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Orbits of Theta Characteristics

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ABSTRACT
The theta characteristics on a Riemann surface are permuted by the induced action of the automorphism group, with the orbit structure being important for the geometry of the curve and associated manifolds. We describe two new methods for advancing the understanding of these orbits, generalizing existing results of Kallel & Sjerve, allowing us to establish the existence of infinitely many curves possessing a unique invariant characteristic as well as determine the number of invariant characteristics for all Hurwitz curves with simple automorphism group. In addition, we compute orbit decompositions for a substantial number of curves with genus ≤ 9, allowing the identification of where current theoretical understanding falls short and the potential applications of machine learning techniques.

KEYWORDS
Theta characteristics; Riemann surfaces; automorphisms; group cohomology; computation

1991 MATHEMATICS SUBJECT CLASSIFICATION
14H37; 14Q05; 14H60; 20J06

1. Introduction

Given a smooth, compact, connected Riemann surface $\mathcal{C}$, a **theta characteristic** on $\mathcal{C}$ is a line bundle $L \to \mathcal{C}$ such that $L^2 := L \otimes L$ is isomorphic to the canonical bundle $K_{\mathcal{C}}$. Given that $\deg K_{\mathcal{C}} = 2g - 2$ is even, where $g = g(\mathcal{C})$ is the genus of the curve, it is necessarily true that theta characteristics exist and moreover that there are exactly 2^{2g} of them. The latter follows for, given a theta characteristic L and a 2-torsion degree-0 line bundle $L' \in \mathrm{Pic}^0(\mathcal{C})[2]$, necessarily $L \otimes L'$ is also a theta characteristic. As such the set of characteristics $S(\mathcal{C})$ is an affine space over $\mathbb{Z}_2$[1] modelled on $H^1(\mathcal{C}, \mathbb{Z}_2)$. Theta characteristics are further distinguished by their **parity**: a theta characteristic L is called odd/even if $\dim H^0(\mathcal{C}, L)$ is odd/even. There are exactly $2^{g-1}(2^g - 1)$ odd theta characteristics and $2^{g-1}(2^g + 1)$ even theta characteristics [3, 36].

Theta characteristics are permuted by the action of the automorphism group $\mathrm{Aut}\,\mathcal{C}$, and in his landmark paper [3] Atiyah showed that every $f \in \mathrm{Aut}(\mathcal{C})$ leaves at least one characteristic invariant. Kallel and Sjerve [48] then greatly expanded upon this result, showing how to compute the orbits of characteristics given knowledge of the rational representation of their automorphism group, and using that to give further results about the number of characteristics invariant under a given automorphism. In Sections 2 and 3 we will briefly review their results, as we will then use this to give a group cohomology description of invariant characteristics. This allows us to generalize the results of [48], giving two of the main new results of this paper

Proposition 1.1. *If there exists $f \in G \leq \mathrm{Aut}(\mathcal{C})$ such that f has odd order, $\langle f \rangle$ is subnormal in G, and $g(\mathcal{C}/\langle f \rangle) = 0$, then $\mathcal{C}$ has a unique theta characteristic invariant under the action of G.*

Theorem 1.2. *There are infinitely many curves $\mathcal{C}$, both non-hyperelliptic and hyperelliptic, with a unique theta characteristic invariant under* $\mathrm{Aut}(\mathcal{C})$.

In Section 4 we will describe tables of curves of genus ≤ 9 and their orbit decompositions, stratified by the automorphism group action of the curve, given in Appendix A. This will serve two purposes: first, it will act as a collation of plane curve models of Riemann surfaces with automorphisms; and second, it will identify cases where the current theory fails to explain why a curve has a unique invariant characteristic. Moreover, by synthesizing the computational tools in Sage with those of [9] we will demonstrate the applicability of machine learning tools in gaining insight.

In particular, this machine learning will lead us to the work in Section 5 where, by extending work of Dolgachev on invariant bundles over modular curves [29], we get the final main result of this paper.

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This article has been corrected with minor changes. These changes do not impact the academic content of the article.
[1]We will, when convenient, use the notation $\mathbb{Z}_n$ to represent the integers modulo n, that is the group $\mathbb{Z}/n\mathbb{Z}$.

Theorem 1.3. *The Hurwitz curves with simple automorphism group that have a unique invariant characteristic are those with automorphism group*

- $\mathrm{PSL}_2(q)$ *for odd* q,
- A_n *when the condition of Corollary 5.18 is satisfied,*
- J_2 *the second Janko group, and*
- *Ru the Rudvalis group.*

All other such curves have no invariant characteristics.

These we can compare with the predictions of the machine learning method.

There are additional approaches to understanding the orbits of characteristics we shall not discuss here, for example explicit representatives of the divisor classes of characteristics on hyperelliptic curves are known [48, 59], Scorza theory allows the computation of orbits of even characteristics on nonhyperelliptic genus-3 curves [28], and effective characteristics on a nonhyperelliptic curve correspond to hyperplanes tangent to the canonical embedding [30]. For a general review of theta characteristics and their applications see [35]; for more recent applications of invariant characteristics and the results of this paper see [19, 53].

Part of this work was completed during the PhD thesis of LDH [27]. The code that was used there is available at https://github.com/DisneyHogg/Riemann_Surfaces_and_Monopoles.

2. Action of the rational representation

Choosing a basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ of the homology group $H_1(\mathcal{C}, \mathbb{Z})$ with canonical intersection matrix $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$, i.e.

$$a_i \circ b_j = \delta_{ij}, \quad a_i \circ a_j = 0 = b_i \circ b_j,$$

yields the **rational representation** $\rho_r : \mathrm{Aut}(\mathcal{C}) \to \mathrm{Sp}_{2g}(\mathbb{Z})$. The starting point for this paper is the work of [48], where the authors use the identification between theta characteristic and spin structures on a Riemann surface to provide a particular choice of isomorphism $S(\mathcal{C}) \cong H_1(\mathcal{C}, \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$ which depends on the choice of homology basis such that $x \in \mathbb{Z}_2^{2g}$ transforms under $f \in \mathrm{Aut}(\mathcal{C})$ as

$$x \mapsto R^T x + v \mod 2, \tag{1}$$

where $R = \rho_r(f)$, and $\cdot^T$ means the transpose. Here the vector v is computed as $v_i = \sum_{j<j'} R_{ji} R_{j'i} J_{jj'}$. One can think of this action on x as matrix multiplication

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} R^T & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Example 2.1. *Suppose $R = I$ is the identity matrix, then*

$$v_i = \sum_{j<j'} \delta_{ji} \delta_{j'i} J_{jj'} = 0.$$

As such, as a sense check we can see that any theta characteristic is invariant under the identity transformation.

Remark 2.2. *While both the rational representation and the isomorphism $S(\mathcal{C}) \cong \mathbb{Z}_2^{2g}$ depend on the choice of canonical homology basis they are such that the overall action of* $\mathrm{Aut}(\mathcal{C})$ *on $S(\mathcal{C})$ implied by equation (1) is independent of this choice. This is what one should expect as the definition of the action in terms of pulling back line bundles does not depend on a choice of homology basis.*

This affine action is a classical result (see, for example, [45, Section V.1]), but one of the strengths of Kallel and Sjerve's approach using spin structures is that the following result is not difficult to prove.

Lemma 2.3. [46, Section 5] *Writing $x = (u, v)$ for $u, v \in \mathbb{Z}_2^g$, the parity of the associated theta characteristic is $q(x) := u \cdot v$.*

As a result of Lemma 2.3 we see the parity of a spin structure is given by a quadratic form on $H_1(\mathcal{C}, \mathbb{Z}_2)$ such that the associated bilinear form $H(x, y) := q(x + y) - q(x) - q(y)$ is the reduction mod 2 of the intersection pairing on $H_1(\mathcal{C}, \mathbb{Z})$.

Given explicitly the rational representation, equation (1) and Lemma 2.3 provide a fast and exact method to compute the orbit decomposition of theta characteristics (as opposed to slow and numerical methods using the analytical representation and the Abel-Jacobi map, see [12, 27]), split by parity. An implementation of this method may be seen in the code from [12] available at https://github.com/DisneyHogg/Brings_Curve, and this is how we shall compute tables of orbit decomposition in Section 4.

We shall at this stage introduce another concept which shall be used later, that of the **signature** of the rational representation.

Definition 2.4. [50] *Given a curve $\mathcal{C}$ and $G \leq \mathrm{Aut}(\mathcal{C})$, denote by Q_i, $i = 1, \ldots, r$, the branch points of the quotient map $\pi : \mathcal{C} \to \mathcal{C}/G$. Moreover, for any $P_i \in \pi^{-1}(Q_i)$ denote $c_i = \left|G_{P_i}\right| \in \mathbb{N}_{>1}$ the order of the associated isotropy subgroup. Finally write $g_0 = g(\mathcal{C}/G)$. We call g_0 the **quotient genus**, and the integer tuple $(g_0; c_1, \ldots, c_r)$ the **signature** of the action. We will use exponents to indicate how many times a value of c_i is repeated as in [16], e.g. the signature $(0; 2, 2, 2, 3)$ is written as $(0; 2^3, 3)$.*

*Given any abstract group G and signature $(g_0; c_1, \ldots, c_r)$ an associated **generating vector** is $\left\{\alpha_1, \beta_1, \ldots, \alpha_{g_0}, \beta_{g_0}, \gamma_1, \ldots, \gamma_r\right\} \subset G$ such that*

- *$\alpha_1, \beta_1, \ldots, \alpha_{g_0}, \beta_{g_0}, \gamma_1, \ldots, \gamma_r$ generate G,*
- *γ_i has order c_i, and*
- *$\prod_{i=1}^{g_0}(\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1})\prod_{j=1}^{r}\gamma_j = 1$.*

Given a genus-g curve acted on by G with signature $(g_0, c_1, \ldots, c_r)$ the Riemann-Hurwitz theorem immediately says that

$$2g - 2 = |G|\left[(2g_0 - 2) + \sum_i \left(1 - \frac{1}{c_i}\right)\right]. \tag{2}$$

The (character of the) rational representation completely determines the corresponding signature [64, p. 401]. Conversely, given a signature satisfying equation (2) and a corresponding generating vector in an abstract group G, there exists a Riemann surface with associated G action, and the generating vector determines the (character of the) rational representation via the Eichler trace formula [13].

3. Invariant characteristics and group cohomology

We now wish to translate the question of whether an invariant characteristic exists into more algebraic language, the starting point of which is the following proposition.

Proposition 3.1. *There is an invariant characteristic on a curve $\mathcal{C}$ if and only if the corresponding affine representation on $\mathbb{Z}_2^{2g}$*

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} R^T & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix}$$

is equivalent by a translation $x \to x' := x - y$ to the linear action

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} R^T & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ 1 \end{pmatrix}$$

for some $y \in \mathbb{Z}_2^{2g}$.

Proof. Suppose we had parameterized our set of theta characteristics differently in terms of $x' \in \mathbb{Z}_2^{2g}$ where $x = x' + y$ for some fixed $y \in \mathbb{Z}_2^{2g}$. This is reparameterization by translation. One can derive the corresponding action on x' by noting

$$\begin{aligned} x = x' + y \mapsto\ & R^T(x' + y) + v, \\ &= R^T x' + \left[v + (R^T - I)y\right] + y, \end{aligned}$$

and so

$$x' \mapsto R^T x' + v_y,$$

where $v_y = v + (R^T - I)y$. If y were fixed by the action (1) then $v_y = 0$ and moreover the converse is true. □

To utilize Proposition 3.1, we note that the question of whether an affine representation can be reduced to a linear one may be presented as a cohomology problem.[2] In particular, given a group G and a (left) G-representation $\rho : G \to \mathrm{GL}(V)$ we have the following results.

Proposition 3.2. *An affine representation of G on V which acts multiplicatively via ρ determines a 1-cocycle in the group cohomology[3] $H^1_{\mathrm{Grp}}(G, V)$ (making V into a G-module in the natural way with ρ) with the standard linear representation $G \times V \to V$, $(g, x) = \rho(g)x$, corresponding to the zero 1-cocycle.*

[2] LDH is grateful to Andrew Beckett for highlighting this to him.

[3] For an introduction to group cohomology with necessary definitions see [70, Section 6]. We shall from here on in drop the Grp subscript as through context it shall not cause confusion with any other cohomologies in this work.

Proof. An affine representation acting multiplicatively by ρ is defined by

$$\begin{aligned} G \times V &\to V, \\ (g, x) &\mapsto g \cdot x := \rho(g)x + v(g). \end{aligned}$$

for some set map $v : G \to V$. By definition the set map v is exactly a 1-cochain in group cohomology, with the linear representation giving the zero 1-cochain. Moreover, to truly get an action we require $\forall g, h \in G,\ x \in V,\ g \cdot (h \cdot x) = (gh) \cdot x$. We can compute from the definition

$$\begin{aligned} g \cdot (h \cdot x) &= \rho(g)(h \cdot x) + v(g), \\ &= \rho(g)[\rho(h)x + v(h)] + v(g), \\ &= \rho(gh)x + \rho(g)v(h) + v(g), \\ &= (gh) \cdot x + \rho(g)v(h) - v(gh) + v(g), \end{aligned}$$

and so we must have

$$\forall g, h \in G,\ 0 = \rho(g)v(h) - v(gh) + v(g). \tag{3}$$

In particular, setting $g = e$ the identity element in equation (3) shows $v(e) = 0$. Equation (3) is exactly the condition that the 1-cochain v is in fact a 1-cocycle [70, Example 6.5.6]. □

Proposition 3.3. *Two affine representations as defined in Proposition 3.2 are equivalent under a translation of V if and only if the associated 1-cocycle is a 1-coboundary.*

Proof. Fixing $y \in V$ and $v : G \to V$ defining an affine representation we have that

$$\begin{aligned} g \cdot (x + y) &= \rho(g)(x + y) + v(g), \\ &= \left\{\rho(g)x + \left[v(g) + (\rho(g) - I)y\right]\right\} + y. \end{aligned}$$

This defines a different affine action on V given by 1-cocycle $v_y(g) := v(g) + (\rho(g) - I)y$. This new affine action is actually linear if and only if

$$\forall g \in G,\ (\rho(g) - I)y + v(g) = 0 \Leftrightarrow \forall g \in G,\ g \cdot y = y \Leftrightarrow y \in V^G, \tag{4}$$

where we have used V^G to denote the subset of V invariant under G. The condition that $v(g) = \rho(g)y - y$ for some $y \in V$ is exactly the condition that v is a 1-coboundary [70, Example 6.5.6]. □

In the case at hand (the group action of the automorphism group on theta characteristics) the representation will be the reduction mod 2 (with the mod 2 reduction of R denoted by $\overline{R}$, following [48]) of the transpose of the rational representation $\rho = \overline{\rho}_r^T$ acting on the $\mathbb{Z}_2$-vector space $V = H_1(\mathcal{C}, \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$. Moreover, we can count the number of invariant characteristics as the size of $H^0(G, V)$, as $H^0(G, V) = V^G$ is exactly the submodule of invariants. This fact gives us an immediate refinement of [48, Corollary 1.3].

