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Chapter 1

Omega-categorical limits of betweenness relations and D-sets

Asma Ibrahim Almazaydeh, Samuel Braunfeld, and Dugald Macpherson

We explore two constructions of oligomorphic Jordan permutation groups preserving a ‘limit of betweenness relations’ and a ‘limit of D-relations’, from Bhattacharjee and Macpherson (2006) and Almazaydeh and Macpherson (2021) respectively. Several issues left open in Almazaydeh and Macpherson (2021) are resolved. In particular it is shown that the ‘limit of D-relations’ is not homogeneous in the given language, but is ‘homogenizable’, that is, there is a homogeneous structure over a finite relational language with the same universe and the same automorphism group. The structure is NIP and indeed distal, but not monadically NIP, its age is not well-quasi-ordered under embeddability, and the growth rate of the sequence enumerating orbits on k-sets grows faster than exponentially. The automorphism group is maximal-closed in the symmetric group. Similar results are shown for the construction in Bhattacharjee and Macpherson (2006).

1.1 Introduction

If G is a transitive permutation group on a set Ω, then a subset Γ of Ω is a *Jordan set* if $|\Gamma| > 1$ and the pointwise stabiliser $G_{(\Omega \setminus \Gamma)}$ is transitive on Γ. It is a *proper* Jordan set if in addition, if G is $(k+1)$-transitive then $|\Omega \setminus \Gamma| \neq k$, that is, the Jordan condition does not arise just from the degree of transitivity. A *Jordan group* is a transitive permutation group with a proper Jordan set. Finite Jordan groups which are *primitive* (i.e. preserve no proper non-trivial equivalence relation on the set acted on) were classified in [29] – see also [21] and the Appendix of [17]. A structure theorem for *infinite* primitive Jordan groups was given in [3], building on earlier work of Adeleke and Neumann in [5]. This structure theorem states that if (G, Ω) is a primitive Jordan group which is not *highly transitive* (i.e. for some $k \geq 1$ is not k-transitive), then G preserves on Ω a structure of one of the following types: a Steiner system (possibly with infinite ‘lines’); a linear order, circular order, linear betweenness relation or separation relation; a semilinear order, ‘general’ betweenness relation, C-relation or D-relation; or a ‘limit’ of Steiner systems, betweenness relations or D-relations. The three ‘limit’ constructions are rather different in that no explicit class of invariant relational structures is specified.

A construction of a group preserving a limit of Steiner systems was given by Adeleke in [1], and developed further by Johnson in [20]. Adeleke also in [2] gave examples of groups preserving a limit of betweenness relations and a limit of D-relations. His constructions are not oligomorphic, at least in the case of a limit of betweenness

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relations, and no invariant relational structure is explicit. The paper [9], which builds on early drafts of [2], constructs an ω-categorical structure $\mathcal{M}_B$ whose automorphism group preserves a limit of betweenness relations, and the same is done (with an analogous structure $\mathcal{M}_D$) for limits of D-relations in [7]. These constructions are intricate, but we believe them to be essentially new treelike constructions which may have significance for other reasons. A number of fine structural issues were left open in [9] and [7], and we here resolve some of these. We note also the paper [11], which takes a rather more general approach to the construction of limits of betweenness relations of [9], extending beyond the ω-categorical context.

A countably infinite relational structure M is said to be *homogeneous* if every isomorphism between finite substructures of M extends to an automorphism. Following Covington [18], we say that M is *homogenizable* if it can be made homogeneous by adapting the language, that is, if there is a homogeneous structure M' over a finite relational language such that M and M' have both the same universe and the same automorphism group.

Notation. Generally, we prefer not to distinguish notationally between a structure and its universe, but for $\mathcal{M}_D$ and $\mathcal{M}_B$ this seems necessary, largely because for $\mathcal{M}_D$ we consider the structure under three different languages, all with the same automorphism group, and similarly for $\mathcal{M}_B$. Our convention is as follows for $\mathcal{M}_D$, and is essentially the same for $\mathcal{M}_B$. When the choice of language is unimportant, and the focus is just on 0-definable sets and orbits, we just write $\mathcal{M}_D$. The structure constructed in [7], in a language with relation symbols L, S, L', S', R, Q will be denoted by $\mathcal{M}_D^1$. Its reduct to the language with just the ternary relation symbol L will be denoted $\mathcal{M}_D^0$. In the language in which it is homogeneous, which has relation symbols $L, S, Q^{\leq}, Q^{\geq}, P, T$, it will be denoted $\mathcal{M}_D^2$. The universe will just be denoted M_D. We may write $\mathcal{M}$ for one of these structures, but except with these structures, we do not distinguish notationally between a structure and its domain.

