \pm a_2$ yielding the desired $a_1^2 = a_2^2$. Note that this also means $\alpha_3 = 0$.

3.1.4 Reduction of the spectral curve; folding

In order to get the curve with D_6 symmetry of Table 1, we must set $\alpha_3 = 0$. We have seen that for $k = 3$ this is a consequence of the symmetry $r : (\zeta, \eta) \rightarrow (1/\zeta, -\eta/\zeta^2)$. For general k this means we keep only the even terms of (2),

$$\eta^k + \alpha_2\eta^{k-2}\zeta^2 + \alpha_4\eta^{k-4}\zeta^4 + \dots + \beta[\zeta^{2k} + (-1)^k] = 0. \tag{9}$$

The full automorphism group of this curve is $D_k \times C_2$; for $k = 3$ this is the curve with full automorphism group $D_6 \cong D_3 \times C_2$ that we are interested in. Setting $x = \eta/\zeta$ in (9) we have

$$x^k + \alpha_2x^{k-2} + \alpha_4x^{k-4} + \dots + \alpha_k + \beta[\zeta^k + \zeta^{-k}] = 0, \quad k \text{ even,}$$

⁷ We have the correspondences between the \mathbb{R}^3 transformations and (ζ, η) actions

$$r := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \leftrightarrow (\zeta, \eta) \mapsto (1/\zeta, -\eta/\zeta^2), \quad rt := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \leftrightarrow (\zeta, \eta) \mapsto (1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2).$$

If the rotation $s := (\eta, \zeta) \rightarrow (\omega\eta, \omega\zeta)$ takes place in the xy -plane then we obtain the point groups $D_k = \langle s, r \rangle$, $C_{kv} \cong D_k = \langle s, t \rangle$ and $C_{kh} \cong C_k \times C_2 = \langle s, rt \rangle$. The prismatic dihedral group $D_{kh} = \langle s, r, t \rangle$ is obtained by adding any two of the above to the rotations, so giving the third. Abstractly $D_{kh} \cong D_k \times C_2$.

$$x^k + \alpha_2 x^{k-2} + \alpha_4 x^{k-4} + \dots + \alpha_{k-1} x + \beta[\zeta^k - \zeta^{-k}] = 0, \quad k \text{ odd.}$$

If $y = \beta[\zeta^k - (-1)^k \zeta^{-k}]$ then $r : (x, y) \rightarrow (-x, (-1)^{k-1} y)$; thus y is invariant under r only for k odd, in which case it will be a function on the quotient curve $\hat{C}/\langle s, r \rangle$; for k -even $v = xy$ is invariant. Thus we have curves

$$\begin{aligned} v^2 &= x^2(x^k + \alpha_2 x^{k-2} + \alpha_4 x^{k-4} + \dots + \alpha_k)^2 - 4\beta^2 x^2 && k \text{ even,} \\ y^2 &= (x^k + \alpha_2 x^{k-2} + \alpha_4 x^{k-4} + \dots + \alpha_{k-1} x)^2 + 4\beta^2 && k \text{ odd.} \end{aligned}$$

Setting $k = 2l$ or $k = 2l - 1$ for the even and odd cases of the curves then with $u = x^2$ we have these curves covering 2 : 1 the curves

$$v^2 = u(u^l + \alpha_2 u^{l-1} + \alpha_4 u^{l-2} + \dots + \alpha_k)^2 - 4\beta^2 u \quad k \text{ even,} \tag{10}$$

$$y^2 = u(u^{l-1} + \alpha_2 u^{l-2} + \alpha_4 u^{l-3} + \dots + \alpha_{k-1})^2 + 4\beta^2 \quad k \text{ odd.} \tag{11}$$

The first has genus l and the second has genus $l - 1$. Under the cyclic transformation, it was shown in [7] that

$$\frac{\eta^{k-2} d\zeta}{\partial_\eta P} = \pi^* \left(-\frac{1}{k} \frac{x^{k-2} dx}{y} \right)$$

for the curve (3) and we observe that this differential is invariant under r for k both even and odd. Further

$$\frac{x^{k-2} dx}{y} = \begin{cases} \frac{x^{2l-2} dx}{y} = \frac{x^{2l-2} du}{2xy} = \frac{u^{l-1} du}{2v}, \\ \frac{x^{2l-3} dx}{y} = \frac{x^{2l-4} du}{2y} = \frac{u^{l-2} du}{2y}. \end{cases}$$

In each case we obtain the maximum degree in u differential on the corresponding hyperelliptic curve and the work of [7] tells us the Ercolani-Sinha vector, if it exists, will reduce to one on the quotient curve.

In particular the $k = 3$ curve $y^2 = (x^3 + \alpha_2 x)^2 + 4\beta^2$ covers the elliptic curve $\mathcal{E} = \mathcal{C}/H$,

$$y^2 = u(u + \alpha_2)^2 + 4\beta^2,$$

with $H = \langle s, r \rangle \cong S_3$. The j -invariant of this curve is $j_{\mathcal{E}} = 16\alpha_2^6 / (\beta^2[\alpha_2^3 - 27\beta^2])$, the value observed earlier. We note that the genus-2 curve also covers the elliptic curve $\mathcal{E}' = \mathcal{C}/H'$,

$$w^2 = u^2(u + \alpha_2)^2 + 4\beta^2,$$

where now $H' = (s, rt) \cong C_6$ with $w = xy$ the invariant coordinate. Because $\pi^*(du/(2w)) = dx/y$ does not pull back to the differential appearing in the Ercolani-Sinha constraint we cannot solve the Hitchin constraints in terms of \mathcal{E}' . We record that the curve is in general distinct $j_{\mathcal{E}'} = (\alpha_2^4 + 48\beta^2)^3 / (\beta^4 [\alpha_2^4 + 64\beta^2])$. We have that \mathcal{E} and \mathcal{E}' are the two quotients identified in Table 1.

We remark that the reduction of the spectral curve we have just described may be understood directly in terms of the Toda equations and ‘folding’. For the $k = 3$ case at hand set $e^{\rho_i} := a_i^2 = \beta^{2/3} e^{q_i - q_{i+1}}$ (so that $a_0 a_1 a_2 = \beta$) and again take $b_j = q'_j$ and Hamiltonian $H = \frac{1}{2} \sum_{j=1}^k b_j^2 - \sum_{j=0}^{k-1} a_j^2 = \frac{1}{2} \sum_{j=1}^k p_j^2 - \beta^{2/3} \sum_{j=0}^{k-1} e^{q_j - q_{j+1}}$. Then the Toda equations take the form

$$\rho_i'' = 2e^{\rho_i} - e^{\rho_{i-1}} - e^{\rho_{i+1}} = \bar{K}_{ij} e^{\rho_j}$$

where \bar{K}_{ij} is the extended Cartan matrix of A_2 . Folding [33] corresponds to the action $\rho_i \rightarrow \rho_{\sigma(i)}$ by a diagram automorphism σ of the extended Dynkin diagram: this retains integrability and here corresponds to identifying $\rho_1 = \rho_2 := \rho_{12}$, equivalently $a_1^2 = a_2^2$. Using $e^{\rho_0} = \beta^2 e^{-2\rho_{12}}$ the equations of motion $\rho_{12}'' = e^{\rho_{12}} - e^{\rho_0}$ and $\rho_0'' = 2(e^{\rho_0} - e^{\rho_{12}})$ reduce to the one equation,

$$\rho_{12}'' = e^{\rho_{12}} - \beta^2 e^{-2\rho_{12}},$$

the ODE reduction of the Bullough-Dodd equation, a known integrable equation. This may be directly integrated. With $u = e^{\rho_{12}}$ we obtain precisely (8). More generally we are seeing the reduction by folding $A_{2l-1}^{(1)} \rightarrow C_l^{(1)}$ for $k = 2l$ even, and $A_{2[l-1]}^{(1)} \rightarrow A_{2[l-1]}^{(2)}$ for $k = 2l - 1$ odd, both coming from an order-2 symmetry of the Dynkin diagram.

3.2 Solving for Nahm data

A number of different arguments lead us then to an elliptic reduction of the Toda equations for $k = 3$ with corresponding ODE (8). The aim of this subsection is to show that from this ODE Nahm data can be constructed. In doing so we will use properties of hypergeometric functions, and we lay out some of these details in Appendix A.

We have seen that the reduction leads to $a_1^2 = a_2^2$ and $b_2 = 0$. In continuing to solve for the Nahm data one finds that the choice of sign of a_2 relative to a_1 does not affect the ability to impose the Hitchin constraints. Indeed, changing the choice of sign merely corresponds to changing the sign of β , and again as we will see this does not restrict the spectral curve. As such we take $a_2 = -a_1$ in what follows. Now setting $\tilde{u} = u - \frac{\alpha_2}{6}$ and $\tilde{s} = s/\sqrt{2}$ we may transform (8) into standard Weierstrass form with solution

$$\tilde{u} = \wp \left((s - s_0)/\sqrt{2}; g_2, g_3 \right),$$

where $g_2 = \frac{\alpha_2^2}{3}$ and $g_3 = \frac{\alpha_2^3}{27} - 2\beta^2$. Here we assume $\Delta := g_2^3 - 27g_3^2 = 4\beta^2(\alpha_2^3 - 27\beta^2) \neq 0$ to avoid nonsingularity, commenting on the singular limits at the appropriate junctures. The j -invariant of the elliptic curve is as we have already seen

$$j = 1728 \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{16\alpha_2^6}{\beta^2(\alpha_2^3 - 27\beta^2)}.$$

To be Nahm data we require that the Nahm matrices have a pole at $s = 0$ which can be achieved by setting $s_0 = 0$. We can then express all the Flaschka variables as

$$a_1 = \pm \sqrt{\wp(s/\sqrt{2}; g_2, g_3) + \frac{\alpha_2}{6}}, \quad a_2 = -a_1, \quad a_0 = \frac{\beta}{a_1 a_2}, \quad (12)$$

$$b_1 = \pm \sqrt{2a_1^2 + a_0^2 - \alpha_2}, \quad b_2 = 0, \quad b_3 = -b_1. \quad (13)$$

We have some signs of the square roots to set above.

- (i) Using that, around $s = 0$, $\wp(s/\sqrt{2}; g_2, g_3) \sim 2s^{-2} \Rightarrow a_1^2 \sim \frac{2}{s^2}$, we have $a_0 \sim \frac{\beta s^2}{2}$. The ODE for a'_0 , with $b_3 = -b_1$, gives

$$b_1 = -\frac{a'_0}{a_0} \sim -\frac{(\beta s)}{(\beta s^2/2)} = -\frac{2}{s}.$$

This requires us to take the negative square root for b_1 around $s = 0$. We will want residues at $s = 2$, and it will turn out by applying similar analysis that we need the positive root around $s = 2$. These swap over when $b_1 = 0$, which corresponds to $a'_1 = 0$. As we see later this must happen at $s = 1$. Alternatively one can see this from the observation that a_1 is even about $s = 1$ by a judicious choice of period, and so $b_1 = \frac{2a'_1}{a_1}$ is odd about the same point.

- (ii) The sign of a_1 is a free choice, and does not affect the geometry of the monopole, hence in what follows below we always take the positive sign.