Proposition 3.4. *The number of characteristics invariant under the action of the whole group is either 0 or 2^k, where $k = \dim H^0(G, V)$ is the dimension[4] of the subspace of invariants.*

Proof. This is immediate from the fact $H^0(G, V)$ is a vector space over $\mathbb{Z}_2$. □

While equation (1) may be used to determine the existence of invariant characteristics, the cohomology condition thus formulated gives an algebraic formulation better suited for the investigations of this article.

We shall now want to consider two simple examples, for which we require the (proof of the) following lemma.

Lemma 3.5. [3, Lemma 5.1] *Let V be a finite-dimensional $\mathbb{Z}_2$ vector space and $q : V \to \mathbb{Z}_2$ a quadratic function fixed under an affine transformation $x \mapsto Ax + b$ whose associated bilinear H defined by*

$$H(x, y) = q(x + y) - q(x) - q(y)$$

is non-degenerate. Then the affine transformation has a fixed point.

[4] When writing dim for the dimension of the group cohomology of a $\mathbb{Z}_2$ vector space, we naturally mean the dimension over $\mathbb{Z}_2$.

Proof. We shall recall the proof of [3]. As the transform preserves q we get

$$q(x) = q(Ax + b) = q(Ax) + q(b) + H(Ax, b).$$

Setting $x = 0$ gives $q(b) = 0$ and hence $q(x) = q(Ax) + H(Ax, b)$. We can thus say

$$\begin{aligned} H(x, y) &= q(x + y) - q(x) - q(y), \\ &= \left[q(A(x + y)) + H(A(x + y), b)\right] - \left[q(Ax) + H(Ax, b)\right] \\ &\quad - \left[q(Ay) + H(Ay, b)\right], \\ &= \left[q(Ax + Ay) - q(Ax) - q(Ay)\right], \\ &= H(Ax, Ay), \end{aligned}$$

and so defining A^* to be the dual of A with respect to the non-degenerate inner product H we have $A^*A = I$. Suppose we have $x \in \mathrm{Ker}(A - I)^*$, then

$$A^*x = x \Rightarrow Ax = x \Rightarrow H(x, b) = 0,$$

the latter implication following from the relation $q(x) = q(Ax) + H(Ax, b)$, and hence we know $b \perp \mathrm{Ker}(A - I)^*$. Now $(\mathrm{Ker}(A - I)^*)^{\perp} = \mathrm{Im}(A - I)$, and so

$$b \in \mathrm{Im}(A - I) \Rightarrow \exists y \in V,\ b = (A - I)y \Rightarrow \exists y \in V,\ Ay + b = y.$$

□

We are now ready to consider two examples.

Example 3.6 (Hyperelliptic Involution). *Suppose we just have $G = C_2 = \{\pm 1\}$, where the generator of the C_2 is the hyperelliptic involution, for which the rational representation is given by $R = -I$. For a cochain $v : G \to V$ to be closed it must satisfy $v(1) = 0 \in V$, and then $v(-1) := x$ is arbitrary. It is a coboundary if in addition there exists $y \in V$, $v(-1) = y - y = 0$, and so with the hyperelliptic representation the cohomology group is given by $H^1(C_2, V) = V$. Note this is what we would expect: if the representation is trivial, then for all $y \in V$ we have $v_y = v$, meaning that the action is always given by the specific shift v. We may now use Lemma 3.5. The quadratic function on $H_1(\mathcal{C}, \mathbb{Z}_2)$ given by the parity is preserved under the affine action of $\mathrm{Aut}(\mathcal{C})$, so the affine transformation given by the hyperelliptic involution has a fixed point. The vector b in the proof of Lemma 3.5 is the value $v(-1)$, and so we must have $v(-1) \in \mathrm{Im}\, 0 = 0$. Hence we have an invariant theta characteristic, and moreover we have $2^{2g} = |V|$ of them.*

Example 3.7 (Cyclic groups). *Let us now try and understand [48, Theorem 1.1] in this language of group cohomology. Suppose we take a cyclic automorphism group $\langle f \rangle = G$, and let n be the order of f. A closed cochain must have $v(1) = 0$ as before, and then it is specified by $v(f)$, which is subject only to the condition that*

$$(I + A + \cdots + A^{n-1})v(f) = v(f^n) = v(1) = 0, \tag{5}$$

where $A = \bar{R}^T = \rho(f)$ (already reduced mod 2). Provided $A \neq I$, given that we have $(A - I)(\sum_{k=0}^{n-1} A^k) = A^n - I = 0$, the sum $\sum_{k=0}^{n-1} A^k$ has nontrivial kernel in which some $v(f) \neq 0$ must lie. A coboundary is given by $v(f) = (A - I)y$ for some $y \in V$, and hence

$$H^1(G, V) \cong \mathrm{Ker}(I + \cdots + A^{n-1}) / \mathrm{Im}(A - I).$$

Note this result is contained in [70, Theorem 6.2.2]. As in Example 3.6, the quadratic function given by the parity is preserved under the affine transformation generating the group action on $H_1(\mathcal{C}, \mathbb{Z}_2)$, so the transformation has a fixed point. As such, the remaining question is just how many invariant characteristics there are, which is given by $\dim H^0(G, V) = \dim \ker(A - I)$ and Proposition 3.4. We see there is a unique characteristic invariant under the action if and only if $A - I$ is invertible. Certainly if $n = 2$ then $A - I$ cannot be invertible as

$$(A - I)^2 = A^2 - 2A + I = 0 \mod 2.$$

This in fact generalizes; if $n = 2^k$ then $A - I$ cannot be invertible as

$$(A - I)^n = \sum_{i=0}^{n} \binom{n}{i} A^i = A^n - I = 0 \mod 2$$

using Lucas' theorem for the binomial coefficient of a prime power mod that prime (see for example [37, Theorem 3], or alternatively prove this with induction). We will therefore restrict to look at the case when n is not a power of 2. [48, Corollary 2.12] shows how to determine if $A - I$ is invertible using cyclotomic polynomials. In particular they define positive integers $1 \leq d_1 < d_2 < \cdots < d_r \leq n$ such that the minimal polynomial of A is $\prod_i \Phi_{d_i}$, where the Φ_d are the cyclotomic polynomials, and the necessary and sufficient condition for a unique invariant characteristic is that all $d_i > 1$ and no d_i is a power of 2. In [63, p. 307] the d_i are identified as exactly the $d|n$ such that $g(\mathcal{C}/\langle f^d \rangle) \neq 0$. The necessary and sufficient conditions are thus that $g(\mathcal{C}/\langle f^{2^l} \rangle) = 0$ for all l such that $2^l | n$.

In Examples 3.6 and 3.7 we saw that we always had an invariant characteristic because of Lemma 3.5. When the group in question is not cyclic we cannot rely on this result, and indeed in Table 1 we will see a case where the group is not cyclic and there are no invariant characteristics. However, if $H^1(G, V) = 0$ existence is always guaranteed. We may therefore on occasion employ techniques such as those in [47] to determine existence of an invariant characteristic.

Remark 3.8. *We can also use [65, Theorem 1] to say more. Namely, suppose that G is generated by elements $\gamma_1, \dots, \gamma_r$ such that $\prod_i \gamma_i = 1$, then*

$$\sum_{i=1}^{r} \left[2g - \dim H^0(\langle\gamma_i\rangle, V)\right] \geq \left[2g - \dim H^0(G, V)\right] + \left[2g - \dim H^0(G, V^*)\right],$$

$$\Rightarrow \dim H^0(G, V) \geq \sum_i \dim H^0(\langle\gamma_i\rangle, V) - 2g(r-2) - \dim H^0(G, V^*),$$

where $V^* = \mathrm{Hom}(V, \mathbb{Z}_2)$ is the dual G-module. Scott shows how to relate these cohomologies to those of the polyhedral group from which G can be constructed [13], suggesting one can use knowledge of the number of characteristics invariant under the $\langle\gamma_i\rangle$ to give information about how many possible invariant characteristics there may be. Moreover, Scott defines the ***homology invariant*** *h to be the nonnegative integer which is the deficit in the inequality, and finds a strong lower bound on h using conditions on the lifting order of elements in G to its double cover, similar to those which will be seen in Section 5.*

A key question we shall want to address using group cohomology is when the action of G has a **Unique Invariant Characteristic (UIC)**. To make more progress, we use the inflation-restriction exact sequence. That is for normal subgroup $N \triangleleft G$ and abelian group V with G action we have [70, p. 196]

$$0 \to H^1(G/N, V^N) \to H^1(G, V) \to H^1(N, V)^{G/N} \to H^2(G/N, V^N) \to H^2(G, V),$$

so named because the 3 inner maps are **inflation**, **restriction**, and **transgression**. Suppose we know $H^1(N, V) = 0$ and $H^0(N, V) = V^N = 0$ (as we have when N is an odd-order cyclic group quotienting to $\mathbb{P}^1$), then we have

$$0 \to H^1(G/N, 0) \to H^1(G, V) \to 0 \to H^2(G/N, 0) \to H^2(G, V).$$

This trivially gives $H^1(G, V) = 0$, and moreover because we have $(V^N)^{(G/N)} \cong V^G$, we get $H^0(G, V) = 0$.

In fact we do not need the restrictive condition that $H^1(N, V) = 0$ in the situation we care about, as we want the case where there is a unique characteristic invariant under N, which is the case when $H^0(N, V) = V^N = 0$ but also when the specific affine representation of G as an element in $H^1(G, V)$ is the zero class when restricted to the action as an element in $H^1(N, V)$. This means that the class in $H^1(G, V)$ is in the kernel of the restriction map, and as we have said that the image of the inflation map is trivial, this means the class in $H^1(G, V)$ must be the zero class. Read together these deductions from the conditions $H^0(N, V) = 0$ and $[v] = [0] \in H^1(N, V)$ tell us that if we have a normal subgroup given with a UIC, then the action of the whole group has a UIC. Moreover, this extends to if we have a subgroup which is subnormal in the original group (that is $H \leq G$ such that there exist H_i with $H \triangleleft H_1 \triangleleft \cdots \triangleleft H_k \triangleleft G$). Taking the subnormal subgroup to be a cyclic group with odd order quotienting $\mathcal{C}$ to $\mathbb{P}^1$ ensures the unique characteristics invariant under this subgroup [48, Theorem 1.1] and so gives the following proposition.

Proposition 3.9. *If there exists $f \in G \leq \mathrm{Aut}(\mathcal{C})$ such that f has odd order, $\langle f\rangle$ is subnormal in G, and $g(\mathcal{C}/\langle f\rangle) = 0$, then $\mathcal{C}$ has a unique theta characteristic invariant under the action of G. We will call a curve with this property Subnormal Odd Cyclic (SOC).*

Remark 3.10. *A more elementary proof of Proposition 3.9 exists without using the language of group cohomology. Given $N \triangleleft G$, the fixed-point set of N naturally acquires a G/N action. When the fixed-point set consists of just a single characteristic the G/N action must be trivial, but then the whole G action must be trivial. Induction on normal subgroups then gives the result. The reason for presenting the proof as it currently is in the main text is to hopefully provide the framework for future developments of the theory.*

This condition captures the existence of unique invariant characteristics in the case of even-order cyclic group actions seen earlier in Example 3.7. In that case, writing $n = 2^k m$ for some odd integer m, the necessary and sufficient conditions previously described are equivalent to the subnormal subgroup C_m having quotient genus 0. In the case of SOC curves, we are also able to determine the parity of the invariant characteristic from the signature.

Proposition 3.11. *Given a SOC curve when the subnormal cyclic group $\langle f\rangle$ has odd order n and signature $(0; c_1, \dots, c_r)$, the parity of the invariant characteristic is*

$$n \sum_{i=1}^{r} \frac{c_i - c_i^{-1}}{8} \mod 2.$$

Proof. Note that the c_i must be odd because they divide n. Hence, using [66, equation 16] we have that the parity is

$$\begin{aligned}\sum_{P\in\mathcal{C}}\frac{|\mathrm{Stab}(P)|^2-1}{8} &= \sum_{Q\in\mathcal{C}/G}\sum_{\substack{P\in\mathcal{C}\\ \pi(P)=Q}}\frac{|\mathrm{Stab}(P)|^2-1}{8} \mod 2,\\ &= n\sum_{\pi(P)\in\mathcal{C}/G}\frac{|\mathrm{Stab}(P)|-|\mathrm{Stab}(P)|^{-1}}{8} \mod 2,\\ &= n\sum_{i=1}^{r}\frac{c_i-c_i^{-1}}{8} \mod 2.\end{aligned}$$

□

We also have a slight generalization of the Proposition 3.9.

Proposition 3.12. *Let $\mathcal{L}$ be a collection of subgroups of $G \leq \mathrm{Aut}(\mathcal{C})$. Suppose that $\mathcal{L}$ has a minimal element (with respect to inclusion), that the subgroups in $\mathcal{L}$ generate G, and that for each $L \in \mathcal{L}$ there is a unique characteristic invariant under L. Then $\mathcal{C}$ has a unique characteristic invariant under the action of G.*

Proof. The proof of [2] directly works for this situation, where we only need to consider the specific extension of $\mathbb{Z}_2G$-modules $0 \to V \to E \to \mathbb{Z}_2 \to 0$ corresponding to the desired element of $H^1(G, V)$ [70, Exercise 6.1.2]. An elementary proof of this is also simply found. □

4. Computation of orbit tables

In Appendix A we shall provide tables of orbit decompositions for many different curves of genera ≤ 9, giving those for all possible curves of genus 2, 3, and 4. In this section we shall make some comments on the computation of these tables and their results.