We prove here the following result, clarifying the model-theoretic properties of $\mathcal{M}_B$ and $\mathcal{M}_D$. We say that a structure M has *trivial algebraic closure* if $\mathrm{acl}(A) = A$ for all $A \subset M$. A structure (or its theory) has the *independence property* if some family of definable sets given by a formula $\phi(\bar{x}, \bar{y})$ has infinite Vapnik-Chervonenkis dimension; it is said to be *NIP* otherwise. See [31] for further background on this, and also on the notion of *distality*, a strengthening of NIP which does not hold for stable theories.

Theorem 1.1.1. *The following hold, for* $\mathcal{M} \in \{\mathcal{M}_B^1, \mathcal{M}_D^1\}$.

(1) *$\mathcal{M}$ is not homogeneous, but is homogenizable.*

(2) *$\mathcal{M}$ is distal, and in particular NIP.*

(3) *$\mathcal{M}$ has trivial algebraic closure.*

We view the family of structures explored in [6] – namely semilinear orders, betweenness relations, and C and D relations – as treelike structures. They are all built from (lower) semilinear orders: betweenness relations are obtained as reducts of semilinear orders by replacing the ordering by an induced betweenness, C-relations have as universe a dense set of maximal chains of a semilinear order, and a D-relation can be obtained from a C-relation by forgetting the downwards direction (and can be viewed as a dense set of ends in a betweenness relation). In this paper we shall refer to such structures as the *basic* treelike structures. The main theorem of [3] says that any primitive but not highly transitive Jordan group (G,Ω) with a proper *primitive* Jordan set preserves on Ω either a linear order, linear betweenness relation, circular order, or separation relation, or a basic treelike structure. There is a uniformity in the method of proof – essentially, in each case, the relational structure is identified by finding a group-invariant family of subsets of Ω with certain specified intersection properties.

The structures $\mathcal{M}_B$ and $\mathcal{M}_D$ are entitled to be called treelike, even though their automorphism groups do not preserve any basic treelike structure on the domain. The universe is the union of an invariant family of subsets which is ordered (under inclusion) by an invariant semilinear order; each of the sets in this family has a unique maximal invariant equivalence relation, with an invariant betweenness relation or D-relation (for $\mathcal{M}_B$ or $\mathcal{M}_D$ respectively) on the quotient; and for both $\mathcal{M} = \mathcal{M}_B$ and $\mathcal{M} = \mathcal{M}_D$, if $a \in M$ then there is an a-definable C-relation on $M \setminus \{a\}$.

There are several strong model-theoretic and combinatorial conditions that, among finitely homogeneous and more generally ω-categorical structures, appear to coincide, and we now describe these. If G is an oligomorphic group on Ω, let $f_k(G)$ denote the number of orbits of G on the collection $\Omega^{[k]}$ of unordered k-subsets of Ω. It was shown in [27] that if (G,Ω) is primitive but not highly homogeneous then $(f_k(G))$ grows at least exponentially, and more detailed results on growth rates, answering several questions from [27] and [26], are obtained in [12]. We shall say that $(f_k(G))$ has *super-exponential growth* if, for every $c > 1$, there is $K \in \mathbb{N}$ such that $f_k(G) > c^k$ for every $k > K$. By [28, Theorem 1.1], if M is ω-categorical and has the independence property, then $f_k(\mathrm{Aut}(M))$ has super-exponential growth; in fact, there is a polynomial $p(k)$ of degree at least 2 such that $f_k(\mathrm{Aut}(M)) > 2^{p(k)}$ for all k. It is shown in [16], extending earlier results, that there are basic treelike structures of all four kinds with non-super-exponential growth – in fact, there are examples of interesting homogeneous *expansions* of such structures with non-super-exponential growth, such as a C-relation equipped with a compatible linear order or a D-relation with a compatible circular order.