The corresponding Nahm matrices (6) have residues at $s = 0$ given by

$$R_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_3 = i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which yield a 3-dimensional irreducible representation.

Next we require a simple pole at $s = 2$ again forming a 3-dimensional irreducible representation. There are two ways to achieve a residue at $s = 2$:

- (i) have that $2/\sqrt{2} = \sqrt{2}$ is in the lattice corresponding to the values g_2, g_3 , or
- (ii) have that around $s = 2$, $\wp(s/\sqrt{2}; g_2, g_3) \sim -\frac{\alpha_2}{6} + \mathcal{O}(s - 2)$.

These correspond to having α_1 and α_0 be singular at $s = 2$, respectively. (Because of the constant β they cannot both be singular.) One can check that the second condition would give a reducible representation at $s = 2$ (as again only one of the α_i have a pole here) and so we discount it.

Focussing then on the first condition, one way to fix the real period of the associated lattice is to invert the j -invariant of the elliptic curve corresponding to g_2, g_3 to give the period τ . Here this is most readily achieved by solving the quadratic (for example, see [5])

$$4\alpha(1 - \alpha) = \frac{1728}{j} = 108(\beta^2/\alpha_2^3) \left[1 - 27(\beta^2/\alpha_2^3) \right],$$

for which we see the two solutions are $\alpha = \frac{27\beta^2}{\alpha_2^3}$, and $1 - \alpha$. The corresponding normalised period is

$$\tau = \tau(\alpha) := i \frac{{}_2F_1(1/6, 5/6, 1; 1 - \alpha)}{{}_2F_1(1/6, 5/6, 1; \alpha)}.$$

Some analytic properties of this function we need are given in Appendix A.

Remark 3.1 If we had taken the other root α in the numerator of the hypergeometric function then this would give the period $-1/\tau$.

As we want the lattice corresponding to g_2, g_3 to be $\sqrt{2}\mathbb{Z} + \sqrt{2}\tau\mathbb{Z}$, we get the transcendental equations

$$\frac{1}{3}\alpha_2^2 = \frac{1}{4}g_2(1, \tau), \quad \frac{1}{27}\alpha_2^3 - 2\beta^2 = \frac{1}{8}g_3(1, \tau).$$

For any given value of $\alpha \in (0, 1)$, let $\alpha_2^2 = \frac{3}{4}g_2(1, \tau)$. We then have two equations defining β :

$$\beta^2 = \frac{\alpha\alpha_2^3}{27}, \quad \beta^2 = \frac{1}{2} \left[\frac{1}{27}\alpha_2^3 - \frac{1}{8}g_3(1, \tau) \right].$$

To have a valid solution we must have that the two equations are consistent with each other, which one can check (see Appendix A) is equivalent to $\text{sgn}(g_3(1, \tau)) = \text{sgn}(\alpha_2) \text{sgn}(1 - 2\alpha)$. A consideration of the information given about τ and g_3 in Appendix A tells us that we only get solutions in the region $\alpha \in [0, 1]$, where $\alpha = 0, 1$ really correspond to the limits $\lim_{\epsilon \rightarrow 0^+} \epsilon, 1 - \epsilon$, respectively.

In order to exclude the possibility of other poles of the Nahm matrices in the region $s \in (0, 2)$, it is necessary that for all $s \in (0, 2)$

$$\wp(s/\sqrt{2}; g_2, g_3) + \frac{\alpha_2}{6} > 0.$$

We know that (i) \wp takes its minimum at $s = 1$; (ii) that the minimum value is the most-positive root of the corresponding cubic $4\wp^3 - g_2\wp - g_3 = 0$; (iii) that this root

is positive [13, §23.5]. Therefore there are no other poles in $(0, 2)$. Further as $\alpha_2 \neq 0$ has the same sign as α , then $\alpha_2 > 0$ so $a_1^2 > 0$. Therefore we know that a_1 is always real, and hence so are all the Flaschka variables, thus giving all the Nahm variables being real as desired.

The remaining condition required for valid Nahm data is that $T_i(s) = T_i(2 - s)^T$. The nature of the Weierstrass \wp is such that $\wp((2-s)/\sqrt{2}; g_2, g_2) = \wp(s/\sqrt{2}; g_2, g_3)$, so we automatically have that $a_1(2 - s) = a_1(s)$, $a_0(s) = a_0(2 - s)$. Moreover, because of the change in the sign of the square root giving b_1 at $s = 1$, we have that $b_1(2 - s) = -b_1(s)$. Taken together these ensure the desired symmetry of the Nahm matrices and we have a one-parameter family of new solutions.

As such, we have now proven the following theorem.

Theorem 3.2 *Given $\alpha \in [0, 1]$, define*

$$\tau = \tau(\alpha) = i \frac{{}_2F_1(1/6, 5/6, 1; 1 - \alpha)}{{}_2F_1(1/6, 5/6, 1; \alpha)}.$$

Solving

$$\frac{1}{3}\alpha_2^2 = \frac{1}{4}g_2(1, \tau), \quad \frac{1}{27}\alpha_2^3 - 2\beta^2 = \frac{1}{8}g_3(1, \tau),$$

with $\text{sgn}(\alpha_2) = \text{sgn}(\alpha)$ yields a monopole spectral curve with D_6 symmetry

$$\eta^3 + \alpha_2\eta\zeta^2 + \beta(\zeta^6 - 1) = 0.$$

Moreover, the Nahm data is given explicitly in terms of \wp -functions by (6) and (12).

3.3 Distinguished curves

Having solved for general $\alpha \in [0, 1]$ we now investigate the special values of $\alpha = 0, 1/2, 1$.

3.3.1 $\alpha = 0^+$

The limit $\alpha \rightarrow 0$ corresponds to $\tau \rightarrow +i\infty$, and we have using the asymptotic expansion of the Eisenstein series that $g_2(1, \tau) \rightarrow \frac{4\pi^4}{3}$, $g_3(1, \tau) \rightarrow \frac{8\pi^6}{27}$, so $\alpha = 0$ is indeed a solution with $\beta = 0$, $\alpha_2 = \pi^2$. This recreates the well known axially-symmetric monopole with spectral curve $\eta(\eta^2 + \pi^2\zeta^2) = 0$ [20, 21].

If we had $\beta = 0$ from the beginning (and so $\Delta = 0$, and for $\alpha_2 \neq 0$ then $\alpha = 0$), we would have found a singular elliptic curve

$$4\tilde{u}^3 - \frac{1}{3}\alpha_2^2\tilde{u} - \frac{1}{27}\alpha_2^3 = 4\left(\tilde{u} + \frac{\alpha_2}{6}\right)^2\left(\tilde{u} - \frac{\alpha_2}{3}\right),$$

with solution to the corresponding ODE (using known integrals) given by

$$\tilde{u} = \frac{\alpha_2}{3} + \frac{\alpha_2}{2} \tan^2 \left[\frac{\sqrt{\alpha_2}}{2} (s - s_0) \right], \quad a_1 = \sqrt{\frac{\alpha_2}{2}} \sec \left[\frac{\sqrt{\alpha_2}}{2} (s - s_0) \right].$$

We could then manufacture the right residue at $s = 0$ by having $s_0 = \frac{\pi}{2} \cdot \frac{2}{\sqrt{\alpha_2}}$. To get the correct periodicity, we would require that $\frac{\pi}{2} = \frac{\sqrt{\alpha_2}}{2} (2 - s_0)$ and consequently that $\alpha_2 = \pi^2$ again giving the axially-symmetric monopole.

3.3.2 $\alpha = 1^-$

To get this limit, we use $\tau(1^-) = -1/\tau(0^+)$, so

$$g_2(1, \tau(1^-)) = g_2(1, -1/\tau(0^+)) = \tau(0^+)^4 g_2(1, \tau(0^+)) = \frac{1}{\tau(1^-)^4} \frac{4\pi^4}{3},$$

and likewise for g_3 . Solving gives

$$\alpha_2 \sim -\left(\frac{\pi}{\tau}\right)^2, \quad \beta \sim \pm \frac{i}{3\sqrt{3}} \left(\frac{\pi}{\tau}\right)^3,$$

or equivalently writing $\tau = i\epsilon$ for $0 < \epsilon \ll 1$,

$$\alpha_2 \sim 3 \left(\frac{\pi}{\sqrt{3}\epsilon}\right)^2, \quad \beta \sim \pm \left(\frac{\pi}{\sqrt{3}\epsilon}\right)^3,$$

The corresponding spectral curve thus factorises as

$$\begin{aligned} 0 &= \eta^3 + 3 \left(\frac{\mp\pi}{\sqrt{3}\epsilon}\right)^2 \eta \zeta^3 - \left(\frac{\mp\pi}{\sqrt{3}\epsilon}\right)^3 (\zeta^6 - 1), \\ &= \left[\eta - \left(\frac{\mp\pi}{\sqrt{3}\epsilon}\right) (\zeta^2 - 1) \right] \left[\eta - \left(\frac{\mp\pi}{\sqrt{3}\epsilon}\right) (\omega \zeta^2 - \omega^2) \right] \left[\eta - \left(\frac{\mp\pi}{\sqrt{3}\epsilon}\right) (\omega^2 \zeta^2 - \omega) \right]. \end{aligned}$$

This corresponds to three well-separated 1-monopoles on the vertices of an equilateral triangle in the x, y -plane with side length $\frac{\pi}{\epsilon}$ [22]. As ϵ tends to zero these three vertices tend to the point ∞ , the singular degeneration to the cuspidal elliptic curve with $\Delta = 0$ and $\alpha = 1$.

3.3.3 $\alpha = 1/2$

In this case $\tau = i$, and the lattice is the square lattice. The values of g_2, g_3 for this lattice are known explicitly [13, 23.5.8], giving the equations

$$\frac{1}{3} \alpha_2^2 = \frac{1}{4} \frac{\Gamma(1/4)^8}{16\pi^2}, \quad \frac{1}{27} \alpha_2^3 - 2\beta^2 = 0 \Rightarrow \alpha_2 = \frac{\sqrt{3}\Gamma(1/4)^4}{8\pi}, \quad \beta = \pm \frac{\Gamma(1/4)^6}{32(\sqrt{3}\pi)^{3/2}}.$$

Remark 3.3 The coefficients seen here are the same, up to a sign, as those of a distinguished monopole found in [26]. This is no accident, but arises because the square lattice is behind the distinguished “twisted figure-of-eight” monopole, as we show later in Sect. 4.2.

3.4 Scattering

To complete our understanding of these monopoles we discuss the corresponding scattering. This has already been described using the rational map approach in [36]. The D_6 -symmetric monopoles described here corresponds to the prismatic subgroup D_{3h} of $O(3)$: this confines the monopoles to lie in a plane, and thus any scattering observed must be planar. Note for each value of $\alpha \neq 0$ there are two choices of β from the defining equations, and these two branches coalesce where $\beta = 0 \Leftrightarrow \alpha = 0$. This gives us a view of scattering from $\alpha = 1$ with three initially well-separated 1-monopoles with a choice of sign. They move inwards along the axes of symmetry of the corresponding equilateral triangle through $\alpha = 0$ where the 3-monopoles instantaneously takes the configuration of the axially-symmetric monopole. Here we change branch (i.e. sign of β), and move back out to $\alpha = 1$ where now because of the change of sign these three well-separated 1-monopoles are deflected by $\pi/3$ radians. Note that as with the planar scattering of 2-monopoles [3], because of symmetry one cannot associate a given in-going monopole with an out-going one but rather interpret the scattering process as the the 3 in-going monopoles splitting into thirds which then recombine to form the out-going monopoles.