The tables are computed using the representation in terms of binary vectors via spin structures as in [48]. We restrict to curves for which we can represent the curve in plane form as $f(x, y) = 0$, whereby we may use the method of [17] implemented in Sage to compute the rational representation. We will also want to compute the signature of the action where possible. Throughout signatures are computed with the help of the LMFDB [54], and in cases of ambiguity were verified by comparing the character of the rational representation found using this signature and the Eichler trace formula [13, p. 41] to that found by computing directly with Sage, or by determining the signature from the rational representation as in [64].

In doing the orbit decomposition computations, we are aided by the fact that we need only choose one representative curve from each class because the decomposition is uniquely determined by the rational representation and two curves in the interior of an equisymmetric family (in the sense of [15]) have equivalent rational representations [62, p. 896]. One can understand this intuitively: if the coefficients of the curve are varied only slightly (and generically) then as the rational representation is given in terms of integer valued matrices these would not be expected to change.

It is shown in [10, 48] that the hyperelliptic involution (if it exists) is the only nonidentity automorphism ι for which $\rho_r(\iota) = I \mod 2$, and so is the only nonidentity automorphism of a curve that fixes every theta characteristic, hence we know that the **reduced automorphism group** [60, 61]

$$\overline{\mathrm{Aut}}(\mathcal{C}) := \begin{cases} \mathrm{Aut}(\mathcal{C})/\langle\iota\rangle, & \mathcal{C} \text{ is hyperelliptic},\\ \mathrm{Aut}(\mathcal{C}), & \text{otherwise},\end{cases}$$

acts faithfully on the theta characteristics. There is no a priori reason to expect $\overline{\mathrm{Aut}}(\mathcal{C})$ to have an action on $\mathcal{C}$ when $\mathcal{C}$ is hyperelliptic [61].

As such we shall give tables of curves of a given genus as a plane curve $f(x, y) = 0$, their reduced automorphism group (with the GAP group ID if the presentation given of the group is not specific [41]), the quotient genus g_0 and signature $\boldsymbol{c}$ of the $\overline{\mathrm{Aut}}$ action on the curve (recall Definition 2.4) when it exists and is known, the $\overline{\mathrm{Aut}}$ orbits of the odd and even characteristics respectively presented as a list of values a_b indicating b orbits of size a, and the total number of characteristics I invariant under the group action. When giving f we shall leave in free parameters where possible, not specifying values that must be avoided. The code to recreate this data (except for the signature) is given in the Sage notebooks `list_of_plane_curves.ipynb` and `theta_characteristic_orbit.ipynb`. Table 1 shows the table constructed in the case of genus-2 curves.

Remark 4.1. *We will use the convention that the dihedral group D_n be of size $2n$.*

Remark 4.2. *The code provided may easily be adapted to compute the orbit decomposition for all subgroups of the automorphism group.*

Every genus-2 curve with a unique invariant characteristic satisfied the conditions of Proposition 3.9, that is it is SOC.
We shall now make some remarks about Tables 6–12.

Table 1. Orbit decomposition, all genus-2 curves.

f	$\overline{\mathrm{Aut}}, \boldsymbol{c}$	Odd	Even	I
$y^2 - (x^2-1)(x^2-a)(x^2-b)$	$C_2, (1; 2^2)$	2_3	$1_4, 2_3$	4
$y^2 - (x^2-1)(x^2-a)(x^2-a^{-1})$	$V_4, (0; 2^5)$	$2_1, 4_1$	$1_2, 2_2, 4_1$	2
$y^2 - (x^5-1)$	$C_5, (0; 5^3)$	$1_1, 5_1$	5_2	1
$y^2 - (x^6 - ax^3 + 1)$	$S_3, (0; 2^2, 3^2)$	6_1	$1_1, 3_3$	1
$y^2 - (x^6-1)$	$D_6, (0; 2^3, 3)$	6_1	$1_1, 3_1, 6_1$	1
$y^2 - x(x^4-1)$	S_4	6_1	$4_1, 6_1$	0

- At genus 3, not every curve with a unique invariant characteristic is SOC. There is a single exception, Klein's curve, whose automorphism group $\mathrm{PSL}_3(\mathbb{F}_2)$ is simple and so cannot have a nontrivial subnormal cyclic group. There is a C_7 subgroup of the automorphism group quotienting to $\mathbb{P}^1$ (as clearly seen by writing Klein's curve in Lefschetz form [51, 52, 75]), but it is not subnormal.
- The non-hyperelliptic genus-3 curve with $\overline{\mathrm{Aut}} = S_4$ is the first example of a curve with large automorphism group without $I \leq 1$.
- At genus 4 one sees the first example of curves of the same genus with the same pair $(\overline{\mathrm{Aut}}, \boldsymbol{c})$, but different orbit decompositions. This highlights the fact that the signature does not fully the determine the rational representation, one must also pick a generating vector [16, Definition 2.2]. The converse, that the (character of the) rational representation determines the signature, is true [64].
- At genus 4 not all curves with a unique invariant characteristic are SOC. The exceptions are

  (1) the curve with $(\overline{\mathrm{Aut}}, \boldsymbol{c}) = (A_4, (0; 2, 3^3))$,
  (2) the curve with $(\overline{\mathrm{Aut}}, \boldsymbol{c}) = (S_5, (0; 2, 4, 5))$, Bring's curve.

  More about the orbit decomposition on Bring's curve is said in [12]; here we only note that similarly to Klein's curve there is an odd order cyclic group quotienting to $\mathbb{P}^1$ (here a C_5) that is not subnormal. On the A_4 curve, we note that the quotient of the curve by the C_3 action has genus 1, and hence presents the first case where the existence of a UIC is not clearly governed by an odd cyclic quotient to $\mathbb{P}^1$.
- At genus 5 the Wiman octic is not SOC; there is a unique characteristic invariant under the $((C_4 \times C_2) \rtimes C_4) \rtimes C_3$ normal subgroup, but not under any subnormal cyclic group. Moreover, again we find that the quotient by C_3 has genus 1, and so the Wiman octic presents the second case where the existence of a UIC is not clearly governed by an odd cyclic quotient to $\mathbb{P}^1$. The curve with automorphism group $C_3 \times D_5$ is clearly SOC.
- At genus 6 every curve written down with a UIC is SOC.
- Edge describes the orbits of some of the odd characteristics of the genus-7 Fricke-Macbeath curve in terms of tangent hyperplanes [31].
- All the genus-7 curves written with a unique invariant characteristic are SOC.
- At genus 8 the four curves which have a UIC are SOC.
- Not all the genus-9 curves written here with a UIC are SOC; the hyperelliptic curve with $\overline{\mathrm{Aut}} \cong A_5$ cannot have a subnormal cyclic group.

At genera greater than 8 the computations were becoming prohibitively slow, with the calculation of the symplectic automorphism group of the Fricke octavic curve in Sage taking just under three hours on a Intel Core i5-8350U CPU at 1.70GHz. Leaving behind the criteria of requiring a plane model of the curve, one can compute additional examples of theta characteristic decompositions using the code from [9], available at https://github.com/rojas-ani/sage-routines,[5] from the data of a group, signature, and choice of generating vector provided the quotient genus is 0. The Sage notebook `genus_order_invariants_data.ipynb` shows how this can be done.

Having computed now many examples in low genera, we can pick out some families of curves with unique invariant characteristics, giving us the following theorem.

Theorem 4.3. *There are infinitely many curves $\mathcal{C}$, both non-hyperelliptic and hyperelliptic, with a unique theta characteristic invariant under* $\mathrm{Aut}(\mathcal{C})$.

Proof. It is sufficient to consider only curves of Lefschetz type, in particular the two families we shall consider are one of the Wiman hyperelliptic curves $y^2 = x^{2g+1} - 1$ and the Lefschetz curves of the form $x^m y^n + y^m + x^n = 0$ for coprime m, n where $p := m^2 - mn + n^2 > 7$ is a prime congruent to 1 mod 3.

[5] LDH is grateful to Anita Rojas for her correspondence on the workings of this code.

The former has automorphism group $C_{2g+1} \times C_2$, which contains the normal subgroup C_{2g+1} of odd order which quotients to $\mathbb{P}^1$. The latter has automorphism group of the form $C_p \rtimes C_3$, which contains the normal subgroup C_p of odd order which quotients to $\mathbb{P}^1$, which is easiest seen by writing the curve in the from $\tilde{y}^p + \tilde{x}^a(1 + \tilde{x}) = 0$ as can always be done for some a [52, p. 464]. □

Remark 4.4. *For the hyperelliptic family in the proof of Theorem 4.3 we can say more in the case that $2g + 1$ is a prime p, as all characteristics that are not invariant are in orbits of order p.*

Using Riemann-Hurwitz we know that the C_p subgroup acts with signature $(0; p^r)$ where

$$2 \times \frac{p-1}{2} - 2 = p\left[-2 + r\left(1 - \frac{1}{p}\right)\right] \Rightarrow r = 3.$$

Using Proposition 3.11 we know the parity of the UIC is $3p(p - p^{-1})/8$, and hence is determined entirely by whether $p = \pm 1$ or $p = \pm 3$ mod 8.

5. Hurwitz curves and the method of Dolgachev

5.1. Machine learning predictions

One use of the code for computing orbit decomposition united with the method of [9] for computing rational representation from signatures is that we can investigate the application of machine learning techniques in illuminating the behavior of orbits. In particular we trained a classifier to predict whether an action of a group G with a given signature $(g_0; c_1, \ldots, c_r)$ had a unique invariant characteristic. To do this we used [9] to generate 1326 topologically inequivalent generating vectors, and for each computed the number of invariant characteristics. With this we constructed input and output data upon which to train a Random Forest classifier[6]: the input data was a vector of length 10 whose entries were

(1) the genus g of the action (determined via Riemann-Hurwitz),
(2) the group order $|G|$,
(3) whether the group action was large (in the sense $|G| > 4(g - 1)$) (given as either 0 or 1),
(4) the maximum power of 2 dividing the group order,
(5) the number of involutions in the group,
(6) the number of involutions up to conjugacy in the group,
(7) the number of entries in the signature r,
(8) the number of even entries in the signature,
(9) the maximum entry of the signature, and
(10) the dimension of the corresponding family of actions $3(g_0 - 1) + r$.

The output data was a single binary value indicating whether there was a unique invariant characteristic or not (i.e. a True or False value for whether I, number of invariant characteristics, is equal to 1). These input features were chosen both because they were easily computable and because the authors felt they contained pertinent information to determining the existence of an invariant characteristic (for example, Example 3.7 shows how involutions can lead to non-uniqueness of invariant characteristics). For all our data the genus of the action was ≤ 12. Machine classification achieved an accuracy of approximately 93% in cross-validation when predicting if the corresponding group action gave a unique invariant characteristic when trained on the data, a far higher accuracy than the approximately 54% one would expect if choosing randomly with prior knowledge of the frequency of actions with a UIC in the dataset. This suggests that from very basic heuristics alone one should be able to get strong results understanding the behavior of UICs. Moreover, estimating feature importance using these methods showed that whether or not a group action was large for a given genus was unimportant in determining whether a given action led to a UIC.

In order to lay a benchmark to guide future research we computed in Table 2 the prediction of the classifier given the corresponding input data for all the simple Hurwitz groups with order $< 10^6$, provided in [20] (the J_i are the first two Janko groups). These results are correct for the two Hurwitz curves in the training set for which the answer is known, namely Klein's curve and the Fricke-Macbeath curve, though removing their data from the training set makes the classifier less accurate. Running a similar pipeline which instead predicts whether a group action has 0, 1, or many invariant characteristics confirms the predictions of Table 2, predicting that those without a unique invariant characteristic have none.

In Sections 5.2–5.6 we will now show how to compute the number of invariant characteristics for the curves in Table 2 analytically using a new method, and in Section 5.7, Table 5, we shall compare the answers.

[6]LDH is very grateful to Jacob Bradley for showing him how to do this.

Table 2. Machine prediction of whether $I = 1$, all simple Hurwitz groups of order $< 10^6$, where True indicates a prediction of a UIC.

G	g	$I = 1$
$PSL_2(7)$	3	True
$PSL_2(8)$	7	False
$PSL_2(13)$	14	True
$PSL_2(27)$	118	True
$PSL_2(29)$	146	True
$PSL_2(41)$	411	False
$PSL_2(43)$	474	True
J_1	2091	False
$PSL_2(71)$	2131	False
$PSL_2(83)$	3404	True
$PSL_2(97)$	5433	False
J_2	7201	False
$PSL_2(113)$	8589	False
$PSL_2(125)$	11626	True

5.2. The method of Dolgachev

Approximately four months after constructing these machine estimates we were alerted by Vanya Cheltsov to the paper [29], which allows one to get analytic solutions to the questions raised in Table 2, which we shall describe now, starting with the appropriate definitions. As before, take $G \leq \mathrm{Aut}(\mathcal{C})$.

Definition 5.1. *Denote with* $\mathrm{Pic}(\mathcal{C})^G$ *the group of G-invariant isomorphism classes of line bundles on* $\mathcal{C}$*, that is line bundles* $L \to \mathcal{C}$ *that admit a collection of isomorphisms* $\phi_g : g^*L \to L$ *for each* $g \in G$.