Recall that the *age* $\mathrm{Age}(M)$ of a relational structure M is the collection of finite structures which embed in M. It is well-known that for certain rather rare homogeneous structures M, $\mathrm{Age}(M)$ is well quasi-ordered under embeddability – that is, there are no infinite antichains (and, as is automatic, no infinite descending chains). We shall say that such structures have *wqo age*. Using Kruskal's Theorem, one can rather easily show that natural examples of basic treelike structures have wqo age.

Monadically NIP structures, for which any expansion by unary predicates remains NIP, were introduced by Baldwin and Shelah in [8], and their study continued in [13]. There, extending [24, Conjecture 2.9], Conjecture 1 states that for an ω-categorical homogeneous relational structure M, the following conditions are equivalent: M is monadically NIP; $(f_k(\mathrm{Aut}(M))$ is bounded above exponentially; and M has wqo age. In this direction, the following is an immediate consequence of [13, Theorem 1.2].

Theorem 1.1.2. [13, Theorem 1.2]*] Let M be an ω-categorical structure that is not monadically NIP. Then $(f_k(Aut(M)))$ is asymptotically greater than $\lfloor k/\ell \rfloor!$ for some $\ell \in \mathbb{N}^{>0}$.*

Originally we had hoped that limits of betweenness relations and D-sets might have provided a counterexample to [13, Conjecture 1], but the result below shows they do not.

Theorem 1.1.3. *Let $\mathcal{M} \in \{\mathcal{M}_B^2, \mathcal{M}_D^2\}$. Then*

(1) *$\mathcal{M}$ is not monadically NIP.*

(2) $(f_k(\mathrm{Aut}(\mathcal{M})))$ *has super-exponential growth rate.*

(3) *$Age(\mathcal{M})$ is not wqo.*

Recall that if M is countably infinite, then $\mathrm{Sym}(M)$ carries a topological group structure, where the basic open sets are cosets of pointwise stabilisers of finite sets, and that the topology is metrizable, giving $\mathrm{Sym}(M)$ (or the symmetric group on any countably infinite set) the structure of a Polish group. Its closed subgroups are exactly the automorphism groups of first order structures with universe M. If M is a countably infinite structure, we say that $\mathrm{Aut}(M)$ is *maximal-closed* in $\mathrm{Sym}(M)$ if for every closed subgroup H of $\mathrm{Sym}(M)$ with $\mathrm{Aut}(M) \le H \le \mathrm{Sym}(M)$ we have $H = \mathrm{Aut}(M)$ or $H = \mathrm{Sym}(M)$. By the Ryll-Nardzewski Theorem, if M is ω-categorical then $\mathrm{Aut}(M)$ is maximal-closed in $\mathrm{Sym}(M)$ if and only if M has no proper non-trivial *reduct* in the sense of [32]. There is a substantial literature on reducts of various homogeneous structures, beginning with the classification of reducts of $(\mathbb{Q}, <)$ by Cameron in [15], and the corresponding result for the random graph in [32]. We prove the following, solving Problem 6.4 of [7].

Theorem 1.1.4. *Let $\mathcal{M} \in \{\mathcal{M}_B, \mathcal{M}_D\}$. Then* $\mathrm{Aut}(\mathcal{M})$ *is maximal-closed in* $\mathrm{Sym}(M)$.

We comment briefly on the motivation of this body of work (the present paper, and also [9] and [7]). First, we believe that the class of Jordan permutation groups is a natural and important class. Several applications of structural results on Jordan groups are described in the introduction to [7]. We do not give detail on these here, but they include: the structure of ω-categorical strictly minimal sets (see the classification given in [17]) and analogous results on permutation groups in [29] (a description of groups acting primitively on a countably infinite set X with no countable orbits on the collection of infinite co-infinite subsets of X); classification results for primitive groups of uncountable degree containing a non-identity element of small support (see [4]); analogues (see [25]) for other cycle types of Wielandt's result that any primitive group of finite degree containing a non-identity finitary permutation is highly transitive; several results showing that certain closed infinite permutation group are maximal-closed in the symmetric group (see e.g. [10] for a non-oligomorphic example of countable degree, and [23] concerning affine and projective groups).

The further motivation to better understand these limit constructions is that they have potential to provide useful counter-examples, in model theory and permutation group theory. In the work in [3] giving a structure theorem for primitive Jordan groups, their existence came initially as a surprise, not anticipated in the classification results in [5] of primitive Jordan groups with a proper *primitive* Jordan set. (We add, though, that Adeleke's initial manuscript giving existence of groups preserving limits of betweenness and D-relations, which led to [2], was from the early 1990s so predates [3].)