4 V_4 monopoles

4.1 Solving for Nahm data

For our curve with V_4 symmetry the generators of the automorphism group are $(\zeta, \eta) \mapsto (-\zeta, -\eta)$ and $(\zeta, \eta) \mapsto (-1/\zeta, \eta/\zeta^2)$; equivalently these correspond to the rotations $\text{diag}(-1, -1, 1)$ and $\text{diag}(-1, 1, -1)$ whose product is the earlier r . If we impose further the involution $(\zeta, \eta) \mapsto (\zeta, -\eta)$ as a symmetry (the composition of inversion with the anti-holomorphic involution) we restrict to the case of the inversion-symmetric 3-monopoles known in [25]. Here they solve for Nahm matrices given in terms of 3 real-valued functions f_i satisfying $f'_1 = f_2 f_3$ (and cyclic), with the corresponding spectral curve being

$$\eta^3 + \eta \left[(f_1^2 - f_2^2) (\zeta^4 + 1) + (2f_1^2 + 2f_2^2 - 4f_3^3) \zeta^2 \right] = 0.$$

We find in Appendix C that the same procedure, now without imposing the extra symmetry, yields Nahm matrices in terms of 3 complex-valued functions satisfying⁸

⁸ These equations are also found in [23, (3.57)] where they are attributed to [2]; they also appear as the x -independent solutions in the description of 3-wave scattering [32, (17) p177]. We thank Pol Vanhaecke and Sasha Mikhailov for this latter reference.

$$\bar{f}_1' = f_2 f_3 \quad (\text{and cyclic}), \tag{14}$$

with the corresponding spectral curve being

$$\eta^3 + \eta \left[a(\zeta^4 + 1) + b\zeta^2 \right] + c\zeta(\zeta^4 - 1) = 0, \tag{15}$$

where

$$a = |f_1|^2 - |f_2|^2, \quad b = 2|f_1|^2 + 2|f_2|^2 - 4|f_3|^3, \quad c = 2(f_1 f_2 f_3 - \bar{f}_1 \bar{f}_2 \bar{f}_3).$$

The Nahm matrices are given by

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\bar{f}_1 \\ 0 & f_1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & f_2 \\ 0 & 0 & 0 \\ -\bar{f}_2 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -\bar{f}_3 & 0 \\ f_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

Remark 4.1 We observe that equations (14) come from the Poisson structure $\{f_i, \bar{f}_j\} = \delta_{ij}$, with Hamiltonian $c/2 = f_1 f_2 f_3 - \bar{f}_1 \bar{f}_2 \bar{f}_3$. This complex extension of the Euler equations is integrable.

Remark 4.2 We have not fully used up the gauge symmetry available to us. Namely, if we conjugate the T_i by $U = \text{diag}(u_1, u_2, u_3)$ where $u_j = e^{i\phi_j}$ and $\sum \phi_j = 0$, we get

$$f_1 \mapsto u_3 u_2^{-1} f_1, \quad f_2 \mapsto u_1 u_3^{-1} f_2, \quad f_3 \mapsto u_2 u_1^{-1} f_3,$$

which preserves the form of the equations.

A consequence of this remark and the form (16) is that for the T_i to have residues which form an irreducible representation of $\mathfrak{su}(2)$ it is sufficient for the f_i to have simple poles at $s = 0, 2$.

In order to find a solution we note that $a_{ij} = |f_i|^2 - |f_j|^2$ and $c = 2(f_1 f_2 f_3 - \bar{f}_1 \bar{f}_2 \bar{f}_3)$ are now constants. As c is imaginary it will be useful to introduce $\tilde{c} := -ic$. Setting $F = |f_1|$ we have

$$\begin{aligned} (F')^2 &= \left\{ \left[(f_1 \bar{f}_1)^{1/2} \right]' \right\}^2 = \left\{ \frac{1}{2} (f_1 \bar{f}_1)' (f_1 \bar{f}_1)^{-1/2} \right\}^2 = \frac{1}{4} (f_1 f_2 f_3 + \bar{f}_1 \bar{f}_2 \bar{f}_3)^2 F^{-2}, \\ &= \frac{1}{4} F^{-2} \left[(c/2)^2 + 4|f_1|^2 |f_2|^2 |f_3|^2 \right], \\ &= \frac{1}{4} F^{-2} \left[(c/2)^2 + 4F^2 (F^2 - a_{12})(F^2 + a_{31}) \right], \end{aligned}$$

and so with $G = F^2$ we get

$$(G')^2 = \frac{1}{4} c^2 + 4G(G - a_{12})(G + a_{31}),$$

which then has solutions in terms of elliptic functions. In terms of the coefficients of the spectral curve we already have $a_{12} = a$, and we can moreover find $a_{31} = \frac{-1}{4}(b + 2a)$, so we can rewrite the equation as

$$(\tilde{G}')^2 = 4\tilde{G}^3 - g_2\tilde{G} - g_3, \tag{17}$$

where $\tilde{G} = G - \frac{b+6a}{12}$, $g_2 = a^2 + \frac{b^2}{12}$, and $g_3 = \frac{b(b^2-36a^2)}{216} + \frac{1}{4}\tilde{c}^2$. Then $\tilde{G} = \wp$, the Weierstrass \wp -function. The j -invariant for this elliptic curve is

$$j = 1728 \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{(12a^2 + b^2)^3}{(a^6 - \frac{1}{2}a^4b^2 + \frac{1}{16}a^2b^4 + \frac{9}{4}a^2b\tilde{c}^2 - \frac{1}{16}b^3\tilde{c}^2 - \frac{27}{16}\tilde{c}^4)}, \tag{18}$$

which is precisely that of the quotient of (15) by the V_4 symmetry. We also note that the pull-back of the invariant differential of this quotient is exactly that needed when discussing the Ercolani-Sinha constraint.

Before going on to solve this completely, let's recall what remains to be shown to get a monopole spectral curve (i.e. to have our Nahm matrices satisfy all the conditions to give Nahm data). We need to have that the \wp -function associated with the above elliptic curve has real period 2, but we will be able to impose this by tuning the coefficients. Also, as the right-hand side of

$$\wp = |f_1|^2 - \frac{b + 6a}{12} \tag{19}$$

is always real this requires \wp to be real and so to be taken on a rectangular or rhombic lattice. Also for reality we need that

$$G(s) = \wp(s) + \frac{b + 6a}{12}, \quad G(s) - a_{12} = \wp(s) + \frac{b - 6a}{12}, \quad G(s) + a_{31} = \wp(s) - \frac{b}{6}$$

are always positive. Once we have achieved this we will have regularity in the region $(0, 2)$, and so get the right pole structure. The final condition is symmetry about $s = 1$, which is enforced on the $|f_j|$ (because $|f_j| \sim \sqrt{\wp}$), and so the remaining Nahm constraint $T_j(s) = T_j(2 - s)^T$ becomes simply $f_j(s) = -\bar{f}_j(2 - s)$: that is we require $\arg f_j(s) = \pm\pi - \arg f_j(2 - s)$.

Indeed writing $f_j = |f_j| e^{i\theta_j}$ we can work out the equations for the angles, using

$$f'_j = \left(|f_j|' + i\theta'_j |f_j| \right) e^{i\theta_j} = \left(\frac{|f_j|'}{|f_j|} + i\theta'_j \right) f_j \Rightarrow \theta'_j = \frac{1}{i} \left[\frac{f'_j}{f_j} - \frac{|f_j|'}{|f_j|} \right] = \frac{-\tilde{c}}{4|f_j|^2}. \tag{20}$$

The θ_j are thus strictly monotonic (unless $\tilde{c} = 0$, in which case they are constant), and symmetry about $s = 1$ of $|f_j|$ then necessitates that $\theta_j(s) - \theta_j(1)$ is antisymmetric about $s = 1$.

We also have that

$$\tilde{c} = 4 |f_1| |f_2| |f_3| \sin(\theta_1 + \theta_2 + \theta_3) = \sqrt{\tilde{c}^2 + 4(G')^2} \sin(\theta_1 + \theta_2 + \theta_3).$$

At $s = 1$ where $G'(s) = 0$ we need $\sin(\theta_1 + \theta_2 + \theta_3) = 1$, and by our gauge freedom we can choose $\theta_1(1) = \pi/2 = \theta_2(1)$ and so $\theta_3(1) = -\pi/2$, thus enforcing our condition of symmetry about $s = 1$. We then see that the anti-symmetry of $\theta_j(s) - \theta_j(1)$ about $s = 1$ enforces the remaining reality condition. We also note that as $|f_j(s)|^2 = \wp(s) - c_j := \wp(u) - \wp(v_j)$ for appropriate s and $v_j = \int_{\infty}^{c_j} [4u^3 - g_2u - g_3]^{-1/2} du$ we have [28, (6.14.6)]

$$\int \frac{du}{\wp(u) - \wp(v)} = \frac{1}{\wp'(v)} \left[2u\zeta(v) + \ln \frac{\sigma(u - v)}{\sigma(u + v)} \right], \tag{21}$$

allowing us to find the $\theta_j(s)$ explicitly which is done in (22) in Appendix A.3.

It remains to fix the real period of the corresponding elliptic curve. We describe two methods. The first makes use of the Jacobi elliptic functions to express the lattice invariants in terms of complete elliptic functions [1, §18.9]. We explain this in Appendix A.4 in which, by showing that we may fix the real period, establishes the following theorem.

Theorem 4.3 *Given $\alpha \in \mathbb{R}$, $m \in [0, 1]$, and $\text{sgn} = \pm 1$, define g_2, g_3 by $g_2 = 12 (K(m)^2/3)^2 q_1(m)$, $g_3 = 4 (K(m)^2/3)^3 (2m - 1)q_2(m)$, where*

$$q_1(m) = \begin{cases} 1 - m + m^2 & \text{sgn} = 1, \\ 1 - 16m + 16m^2 & \text{sgn} = -1, \end{cases} \quad q_2(m) = \begin{cases} (m - 2)(m + 1) & \text{sgn} = 1, \\ 2(32m^2 - 32m - 1) & \text{sgn} = -1. \end{cases}$$

If m is such that $g_2 > 0$ and the polynomial $(4 - 2\alpha)x^3 - g_2x - g_3$ has a real root x_ with $|x_*| < \sqrt{g_2/3}$ and $\text{sgn}(x_*) = -\text{sgn}(\alpha)$, then we may solve*

$$a^2 + \frac{b^2}{12} = g_2, \quad \frac{b(b^2 - 36a^2)}{216} + \frac{\tilde{c}^2}{4} = g_3$$

for $a, b, \tilde{c} \in \mathbb{R}$ taking $\alpha = \frac{-27\tilde{c}^2}{b^3}$. Then

$$\eta^3 + \eta \left[a(\zeta^4 + 1) + b\zeta^2 \right] + i\tilde{c}\zeta(\zeta^4 - 1) = 0$$

is a monopole spectral curve with V_4 symmetry. Moreover the Nahm data is given explicitly in terms of elliptic functions by (16), (19) and (22).