Definition 5.2. *Define* $\mathrm{Pic}(G;\mathcal{C})$ *to be the group of G-linearized line bundles on* $\mathcal{C}$ *and corresponding linearizations, that is G-invariant line bundles that admit a collection of isomorphisms* $\phi_g : g^*L \to L$ *such that*

$$\forall g, h \in G, \quad \phi_{g\circ h} = \phi_h \circ h^*\phi_g.$$

Thinking in terms of divisors $\mathrm{Div}(\mathcal{C})$, principal divisors $\mathrm{PDiv}(\mathcal{C})$, and nonzero meromorphic functions $\mathcal{M}(\mathcal{C})^\times$ instead of line bundles, $\mathrm{Pic}(\mathcal{C})^G$ corresponds to $(\mathrm{Div}(\mathcal{C})/\mathrm{PDiv}(\mathcal{C}))^G$, while $\mathrm{Pic}(G;\mathcal{C})$ corresponds to $\mathrm{Div}(\mathcal{C})^G/[\mathcal{M}(\mathcal{C})^\times]^G$ [29, Corollary 2.3]. The key results of Dolgachev are the following.

Proposition 5.3. [29, Proposition 2.2] *We have a short exact sequence of abelian groups*

$$0 \to \mathrm{Hom}(G, \mathbb{C}^\times) \to \mathrm{Pic}(G;\mathcal{C}) \to \mathrm{Pic}(\mathcal{C})^G \to H^2(G, \mathbb{C}^\times) \to 0. \quad (6)$$

Here $H^2(G, \mathbb{C}^\times)$ *is the group cohomology viewing* $\mathbb{C}^\times$ *as a trivial G-module, which is known as the Schur multiplier of the group G.*

Proof. Dolgachev proves this using the spectral sequence of [42, §5.2], but we shall give a more elementary approach here following [40].

Observe that the Short Exact Sequence (SES)

$$0 \to \mathrm{PDiv}(\mathcal{C}) \to \mathrm{Div}(\mathcal{C}) \to \mathrm{Pic}(\mathcal{C}) \to 0,$$

gives rise to the Long Exact Sequence (LES)

$$0 \to \mathrm{PDiv}(\mathcal{C})^G \to \mathrm{Div}(\mathcal{C})^G \to \mathrm{Pic}(\mathcal{C})^G \to H^1(G, \mathrm{PDiv}(\mathcal{C})) \to H^1(G, \mathrm{Div}(\mathcal{C})).$$

We can use [40, Lemma 4.1] to say

- $H^1(G, \mathrm{Div}(\mathcal{C})) = 0$, and
- $H^1(G, \mathrm{PDiv}(\mathcal{C})) \cong H^2(G, \mathbb{C}^\times)$.

Moreover, from the SES

$$0 \to \mathbb{C}^\times \to \mathcal{M}(\mathcal{C})^\times \to \mathrm{PDiv}(\mathcal{C}) \to 0,$$

where $\mathcal{M}(\mathcal{C})$ are the meromorphic functions on the curve, we get the portion of the LES

$$0 \to \mathbb{C}^\times \to [\mathcal{M}(\mathcal{C})^\times]^G \to \mathrm{PDiv}(\mathcal{C})^G \to \mathrm{Hom}(G, \mathbb{C}^\times) \to 0,$$

where we have used [70, Exercise 6.1.6] to equate $H^1(G, \mathbb{C}^\times) = \mathrm{Hom}(G, \mathbb{C}^\times)$. As such, quotienting by $[\mathcal{M}(\mathcal{C})^\times]^G$ we are done. □

Remark 5.4. *The map* $\mathrm{Pic}(G;\mathcal{C}) \to \mathrm{Pic}(\mathcal{C})^G$ *sends any G-linearized line bundle L and its choice of linearization to the isomorphism class of L. In this way, we think of* $\mathrm{Hom}(G,\mathbb{C}^\times)$ *as parameterizing the possible G-linearizations of the trivial bundle* $\mathcal{O}_\mathcal{C}$.

Proposition 5.5. [29] *Suppose that G acts on* $\mathcal{C}$ *with signature* $(0;c_1,\dots,c_r)$. *Then we have*

$$\mathrm{Pic}(G;\mathcal{C}) \cong \mathbb{Z} \oplus \left[\bigoplus_{i=1}^{r-1} \mathbb{Z}/(d_i/d_{i-1})\mathbb{Z}\right], \tag{7}$$

where

$$d_i = \gcd\left\{\prod_{j\in S} c_j \,|\, S \subset \{1,\dots,r\}, |S| = i\right\}, \quad \text{taking} \quad d_0 = 1.$$

The generator of the $\mathbb{Z}$ *factor* γ *has underlying G-invariant line bundle (isomorphism class)* Γ *determined by*

$$K_\mathcal{C} = N\left(r - 2 - \sum_{i=1}^{r}\frac{1}{c_i}\right)\Gamma,$$

where $N = \mathrm{lcm}(c_i)$.

Proof. We shall not recreate the entire proof but note that, as it shall be helpful later, letting $Q_1,\dots,Q_r \in \mathbb{P}^1$ be the branch points of the projection $\pi : \mathcal{C} \to \mathcal{C}/G$, then $\mathrm{Pic}(G;\mathcal{C})$ is generated by the G-invariant divisors

$$D_i = \pi^{-1}(Q_i) = \sum_{\substack{P\in\mathcal{C}\\ \pi(P)=Q_i}} P.$$

These are subject to the equivalence relation $c_iD_i - c_jD_j = \pi^*(Q_i - Q_j) \sim 0$. Computing the Smith normal form of the matrix of generators and relations yields the d_i. □

Corollary 5.6. *In the situation of Proposition 5.5, letting* $e = |\{\text{even } c_i\}|$,

$$\mathrm{Pic}(G;\mathcal{C})/2\,\mathrm{Pic}(G;\mathcal{C}) = \begin{cases} \mathbb{Z}_2, & e = 0,\\ \mathbb{Z}_2^e, & e \neq 0. \end{cases},$$

$$\mathrm{Pic}(G;\mathcal{C})[2] = \begin{cases} 0, & e = 0,\\ \mathbb{Z}_2^{e-1}, & e \neq 0. \end{cases}.$$

Proof. The first equality follows from writing

$$\mathrm{Pic}(G;\mathcal{C})/2\,\mathrm{Pic}(G;\mathcal{C}) = \left\langle D_i,\ i = 1,\dots,r \,|\, \forall i,j,\ c_iD_i - c_jD_j = 0 = 2D_i\right\rangle.$$

If all c_i are odd, then we may use the second relation to reduce the first relation to $D_i = D_j$, and then the quotient has presentation $\langle D_1 \,|\, 2D_1 = 0\rangle$. If there are c_i which are even, say $c_1,\dots,c_e$ then they will drop out of the first relation to just give $D_i = 0$ for $i > e$ and then the quotient has presentation $\langle D_i,\ i = 1,\dots,e \,|\, \forall i,\ 2D_i = 0\rangle$.

The second follows from observing that

$$\mathbb{Z}/N\mathbb{Z} \oplus \left[\bigoplus_{i=1}^{r-1}\mathbb{Z}/(d_i/d_{i-1})\mathbb{Z}\right] = \bigoplus_{i=1}^{r}\mathbb{Z}/(d_i/d_{i-1})\mathbb{Z} = \bigoplus_{i=1}^{r}\mathbb{Z}/c_i\mathbb{Z}. \tag{8}$$

The 2-torsion part of the left hand side is $\mathbb{Z}_{\gcd(2,N)} \oplus \mathrm{Pic}(G;\mathcal{C})[2]$, while 2-torsion of the right hand side is $\mathbb{Z}_2^e$. The result follows using

$$\mathbb{Z}_{\gcd(2,N)} = \begin{cases} 0, & e = 0,\\ \mathbb{Z}_2 & e \neq 0. \end{cases}$$

□

As invariant theta characteristics would give elements of $\mathrm{Pic}(\mathcal{C})^G$ which square to $K_\mathcal{C}$, we can hope that in particular circumstances (7) may give us enough information to determine the number of invariant characteristics. We have the following characterizations.

Lemma 5.7. *Suppose we are in the situation of Proposition 5.5. Then*

- *an invariant characteristic exists if and only if the image of* $K_\mathcal{C} \in \mathrm{Pic}(G;\mathcal{C})$ *under the map* $\mathrm{Pic}(G;\mathcal{C}) \to \mathrm{Pic}(\mathcal{C})^G$ *is a square, and*
- *if exactly* $m > 0$ *invariant characteristic exists, then the 2-torsion subgroup* $\mathrm{Pic}(\mathcal{C})^G[2]$ *has rank* $m - 1$.

As computing the 2-torsion subgroup of $\mathrm{Pic}(\mathcal{C})^G$ will be important, we recall briefly a quick fact from homological algebra

Lemma 5.8. *Given a SES $0 \to A \to B \to C \to 0$ there is an associated LES*

$$0 \to A[2] \to B[2] \to C[2] \to A/2A \to B/2B \to C/2C \to 0 \tag{9}$$

where $A[2]$ is the 2-torsion part of A (and likewise for B, C).

Example 5.9 (Cyclic Groups). *Consider the case $G = C_n$ with quotient genus 0, which gives*

$$\mathrm{Hom}(G, \mathbb{C}^\times) \cong C_n, \quad H^2(G, \mathbb{C}^\times) \cong 0,$$

and

$$\mathrm{Pic}(G; \mathcal{C}) \cong \mathbb{Z} \oplus \left[\bigoplus_{i=1}^{r-1} \mathbb{Z}/(d_i/d_{i-1})\mathbb{Z}\right],$$

with

$$K_{\mathcal{C}} = N\left(r - 2 - \sum_i \frac{1}{c_i}\right)\Gamma.$$

Equation (6) thus becomes

$$0 \to \mathbb{Z}_n \to \mathrm{Pic}(G; \mathcal{C}) \to \mathrm{Pic}(\mathcal{C})^G \to 0. \tag{10}$$

In the case of a cyclic group action which quotients to $\mathbb{P}^1$, [43] shows that $N = n$. Then Riemann-Hurwitz gives that $K_{\mathcal{C}} = 2(g-1)\Gamma$ which shows that $(g-1)\Gamma$ is an invariant theta characteristic. We now need to count how many exist.

Applying Lemma 5.8 to equation (10) we get

$$0 \to \mathbb{Z}_{\gcd(2,n)} \to \mathrm{Pic}(G; \mathcal{C})[2] \to \mathrm{Pic}(\mathcal{C})^G[2] \to \mathbb{Z}_n/2\mathbb{Z}_n \to \mathrm{Pic}(G; \mathcal{C})/2\,\mathrm{Pic}(G; \mathcal{C}), \tag{11}$$

We analyze this in the case of n odd and n even. Let e be the number of c_i which are even as before. We will use Corollary 5.6.

- *When n is odd, $e = 0$, and so equation (11) becomes*

  $$0 \to 0 \to 0 \to \mathrm{Pic}(\mathcal{C})^G[2] \to 0 \to \mathbb{Z}_2,$$

  and immediately we see $\mathrm{Pic}(\mathcal{C})^G[2] = 0$, i.e. the invariant characteristic is unique.
- *When n is even equation (11) becomes*

  $$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2^{e-1} \to \mathrm{Pic}(\mathcal{C})^G[2] \to \mathbb{Z}_2 \to \mathbb{Z}_2^e.$$

  At this stage we claim that the map $\mathrm{Pic}(\mathcal{C})^G[2] \to \mathbb{Z}_2$ is the zero map, which is equivalent to the map $\mathbb{Z}_2 \to \mathbb{Z}_2^e$ being injective. This would mean that the generator of $\mathrm{Hom}(G, \mathbb{C}^\times)$, which itself is not a square, call it ψ, does not map to a square in $\mathrm{Pic}(G; \mathcal{C})$. Now the generator of $\mathrm{Hom}(G, \mathbb{C}^\times)$ determines a linearization $\{\psi_g\}$ of the trivial line bundle $\mathcal{O}_{\mathcal{C}}$. Suppose this is a square in $\mathrm{Pic}(G; \mathcal{C})$, meaning the existence of a line bundle $L \in \mathrm{Pic}(\mathcal{C})^G$ and linearization $\{\phi_g\}$ of this such that $[L^2] = [\mathcal{O}_{\mathcal{C}}]$ and $\phi_g^2 \sim \psi_g$, but the injectivity of the map $\mathrm{Hom}(G, \mathbb{C}^\times) \to \mathrm{Pic}(G; \mathcal{C})$ means that $\phi \in \mathrm{Hom}(G, \mathbb{C}^\times)$ which mapped to $\{\phi_g\}$ satisfies $\phi^2 = \psi$, giving a contradiction. As a result of this, the we find that the number of invariant characteristics is 2^{e-2}.

Remark 5.10. *More generally, we could apply Lemma 5.8 to the exact sequence*

$$0 \to {}^{\mathrm{Pic}(G;\mathcal{C})}\!\big/_{\mathrm{Hom}(G, \mathbb{C}^\times)} \to \mathrm{Pic}(\mathcal{C})^G \to H^2(G, \mathbb{C}^\times) \to 0,$$

which follows from equation (6). The free part of $\mathrm{Pic}(G; \mathcal{C})$, whose image in $\mathrm{Pic}(\mathcal{C})^G$ is our interest, is not affected by the quotient. We shall not pursue this avenue further in this work.

5.3. Hurwitz curves

When $\mathcal{C}$ is a Hurwitz curve and G is the full automorphism group, we know $|G| = 84(g-1)$ and the signature is $(0; 2, 3, 7)$, which means that simply $\mathrm{Pic}(G; \mathcal{C}) = \mathbb{Z}\gamma$, and moreover that $\Gamma = K_{\mathcal{C}}$. Moreover, all Hurwitz groups are perfect (that is $[G, G] = G$) [21], and so $\mathrm{Hom}(G, \mathbb{C}^\times) = 0$. This reduces equation (7) to

$$0 \to \mathbb{Z}K_{\mathcal{C}} \to \mathrm{Pic}(\mathcal{C})^G \to H^2(G, \mathbb{C}^\times) \to 0. \tag{12}$$

Obviously when $H^2(G, \mathbb{C}^\times) = 0$, equation (12) is just

$$0 \to \mathbb{Z}K_{\mathcal{C}} \to \mathrm{Pic}(\mathcal{C})^G \to 0,$$

and no invariant characteristics can exist (by Lemma 5.7), so we assume that the multiplier group is nontrivial from hereon in.