Several questions on these limit constructions are posed in [7, Section 6]. This paper answers Problem 6.4 and sheds light on Problem 6.6, but the other problems remain. We also mention that a more precise classification of closed *oligomorphic* primitive Jordan groups may be feasible. There are further questions on the model theory of $\mathcal{M}_B$ and $\mathcal{M}_D$ – for example we expect that the analysis of indiscernible sequences in Lemmas 1.3.4 and 1.3.5 should yield that they are dp-minimal. We also ask whether the construction of $\mathcal{M}_B$ can be adapted to put $\mathbb{R}$-tree structure on each betweenness relation, and whether there are analogues in which these structures are discrete, possibly with interesting group-theoretic consequences.

We give background on limits of betweenness relations and D-relations in Section 2. Theorem 1.1.1 and Theorem 1.1.3 are proved in Section 3, and Theorem 1.1.4 in Section 4. For the relevant model-theoretic background (in particular, NIP theories, distality, and dp-minimality) [31] should suffice as a reference.

1.2 Background on the constructions

In this section we give an overview of the structures $\mathcal{M}_B$ and $\mathcal{M}_D$. We shall assume some familiarity with the treelike structures examined in detail in [6], namely semilinear orders, betweenness relations, C-relations and D-relations. Some background on these is also given in [9] and [7].

First, we recall some standard permutation group terminology. A permutation group (G, X) is *primitive* if G preserves no proper non-trivial equivalence relation on X. For $k > 1$ we say G is k-*transitive* (respectively, k-*homogeneous*) if it is transitive on the collection of ordered (respectively, unordered) k-subsets of X, and that (G, X) is *highly transitive* if it is k-transitive for all k. If $k \geq 2$ we say that (G, X) is k-*primitive* if it is $(k-1)$-transitive and for distinct $x_1, \ldots, x_{k-1} \in X$, $G_{x_1,\ldots,x_{k-1}}$ acts primitively on $X \setminus \{x_1, \ldots, x_{k-1}\}$. If $A \subset X$, we write $G_{(A)}$ for the pointwise stabiliser of A, and $G_{\{A\}}$ for the setwise stabiliser of A.

Next, we give the key definition from [3].

Definition 1.2.1. *([3], Definition 2.1.9)* If (G, X) is an infinite Jordan group we say that G preserves a *limit of D-relations* if there are: a linearly ordered set $(J, \leq)$ with no least element, a chain $(Y_i : i \in J)$ of subsets of X and chain $(H_i : i \in J)$ of subgroups of G with $Y_i \supset Y_j$ and $H_i > H_j$ whenever $i < j$, such that the following hold:

(i) for each i, $H_i = G_{(X \setminus Y_i)}$, and H_i is transitive on Y_i and has a unique non-trivial maximal congruence σ_i on Y_i;

(ii) for each i, $(H_i, Y_i/\sigma_i)$ is a 2-transitive but not 3-transitive Jordan group preserving a D-relation;

(iii) $\bigcup(Y_i : i \in J) = X$;

(iv) $(\bigcup(H_i : i \in J), X)$ is a 2-primitive but not 3-transitive Jordan group;

(v) $\sigma_j \supseteq \sigma_i|_{Y_j}$ if $i < j$;

(vi) $\bigcap(\sigma_i : i \in J)$ is equality in X;

(vii) $(\forall g \in G)(\exists i_0 \in J)(\forall i < i_0)(\exists j \in J)(Y_i^g = Y_j \wedge g^{-1}H_i g = H_j)$;

(viii) for any $x \in X$, G_x preserves a C-relation on $X \setminus \{x\}$.

We say that G preserves a *limit of betweenness relations* if the same condition holds, except that in (ii), 'D-relation' is replaced by 'betweenness relation'.

We suspect this definition is not optimal, and in particular that the *sequence* of sets Y_i indexed by J should be replaced by a G*-invariant* collection of sets Y_i which is semilinearly ordered under inclusion – as happens for the constructions in this paper. As commented in [7, Problem 6.2], the problem then is to show that the classification of primitive Jordan groups from [3] still holds with this modified definition.

Since it is analogous to the above, and will be needed in Section 4, we also give the definition of the notion *limit of Steiner systems* which also occurs in the main theorem of [3].