A second approach to fixing the correct real period to give Nahm data is to invert the j -invariant (18) as done in the earlier D_6 case. Though we are unable to invert in terms of a single rational α as with the D_6 -symmetric monopole, we may use [15, (4)] which gives

$$\tau = i \left[\frac{2\sqrt{\pi}}{\Gamma(7/12)\Gamma(11/12)} \frac{{}_2F_1(1/12, 5/12, 1/2; x)}{{}_2F_1(1/12, 5/12, 1; 1 - x)} - 1 \right],$$

where $x = 1 - \frac{1728}{j} = \frac{(1-2\alpha-3\gamma)^2}{(1+\gamma)^3}$, with $\alpha = -\frac{27\tilde{c}^2}{b^3}$, $\gamma = \frac{12a^2}{b^2}$. One may then fix the real period of the lattice, which will give solutions consistent with the definition of x for some range of the parameters α, γ . We investigate one particular restriction of this kind in Sect. 4.2. We remark that [23] solved the associated Nahm data only for the (one-parameter) case $\Delta = 0$ in which the elliptic curve degenerates and has trigonometric solutions.

4.2 D_4 monopoles

In [26] a subfamily of (15) with D_4 symmetry was studied. To the existing V_4 symmetries is appended the order-4 element $(\zeta, \eta) \mapsto (i\zeta, -i\eta)$ (corresponding to the composition of inversion with a rotation of $\pi/2$ in the xy -plane). This symmetry then requires $a = 0$. By a dimension argument we expect the j -invariant inversion to yield a geodesic 1-parameter family for the enlarged symmetry group, and this was the case considered in [26] where the C_4 quotient yields an elliptic curve. Placing this curve in our V_4 family allows us a different approach to this family of curves. The restriction $a = 0$ means that $\frac{1728}{j} = 4\alpha(1 - \alpha)$ with $\alpha = -\frac{27\tilde{c}^2}{b^3}$ and we can then fix the real period via the same approach as for the D_6 monopole. The equations we get are

$$\frac{b^2}{3} = \frac{1}{4}g_2(1, \tau), \quad \frac{b^3}{27} + 2\tilde{c}^2 = \frac{1}{8}g_3(1, \tau),$$

with these being consistent with the definition of α provided $\text{sgn}(g_3(1, \tau)) = \text{sgn}(b) \text{sgn}(1 - 2\alpha)$. To also have that \tilde{c} is real, we must have $\text{sgn}(b) = -\text{sgn}(\alpha)$ and hence our consistency condition is $\text{sgn}(g_3(1, \tau)) = -\text{sgn}(\alpha) \text{sgn}(1 - 2\alpha)$. We thus have solutions in the region $\alpha \in (0, 1/2)$ if $\text{sgn}(g_3(1, \tau)) < 0$, which requires $\tau = -1/\tau(\alpha)$. We can extend this to $\alpha \in (1/2, 1)$ still taking $\tau = -1/\tau(\alpha)$. Moreover, for $\alpha < 0$, we require $g_3(1, \tau) > 0$, which can be achieved taking $\tau = \tau(\alpha)$. Finally, for $\alpha > 0$, we require $\text{sgn}(g_1(1, \tau)) > 0$, achievable with $\tau = -1/\tau(\alpha)$. As such the parameter region in this case is the whole of \mathbb{R} . A case-by-case consideration shows that $G, G - a_{12}, G + a_{31}$ are always positive on the interval $[0, 2]$, so we do indeed get Nahm data as desired.

As with the D_6 -symmetric monopoles we may identify special values of α and the curves they give. A similar analysis gives those found in [26], namely

- $\alpha = \pm\infty$ gives the tetrahedrally-symmetric monopole,
- $\alpha = 0^+, 0^-$ gives three well-separated 1-monopoles and the axially-symmetric monopole, respectively,
- $\alpha = 1/2$ gives the ‘‘twisted figure-of-eight’’ monopole. Note $\alpha = 1/2$ corresponds to the square lattice we saw as distinguished for the D_6 monopole.

We additionally see the curve with $\alpha = 1$ as distinguished in our parametrisation, which gives the curve

$$\eta^3 - \pi^2\eta\zeta^2 \pm \frac{i}{\sqrt{27}}\pi^3\zeta(\zeta^4 - 1) = 0.$$

In terms of the parameters a, ϵ of [26], this curve is given by $a = 2\sqrt{2}, \epsilon = -1$.

4.2.1 Scattering

As such we can now understand our scattering as starting at $\alpha = 0^+$ with three well-separated 1-monopoles. As α increases to ∞ we have to pick a choice of \tilde{c} continuously (though there is no specific choice at $\alpha = 0^+$ as the map $\zeta \mapsto -\zeta$ which swaps the choice of \tilde{c} is a symmetry of our well separated configuration), and we pass through two distinguished curves, arriving at the tetrahedrally-symmetric monopoles in one orientation. We match that to $\alpha = -\infty$ taking the tetrahedrally-symmetric monopole with the same orientation there, allowing α to then increase up to 0^- where it takes the configuration of the axially-symmetric monopole. Here the two branches of \tilde{c} coalesce, we change branch and do the process in reverse.

5 Conclusion

In this letter, we have begun systematising the classification of charge-3 monopole spectral curves with automorphisms, providing an exhaustive list of candidate curves; we nevertheless expect this list to contain curves that do not correspond to monopole spectral curves. We have also identified how one may use group theory to identify the subset of these candidates that quotient to an elliptic curve. This was done because such curves are amenable to the construction of Nahm matrices in terms of elliptic functions using the procedure of [22]. Here the imposition of Hitchin's conditions (or equivalently those of Ercolani-Sinha) reduces to questions about the real periods of elliptic functions. Having provided new candidate spectral curves we solved for the Nahm data in two new cases, those of D_6 and V_4 symmetry. The latter led us to an integrable system (14) that may be viewed as the complexified Euler equations. Given Nahm matrices and the corresponding group action what is not yet clear is how to methodically extract from the resulting coupled ODE's the relevant elliptic equations; providing such an understanding would simplify the construction of the solutions to Nahm's equations from the spectral data. This is the reason for our not treating the C_2 -symmetric monopoles here: in this case we have 13 coupled ODE's with 7 conserved quantities.

One can generalise to higher charge several of the viewpoints put forward in this paper. We have seen that compactifying mini-twistor space in $\mathbb{P}^{1,1,2}$ and then looking at its image in \mathbb{P}^3 a possible charge- k spectral curve is represented by the intersection of the cone and a degree- k hypersurface. There may be value in this viewpoint for providing a candidate list of monopole spectral curves in higher charge. Further, the methods used to calculate the group-signature pairs giving elliptic quotients in genus 4 extends to higher genus, and so may be used to provide candidate spectral curves potentially amenable to solutions in terms of elliptic functions at higher charges. At present, these data have not been computed in the LMFDB, and so a first step would be the tabulation of those results. In the event that such a computation produced too extensive a list we suggest restricting to the case where $\delta(g, G, c) = 1, 2$, for which

we expect any corresponding monopole spectral curves to be either isolated points in the moduli space or to correspond to geodesic motion, respectively, as we conjectured.

Finally, the geometry we introduced here may have applications for the understanding of spectral curves of hyperbolic monopoles. Spectral curves corresponding to hyperbolic monopoles live in the mini-twistor space of hyperbolic space, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and specifically charge- k hyperbolic monopoles are bidegree- (k, k) curves in this surface [3]. As $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the non-singular quadric in \mathbb{P}^3 , and bidegree- $(3, 3)$ curves in this correspond to the other class of non-hyperelliptic curves classified by Wiman, our work highlights the potential of classifying certain hyperbolic monopole spectral curves.

Acknowledgements We are grateful to Conor Houghton for correspondence and to Paul Sutcliffe for discussions and providing code (discussed below) that we have modified to plot the accompanying monopole solutions. The research of LDH is supported by a UK Engineering and Physical Sciences Research Council (EPSRC) studentship.

Data availability The datasets generated during the current study and the code for their creation/analysis are available from the second author upon reasonable request.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Appendix A: Hypergeometric functions and lattice invariants

We gather here some of properties of elliptic and related functions used in the text and prove those statements noted in the text. We follow the conventions of [13, Chapter 23]. First we recall the Weierstrass- \wp function is defined by

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

Here $g_k = g_k(\omega, \omega')$ are defined by the lattice $\Lambda := 2\omega\mathbb{Z} + 2\omega'\mathbb{Z}$. Let $\tau = \omega'/\omega$. We have

- (i) $g_k(\lambda\omega, \lambda\omega') = \lambda^{-2k}g_k(\omega, \omega')$,
- (ii) given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$, $g_k(1, (a\tau + b)(c\tau + d)^{-1}) = (c\tau + d)^{2k}g_k(1, \tau)$,
- (iii) $\lim_{\text{Im } \tau \rightarrow \infty} g_2(1, \tau) = \frac{4\pi^4}{3}$, $\lim_{\text{Im } \tau \rightarrow \infty} g_3(1, \tau) = \frac{8\pi^6}{27}$.
- (iv) When $\tau = i$, $g_2(1, \tau) = \frac{\Gamma(1/4)^8}{256\pi^2}$, $g_3(1, \tau) = 0$.

(v) When $\tau = e^{2\pi i/3}$, $g_2(1, \tau) = 0$, $g_3(1, \tau) = \frac{\Gamma(1/3)^{18}}{(2\pi)^6}$.

Our parametrization of the Nahm matrices requires us to know the reality properties of the \wp -function. We have [27, Theorem 3.16.2]

$$g_2, g_3 \in \mathbb{R} \Leftrightarrow \forall z \in \mathbb{C}, \wp(\bar{z}; g_2, g_3) = \overline{\wp(z; g_2, g_3)} \Leftrightarrow \Lambda = \bar{\Lambda}.$$

Lattices for which $\Lambda = \bar{\Lambda}$ are called real lattices and they fall into two classes; rectangular lattices ($\omega \in \mathbb{R}, \omega' \in i\mathbb{R}$), and rhombic lattices ($\bar{\omega} = \omega'$). The rhombic lattices correspond to τ being on the boundary of the fundamental domain of the $SL(2, \mathbb{Z})$ action on the upper half plane while the rectangular lattices correspond to τ on the imaginary axis with $\Im(\tau) \geq 1$. When restricted to rectangular or rhombic lattices we can say more about the values of $g_2(1, \tau)$ and $g_3(1, \tau)$. This is done by relating the g_k to the roots e_i of the corresponding cubic equation by

$$g_2(1, \tau) = 2(e_1^2 + e_2^2 + e_3^2), \quad g_3(1, \tau) = -4e_1e_2e_3.$$

- (i) On a rectangular lattice have $e_i \in \mathbb{R}$ so $g_2 > 0$; further, $g_3 > 0$ if $|\tau| > 1$, $g_3 < 0$ if $|\tau| < 1$.
- (ii) On a rhombic lattice, $e_1 \in \mathbb{R}, e_2 = \bar{e}_3$, and $\text{sgn}(e_1) = \text{sgn}(g_3)$.