We can immediately apply Lemma 5.8 to get

$$0 \to \mathrm{Pic}(G; \mathcal{C})[2] \to \mathrm{Pic}(\mathcal{C})^G[2] \to H^2(G, \mathbb{C}^\times)[2] \to \mathrm{Pic}(G; \mathcal{C})/2\,\mathrm{Pic}(G; \mathcal{C}),$$

which can be rewritten as

$$0 \to 0 \to \mathrm{Pic}(\mathcal{C})^G[2] \to H^2(G, \mathbb{C}^\times)[2] \to \mathbb{Z}_2. \tag{13}$$

Dolgachev provides the following helpful interpretation.

Lemma 5.11. [29] *Let G be a perfect group acting with signature containing only a single even element $c_1 = 2$. The map $H^2(G, \mathbb{C}^\times)[2] \to \mathrm{Pic}(G; \mathcal{C})/2\,\mathrm{Pic}(G; \mathcal{C})$ is the restriction map $H^2(G, C_2) \to H^2(G_{P_1}, C_2)$ where $P_1 \in \mathcal{C}$ is a ramification point whose isotropy subgroup $G_{P_1} \cong C_2$.*

When we restrict to looking at simple Hurwitz groups, we may use the fact that the Schur multipliers have been computed in the Atlas [25] and are cyclic groups. In particular, for the finite simple groups which are Hurwitz groups [21, 71, 72] we give the relevant multipliers in Table 3.

Our earlier discussion in Section 5.3 of the case where $H^2(G, \mathbb{C}^\times) = 0$ immediately shows that the Hurwitz curves with automorphism group $\mathrm{PSL}_2(8)$, ${}^2G_2(3^p)$, J_1, Co_3, He, HN, Ly, Th, J_4, or M have no invariant theta characteristics.

Now because $\mathrm{Hom}(G, \mathbb{C}^\times) = 0$ we have that $\mathrm{Pic}(G; \mathcal{C}) \cong \mathbb{Z}\gamma$ is a subgroup of $\mathrm{Pic}(\mathcal{C})^G$ of index $n := \left|H^2(G, \mathbb{C}^\times)\right|$. Set $A = \mathrm{Pic}(G; \mathcal{C}) \cong \mathbb{Z}$, $B = \mathrm{Pic}(\mathcal{C})^G \cong \mathbb{Z} \oplus T$, where T is the torsion subgroup, $C = H^2(G, \mathbb{C}^\times) \cong C_n$, and write our exact sequence as

$$0 \to \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \oplus T \xrightarrow{\psi} C_n \to 0, \qquad \phi(1) = (k, l).$$

Now if $0 \to A \to B \to C$ is exact then so is $0 \to \mathrm{Tor}(A) \to \mathrm{Tor}(B) \to \mathrm{Tor}(C)$. Thus T injects into the cyclic group C_n and so $T \cong C_c$ for some $c|n$. The matrix of generators and relations of the quotient $B/A = \mathrm{Pic}(\mathcal{C})^G/\,\mathrm{Pic}(G; \mathcal{C})$ is then $\left(\begin{smallmatrix} 0 & c \\ k & l \end{smallmatrix}\right)$. The Smith normal form of this will be $C_{d_1} \times C_{d_2/d_1}$ where $d_1 = \gcd(c, k, l)$, $d_2 = ck$. For this quotient to be isomorphic to C_n, it certainly must be true that $ck = n$.

Now an invariant theta characteristic exists if and only if k is even and there exists some m such that $l = 2m \mod c$, which is to say that an invariant characteristic can never exist when n is odd as then necessarily k is odd. This tells us that Hurwitz curves with automorphism group Fi'_{24} have no invariant theta characteristics, and hereon we assume n is even. Looking at Table 3 we see that when n is even, the maximum power of 2 dividing n is just 2, and so we have

$$k \text{ even} \Leftrightarrow c \text{ odd} \Leftrightarrow \mathrm{Pic}(\mathcal{C})^G[2] = 0.$$

As c being odd is sufficient to be able to solve $l = 2m \mod c$ for some m, we see that we have an invariant characteristic, and moreover a unique one, when $\mathrm{Pic}(\mathcal{C})^G[2] = 0$. By using Lemma 5.11 with equation (13) this is equivalent to having the map $H^2(G, C_2) \to H^2(G_{P_1}, C_2)$ being injective, equivalently surjective.

Table 3. All finite simple Hurwitz groups and their Schur multiplier.

G	$H^2(G, \mathbb{C}^\times)$	Comments
$\mathrm{PSL}_2(q)$	$C_{\gcd(2,q-1)}$	$q = 7, q = p$ for prime $p = \pm 1 \mod 7$, or $q = p^3$ for prime $p = \pm 2, \pm 3 \mod 7$.
A_n	C_2	A_n Hurwitz for all but finitely many n
${}^2G_2(3^p)$	trivial	p a prime > 3
J_1	trivial	
J_2	C_2	
Co_3	trivial	
He	trivial	
Ru	C_2	
HN	trivial	
Ly	trivial	
Fi'_{24}	C_3	
Th	trivial	
Fi_{22}	C_6	
J_4	trivial	
M	trivial	

Proposition 5.12. *Let $\mathcal{C}$ be a Hurwitz curve with simple automorphism group G. Then $\mathcal{C}$ has a unique invariant characteristic exactly when $H^2(G,\mathbb{C}^\times) \cong C_n$ has $n > 0$ even and the map $H^2(G,C_2) \to H^2(G_{P_1},C_2)$ is surjective. Otherwise, $\mathcal{C}$ has no invariant characteristics.*

One way to understand the map $H^2(G,C_2) \to H^2(G_{P_1},C_2)$ is using the fact that isomorphism classes of central extensions of G by an abelian group A, $0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 0$, are in bijection with elements of $H^2(G,A)$ where A is a trivial G-module. Given the inclusion $\iota : H \hookrightarrow G$, the restriction morphism $H^2(G,A) \to H^2(H,A)$ works via the diagram [70, Exercise 6.6.4]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow{\iota} & & \\ 0 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{\pi} & G & \longrightarrow & 0, \end{array}$$

sending the cocycle corresponding to the extension E to the cocycle corresponding to the extension

$$E' := E \times_G H = \{(e,h) \in E \times H \,|\, \pi(e) = \iota(h)\}\,.$$

There are two central extensions of $G = C_2$ by $A = C_2$, these are $E = V_4$ and $E = C_4$, with the latter corresponding to the generator of $H^2(C_2,C_2)$.

Restricting to the case where $H^2(G,\mathbb{C}^\times)[2] = H^2(G,C_2) \cong C_2$, suppose $g = \gamma_1 \in G$ generates the subgroup $H = G_{P_1}$ and $\tilde{g}$ is one of the elements of $\pi^{-1}(g) \in E$ where E is the unique nontrivial C_2 central extension of G, often denoted $2 \cdot G$. We can define likewise $\tilde{e}$ to be the order-2 lift of the identity $e \in G$. We then have

$$E' = \left\{(e,e), (\tilde{e},e), (\tilde{g},g), (\tilde{e}\tilde{g},g)\right\}.$$

We define the **lifting order** of g to be the order of $\tilde{g}$ in E, and it depends only on the conjugacy class of g [25]. There are two possible cases:

(1) the lifting order is 2, so $\tilde{g}^2 = e$, and $E' \cong V_4$, or
(2) the lifting order is 4, so $\tilde{g}^2 = \tilde{e}$, and $E' \cong C_4$.

The latter is equivalent to the map $H^2(G,C_2) \to H^2(G_{P_1},C_2)$ being surjective. As such, we have proven the following result.

Proposition 5.13. *Given a Hurwitz curve $\mathcal{C}$ with simple automorphism group G, and $g \in G$ generating the stabilizer group $G_{P_1} \cong C_2$, either:*

(1) $H^2(G,\mathbb{C}^\times)$ is cyclic of odd order, and $\mathcal{C}$ has no invariant characteristics,
(2) $H^2(G,\mathbb{C}^\times)$ is cyclic of even order, the lifting order of g in $2 \cdot G$ is 2, and $\mathcal{C}$ has no invariant characteristics, or
(3) $H^2(G,\mathbb{C}^\times)$ is cyclic of even order, the lifting order of g in $2 \cdot G$ is 4, and $\mathcal{C}$ has exactly one invariant characteristic.

Remark 5.14. *Using the identification between theta characteristics and spin structures [3, Proposition 3.2], a theta characteristic in $\mathrm{Pic}(\mathcal{C})^G$ is equivalent to a spin structure which admits a lift of the action of any $f \in G$. If f has even order 2m and f^m has a fixed point then it is known that the lift has order 4m [4, 53].*

We will now distinguish between the latter two cases for the curves in Table 3.

5.4. $\mathrm{PSL}_2(q)$

First consider the case of $G = \mathrm{PSL}_2(q)$. The only even value of q which gives a Hurwitz group is $q = 8$, corresponding to the Fricke-Macbeath curve. In this case the Schur multiplier group is trivial, and so we have analytically shown that the Fricke-Macbeath curve has 0 invariant characteristics.

Restrict then to the case when q is odd. The generator of the group

$$H^2(\mathrm{PSL}_2(q),C_2) \cong C_2$$

corresponds to the extension

$$1 \to C_2 \to \mathrm{SL}_2(q) \xrightarrow{\pi} \mathrm{PSL}_2(q) \to 1.$$

Proposition 5.15. *The lifting order of any involution in $\mathrm{PSL}_2(q)$ to $\mathrm{SL}_2(q)$ for q odd is 4.*

Proof. Suppose $\tilde{g} \in \mathrm{SL}_2(q)$ satisfies $\tilde{g}^2 = 1$, then its minimal polynomial $m(t)$ divides $t^2 - 1 = (t-1)(t+1)$. If $q \neq 2$ this latter polynomial is square free, so the minimal polynomial is, and hence g is diagonalizable with eigenvalues ± 1. Because $\det \tilde{g} = 1$ the eigenvalues must be the same, and hence $\tilde{g} = \pm 1 \Rightarrow g$ is the identity in $\mathrm{PSL}_2(q)$. Hence any involution must have lifting order 4. □

Corollary 5.16. *Hurwitz curves with automorphism group $\mathrm{PSL}_2(q)$, q odd, have a unique invariant characteristic.*

5.5. A_n

The alternating group A_n is a Hurwitz group for all n except 64 values in the range $1 \leq n \leq 167$ [22]. The group S_n has the n-dimensional permutation representation which splits into a sum of a 1-dimensional representation acting on vectors with equal entries and a $(n-1)$-dimensional representation acting on vectors whose entries sum to 0 (that is on an $n-1$ simplex), and so this gives an embedding $A_n \hookrightarrow \mathrm{SO}(n-1)$. Restricting to $n > 4$, the nontrivial double cover of $\mathrm{SO}(n-1)$ is the well known group $\mathrm{Spin}(n-1)$, and so we get a nontrivial double cover of A_n denoted $2 \cdot A_n$ as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_2 & \longrightarrow & 2 \cdot A_n & \longrightarrow & A_n & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow{\iota} & & \\ 0 & \longrightarrow & C_2 & \longrightarrow & \mathrm{Spin}(n-1) & \xrightarrow{\pi} & \mathrm{SO}(n-1) & \longrightarrow & 0. \end{array}$$

This is the nontrivial element of $H^2(A_n, C_2) \cong C_2$.

We can then reduce the question to asking about whether, given nontrivial $g \in \mathrm{SO}(n-1)$ such that $g^2 = 1$, under what conditions can we have $\tilde{g}^2 = 1$ where $\tilde{g} \in \mathrm{Spin}(n-1)$ is a lift of g.

Lemma 5.17. [6] $\tilde{g}^2 = 1$ *if and only if g is a product of 4k disjoint transpositions for some integer k.*

Corollary 5.18. *The lifting order of g is 4, and hence the corresponding A_n Hurwitz curve has a UIC, if and only if g is a product of 2l disjoint transpositions for some odd integer l.*

We may give a relatively simple criterion to identify the two cases. In particular, when A_n is a Hurwitz group let γ_i, $i = 1, 2, 3$, be the generators of order $2, 3, 7$, respectively in the corresponding generating vector, and denote $f_i = |\mathrm{Fix}(\gamma_i)|$. Then [22, 24] there exists $h \in \mathbb{Z}_{\geq 0}$ such that

$$n = 84(h-1) + 21f_1 + 28f_2 + 36f_3.$$

Noting that $n - f_1 = 4l$, we get a simple lemma.

Lemma 5.19. *The parity of l is determined by*

$$2|l \Leftrightarrow h - 1 + f_1 + f_2 + f_3 = 0 \mod 2.$$

The five smallest values of n for which we get that A_n is a Hurwitz group are found in [34] and yield Table 4 giving the number of invariant characteristics I.

This shows that the question of whether an A_n Hurwitz curve has a UIC depends on n in a way that is not immediately clear. One may initially conjecture that the existence of a UIC depends on n only and not on the choice of generating vector, as occurs for $n = 15, 21, 22, 28, 29$, akin to how the existence of a UIC for a $\mathrm{PSL}_2(q)$ Hurwitz curve depended on q only. This is in fact false;[7] we may verify using the functions "LowIndexSubgroupsFpGroup" and "FactorCosetAction" in GAP [41] that $n = 36$ gives the first example where both parities of l are possible. Moreover, we have the following theorem for generic n.

Theorem 5.20 (Conder, Private communication). *For all but finitely many n, there exist Hurwitz curves with automorphism group A_n with either of $I = 0$ or $I = 1$ invariant characteristics.*

Proof. The proof is given in Appendix B. □

5.6. Sporadic examples

It remains to determine the number of invariant characteristics for the sporadic simple groups J_2, Ru, and Fi_{22}. The Atlas [25] records the lifting orders of conjugacy classes of G to the double cover $2 \cdot G$, and so we need only to identify the conjugacy class of the generator of G_{P_1}.

Table 4. First five A_n Hurwitz groups and the number of invariant characteristics.

$(n; h, f_1, f_2, f_3)$	$I = h - 1 + f_1 + f_2 + f_3 \mod 2$
$(15; 0, 3, 0, 1)$	1
$(21; 0, 1, 3, 0)$	1
$(22; 0, 2, 1, 1)$	1
$(28; 0, 4, 1, 0)$	0
$(29; 0, 1, 2, 1)$	1

[7] LDH is grateful to Marston Conder for his correspondence on code to determine low n examples and the proof of Theorem 5.20.