Definition 1.2.2. *([3], Definition 2.1.10, with notation changed)* If (H, X) is an infinite Jordan group we say that H preserves a *limit of Steiner systems* on X if for some $n > 2$, (H, X) is n-transitive but not $(n+1)$-transitive, and there is a totally ordered index set $(K, \leq)$ with no greatest element, and an increasing chain $(S_k : k \in K)$ of subsets of X such that:

(i) $\bigcup(S_k : k \in K) = X$;

(ii) for each $k \in K$, $H_{\{S_k\}}$ is $(n-1)$-transitive on S_k and preserves a non-trivial Steiner $(n-1)$-system on S_k;

(iii) if $i < k$ then S_i is a subset of a block of the $H_{\{S_k\}}$-invariant Steiner $(n-1)$-system on S_k.

(iv) for all $g \in H$ there is $i_0 \in K$, dependent on h, such that for every $i > i_0$ there is $k \in K$ such that $S_i^h = S_k$ and the image under h of every block of the Steiner system on S_i is a block of the Steiner system on S_k;

(v) for every $k \in K$, the set $X \setminus S_k$ is a Jordan set for (H, X).

Next, we give an informal description of the structure $\mathcal{M}_B$ from [9] – see e.g. Proposition 5.3. The structure may be viewed as of form $\mathcal{M}_B^0$ having a single ternary relation L, though it is in fact built by an amalgamation construction in a rather richer language, so is called $\mathcal{M}_B^1$ in our notation here. We describe $\mathcal{M}_B$ in terms of relations preserved by the automorphism group. Let $G := \mathrm{Aut}(\mathcal{M}_B)$.

First, there is (in $\mathcal{M}_B^{\mathrm{eq}}$) a 0-interpretable dense lower semilinear order $(J, <)$ which is a meet-tree; this is a partial order such that for all $a \in J$, $\{x \in J : x < a\}$ is totally ordered by $<$, and such that any two elements (called 'vertices') of J have a greatest lower bound. For each $a \in J$ the set C_a of 'cones' at a is infinite. (Here a *cone* is an equivalence class of the equivalence relation ρ_a on $\{x \in J : a \leq x\}$ whereby $\rho_a xy$ if and only if there is $z \in J$ with $a < z \wedge z < x \wedge z < y$.) The set J is an index set for a G-invariant family $(X_j : j \in J)$ of subsets of M_B, with $X_j \supset X_k$ whenever $j < k$, for $j, k \in J$. We have $\bigcup_{j \in J} X_j = M_B$ and $\bigcap_{j \in J} X_j = \emptyset$.

The group G_j (the stabiliser of j) fixes X_j setwise and preserves a j-definable equivalence relation $\sim_j$ on X_j. There is a j-definable betweenness relation B_j on $Z_j := X_j/{\sim_j}$. This is *dense* in the sense that for any distinct $x, y \in Z_j$ there is $z \in Z_j \setminus \{x, y\}$ with $B_j(z; x, y)$. It has *positive type*, that is, satisfies the axiom (B7) of [6, Ch. 15], namely

$$(\forall x, y, z)(\exists w)(B_j(w; x, y) \wedge B_j(w; x, z) \wedge B_j(w; y, z)).$$

If $a \in Z_j$ there is an equivalence relation $\equiv_a$ on $\{z \in Z_j : z \neq a\}$, whereby, for distinct $x, y \in Z_j \setminus \{a\}$ we have $x \equiv_a y$ if and only if $\neg B_j(a; x, y)$. We shall call the $\equiv_a$-classes *branches* at a (they are called *components* in [6]), and write W_a for the set (which is infinite) of branches at a.

For $a \in X_j$ the set $\{x \in M_B : a \sim_j x\}$ is called a *pre-node* of Z_j, or *pre-node* at j, and denoted by $[a]_j$, or $[a]$ when j is clear. The $\sim_j$-class containing a when viewed as an imaginary is called a *node* of Z_j, and is denoted by $a/{\sim_j}$. Slightly abusing notation, we shall use the word 'branch' both for a set in Z_j and for the corresponding subset of M_B.