A.1 Properties of $\tau(\alpha)$

In order to use the process of j -invariant inversion to impose the correct periodicity constraints and to give the limiting behaviours noted in the text enabling us to find certain distinguished monopoles we require the properties of

$$\tau = \tau(\alpha) = i \frac{{}_2F_1(1/6, 5/6, 1; 1 - \alpha)}{{}_2F_1(1/6, 5/6, 1; \alpha)}.$$

This is multi-valued when $\alpha < 0$ [13, 15.2.3], with a principal branch τ_p and second branch $\tau_p + 1$, but for our purposes this difference will not be important. The specific properties we require are that

- (i) $\forall \alpha \in (0, 1), \tau(\alpha) \in i\mathbb{R}_{>0}$,
- (ii) $\tau(0^+) = +i\infty, \tau(1/2) = i, \tau(1^-) = 0$,
- (iii) $\forall \alpha < 0, \text{Re}(\tau(\alpha)) \equiv 1/2 \pmod{1}$,
- (iv) $\tau(-\infty) = e^{2\pi i/3}, \tau(0^-) = \frac{1}{2} + i\infty$.

Evaluated at the specific $\tau(\alpha)$ above we find that

$$\text{sgn}(g_3(1, \tau(\alpha))) = \begin{cases} 1 & \alpha < 1/2, \\ -1 & \alpha \in (1/2, 1). \end{cases}$$

Here we provide the necessary definitions and proofs.

We can understand the behaviour of τ using known results about hypergeometric functions (see for example [1, §15]). First, in the region $\alpha \in (0, 1)$, we may use the

series expression for ${}_2F_1(a, b, c; z)$ when $|z| < 1$:

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(a)_n$ is the rising Pochhammer symbol

$$(a)_n = \begin{cases} 1 & n = 0, \\ a(a + 1) \dots (a + n - 1) & n \geq 1. \end{cases}$$

This means we have (i) that for all $\alpha \in (0, 1)$, $\tau(\alpha) \in i\mathbb{R}_{>0}$. This is important as it makes the lattice rectangular, which forces the Weierstrass \wp -function to be real on the real axis [13, §23.5]. Moreover, as ${}_2F_1(a, b, c; z)$ is increasing in $z \in (0, 1)$, $\text{Im } \tau(\alpha)$ is strictly decreasing in α . We can calculate the limits to be

$$\tau(0^+) = +i\infty, \quad \tau(1^-) = 0,$$

so giving (ii). We may use [1, 15.3.10] which says that when $|1 - z| < 1$, $|\arg(1 - z)| < \pi$,

$$\begin{aligned} {}_2F_1(a, b, a + b; z) &= \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} \\ &\quad [2\psi(n + 1) - \psi(a + n) - \psi(n + b) - \log(1 - z)] (1 - z)^n, \end{aligned}$$

where ψ is the digamma function, to understand exactly this limiting behaviour, namely that the divergence is logarithmic. We can also highlight a special value in this regions, namely $\tau(1/2) = i$.

For $\alpha \notin [0, 1]$ we no longer have that τ lies on the imaginary axis, and we would thus need to get a rhombic lattice (that is $\text{Re } \tau = 1/2$) for the reality of \wp . Numerical tests suggest that while this happens for $\alpha < 0$ for $\alpha > 1$ we instead get $\text{Re}(-1/\tau) = 1/2$. Indeed we may use [13, 15.10.29] to say⁹

$$\begin{aligned} {}_2F_1(1/6, 5/6, 1; 1 - \alpha) &= e^{5\pi i/6} \frac{\Gamma(1)\Gamma(1/6)}{\Gamma(1)\Gamma(1/6)} {}_2F_1(1/6, 5/6, 1; \alpha) \\ &\quad + e^{-\pi i/6} \frac{\Gamma(1)\Gamma(1/6)}{\Gamma(5/6)\Gamma(1/3)} \alpha^{-1/6} {}_2F_1(1/6, 1/6, 1/3; 1/\alpha), \\ &= e^{5\pi i/6} {}_2F_1(1/6, 5/6, 1; \alpha) \\ &\quad + (-\alpha)^{-1/6} \frac{\Gamma(1/6)}{\Gamma(5/6)\Gamma(1/3)} {}_2F_1(1/6, 1/6, 1/3; 1/\alpha), \end{aligned}$$

⁹ We are very grateful to Adri Olde Daalhuis for this argument.

and hence when $\alpha < 0$ (and taking the principal branch of the hypergeometric function) we get

$$\tau(\alpha) = i \left[e^{5\pi i/6} + T(\alpha) \right]$$

with

$$T(\alpha) = (-\alpha)^{-1/6} \frac{\Gamma(1/6)}{\Gamma(5/6)\Gamma(1/3)} \frac{{}_2F_1(1/6, 1/6, 1/3; 1/\alpha)}{{}_2F_1(1/6, 5/6, 1; \alpha)} \in \mathbb{R}.$$

This means $\text{Re}(\tau(\alpha)) \equiv 1/2 \pmod{1}$, which yields (iii).

To get the asymptotics as $\alpha \rightarrow -\infty$, we use [1, 15.3.7]

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} {}_2F_1(a, a+1-c, a+b-1; z^{-1}) \\ &\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} {}_2F_1(b, b+1-c, b+a-1; z^{-1}). \end{aligned}$$

Taking $\alpha = -\epsilon^{-1}$, this gives that as $\epsilon \rightarrow 0^+$,

$${}_2F_1(a, b, c; -\epsilon^{-1}) \sim \frac{\Gamma(2/3)}{\Gamma(5/6)^2} \epsilon^{1/6}, \quad {}_2F_1(a, b, c; 1 + \epsilon^{-1}) \sim \frac{\Gamma(2/3)}{\Gamma(5/6)^2} (-\epsilon)^{1/6},$$

and so $\tau(-\infty) = e^{2\pi i/3} = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$. To get the remaining asymptotics of (iv), as $\alpha \rightarrow 0^-$ we write $\alpha = -\epsilon$. Then

$${}_2F_1(a, b, c; -\epsilon) \sim 1, \quad {}_2F_1(a, b, c; 1 + \epsilon) \sim \frac{-\Gamma(1)}{\Gamma(1/6)\Gamma(5/6)} \log(-\epsilon) = \frac{-1}{2\pi} (i\pi + \log \epsilon),$$

and so $\tau(0^-) = \frac{1}{2} + i\infty$.

To get the asymptotics as $\alpha \rightarrow 1^+$ we recognise that $\tau(1 - \alpha) = -1/\tau(\alpha)$ and so $-1/\tau(1^+) = \frac{1}{2} + i\infty$. Finally to get the asymptotics as $\alpha \rightarrow \infty$ we do the same, so $-1/\tau(\infty) = \frac{1}{2} + \frac{i\sqrt{3}}{2}$.

A.2 Check of consistency

We see that α_2 must be the same sign as α to get $\beta \in \mathbb{R}$. Moreover, as $g_2(1, \tau) > 0$ because α_2 is real, we can check that

$$\begin{aligned} \frac{1}{2} \left[\frac{1}{27} \alpha_2^3 - \frac{1}{8} g_3(1, \tau) \right] &= \frac{1}{2} \left[\frac{\text{sgn}(\alpha_2)}{27} (3g_2(1, \tau)/4)^{3/2} - \frac{1}{8} g_3(1, \tau) \right], \\ &= \frac{\text{sgn}(\alpha_2) g_2(1, \tau)^{3/2}}{16\sqrt{27}} \left[1 - \frac{\text{sgn}(g_3(1, \tau))}{\text{sgn}(\alpha_2)} \sqrt{\frac{27g_3^2}{g_2^3}} \right], \end{aligned}$$

$$\begin{aligned}
 &= \frac{\operatorname{sgn}(\alpha_2) (4\alpha_2^2/3)^{3/2}}{16\sqrt{27}} \left[1 - \frac{\operatorname{sgn}(g_3(1, \tau))}{\operatorname{sgn}(\alpha_2)} \left(1 - \frac{1728}{j} \right)^{1/2} \right], \\
 &= \frac{\alpha_2^3}{2 \times 27} \left[1 - \frac{\operatorname{sgn}(g_3(1, \tau))}{\operatorname{sgn}(\alpha_2)} (1 - 4\alpha(1 - \alpha))^{1/2} \right], \\
 &= \frac{\alpha_2^3}{2 \times 27} \left[1 - \frac{\operatorname{sgn}(g_3(1, \tau))}{\operatorname{sgn}(\alpha_2) \operatorname{sgn}(1 - 2\alpha)} (1 - 2\alpha) \right], \\
 &= \frac{\alpha\alpha_2^3}{27} \quad \text{if} \quad \operatorname{sgn}(g_3(1, \tau)) = \operatorname{sgn}(\alpha_2) \operatorname{sgn}(1 - 2\alpha).
 \end{aligned}$$

Hence the two equations are consistent, provided the stated sign condition holds, or if $\alpha = 0$.

A.3 The theta integration

We note that $|f_j(s)|^2 = \wp(s) - c_j := \wp(s) - \wp(v_j)$ does not fix the sign of v_j for $\wp(\pm v_j) = c_j$. We fix the sign as follows. First observe that

$$(\tilde{G}'(s))^2 = (G'(s))^2 = \frac{1}{4}c^2 + 4[\wp(s) - \wp(v_1)][\wp(s) - \wp(v_2)][\wp(s) - \wp(v_3)],$$

and so $\wp'^2(v_i) = c^2/4$; we fix the sign so that $\wp'(v_i) = c/2 = i\tilde{c}/2$. Further consider the elliptic function $\wp'(s) - c/2$ with three zeros (at $s \in \{v_1, v_2, v_3\}$) and three poles (at $s = 0$). Then with the base of the Abel-Jacobi map at $s = 0$ (as is standard) we have that $\sum_i v_i$ is a lattice point. Also observe that

$$\zeta(v_i) + \zeta(v_j) = \zeta(v_i + v_j).$$

We find from (20, 21) that

$$\theta_i(s) := \theta_i(1) + i \left[s\zeta(v_j) + \frac{1}{2} \ln \frac{\sigma(s - v_j)\sigma(1 + v_j)}{\sigma(s + v_j)\sigma(1 - v_j)} \right], \tag{22}$$

where $\theta_i(1)$ is a constant of integration and chosen as described in the text. Then $\theta_i(-s) - \theta_i(1) = -[\theta_i(s) - \theta_i(1)]$ is anti-symmetric as required. Using the Legendre relation we find that $\sin(\theta_1 + \theta_2 + \theta_3)$ is periodic in s as required for consistency.

A.4 Restrictions on elliptic function parameters

Here we prove Theorem 4.3. Given the discriminant Δ for the cubic defining \wp [1, §18.9] gives equations for the lattice invariants in terms of complete elliptic functions. With our earlier definitions, $g_2 = 12(K(m)^2/3)^2 q_1(m)$ and $g_3 =$

$4 \left(K(m)^2/3 \right)^3 (2m - 1)q_2(m)$ then with $\text{sgn} = \text{sgn}(\Delta)$ we have for $\Delta > 0$

$$g_2 = 12 \left(\frac{K^2}{3\omega_1} \right)^2 (1 - m + m^2), \quad g_3 = 4 \left(\frac{K^2}{3\omega_1} \right)^3 (m - 2)(2m - 1)(m + 1);$$

whereas for $\Delta < 0$

$$g_2 = 12 \left(\frac{K^2}{3\omega_2} \right)^2 (1 - 16m + 16m^2), \quad g_3 = 8 \left(\frac{K^2}{3\omega_2} \right)^3 (2m - 1)(32m^2 - 32m - 1).$$

Here $m = k^2 \in (0, 1)$ is the argument of K , the underlying lattice has periods $2\omega, 2\omega', \omega_1 = \omega$ and $\omega_2 = \omega + \omega'$. Fixing 2 as a period of the lattice, and that the lattice is real, sets $\omega_1 = 1$ for $\Delta > 0$ and $\omega_2 = 1$ for $\Delta < 0$. Observe that for $\text{sgn}(\Delta) = \pm 1$ that $g_2(m)$ takes it's minimum value at $m = 1/2$ while for $m \in (0, 1/2)$ we have $g_3(m) > 0$.