Table 5. Machine prediction of whether $I = 1$, all simple Hurwitz groups of order $< 10^6$, and true value of I.

G	g	$I = 1$ prediction	I
$\mathrm{PSL}_2(7)$	3	True	1
$\mathrm{PSL}_2(8)$	7	False	0
$\mathrm{PSL}_2(13)$	14	True	1
$\mathrm{PSL}_2(27)$	118	True	1
$\mathrm{PSL}_2(29)$	146	True	1
$\mathrm{PSL}_2(41)$	411	False	1
$\mathrm{PSL}_2(43)$	474	True	1
J_1	2091	False	0
$\mathrm{PSL}_2(71)$	2131	False	1
$\mathrm{PSL}_2(83)$	3404	True	1
$\mathrm{PSL}_2(97)$	5433	False	1
J_2	7201	False	1
$\mathrm{PSL}_2(113)$	8589	False	1
$\mathrm{PSL}_2(125)$	11626	True	1

- J_2 has two conjugacy classes of involutions, $2A$ and $2B$, with lifting order 2 and 4 respectively in $2 \cdot J_2$. It is known that the $2B$ conjugacy class is that which generates J_2 as a Hurwitz group [16, 38], and so a J_2 Hurwitz curve has a UIC.
- Ru has two conjugacy classes of involutions, $2A$ and $2B$, which have lifting order 2 and 4, respectively in $2 \cdot Ru$. It is shown in [26] that Ru is $(2B, 3, 7)$-generated, but they do not answer the question of whether it is $(2A, 3, 7)$-generated. Using the “Brauer trick" argument of [14] with respect to the character χ_4 of [25] shows that in fact Ru is not $(2A, 3, 7)$-generated, and so any Hurwitz curve with automorphism group Ru has a UIC.
- Fi_{22} has three conjugacy classes of involutions, but all of them have lifting order 2 in $2 \cdot Fi_{22}$, and so any Hurwitz curve with automorphism group Fi_{22} has no invariant theta characteristics.

Remark 5.21. *Had the lifting orders not been computed in the Atlas, we would have been able to work them out by taking a presentation of the group in an appropriate alternating group, and then applying the method of §5.5.*

5.7. Comparison to machine prediction

We can now make a copy of Table 2 which gives also the true value of I. We see that the machine estimation had an accuracy of $9/14 \approx 64\%$, with zero false positives. It is not surprising that the machine classification performed poorly on the subset of simple Hurwitz groups, as these are acting in a range of genera and signatures poorly represented in the training dataset.

For all the groups considered in Table 5 the number of invariant characteristics on a Hurwitz curve just depends on the group and not its generating vector, i.e. the topological type of its action. We have seen in Theorem 5.20 that this need to be true, and in such cases machine predictions based solely on the signature cannot expect to be correct.

6. Outlook

In this work we’ve laid the groundwork for a more developed theory of invariant characteristics, moving beyond the work of [3, 8, 48] which considered only those invariant under a single automorphism. We have approached this from two separate directions: by rephrasing the question of invariant characteristics in terms of group cohomology of an affine action; and by understanding the whole group of invariant line bundles. At present, it is not clear how the two may be linked, and clarifying this will undoubtedly serve to help the development of theory.

At present, sufficient conditions for unique invariant characteristics have been found, but not any strong necessary conditions. In our current use of the group cohomology method, we have not utilized much of the theory of the rational representation, and it seems likely that better understanding this will be helpful. This is especially pressing as we provide examples through our computation of curves for which the SOC property does not explain the existence of a UIC, though the property is enough to cover the majority of curves seen.

With regards to Hurwitz curves with a simple automorphism group, we have provided a criterion for checking whether certain generating vectors of A_n Hurwitz curves yield a UIC, and shown that generically the value of n is not enough and the full generating vector is required. These examples where the signature alone does not determine the orbit decomposition, along with those seen in genus 4, warrant further research.

For future work, one could also generalize to consider r-**spin structures**, which are rth-roots of the canonical bundle. These exist when $r|2g-2$, and in principle our methods can be extended to study the orbits of these.

Table 6. Orbit decomposition, all non-hyperelliptic and hyperelliptic genus-3 curves.

f	$\overline{\text{Aut}}$, c	Odd	Even	I
$1 + L_2(x,y) + L_4(x,y)$	$C_2, (1;2^4)$	$1_4, 2_{12}$	$1_{12}, 2_{12}$	16
$L_1(x,y) + L_3(x,y)$	$C_3, (0;3^5)$	$1_1, 3_9$	3_{12}	1
$1 + y^4 + x^4 + (ay^2 + bx^2) + cx^2y^2$	$V_4, (0;2^6)$	$2_6, 4_4$	$1_8, 2_6, 4_4$	8
$bx^2y^2 + x^3 + y^3 + axy + 1$	$S_3, (0;2^4,3)$	$1_1, 3_3, 6_3$	$1_3, 3_9, 6_1$	4
$y^4 + x^3 + ay^2 + 1$	$C_6, (0;2,3^2,6)$	$1_1, 3_1, 6_4$	$3_4, 6_4$	1
$1 + y^4 + x^4 + a(x^2 + y^2) + bx^2y^2$	$D_4, (0;2^5)$	$4_5, 8_1$	$1_4, 2_4, 4_4, 8_1$	4
$xy^3 + x^3 + 1$	$C_9, (0;3,9^2)$	$1_1, 9_3$	9_4	1
$y^4 + ay^2 + x^4 + 1$	$(C_4 \times C_2) \rtimes C_2 \cong (16,13), (0;2^3,4)$	$4_1, 8_3$	$2_6, 8_3$	0
$1 + y^4 + x^4 + a(y^2 + x^2 + y^2x^2)$	$S_4, (0;2^3,3)$	$4_1, 12_2$	$1_2, 3_2, 4_1, 6_2, 12_1$	2
$x^4 + y^4 + x$	$((C_4 \times C_2) \rtimes C_2) \rtimes C_3 \cong (48,33), (0;2,3,12)$	$4_1, 24_1$	$6_2, 24_1$	0
$y^4 + x^4 + 1$	$(C_4^2 \rtimes C_3) \rtimes C_2 \cong (96,64), (0;2,3,8)$	$12_1, 16_1$	$4_2, 12_1, 16_1$	0
$xy^3 + x^3 + y$	$\mathrm{PSL}_3(\mathbb{F}_2), (0;2,3,7)$	28_1	$1_1, 7_2, 21_1$	1
$y^2 - (x^8 + ax^6 + bx^4 + cx^2 + 1)$	$C_2, (1;2^4)$	$1_4, 2_{12}$	$1_{12}, 2_{12}$	16
$y^2 - x(x^2-1)(x^4 + ax^2 + b)$	C_2	$1_4, 2_{12}$	$1_4, 2_{16}$	8
$y^2 - (x^4 + ax^2 + 1)(x^4 + bx^2 + 1)$	$V_4, (0;2^6)$	$2_6, 4_4$	$1_8, 2_6, 4_4$	8
$y^2 - (x^4 - 1)(x^4 + ax^2 + 1)$	V_4	$1_2, 2_3, 4_5$	$1_2, 2_7, 4_5$	4
$y^2 - x(x^6 + ax^3 + 1)$	$S_3, (0;2^4,3)$	$1_1, 3_3, 6_3$	$1_3, 3_9, 6_1$	4
$y^2 - (x^8 + ax^4 + 1)$	$D_4, (0;2^2,4^2)$	$4_5, 8_1$	$1_4, 2_4, 4_4, 8_1$	4
$y^2 - (x^7 - 1)$	$C_7, (0;7^3)$	7_4	$1_1, 7_5$	1
$y^2 - x(x^6 - 1)$	D_6	$1_1, 3_1, 6_2, 12_1$	$1_1, 2_1, 3_1, 6_5$	2
$y^2 - (x^8 - 1)$	D_8	$4_1, 8_3$	$1_2, 2_1, 4_2, 8_3$	2
$y^2 - (x^8 + 14x^4 + 1)$	$S_4, (0;3,4^2)$	$4_1, 12_2$	$1_2, 3_2, 4_1, 6_2, 12_1$	2

Appendix A: Tables of Orbits

In this appendix we provide all the tables of orbit decompositions described in Section 4. We shall use the notation of [5, 7] that $L_d(x, y, \dots)$ is a generic homogeneous degree-d polynomial in the arguments. Moreover, we shall make some comment about the completeness of the data.

- The complete list of genus-2 curves with nontrivial reduced automorphism group comes from [11].
- The complete list of non-hyperelliptic genus-3 curves with nontrivial (reduced) automorphism group comes from [7]. Bars attributes the first work completing this to Henn in 1976, but Wiman appears to have completed the calculation earlier in [73]. Bars and Dolgachev disagree on the automorphism group of the curve given by $f = 1 + y^4 + x^4 + a(x^2 + y^2) + bx^2y^2$; using Sage we find agreement with Bars.
- The list of hyperelliptic curves with many automorphisms comes from [58, 67]. The latter reference will also be used for higher genera.
- [67, Table 4] was used to verify the signatures of the non-hyperelliptic actions.
- The complete list of non-hyperelliptic genus-4 curves with nontrivial (reduced) automorphism group comes from [74]. Wiman distinguishes his curves by whether they lie on the nonsingular quadric or the cone, and we have followed this putting those that lie on the cone in the first portion of the table. For the curves which lie on the nonsingular quadric Wiman described the curve by providing the quadric and cubic, hence one must use the resultant to get a single plane equation, and this may require a projective linear transformation to find a nondegenerate coordinate system. We find a typo in Wiman's curve (8) with octahedral symmetry.
- Above genus 4 we are unaware of any complete lists classifying curves by their automorphism groups and giving plane models of the curves. Curves of genus $(d-1)(d-2)/2$ for $d \geq 3$ stand out because of the Plücker formula: any nonsingular plane curve of degree-d gives a curve of this genus. In order to implement the numerical method for computing the rational representation it is also necessary to have a model of the curve with coefficients in $\mathbb{Q}$, and so this further limits the possible curves we may investigate.
- As the LMFDB does not contain the data of signatures with quotient genus > 0 at genera > 4, we shall sometimes omit the signature of the action where it is unknown.
- The first non-hyperelliptic curve of genus 5 used in Table 8 is the family of Humbert curves given in [49, (5.9)], the second from [69], and the third comes from taking resultants of the polynomials provided in [74], hence one may choose to call it the Wiman octic for want of a better name.
- The non-hyperelliptic genus-6 curves come from a list of all nonsingular plane quintics in [5], with the exception of the curve with automorphism group $(48, 15)$ which is from [69]. The curve with automorphism group of order 150 is the Fermat quintic curve which has maximal automorphism group for a genus-6 curve [56], the curve with automorphism group S_5 is the Wiman sextic [32, 74], and the curve with automorphism group of order 72 is given in the LMFDB with label 6.72-15.0.2-4-9.1. The curve with automorphism group of order 39 is attributed to Snyder in [52, p. 464], where it is constructed in a manner similar to Klein's curve.
- The curve with automorphism group $\mathrm{PSL}_2(\mathbb{F}_8)$ is the Fricke-Macbeath curve [39, 55], the unique Hurwitz curve of genus 7, the rational plane model of which is a attributed to Brock in [44]. The remaining curves come from [75, Tables 2 and 5]. Table 5 in Zomorrodian gives all possible automorphism groups of non-hyperelliptic genus-7 curves where $|\mathrm{Aut}| < 65$, and a plane curve form for each; this list contains some errors, for example a typo in curve 4 and the fact that curve 8 is hyperelliptic (as checked with Maple [57]).
- The non-hyperelliptic curve genus-8 with automorphism group $S_3 \times D_5$ was constructed by ourselves and David Swinarksi using the work of [1, 18, 56]; it has the largest automorphism group of any trigonal genus-8 curves as can be deduced from [18, Theorem 2.1] by checking cases, with the curve at hand corresponding to the C3.4 family when $t = 0$, $N = 20$. The curve with automorphism group C_{20} was constructed by Thomas Bouchet. One can in principle get additional such models from the methods of [68], using Sage's modular symbol functionality, but in

Table 7. Orbit decomposition, all non-hyperelliptic genus-4 curves stratified by whether the corresponding quadric is singular, and separately some hyperelliptic genus-4 curve with many automorphisms.