There is for each $i \in J$ an i-definable bijection f_i from the set A_i of cones (of the semilinear order $(J, <)$) at i to the set Z_i. This puts i-definably the betweenness structure of Z_i onto the set of cones (of $(J, <)$) at i. For $i < j_0$ let $C_i(j_0)$ be the cone of $(J, <)$ at i containing j_0. If $[a]$ is a pre-node at j_0 then there is a unique set U of branches at the node $f_i(C_i(j_0))$ of Z_i such that $[a]$ equals the union of the branches corresponding to members of U. We shall write $g_{ij_0}(a) = U$; for distinct pre-nodes $[a], [b]$ at j_0 we have $g_{ij_0}(a) \cap g_{ij_0}(b) = \emptyset$. In particular, it follows that if $i < j_0$ then $\bigcup\{g_{ij_0}(a) : [a]$ prenode at $j_0\}$ is a union of branches at $f(C_i(j_0))$, and is denoted by $g_i(j_0)$. These maps ensure that, in an iterative way, the betweenness structure of the set Z_{j_0} is imposed definably on (a quotient of a subset of) the set of branches at the node $f_i(C_i(j_0))$ of Z_i. Thus, 'higher' sets Z_{j_0} impose structure on 'lower' sets Z_i. The maps g_{ij} compose in a natural way: if $i, j_0, j_1 \in J$ with $i < j_0 < j_1$ and $b \in X_{j_1}$, then

$$g_{ij_1}(b) = \bigcup\{g_{ij_0}(x) : x \in \bigcup\bigcup g_{j_0 j_1}(b)\}.$$

This gives essentially all the structure. The relation L is interpreted in $\mathcal{M}_B$ as follows: $\mathcal{M}_B \models L(x; y, z)$ if and only if there is $j \in J$ such that $x/{\sim_j}, y/{\sim_j}, z/{\sim_j}$ are distinct elements of Z_j and the relation $B_j(x/{\sim_j}; y/{\sim_j}, z/{\sim_j})$ holds. This ensures that $L(x; y, z) \leftrightarrow L(x; z, y)$ holds, and that $L(x; y, z) \rightarrow \neg L(y; x, z)$. It is shown in [9, Section 5] that all the above structure is interpretable without parameters in $\mathcal{M}_B^0 := (M_B, L)$. For example, J is identified with the quotient of a 0-definable subset of M_B^3 by a 0-definable equivalence relation. (It should be mentioned that there was an error in the presentation in [9], pointed out to us by the authors of [11] and corrected for the construction of $\mathcal{M}_D$ – with a correction for $\mathcal{M}_B$ also mentioned before the lemma – in [7, Lemma 4.10]; similarly there is an inaccuracy in [9] in the proof of Proposition 1.2.3(4) below, corrected for $\mathcal{M}_D$ and implicitly for $\mathcal{M}_B$ in [7, Lemma 5.7].)

The structure $\mathcal{M}_B^1$ is constructed by Fraïssé amalgamation of a class of finite structures in a richer finite relational language (though this class is not closed under substructure, and $\mathcal{M}_B^1$ is not homogeneous and so does not have quantifier elimination in this language – see Example 1.3.1). It follows from the amalgamation construction that $\mathcal{M}_B^1$ is ω-categorical – see [9, Section 4]. Furthermore, $\mathrm{Aut}(\mathcal{M}_B^0) = \mathrm{Aut}(\mathcal{M}_B^1)$.

The basic symmetry properties of $\mathcal{M}_B$ are described below.

Proposition 1.2.3. *Let* $G = \mathrm{Aut}(\mathcal{M}_B^1)$. *Then the following hold.*

(1) *G is 2-primitive and 3-homogeneous but not 3-transitive on M_B, and is transitive on the set* $\{(x, y, z) \in M_B^3 : \mathcal{M}_B^1 \models L(x; y, z)\}$;

(2) *G is transitive on the set* $\{(i, j) \in J^2 : i < j\}$;

(3) *for each $j \in J$, G_j is transitive on the set of triples* $\{(x, y, z) : B_j([z]; [y], [z])$ *holds*$\}$, *and induces a 2-transitive Jordan group on Z_j in which branches are Jordan sets;*

(4) *each pre-node and branch (viewed as subsets of M_B) is a Jordan set for G, as is each set X_j;*

(5) *the stabiliser in G of any 3-set $\{x, y, z\}$ induces C_2 on $\{x, y, z\}$ and fixes x if $L(x; y, z)$ holds (in particular, there is a unique choice of first element such that L holds of the triple).*

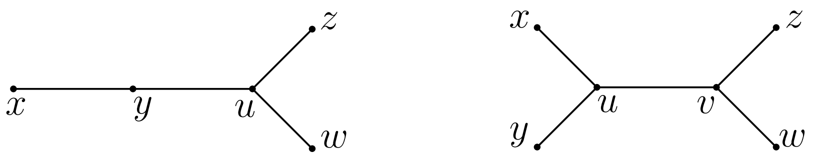

Figure 1.1. The configurations witnessing $N(x, y; z, w)$ on the left and $S(x, y; z, w)$ on the right.