Our elliptic curve gave the equations $a^2 + \frac{b^2}{12} = g_2, \frac{b(b^2 - 36a^2)}{216} + \frac{1}{4}\tilde{c}^2 = g_3$. These equations are underdetermined, but we may substitute for a^2 and take $\alpha = -27\tilde{c}^2/b^3$ to find

$$(4 - 2\alpha)\tilde{b}^3 - g_2\tilde{b} - g_3 = 0, \tag{23}$$

where $\tilde{b} = \frac{b}{6}$. The discriminant of this cubic is

$$\Delta_\alpha(m) = 4(4 - 2\alpha)g_2^3 - 27(4 - 2\alpha)^2g_3^2 = 4(4 - 2\alpha) \left[g_2^3 - 27(1 - \alpha/2)g_3^2 \right].$$

Note $\Delta_0 = \Delta$. For a given generic value of α in some region we may solve (23), determining b, \tilde{c} and a in turn.

In order to get Nahm data, we require that b, \tilde{c} , and a are real. We know that this cubic has real coefficients, and so there will always be a real root of the cubic. To get reality of \tilde{c} , we need that this real root \tilde{b} satisfies $\text{sgn}(\tilde{b}) = -\text{sgn}(\alpha)$, and for reality of a we need $|\tilde{b}| \leq g_2/3$. Necessary conditions to find such solutions are as follows.

First consider $\Delta > 0$. Then $g_2 > 0$ and g_3 is monotonically decreasing for $m \in (0, 1)$ with $\text{sgn}(g_3) = -\text{sgn}(m - 1/2)$. We have the following properties:

- (i) If $\alpha > 2$ the discriminant $\Delta_\alpha(m) < 0$. Then (23) has one real root whose sign is opposite that of g_3 . Now $\text{sgn}(\tilde{b}) = -\text{sgn}(\alpha) < 0$ is opposite that of g_3 ; hence we require $g_3 > 0$ and so $m \in (0, 1/2)$.
- (ii) For $\alpha \in (0, 2)$ the discriminant $\Delta_\alpha(m) > 0$ upon comparison with $\Delta = g_2^3 - 27g_3^2 > 0$. Then, because the sum of the roots is zero, they cannot all be the same sign.
- (iii) When $\alpha < 0$, from the derivative of the cubic we know it will have a local maxima and minima at $\tilde{b} = \pm \sqrt{\frac{g_2}{3(4-2\alpha)}}$; it is the minima when the sign is positive. Recalling that we require a root with sign $\text{sgn} \tilde{b} = -\text{sgn} \alpha = 1$, the local minima must be

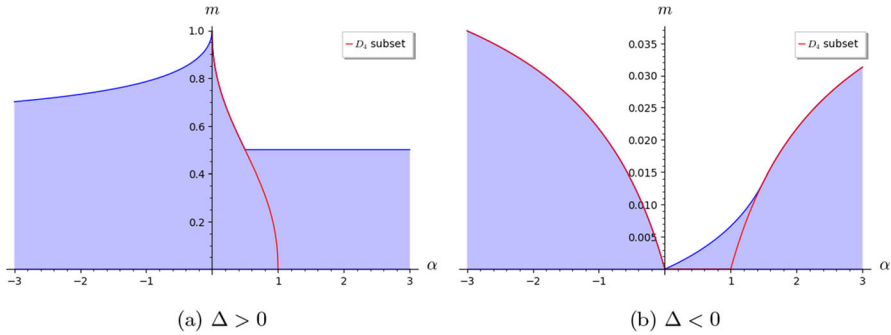


Fig. 4 Valid parameter regions for V_4 monopoles, with the subset corresponding to D_4 monopoles highlighted

non-positive, and the value at this \tilde{b} is $-\frac{2}{3}\tilde{b}g_2 - g_3$. As the value at this minima is monotonically increasing for $m > 1/2$, and negative at $m = 1/2$, then the value at the minima is negative for all $m < m_*$, the value for which it is zero. Solving, one gets the condition $\Delta_\alpha(m_*) = 0$, taking the root greater than $1/2$.

Therefore necessary conditions for a real root of the right sign to exist for $\Delta > 0$ are that

- if $\alpha > 2, m < 1/2$,
- if $\alpha \in (0, 2)$, any m is valid
- if $\alpha < 0, m < m_2(\alpha)$, where m_2 is the root of $\Delta_\alpha(m) = 0$ in $(1/2, 1)$.

To get Nahm data we require that this real root is bounded in magnitude by $\sqrt{g_2/3}$, with the case that it is equal corresponding to $a = 0$, i.e. to the D_4 monopoles of [26]. Figures showing these parameter regions are given in Fig. 4.

In the case $\Delta < 0$, in order to get real roots of the right sign one analogously gets restrictions on m relative to α such that

- if $\alpha < 0, m < 1/2$,
- if $\alpha \in (0, 2), m > 1/2$ or $m < m_1(\alpha)$, defined to be the root $< 1/2$ of the polynomial $\Delta_\alpha(m) = 0$.
- if $\alpha > 2, m < m_2(\alpha)$, now defined to be the root $> 1/2$ of the polynomial $\Delta_\alpha(m) = 0$.

Fixing the size of the root in this case requires more work, complicated by the fact that g_2 is real only if $|m - 1/2| > \sqrt{3}/4$. Using explicitly formulas for the roots \tilde{b} from Cardano’s formula one can achieve explicit bounds, but here we omit these. In practice, when using this approach to plot monopoles, numerical methods can be used to find the appropriate m region for a given α , as done to generate Fig. 4.

Note that, for certain admissible α, m there may be two possible monopoles because two roots of the cubic defining b satisfy the required conditions. Numerical investigations indicates that this phenomenon only occurs for $\Delta > 0$. We plotted two examples of this, seen in Fig. 5 to investigate the difference in the associated monopoles. These regions stitch together so as to make the moduli space connected.

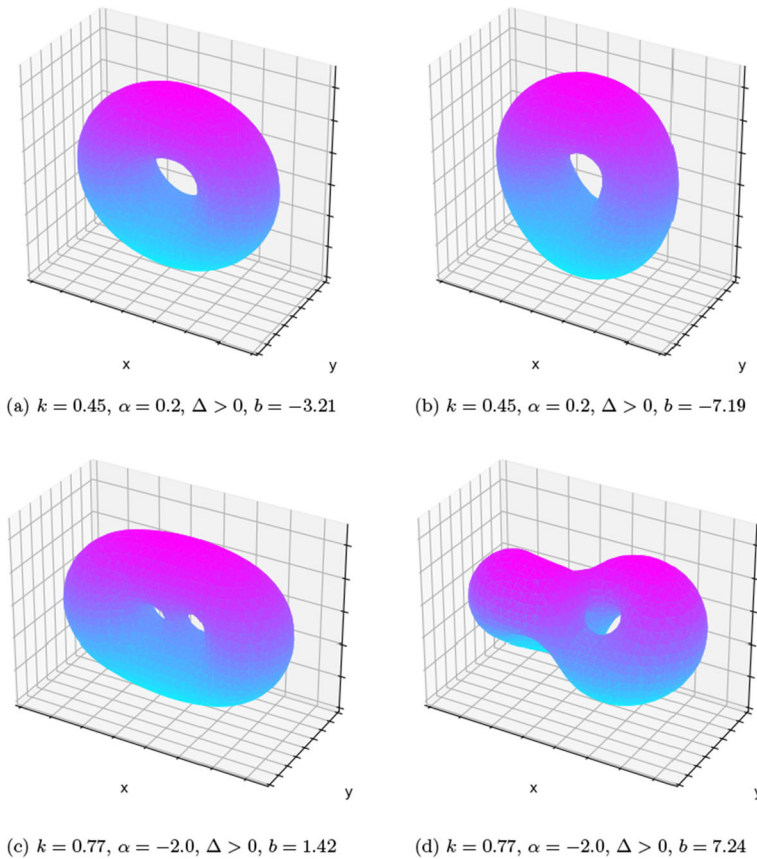


Fig. 5 Comparison of two pairs of monopoles with equal values of α, m , taking $\mathcal{E} = 0.17$

Appendix B: Initial computation of D_6 Nahm matrices

Here we will take the procedure introduced in [22] and developed in [24–26] and apply this to the C_k symmetric monopole. In brief, this procedure uses representation theory to construct Nahm matrices with a given symmetry group $G \leq SO(3)$. This is done by establishing an isomorphism between the $SO(3)$ -representation space $\mathbb{R}^3 \otimes \mathfrak{su}(k)$ containing the matrix triples and a $SO(3)$ -representation space described by homogeneous bivariate polynomials acted on naturally via the isomorphism $SO(3) \cong PSU(2)$. On the latter the action of the symmetry group is well understood, and so one can take a polynomial Q invariant under G and return via the isomorphism a triple of matrices (S_i) also invariant under the symmetry. The number of possible matrix triples one can get from the input polynomial is determined by the representation theory, and there is always a triple (ρ_i) invariant under the action of $SO(3)$ corresponding to the polynomial 1. The matrices returned by this isomorphism need not be anti-Hermitian, but they can be fixed to be so.

We describe the procedure for general k before applying this in the $k = 3$ context. The steps of the procedure are:

- (1) Take the input polynomials to be $Q_l = \zeta_0^l \zeta_1^l, i = l, \dots, k$ and $Q_{k+1} = \zeta_0^{2k} - \zeta_1^{2k}$.
- (2) Construct the invariant matrix triples $(\rho_i), (S_i^{(j)})$ corresponding to the polynomials 1, Q_l , respectively, and scale them so they are all anti-Hermitian. The degree of the input polynomial d determines how many direct summands of the $(d + 1)$ -dimensional irreducible representation space of $SO(3)$ there are in $\mathbb{R}^3 \otimes \mathfrak{gl}(k)$ when decomposed into irreducibles. There are $3k - 2$ invariant S -vectors and so variables y_j associated with them; together with (ρ_i) and the associated variable x we have $3k - 1$ variables.
- (3) Set now $T_i = x\rho_i + \sum_j S_i^{(j)}$, where x, y_j are real functions and so T_i is anti-Hermitian.
- (4) Diagonalise the matrix T_3 with a unitary matrix U whose columns are the normalised eigenvectors of T_3 , that is construct $U^{-1}T_3U$. As T_3 is anti-hermitian and linear in the invariant vectors, the diagonal entries which are the eigenvalues will be pure-imaginary, linear in $\{x, y_j\}$.
- (5) Now from [7], conjugating by the same unitary matrix will give

$$U^{-1}(T_1 + iT_2)U = \sum_{j=1}^k \alpha_j E_{j,j+1}, \quad U^{-1}(T_1 - iT_2)U = \sum_{j=1}^k -\bar{\alpha}_j E_{j+1,j},$$

for some $\alpha_1, \dots, \alpha_n \in \mathbb{C}$.