f	$\overline{\mathrm{Aut}}$	Odd	Even	I
$y^3+y(ax^4+bx^2+c)+(dx^6+ex^4+fx^2+g)$	$C_2, (1;2^6)$	$1_{24}, 2_{48}$	$1_{40}, 2_{48}$	64
$y^3+y(ax^4+bx^2+c)+dx(x^4+ex^2+f)$	$C_2, (2;2^2)$	2_{60}	$1_{16}, 2_{60}$	16
$y^3+y[a(x^4+1)+bx^2]+x[c(x^4+1)+dx^2]$	$V_4, (1;2^3)$	$2_{24}, 4_{18}$	$1_{16}, 2_{24}, 4_{18}$	16
$y^3+y[a(x^4+1)+bx^2]+x(x^4-1)$	$V_4, (1;2^3)$	4_{30}	$1_4, 2_{18}, 4_{24}$	4
$y^3+ayx^2+x(x^4+1)$	$D_4, (0;2^4,4)$	$4_{12}, 8_9$	$1_4, 2_6, 4_{18}, 8_6$	4
$y^3+y(x^4+a)+(bx^4+c)$	$C_4, (0;2,4^4)$	$1_4, 2_{10}, 4_{24}$	$1_4, 2_{18}, 4_{24}$	8
$y^3+ayx^2+(x^6+bx^3+1)$	$S_3, (0;2^6)$	$1_6, 3_{18}, 6_{10}$	$1_{10}, 3_{30}, 6_6$	16
$y^3+ayx^2+(x^6+1)$	$D_6, (0;2^5)$	$2_3, 6_{13}, 12_3$	$1_4, 2_3, 3_{12}, 6_9, 12_3$	4
$y^3+y(ax^3+b)+(x^6+cx^3+d)$	$C_3, (1;3^3)$	$1_3, 3_{39}$	$1_1, 3_{45}$	4
$y^3+ay(x^3+1)+(x^6+20x^3-8)$	$A_4, (0;2,3^3)$	$4_3, 12_9$	$1_1, 3_1, 6_6, 12_8$	1
$y^3+ay+(x^6+b)$	$C_6, (0;2,6^3)$	$1_3, 3_7, 6_{16}$	$1_1, 3_{13}, 6_{16}$	4
y^3+y+x^6	$C_{12}, (0;4,6,12)$	$1_1, 2_1, 3_1, 6_3, 12_8$	$1_1, 3_1, 6_6, 12_8$	2
$y^3+ay+(x^5+b)$	$C_5, (0;5^4)$	5_{24}	$1_1, 5_{27}$	1
y^3+y+x^5	$C_{10}, (0;5,10^2)$	10_{12}	$1_1, 5_3, 10_{12}$	1
$y^3-(x^6+ax^5+bx^4+cx^3+dx^2+ex+f)$	$C_3, (0;3^6)$	3_{40}	$1_1, 3_{45}$	1
$y^3-(x^6+ax^4+bx^2+1)$	$C_6, (0;2^2,3^3)$	$3_8, 6_{16}$	$1_3, 3_{13}, 6_{16}$	1
$y^3-x(x^4+ax^2+1)$	$C_6\times C_2, (0;2^2,3,6)$	$6_8, 12_6$	$1_1, 3_5, 6_8, 12_6$	1
$y^3-(x^6+ax^3+1)$	$C_3\times S_3, (0;2^2,3^2)$	$3_2, 6_1, 9_6, 18_3$	$1_1, 3_3, 9_{10}, 18_2$	1
$y^3-(x^5+1)$	$C_{15}, (0;3,5,15)$	15_8	$1_1, 15_9$	1
$y^3-(x^6+1)$	$C_6\times S_3, (0;2,6^2)$	$6_2, 18_4, 36_1$	$1_1, 3_1, 6_1, 9_4, 18_3, 36_1$	1
$y^3-x(x^4+1)$	$C_3\times S_4, (0;2,3,12)$	$12_2, 24_1, 36_2$	$1_1, 3_1, 12_2, 18_2, 36_2$	1
$y^4(x+1)+y^3(x^2+ax+1)+y^2[b(x^3+1)+cx(x+1)]+y[dx(x^2+1)+ex^2]+fx^2(x+1)$	$C_2, (1;2^6)$	$1_{24}, 2_{48}$	$1_{40}, 2_{48}$	64
$y^6+y^4(x^2+ax+1)+y^2x(dx^2+bx+e)+cx^3$	$C_2, (2;2^2)$	2_{60}	$1_{16}, 2_{60}$	16
$y^6+y^4(x^2+ax+1)+y^2x[d(x^2+1)+bx]+cx^3$	$V_4, (1;2^3)$	$2_{24}, 4_{18}$	$1_{16}, 2_{24}, 4_{18}$	16
$y^6+y^2[cx(xy^2+1)+b(x^3+y^2)+ax(y^2+x)]+x^3$	$V_4, (1;2^3)$	4_{30}	$1_4, 2_{18}, 4_{24}$	4
$by^2(y^2-x)(x^2-1)-y^6-axy^2(y^2+x^2)-x^3$	$D_4, (0;2^4,4)$	$4_{12}, 8_9$	$1_4, 2_6, 4_{18}, 8_6$	4
$y^6+ay^3(x^3+1)+bxy^4+cx^2y^2+x^3$	$S_3, (0;2^6)$	$1_6, 3_{18}, 6_{10}$	$1_{10}, 3_{30}, 6_6$	16
$y^6+ay^3(x^3+1)+bxy^4+bx^2y^2+x^3$	$D_6, (0;2^5)$	$2_3, 6_{13}, 12_3$	$1_4, 2_3, 3_{12}, 6_9, 12_3$	4
$by^2(y^2-x)(x^2-1)-(y^2+x)^3$	$S_4, (0;2^3,4)$	$4_6, 12_6, 24_1$	$1_4, 4_6, 6_6, 12_6$	4
$x^3y^3+y^6+(a+b+1)y^2(x^3-y^3)+(ab+a+b)y(x^3-y^3)+ab(x^3-y^3)$	$S_3, (0;2^2,3^3)$	6_{20}	$1_1, 3_{15}, 6_{15}$	1
$y^4(a+y^2)+x^3(1+ay^2)$	$D_6, (0;2^2,3,6)$	12_{10}	$1_1, 3_3, 6_{13}, 12_4$	1
$x^3y^3+y^6+ax^3+y^3$	$S_3\times S_3, (0;2^3,3)$	$6_2, 18_6$	$1_1, 3_6, 9_9, 36_1$	1
$x^3y^3+y^6-x^3+y^3$	$(S_3\times S_3)\rtimes C_2 \cong (72,40), (0;2,4,6)$	$12_1, 36_3$	$1_1, 6_3, 9_3, 18_3, 36_1$	1
$x^2y^3+y^4+a^5x^3+xy$	$D_5, (0;2^2,5^5)$	10_{12}	$1_1, 5_{15}, 10_6$	1
$xy-x^3+y^4+x^2y^3$	$S_5, (0;2,4,5)$	$20_3, 60_1$	$1_1, 5_3, 10_3, 30_3$	1
$y^2-(x^9-1)$	$C_9, (0;9^3)$	$3_1, 9_{13}$	$1_1, 9_{15}$	1
$y^2-x(x^4-1)(x^4+2i\sqrt{3}x^2+1)$	A_4	$4_3, 6_4, 12_7$	$4_1, 6_4, 12_9$	0
$y^2-x(x^8-1)$	D_8	$8_5, 16_5$	$2_2, 4_1, 8_{10}, 16_3$	0
$y^2-(x^{10}-1)$	D_{10}	$10_4, 20_4$	$1_1, 5_3, 10_8, 20_2$	1

Table 8. Orbit decomposition, three non-hyperelliptic genus-5 curves, and separately all hyperelliptic genus-5 curves with many automorphisms.

f	$\overline{\mathrm{Aut}}, \boldsymbol{c}$	Odd	Even	I
$y^4-4(x^4-ax^2+1)y^2+b^2x^4$	C_2^4	$4_{40}, 8_{30}, 16_6$	$1_{32}, 4_{40}, 8_{30}, 16_6$	32
$y^3-x^2(x^5-1)$	$C_3\times D_5, (0;2,6,15)$	$1_1, 15_5, 30_{14}$	$3_1, 15_{15}, 30_{10}$	1
$4x^8+36x^4y^4+81y^8+8x^6+30x^2y^4+5x^4+14y^4+2x^2+1$	$(((C_4\times C_2)\rtimes C_4)\rtimes C_3)\rtimes C_2 \cong (192,181), (0;2,3,8)$	$16_1, 24_2, 48_1, 96_4$	$1_1, 3_1, 4_1, 6_2, 12_1, 16_1, 24_4, 48_4, 96_2$	1
$y^2-(x^{11}-1)$	$C_{11}, (0;11^3)$	$1_1, 11_{45}$	11_{48}	1
$y^2-x(x^{10}-1)$	D_{10}	$1_1, 5_3, 10_{12}, 20_{18}$	$1_1, 2_1, 5_3, 10_{27}, 20_{12}$	2
$y^2-(x^{12}-1)$	D_{12}	$1_1, 3_1, 6_2, 12_{12}, 24_{14}$	$1_1, 2_1, 3_1, 6_5, 12_{25}, 24_8$	2
$y^2-(x^{12}-33x^8-33x^4+1)$	S_4	$1_1, 3_1, 6_2, 12_8, 24_{16}$	$3_2, 4_1, 6_3, 8_1, 12_{13}, 24_{14}$	1
$y^2-x(x^{10}+11x^5-1)$	$A_5, (0;3^2,5)$	$1_1, 15_1, 30_6, 60_5$	$6_3, 10_3, 15_4, 30_{12}, 60_1$	1

practice the process of going from a canonical embedding to a plane model becomes infeasible. Likewise, one could use the methods of [69] to get the canonical embedding, but the problem of finding a plane form from this remains.

- The genus-9 curve with $\overline{\mathrm{Aut}} = S_5$ is the Fricke octavic curve, defined in [33], constructed similarly to Bring's curve in $\mathbb{P}^3$ and so a plane form of the curve is found using resultants and a judicious choice of projective transformation to find a well conditioned coordinate system.
- The genus-9 curve with automorphism group of order 57 is a generalization of Klein's curve and Snyder's curve [52, p. 464].

Table 9. Orbit decomposition, some non-hyperelliptic genus-6 curves, and separately all hyperelliptic genus-6 curves with many automorphisms.

f	$\overline{\mathrm{Aut}}, \boldsymbol{c}$	Odd	Even	I
$L_5(x,y) + L_3(x,y) + L_1(x,y)$	C_2	$1_{96}, 2_{960}$	$1_{160}, 2_{960}$	256
$x^5 + ax^2y^3 + bx^3y + y^4 + cxy^2 + dx^2 + ey$	C_3	$1_6, 3_{670}$	$1_{10}, 3_{690}$	16
$L_5(x,y) + L_1(x,y)$	C_4	$1_{16}, 2_{40}, 4_{480}$	$2_{80}, 4_{480}$	16
$x^5 + axy^4 + bx^2y^2 + cx^3 + dy^2 + ex$	C_4	$1_8, 2_{44}, 4_{480}$	$1_8, 2_{76}, 4_{480}$	16
$1 + L_5(x,y)$	C_5	$1_1, 5_{403}$	5_{416}	1
$x^5 + ax^2y^3 + bx^3y + y^4 + cxy^2 + dx^2 + y$	S_3	$1_6, 3_{90}, 6_{290}$	$1_{10}, 3_{150}, 6_{270}$	16
$x^5 + y^4 + ax^3 + x$	C_8	$1_4, 2_6, 4_{20}, 8_{240}$	$4_{40}, 8_{240}$	4
$x^5 + y^5 + axy^3 + bx^2y + 1$	D_5	$1_6, 5_{90}, 10_{156}$	$1_{10}, 5_{150}, 10_{132}$	16
$x^5 + y^5 + ax^3 + x$	C_{10}	$1_1, 5_{19}, 10_{192}$	$5_{32}, 10_{192}$	1
$x^5 + y^4 + x$	C_{16}	$1_2, 2_1, 4_3, 8_{10}, 16_{120}$	$8_{20}, 16_{120}$	2
$x^5 + y^5 + x$	C_{20}	$1_1, 5_3, 10_8, 20_{96}$	$10_{16}, 20_{96}$	1
$x^5 + y^4 + y$	$C_5 \times S_3$, $(0; 2, 10, 15)$	$1_1, 5_1, 15_{18}, 30_{58}$	$5_2, 15_{30}, 30_{54}$	1
$x^4y + y^4 + x$	$C_{13} \rtimes C_3 \cong (39, 1)$, $(0; 3^2, 13)$	$1_1, 13_5, 39_{50}$	$13_{10}, 39_{50}$	1
$y^3 - (x^4 - 1)^2(x^4 + 1)$	$C_3 \rtimes D_8 \cong (48, 15)$, $(0; 2, 6, 8)$	$24_4, 48_{40}$	$1_1, 3_1, 6_2, 12_8, 24_{34}, 48_{24}$	1
$y^3 - (x^4 - 2i\sqrt{3}x^2 + 1)(x^4 + 2i\sqrt{3}x^2 + 1)^2$	$(V_4 \rtimes C_9) \rtimes C_2 \cong (72, 15)$, $(0; 2, 4, 9)$	$18_4, 36_6, 72_{24}$	$1_1, 9_7, 18_4, 36_{30}, 72_{12}$	1
$(x^6 + y^6 + 1) + (x^2 + y^2 + 1)(x^4 + y^4 + 1) - 12x^2y^2$	S_5, $(0; 2, 4, 6)$	$6_2, 12_2, 20_3, 30_2, 60_{17}, 120_7$	$1_2, 2_1, 10_2, 12_3, 15_6, 20_2, 30_9, 60_{21}, 120_3$	2
$x^5 + y^5 + 1$	$C_5^2 \rtimes S_3 \cong (150, 5)$, $(0; 2, 3, 10)$	$1_1, 15_1, 25_5, 75_{13}, 150_6$	$15_2, 25_{10}, 75_{20}, 150_2$	1
$y^2 - (x^{13} - 1)$	C_{13}, $(0; 13^3)$	$1_1, 13_{155}$	13_{160}	1
$y^2 - x(x^{12} - 1)$	D_{12}	$2_1, 4_1, 6_1, 12_{17}, 24_{75}$	$2_1, 4_2, 6_3, 12_{43}, 24_{64}$	0
$y^2 - x(x^4 - 1)(x^8 + 14x^4 + 1)$	S_4	$6_4, 8_3, 12_6, 24_{79}$	$4_4, 6_4, 8_3, 12_{30}, 24_{69}$	0
$y^2 - (x^{14} - 1)$	D_{14}	$14_{16}, 28_{64}$	$1_1, 7_7, 14_{41}, 28_{52}$	1

Table 10. Orbit decomposition, some non-hyperelliptic genus-7 curves, and separately some hyperelliptic genus-7 curves including all with many automorphisms.