All the symmetry properties apart from those involving Jordan groups follow just from the homogeneity properties ensured by the Fraïssé construction. The Jordan condition is more subtle, as it involves extending partial isomorphisms with infinite domain, as the complement of the Jordan sets arising is infinite. One first shows that any pre-node $[a]$ is a Jordan set, essentially by showing that a certain group embeds in G which fixes the complement of $[a]$ in M_B and acts on $[a]$ as an iterated wreath product in the manner, for example, of [16, Section 6]. The point here is essentially that if $[a]$ is a pre-node at j_0 and $i < j_0$, then because $[a]$ is the union of a set U of branches at $f(C_i(j_0))$, there are two $ij_0a/{\sim_{j_0}}$-definable equivalence relations E_i and F_i on $[a]$. The F_i-classes are the branches (as subsets of M_B) at $f(C_{j_0}(i))$ contained in U, and the E_i-classes are the pre-nodes at i contained in U. In particular, E_i refines F_i and for $i < k < j_0$ we have that $F_k \supset E_k \supset F_i \supset E_i$. There is an a/E_{j_0}-definable C-relation $C_{[a]}$ on $[a]$, with $C_{[a]}(x; y, z)$ holding if either $y = z \neq x$ or x, y, z are distinct and for some $i < j_0$, some F_i-class contains y, z and omits x.

In [9], the actual amalgamation construction is carried out in a richer language $\mathcal{L}_B^1$ with, as well as the ternary relation symbol L, three arity 4 relation symbols L', N, S. In $\mathcal{M}_B^1$, $L'(x; y, z; w)$ holds if there is $j \in J$ (which will necessarily be unique) such that $w \notin X_j$ and x, y, z are $\sim_j$-inequivalent elements of X_j with $B_j(x/{\sim_j}, y/{\sim_j}, z/{\sim_j})$ holding. We have $\mathcal{M}_B^1 \models N(x, y; z, w)$ if there is $u \in M_B$ and some $j \in J$ with x, y, z, w, u all $\sim_j$-inequivalent such that their $\sim_j$-classes carry the structure in (Z_j, B_j) depicted in Figure 1.1. Likewise, $\mathcal{M}_B^1 \models S(x, y; z, w)$ if there are $u, v \in \mathcal{M}_B$ and some $j \in J$ with x, y, z, w, u, v all $\sim_j$-inequivalent and as depicted in Figure 1.1 in (Z_j, B_j).

For the structure $\mathcal{M}_D$, essentially the same assertions hold, except that the ternary betweenness relation B_j on Z_j is replaced by an (arity four) D-relation D_j, and pre-nodes are replaced by 'pre-directions'. There are some differences in the proof, and also some inaccuracies in the proof in [9] are corrected in [7].