- (6) Writing $\alpha_j = r_j e^{i\phi_j}$ (as generically $\alpha_j \neq 0$) and solving

$$\phi_j + \theta_{j+1} - \theta_j = 0, \quad j = 1, \dots, k - 1, \quad \sum_j \theta_j = 0,$$

for $\theta_1, \dots, \theta_k \in \mathbb{R}$, then conjugating by the unitary matrix $D = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_k})$ preserves the form of T_3 , but acts to make each α_j real for $j = 1, \dots, k - 1$ (as it multiplies α_j by $e^{i(\theta_{j+1} - \theta_j)}$). The effect on α_k is to multiply this by $e^{i(\theta_1 - \theta_k)} = e^{i(\phi_1 + \dots + \phi_{k-1})} = \prod_{j=1}^{k-1} (\alpha_j / r_j)$. After quotienting by this action the number of independent variables we have is $3k - 2 + 1 - (k - 1) = 2k$.

We now apply the algorithm in the case $k = 3$. This yields Nahm matrices

$$T_1 = \begin{pmatrix} 0 & iy_0 & iy_1 + y_5 + iy_6 \\ iy_0 & 0 & -2x - y_2 + iy_3 \\ iy_1 - y_5 + iy_6 & 2x + y_2 + iy_3 & 0 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} iy_0 & 0 & 2x + y_2 - iy_3 \\ 0 & -iy_0 & iy_1 + y_5 + iy_6 \\ -2x - y_2 - iy_3 & iy_1 - y_5 + iy_6 & 0 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} iy_1 + iy_4 - \frac{2}{3}iy_6 & -2x + 2y_2 & 0 \\ 2x - 2y_2 & iy_1 + iy_4 - \frac{2}{3}iy_6 & 0 \\ 0 & 0 & -2iy_1 + iy_4 + \frac{4}{3}iy_6 \end{pmatrix}.$$

with accompanying ODEs in 8 real variables

$$\begin{aligned}
 x' &= 2x^2 - \frac{1}{3}y_0^2 - \frac{5}{6}y_1^2 - \frac{1}{2}y_2^2 + \frac{1}{6}y_3^2 + \frac{1}{6}y_5^2 + \frac{5}{6}y_6^2, \\
 y_0' &= -4xy_0 + 4y_0y_2, \\
 y_1' &= -4xy_1 - \frac{16}{5}y_1y_2 - \frac{6}{5}y_3y_5 - \frac{6}{5}y_2y_6, \\
 y_2' &= \frac{2}{3}y_0^2 - \frac{4}{3}y_1^2 - 2xy_2 - y_2^2 - \frac{1}{3}y_3^2 - \frac{1}{3}y_5^2 - y_1y_6 + \frac{1}{3}y_6^2, \\
 y_3' &= 2xy_3 - 2y_2y_3 - 3y_1y_5 + 2y_5y_6, \\
 y_4' &= 0, \\
 y_5' &= -3y_1y_3 + 2xy_5 - 2y_2y_5 + 2y_3y_6, \\
 y_6' &= -\frac{9}{5}y_1y_2 + \frac{6}{5}y_3y_5 + 6xy_6 + \frac{6}{5}y_2y_6.
 \end{aligned}$$

The associated spectral curve is

$$\eta^3 + \alpha_1\eta^2\zeta + \alpha_2\eta\zeta^2 + \alpha_3\zeta^3 + \beta\zeta^6 - \bar{\beta} = 0$$

where

$$\begin{aligned}
 \alpha_1 &= -6y_4, \\
 \alpha_2 &= 4y_0^2 - 8y_1^2 + 48xy_2 - 12y_2^2 + 4y_3^2 + 12y_4^2 + 4y_5^2 + 24y_1y_6 - \frac{4}{3}y_6^2, \\
 \alpha_3 &= -160x^2y_1 + 16y_0^2y_1 + 8y_1^3 + 128xy_1y_2 - 40y_1y_2^2 - 8y_1y_3^2 - 8y_0^2y_4 \\
 &\quad + 16y_1^2y_4 - 96xy_2y_4 + 24y_2^2y_4 - 8y_3^2y_4 - 8y_4^3 - 32xy_3y_5 + 32y_2y_3y_5 \\
 &\quad - 8y_1y_5^2 - 8y_4y_5^2 - \frac{32}{3}y_0^2y_6 - \frac{128}{3}y_1^2y_6 - 32xy_2y_6 + 80y_2^2y_6 \\
 &\quad + \frac{16}{3}y_3^2y_6 - 48y_1y_4y_6 + \frac{16}{3}y_5^2y_6 + 24y_1y_6^2 + \frac{8}{3}y_4y_6^2 + \frac{16}{27}y_6^3, \\
 \beta &= -16x^2y_0 + 4y_0y_1^2 - 16xy_0y_2 - 4y_0y_2^2 + 8jy_0y_1y_3 - 4y_0y_3^2 - 16ixy_0y_5 \\
 &\quad - 8iy_0y_2y_5 + 4y_0y_5^2 + 8y_0y_1y_6 + 8iy_0y_3y_6 + 4y_0y_6^2.
 \end{aligned}$$

In order to make the variables real we have imposed the anti-Hermiticity condition required of the Nahm matrices at the beginning, by making the invariant vectors corresponding to each variable anti-Hermitian.

We may consistently set $y_3 = 0 = y_5$, which we may view as using the conjugation action of diagonal matrices $\text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})$, $\theta_1 + \theta_2 + \theta_3 = 0$. This leaves us with the $2 \times 3 = 6$ real variables we would expect to have from the corresponding Toda. Note that because $\alpha_1' = 0$, the centre of mass of the Toda system is already fixed. Moreover, we may centre to consistently set $y_4 = 0$, and so we now have the equations in the

remaining 5 variables as

$$\begin{aligned}x' &= 2x^2 - \frac{1}{3}y_0^2 - \frac{5}{6}y_1^2 - \frac{1}{2}y_2^2 + \frac{5}{6}y_6^2, \\y_0' &= -4xy_0 + 4y_0y_2, \\y_1' &= -4xy_1 - \frac{16}{5}y_1y_2 - \frac{6}{5}y_2y_6, \\y_2' &= \frac{2}{3}y_0^2 - \frac{4}{3}y_1^2 - 2xy_2 - y_2^2 - y_1y_6 + \frac{1}{3}y_6^2, \\y_6' &= -\frac{9}{5}y_1y_2 + 6xy_6 + \frac{6}{5}y_2y_6,\end{aligned}$$

with conserved quantities

$$\begin{aligned}\alpha_2 &= 4y_0^2 - 8y_1^2 + 48xy_2 - 12y_2^2 + 24y_1y_6 - \frac{4}{3}y_6^2, \\ \alpha_3 &= -160x^2y_1 + 16y_0^2y_1 + 8y_1^3 + 128xy_1y_2 - 40y_1y_2^2 \\ &\quad - \frac{32}{3}y_0^2y_6 - \frac{128}{3}y_1^2y_6 - 32xy_2y_6 + 80y_2^2y_6 \\ &\quad + 24y_1y_6^2 + \frac{16}{27}y_6^3, \\ \beta &= -16x^2y_0 + 4y_0y_1^2 - 16xy_0y_2 - 4y_0y_2^2 \\ &\quad + 8y_0y_1y_6 + 4y_0y_6^2.\end{aligned}$$

At this stage the resulting ODEs are somewhat opaque and we may use the connection to Toda to clarify. Following the steps of the procedure from [7] outlined earlier we may put the Nahm Lax pair in Toda form, namely with

$$\begin{aligned}T_1 + iT_2 &= \begin{pmatrix} 0 & -2\sqrt{2}x - \sqrt{2}y_1 - \sqrt{2}y_2 - \sqrt{2}y_6 & & 0 \\ 0 & & 0 & 2\sqrt{2}x - \sqrt{2}y_1 + \sqrt{2}y_2 - \sqrt{2}y_6 \\ 2y_0 & & 0 & 0 \end{pmatrix}, \\ T_1 - iT_2 &= \begin{pmatrix} & & 0 & 0 & -2y_0 \\ 2\sqrt{2}x + \sqrt{2}y_1 + \sqrt{2}y_2 + \sqrt{2}y_6 & & & 0 & 0 \\ & 0 & -2\sqrt{2}x + \sqrt{2}y_1 - \sqrt{2}y_2 + \sqrt{2}y_6 & & 0 \end{pmatrix}, \\ -2iT_3 &= \begin{pmatrix} -4x + 2y_1 + 4y_2 - \frac{4}{3}y_6 & & 0 & 0 \\ 0 & -4y_1 + \frac{8}{3}y_6 & & 0 \\ 0 & 0 & 0 & 4x + 2y_1 - 4y_2 - \frac{4}{3}y_6 \end{pmatrix}.\end{aligned}$$

This gives us variables

$$\begin{aligned} a_0 &= 2y_0, & a_1 &= -2\sqrt{2}x - \sqrt{2}y_1 - \sqrt{2}y_2 - \sqrt{2}y_6, \\ a_2 &= 2\sqrt{2}x - \sqrt{2}y_1 + \sqrt{2}y_2 - \sqrt{2}y_6, \\ b_1 &= 4x - 2y_1 - 4y_2 + \frac{4}{3}y_6, & b_2 &= 4y_1 - \frac{8}{3}y_6, \\ b_3 &= -4x - 2y_1 + 4y_2 + \frac{4}{3}y_6. \end{aligned}$$

These variables are the Flaschka coordinates for the periodic Toda system. (Any 6-tuple satisfying $\sum_i b_i = 0$ gives valid x, y_j .) In these new variables we have

$$\begin{aligned} a'_0 &= \frac{1}{2}a_0(b_3 - b_1), & a'_1 &= \frac{1}{2}a_1(b_1 - b_2), & a'_2 &= \frac{1}{2}a_2(b_2 - b_3), \\ b'_1 &= a_1^2 - a_0^2, & b'_2 &= a_2^2 - a_1^2, & b'_3 &= a_0^2 - a_2^2, \end{aligned} \tag{24}$$

together with the constants

$$\begin{aligned} \alpha_2 &= b_1b_2 + b_1b_3 + b_2b_3 + a_0^2 + a_1^2 + a_2^2, \\ \alpha_3 &= b_1b_2b_3 + b_1a_2^2 + b_2a_0^2 + b_3a_1^2, & \beta &= a_0a_1a_2. \end{aligned}$$

At this stage we have 6 variables and 3 constraints. One could in principle solve these explicitly using the fact that the flow linearises on the Jacobian of the associated hyperelliptic curve as in [38, Theorem 5.1]. Are simplifications possible? We may use Gröbner bases in Sage to utilise the constants $\alpha_2, \alpha_3, 0 = \sum b_i$ to eliminate the b_i , and we get the equations described in the text,

$$\begin{aligned} 0 &= \sum_{i=0}^2 a_i^2 - \alpha_2 - \frac{1}{3}(d_1^2 + d_1d_2 + d_2^2), \\ 0 &= a_1^2d_2 - a_2^2d_1 + \alpha_3 + \frac{1}{3}\alpha_2(d_1 - d_2) + \frac{1}{27}(d_1 - d_2)^3, \end{aligned}$$

where we have introduced $d_i = \frac{2a'_i}{a_i}$. This in principle is the maximal reduction one can achieve with the variables provided when the α_i and β are generic.