f	$\overline{\mathrm{Aut}}, \boldsymbol{c}$	Odd	Even	I
$(x^3 + y^3)^2 - x^2y^2 - 1$	D_6	$1_4, 2_{12}, 3_{12}, 6_{232}, 12_{556}$	$1_{12}, 2_{12}, 3_{36}, 6_{336}, 12_{508}$	16
$x^6 + y^6 - x^3 - y^3$	$C_3 \times S_3$	$1_1, 3_1, 6_7, 9_{20}, 18_{439}$	$3_4, 6_6, 9_{60}, 18_{426}$	1
$x^6 + y^4 - 1$	$C_{12} \times C_2$, $(0; 4, 6, 12)$	$1_1, 3_1, 6_6, 12_{42}, 24_{316}$	$1_1, 2_1, 3_1, 6_{11}, 12_{66}, 24_{308}$	2
$(x^4 + y^4)^2 - x^3y^3 - x^2y^2$	$C_8 \rtimes V_4 \cong (32, 43)$, $(0; 2^3, 8)$	$8_{16}, 16_{84}, 32_{208}$	$2_4, 4_{22}, 8_{42}, 16_{129}, 32_{180}$	0
$x^7 + y^7 - x^2y^2$	$C_3 \times D_7$	$1_1, 21_{21}, 42_{183}$	$3_1, 21_{63}, 42_{165}$	1
$y^{21} - x(x + 1)^{13}(x - 1)^7$	$C_3 \times D_7$, $(0; 2, 6, 21)$	$1_1, 21_{21}, 42_{183}$	$3_1, 21_{63}, 42_{165}$	1
$x^9 + y^9 - x^6$	$C_3 \times D_9$, $(0; 2, 6, 9)$	$1_1, 18_3, 27_{21}, 54_{139}$	$3_1, 18_4, 27_{63}, 54_{120}$	1
$x^9 + y^9 - x^3y^3$	$C_3 \times D_9$, $(0; 2, 6, 9)$	$1_1, 18_3, 27_{21}, 54_{139}$	$3_1, 18_4, 27_{63}, 54_{120}$	1
$y^8 - (x^2 - 1)(x^2 + 1)^3$	$(C_{16} \rtimes C_2) \rtimes C_2 \cong (64, 41)$, $(0; 2, 4, 16)$	$16_8, 32_{42}, 64_{104}$	$4_2, 8_9, 16_{15}, 32_{64}, 64_{92}$	0
$y^{16} - x(x - 1)^9(x + 1)^6$	$(C_8 \rtimes C_4) \rtimes C_2 \cong (64, 41)$, $(0; 2, 4, 16)$	$16_8, 32_{42}, 64_{104}$	$4_2, 8_9, 16_{15}, 32_{64}, 64_{92}$	0
$y^9 - 6y^6 + 3(9x^4 - 5)y^3 - 8$	$((C_4 \times S_3) \rtimes C_2) \rtimes C_3 \cong (144, 127)$, $(0; 2, 3, 12)$	$4_1, 12_1, 24_2, 36_2, 72_{41}, 144_{35}$	$6_2, 12_2, 18_6, 24_2, 36_{28}, 72_{28}, 144_{35}$	0
$28x^4y^4 + 2x^7 + 2y^7 + 35x^3y^3 + 21x^2y^2 + 7xy + 1$	$\mathrm{PSL}_2(\mathbb{F}_8)$, $(0; 2, 3, 7)$	$28_1, 36_1, 252_{14}, 504_9$	$28_3, 36_3, 126_{16}, 252_{18}, 504_3$	0
$y^2 - x(x^6 - 1)(x^8 - 2)$	C_2	$1_{64}, 2_{4032}$	$1_{64}, 2_{4096}$	128
$y^2 - x(x^7 - 1)(x^7 - 2)$	D_7	$1_1, 7_{63}, 14_{549}$	$1_3, 7_{189}, 14_{495}$	4
$y^2 - (x^{15} - 1)$	C_{15}, $(0; 15^3)$	$3_1, 5_2, 15_{541}$	$1_1, 5_1, 15_{550}$	1
$y^2 - (x^8 - 1)(x^8 - 2)$	D_8	$8_{72}, 16_{472}$	$1_4, 2_4, 4_{25}, 8_{170}, 16_{424}$	4
$y^2 - x(x^{14} - 1)$	D_{14}	$1_1, 7_7, 14_{57}, 28_{260}$	$1_1, 2_1, 7_7, 14_{120}, 28_{233}$	2
$y^2 - (x^{16} - 1)$	D_{16}	$8_4, 16_{62}, 32_{222}$	$1_2, 2_1, 4_3, 8_{14}, 16_{112}, 32_{198}$	2
$y^2 - (x^{16} + 1)$	D_{16}	$8_4, 16_{62}, 32_{222}$	$1_2, 2_1, 4_3, 8_{14}, 16_{112}, 32_{198}$	2

Table 11. Orbit decomposition, all hyperelliptic curves of genus 8 with many automorphisms.

f	$\overline{\mathrm{Aut}}, \boldsymbol{c}$	Odd	Even	I
$y^5 - x^2(x^4 + 1)$	C_{20}	20_{1632}	$1_1, 5_3, 10_{24}, 20_{1632}$	1
$y^3 - (x^5 - 1)(x^5 + 1)^2$	$S_3 \times D_5$, $(0; 2, 6, 10)$	60_{544}	$1_1, 15_{15}, 30_{173}, 60_{458}$	1
$y^2 - (x^{17} - 1)$	C_{17}, $(0; 17^3)$	17_{1920}	$1_1, 17_{1935}$	1
$y^2 - x(x^4 - 1)(x^{12} - 33x^8 - 33x^4 + 1)$	S_4	$4_2, 8_2, 12_{94}, 24_{1312}$	$4_2, 6_8, 8_4, 12_{90}, 24_{1322}$	0
$y^2 - x(x^{16} - 1)$	D_{16}	$16_{72}, 32_{984}$	$2_2, 4_1, 8_7, 16_{188}, 32_{932}$	0
$y^2 - (x^{18} - 1)$	D_{18}	$6_1, 18_{63}, 36_{875}$	$1_1, 3_1, 6_1, 9_{14}, 18_{182}, 36_{819}$	1

Table 12. Orbit decomposition, two non-hyperelliptic curves of genus 9, and all hyperelliptic curves of genus 9 with many automorphisms.

f	$\overline{\text{Aut}}, \boldsymbol{c}$	Odd	Even	l
$x^5y^2+y^5+x^2$	$C_{19} \rtimes C_3 \cong (57,1), (0;3^2,19)$	$1_1, 19_{27}, 57_{2286}$	$19_{36}, 57_{2292}$	1
$y-x^3-x^4y^3+xy^4+3x^2y^2$	$S_5, (0;2,5,6)$	$6_1, 10_4, 20_{12}, 30_{35}, 60_{300}, 120_{929}$	$1_2, 5_6, 6_1, 10_{10}, 15_{24}, 20_6, 30_{85}, 60_{402}, 120_{867}$	2
$y^2-(x^{19}-1)$	$C_{19}, (0;19^3)$	$1_1, 19_{6885}$	19_{6912}	1
$y^2-(x^{12}-33x^8-33x^4+1)(x^8+14x^4+1)$	S_4	$1_2, 2_1, 3_2, 4_2, 6_{13}, 8_{11}, 12_{172}, 24_{5357}$	$3_4, 4_4, 6_{18}, 8_{16}, 12_{290}, 24_{5316}$	2
$y^2-x(x^{18}-1)$	D_{18}	$1_1, 3_1, 6_2, 9_{14}, 12_1, 18_{245}, 36_{3507}$	$1_1, 2_1, 3_1, 6_5, 9_{14}, 18_{497}, 36_{3395}$	2
$y^2-(x^{20}-1)$	D_{20}	$1_1, 5_3, 10_{12}, 20_{246}, 40_{3144}$	$1_1, 2_1, 5_3, 10_{27}, 20_{504}, 40_{3024}$	2
$y^2-(x^{20}-228x^{15}+494x^{10}+228x^5+1)$	$A_5, (0;3,5^2)$	$1_1, 5_3, 10_3, 15_4, 20_9, 30_{117}, 60_{2117}$	$6_3, 10_{12}, 15_{16}, 20_{12}, 30_{345}, 60_{2006}$	1

Appendix B: Proof of Theorem 5.20

We prove Theorem 5.20 by showing that for all but finitely many n the alternating group is $(2,3,7)$-generated in at least two different ways with opposite parities of l. This will be a modification of the proof of the main theorem in [23] in the case $k=7$, and was originally communicated to the authors by Marston Conder via email.

Conder shows that for all but finitely many n the alternating group A_n is a factor group of

$$\Delta(2,3,7) := \left\langle x, y \mid x^2 = y^3 = (xy)^7 = 1 \right\rangle.$$

This is done by constructing coset diagrams for the group

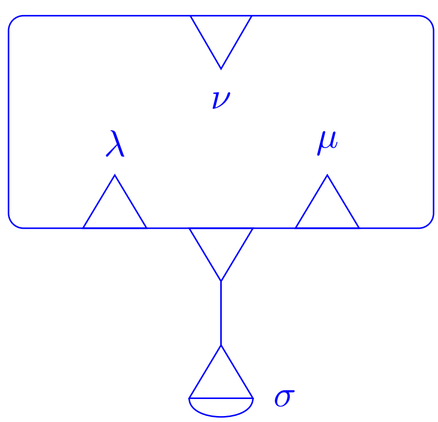
$$\Delta^*(2,3,7) := \left\langle x, y, t \mid x^2 = y^3 = (xy)^7 = t^2 = (xt)^2 = (yt)^2 = 1 \right\rangle,$$

where a diagram on n vertices corresponds to a transitive degree-n permutation representation of $\Delta^*(2,3,7)$. The key aspects of how these diagrams correspond to the action are as follows.

- Fixed points of y are indicated by a filled circle at that vertex.
- Triangles in the diagram correspond to 3-cycles of y, with the vertices permuted anticlockwise.
- Vertices that are swapped by the action of x are joined with an (additional) edge.
- Diagrams have a vertical axis of symmetry, and the action of t is reflection in this axis.

From a diagram one can read off the number of fixed points of x, denoted f_1, as the number of vertices which are only joined to other vertices by edges of a triangle. We can then calculate $l = (n - f_1)/4$.

Example B.1. *Consider the following diagram from [22].*



This is a 15-vertex diagram which corresponds to the $(2,3,7)$-generation of A_{15} seen in [34]. We visually spot the 3 fixed points of x (denoted λ, μ, ν), the lack of fixed points of y, and the single fixed point of xy (denoted σ). This means $f_1 = 3 \Rightarrow l = 3$.

Conder constructs families of basic diagrams with certain desired properties sufficient for the subgroup $\Delta(2,3,7) \le \Delta^*(2,3,7)$ to act as the alternating group A_n, and defines a method for composing diagrams which preserves these properties. By showing that for n sufficiently large one can always construct a diagram with the desired properties through successive composition, the $(2,3,7)$ generation of A_n is achieved.

We shall be interested in the diagrams of [23] denoted $S(7,0)$, $T(7,0)$, and $U(7,0)$, reproduced below in Figures 1–3. These diagrams have structures called (1)-handles denoted $[\lambda,\mu]_1$ which correspond to two fixed points λ, μ of x, such that both xy and t map λ to μ; $S(7,0)$ has three (1)-handles, $T(7,0)$ has one, and $U(7,0)$ has 1. These diagrams are composed by adding edges between (1)-handles, in particular $[\lambda,\mu]_1$ is joined to $[\lambda',\mu']_1$ by adding edges from λ to λ' and μ to μ'. This corresponds to removing fixed points from the action of x by adding the transpositions $(\lambda\lambda')(\mu\mu')$. The desired properties required are the following.

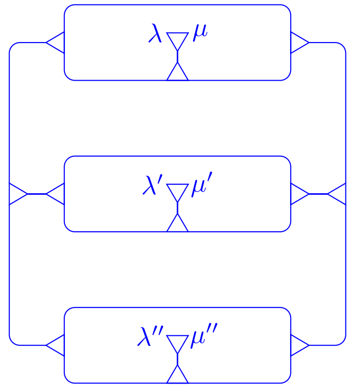


Figure 1. $S(7,0)$.

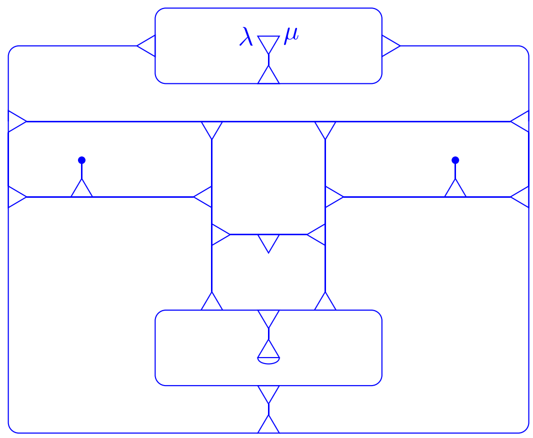


Figure 2. $T(7,0)$.

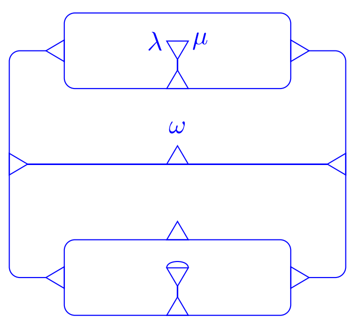


Figure 3. $U(7,0)$.

- There is a vertex ω fixed by t whose orbit under xyt is of prime length p, $p < n-2$, such that p does not divide length of any other orbit of xyt.
- t acts as an odd permutation.

In Figures 1–3 we have labeled the vertices of the (1)-handles and the distinguished vertex ω which exists in $U(7,0)$.

Conder argues that just by joining diagrams of the form $S(7,0)$, $T(7,0)$, and $U(7,0)$ along (1)-handles one can construct a diagram with the desired properties to have $\Delta(2,3,7)$ act as A_n for all but finitely many n. Now suppose we have followed the procedure of Conder to construct a large diagram of degree n with the desired properties. By including two additional copies of $S(7,0)$ in the joining process, each copy joined along two of its three (1)-handles, we now have a diagram (denoted V_n) of degree $n+84$ (as $S(7,0)$ has 42 vertices) which still has the desired properties, but now has two free (1)-handles. If a diagram has two (1)-handles of its own we can join these, and this reduces the number of fixed points by 4 thus increasing the value of l by one (i.e. changing the parity of l). Joining the two (1)-handles on V_n gives a new diagram V'_n of the same degree but for which l has the opposite parity. One can check that this last join process has not broken the properties of V_n which meant that $\Delta(2,3,7)$ acted as an alternating group, and so for all but finitely many n we have $(2,3,7)$-generated A_n in two ways with opposite parities of l.

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Declaration of Interest

The authors have no relevant financial or non-financial interests to disclose.

Data availability statement

The datasets generated during the current study and the code for their creation/analysis are openly available at https://github.com/DisneyHogg/Riemann_Surfaces_and_Monopoles.

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