One simplification which can be achieved is by attempting to make the second equation a polynomial in $d_1 - d_2$. To do this we would need $a_1^2 = a_2^2$. We can calculate that

$$\begin{aligned} \frac{d}{ds}(a_1^2 - a_2^2) &= 2 \left[a_1 \left(\frac{1}{2}a_1(b_1 - b_2) \right) - a_2 \left(\frac{1}{2}a_2(b_2 - b_3) \right) \right], \\ &= a_1^2(b_1 - b_2) - a_2^2(b_2 - b_3), \\ &= a_1^2(b_1 - 2b_2 + b_3) + (a_1^2 - a_2^2)(b_2 - b_3), \\ &= -3b_2a_1^2 + (a_1^2 - a_2^2)(b_2 - b_3). \end{aligned}$$

Hence we can consistently set $a_1^2 - a_2^2 = 0$ provided $b_2 a_1^2 = 0$. As $b_2' = a_2^2 - a_1^2$, this means we can consistently set $a_1^2 = a_2^2$ and $b_2 = 0$. Making these restrictions we can now eliminate the one remaining equation to find

$$0 = a_0^2 + 2a_1^2 - \alpha_2 - d_1^2 \Rightarrow a_1^2 \left(2 \frac{da_1}{ds} \right)^2 = \beta^2 + 2a_1^6 - \alpha_2 a_1^4.$$

Appendix C: Initial computation of V_4 Nahm matrices

Taking the polynomials $\zeta_0 \zeta_1 (\zeta_0^4 - \zeta_1^4)$, $\zeta_0^2 \zeta_1^2$, and $\zeta_0^4 + \zeta_1^4$ as the inputs to the procedure of outlined in Appendix B gives the ODES in the six real-valued variables

$$\begin{aligned} x' &= 2x^2 - \frac{1}{6}y_0^2 + \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 + \frac{1}{6}y_3^2 - \frac{1}{2}y_4^2, & y_2' &= \frac{1}{3}y_0^2 + y_1^2 - 2xy_2 - y_2^2 - \frac{1}{3}y_3^2 - y_1y_4, \\ y_0' &= -2xy_0 + 2y_0y_2 + 2y_1y_3 + y_3y_4, & y_3' &= 2y_0y_1 + 2xy_3 - 2y_2y_3 + y_0y_4, \\ y_1' &= 2xy_1 + 2y_1y_2 + \frac{2}{3}y_0y_3 - y_2y_4, & y_4' &= -2y_1y_2 + \frac{2}{3}y_0y_3 - 4xy_4, \end{aligned}$$

with the corresponding spectral curve

$$C : \eta^3 + \eta \left[a(\zeta^4 + 1) + b\zeta^2 \right] + c\zeta(\zeta^4 - 1) = 0,$$

where

$$\begin{aligned} a &= 8xy_0 + 4y_0y_2 - 4y_1y_3 + 4y_3y_4, \\ b &= 4y_0^2 - 12y_1^2 + 48xy_2 - 12y_2^2 + 4y_3^2 - 24y_1y_4, \\ c &= -8iy_0^2y_1 - 8iy_1^3 + 48ixy_1y_2 + 24iy_1y_2^2 - 16ixy_0y_3 + 16iy_0y_2y_3 \\ &\quad + 8iy_1y_3^2 + 48ix^2y_4 - 4iy_0^2y_4 + 12iy_1^2y_4 - 12iy_2^2y_4 + 4iy_3^2y_4 - 4iy_4^3. \end{aligned}$$

The full Nahm matrices are

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\bar{f}_1 \\ 0 & f_1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & f_2 \\ 0 & 0 & 0 \\ -\bar{f}_2 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -\bar{f}_3 & 0 \\ f_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

where the f_i are given by

$$\begin{aligned} f_1 &= 2x + y_0 - iy_1 + y_2 + iy_3 + iy_4, \\ f_2 &= 2x - y_0 - iy_1 + y_2 - iy_3 + iy_4, \\ f_3 &= 2x + 2iy_1 - 2y_2 + iy_4, \end{aligned}$$

One can check that setting $y_1 = y_3 = y_4 = 0$ is consistent, and corresponds to the inversion symmetric case. Note the condition on the residues of the Nahm data now become that the residue of each f_i at the poles is 1.

References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. 10th printing, with corrections. National Bureau of Standards. A Wiley-Interscience Publication. New York etc.: Wiley. xiv, 1046 pp. £ 42.70 (1972) (1972)
2. Armstrong, J.A., Bloembergen, N., Ducuing, J., Pershan, P.S.: Interactions between light waves in a nonlinear dielectric. *Phys. Rev.* **127**, 1918–1939 (1962)
3. Atiyah, M.F.: Magnetic monopoles in hyperbolic spaces. In: Vector bundles on Algebraic varieties, pp. 1–29 (1984)
4. Atiyah, M.F., Hitchin, N.J.: The Geometry and Dynamics of Magnetic Monopoles. Princeton University Press, Princeton (1988)
5. Berndt, B.C., Bhargava, S., Garvan, F.G.: Ramanujan's theories of elliptic functions to alternative bases. *Trans. Am. Math. Soc.* **347**(11), 4163–4244 (1995)
6. Bielawski, R.: A hyperkähler submanifold of the monopole moduli space. *Ann. Mat. Pura Appl.* (1923-) **199**(1), 401–407 (2020)
7. Braden, H.W.: Cyclic monopoles, affine Toda and spectral curves. *Commun. Math. Phys.* **308**(2), 303–323 (2011)
8. Braden, H.W.: Spectral curves are transcendental. *Lett. Math. Phys.* **111**(1), 9 (2021)
9. Braden, H.W., D'Avanzo, A., Enolski, V.Z.: On charge-3 cyclic monopoles. *Nonlinearity* **24**(3), 643 (2011)
10. Braden, H.W., Enolski, V.Z.: On the tetrahedrally symmetric monopole. *Commun. Math. Phys.* **299**(1), 255–282 (2010)
11. Braden, H.W., Enolski, V.Z.: The construction of monopoles. *Commun. Math. Phys.* **362**(2), 547–570 (2018)
12. Braden, H.W., Enolski, V.Z.: The charge 2 monopole via the ADHMN construction (2019)
13. Olver, F.W.J., Olde Daalhuis, A.B., Lozier, D.W., Schneider, B.I., Boisvert, R.F., Clark, C.W., Miller, B.R., Saunders, B.V., Cohl, H.S., McClain, M.A. eds.: NIST Digital Library of Mathematical Functions. <http://dlmf.nist.gov/>, Release 1.1.8 of -12-15 (2022)
14. Donaldson, S.K.: Nahm's equations and the classification of monopoles. *Commun. Math. Phys.* **96**(3), 387–407 (1984)
15. Duke, W., Imamoglu, Ö.: The zeros of the Weierstrass \wp -function and hypergeometric series. *Math. Ann.* **340**(4), 897–905 (2008)
16. Earle, C.J.: On the moduli of closed riemann surfaces with symmetries. In: Advances in the Theory of Riemann Surfaces. AM-66, vol. 66, pp. 119–130. Princeton University Press, Princeton (1971)
17. Ercolani, N., Sinha, A.: Monopoles and Baker functions. *Commun. Math. Phys.* **125**(3), 385–416 (1989)
18. Hartshorne, R.: Algebraic Geometry. Graduate Texts in Mathematics, No. 52. Springer, New York-Heidelberg (1977)
19. Helgason, S.: Differential geometry: Lie groups, and symmetric spaces. Pure and applied mathematics. A series of monographs and textbooks, vol. 80. Academic Press, New York (1978)
20. Hitchin, N.J.: Monopoles and geodesics. *Commun. Math. Phys.* **83**(4), 579–602 (1982)
21. Hitchin, N.J.: On the construction of monopoles. *Commun. Math. Phys.* **89**(2), 145–190 (1983)
22. Hitchin, N.J., Manton, N.S., Murray, M.K.: Symmetric monopoles. *Nonlinearity* **8**(5), 661–692 (1995)
23. Houghton, C.J.: Multimonopoles. PhD thesis, University of Cambridge (1997)
24. Houghton, C.J., Sutcliffe, P.M.: Tetrahedral and cubic monopoles. *Commun. Math. Phys.* **180**(2), 343–361 (1996)
25. Houghton, C.J., Sutcliffe, P.M.: Inversion symmetric 3-monopoles and the Atiyah–Hitchin manifold. *Nonlinearity* **9**(6), 1609 (1996)
26. Houghton, C.J., Sutcliffe, P.M.: Monopole scattering with a twist. *Nucl. Phys. B* **464**(1), 59–84 (1996)
27. Jones, G.A., Singerman, D.: Complex Functions: An Algebraic and Geometric Viewpoint. Cambridge University Press, Cambridge (1987)
28. Lawden, D.F.: Elliptic Functions and Applications. Applied Mathematical Sciences, vol. 80. Springer, New York (1989)
29. The LMFDB Collaboration. The L-functions and modular forms database. <http://www.lmfdb.org> (2022). Accessed 12 July 2022

30. Magaard, K., Shaska, T., Shpectorov, S., Voelklein, H.: The locus of curves with prescribed automorphism group. In Nakamura, H. (ed.) *Communications on Arithmetic Fundamental Groups and Galois Theory*, Kyoto Technical Report Series (2002)
31. Manton, N., Sutcliffe, P.: *Topological Solitons*. Cambridge University Press, Cambridge (2004)
32. Novikov, S., Manakov, S.V., Pitaevskii, L.P., Zakharov, V.E.: *Theory of solitons. Contemporary Soviet Mathematics*. Consultants Bureau [Plenum], New York, *The inverse scattering method*, Translated from the Russian (1984)
33. Olive, D., Turok, N.: The symmetries of Dynkin diagrams and the reduction of Toda field equations. *Nucl. Phys. B* **215**(4), 470–494 (1983)
34. Stuart, D.: The geodesic approximation for the Yang–Mills–Higgs equations. *Commun. Math. Phys.* **166**(1), 149–190 (1994)
35. Sutcliffe, P.M.: Seiberg–Witten theory, monopole spectral curves and affine Toda solitons. *Phys. Lett. B* **381**(1), 129–136 (1996)
36. Sutcliffe, P.M.: Cyclic monopoles. *Nucl. Phys. B* **505**(1), 517–539 (1997)
37. Vakil, R.: *The Rising Sea: Foundations of Algebraic Geometry* (2010)
38. van Moerbeke, P.: The spectrum of Jacobi matrices. *Inventiones mathematicae* **37**(1), 45–81 (1976)
39. Wiman, A.: Ueber die algebraischen curven von den geschlechtern $p=4, 5$ und 6 , welche eindeutigen transformationen in sich besitzen. *Bihang till Kongl. Svenska vetenskaps-akademiens handlingar*, 21:1–41, English translation available at [arXiv:2204.01656](https://arxiv.org/abs/2204.01656) (1895)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.