A D-set (N, D) can be viewed as a set of 'directions' of an underlying (and interpretable) B-set (P, B), where P is a 0-interpretable quotient of N^3. We shall refer to the elements of a B-set as 'nodes', and talk of 'branches' at a node as above, except that now a branch at a node a of such P may be viewed either as a subset of P, or (in a natural way) as a subset of N. Given distinct $x, y, z \in N$, we write $\mathrm{ram}(x, y, z)$ for the unique node a of P such that x, y, z lie in different branches at a, and refer to such nodes as ‘ramification points’. The structure $\mathcal{M}_D$ has an interpretable semilinear order $(J, \leq)$ indexing a family $(X_j : j \in J)$ of subsets of $\mathcal{M}_D$ semilinearly-ordered by reverse inclusion, again with $\bigcup_{j\in J} X_j = \mathcal{M}_D$ and $\bigcap_{j\in J} X_j = \emptyset$. Again, there is a j-definable equivalence relation $\sim_j$ on X_j, and $Z_j = X_j/{\sim_j}$ carries the structure of a D-relation D_j. However, there is an extra phenomenon: if (P_j, B_j) is the betweenness relation arising as above from the D-set (Z_j, D_j), and $a \in P_j$, then there is a unique (a, j)-definable branch U_j at a, called the ‘special branch’ at a. As with $\mathcal{M}_B$, there is for each $i \in J$ a (uniformly in i) i-definable bijective map f_i from the set of cones at i (in $(J, \leq)$) to the set of ramification points of the D-set (Z_i, D_i). Also as for $\mathcal{M}_B$, for each $j > i$ in J with j in the cone C at i, there is an ij-definable map g_{ij} from Z_j to sets of non-special branches at node $f_i(C)$ of the D-set on Z_i, with the sets of branches $g_{ij}(a)$ and $g_{ij}(a')$ disjoint for distinct nodes a, a' of Z_j. We have that for $a \in Z_j$, the subset $a/{\sim_j}$ of M_D is exactly the union of the $\sim_i$-classes of the elements of Z_i lying in the union of the branches in $g_{ij}(a)$.

In [7] the amalgamation construction of $\mathcal{M}_D$ takes place in a language $\mathcal{L}^1_D$ which has a ternary relation symbol L, relation symbols L', S of arity 4, and S' of arity 5, R of arity 6, and Q of arity 7. In the structure $\mathcal{M}^1_D$ in this language, we write $L(x; y, z)$ if there is $j \in J$ with $x, y, z \in X_j$ inequivalent modulo $\sim_j$ such that $x/{\sim_j}$ lies in the special branch at $\mathrm{ram}(x/{\sim_j}, y/{\sim_j}, z/{\sim_j})$; such j will be unique, and we put $L'(x; y, z; w)$ if in addition such X_j does not contain w. We put $\mathcal{M}^1_D \models S(x, y; z, w)$ if there is $j \in J$ (again unique) with x, y, z, w distinct in X_j modulo $\sim_j$, and with $D_j(x/{\sim_j}, y/{\sim_j}; z/{\sim_j}, w/{\sim_j})$ holding; we have $\mathcal{M}^1_D \models S'(x, y; z, w; u)$ if the above holds and in addition $u \notin X_j$. In the above setting, we say that $L(x; y, z)$ and $S(x, y; z, w)$ are *witnessed* in the D-set (Z_j, D_j). Now $\mathcal{M}^1_D \models R(x; y, z : u; v, w)$ if $L(x; y, z)$ and $L(u; v, w)$ hold and are witnessed in the same D-set, and likewise $Q(x, y; z, w : p; q, s)$ holds if $S(x, y; z, w)$ and $L(p; q, s)$ hold and are witnessed in the same D-set.

In [7, Lemma 4.1], it is shown that L', S', Q, R are 0-definable in (M_D, L, S). In fact, we have the following lemma, which yields that we may view $\mathcal{M}_D$ as a structure, denoted $\mathcal{M}^0_D$, with just the single ternary relation symbol L.

Lemma 1.2.4. *In the structure $\mathcal{M}_D$, the relation S is definable without parameters just from the relation L.*

Proof. We refer to Figure 1.2 for illustrations of several of the configurations in this proof. Let $\chi(u, v, w, z)$ be the formula

$$L(u; v, w) \wedge L(u; v, z) \wedge L(w; z, u) \wedge L(w; z, v).$$

Also let $\psi(u, v, w, z)$ be the formula

$$(L(v; w, z) \wedge L(u; v, w) \wedge L(u; v, z) \wedge L(u; w, z))\wedge$$
$$\wedge\exists t(L(t; u, w) \wedge L(t; u, z) \wedge L(t; v, w) \wedge L(t; v, z) \wedge L(u; v, t) \wedge L(t; w, z)).$$

It suffices to check that if $u, v, w, z \in M_D$, then $\mathcal{M}_D \models S(u, v; w, z)$ if and only if the following formula holds.

$$\begin{aligned}(u, v, w, z \text{ are distinct }) \wedge ([L(u; w, z) \wedge L(v; w, z) \wedge L(w; u, v) \wedge L(z; u, v)]\vee\\ \vee[\chi(u, v, w, z) \vee \chi(u, v, z, w) \vee \chi(v, u, w, z) \vee \chi(v, u, z, w)]\vee\\ \vee[\psi(u, v, w, z) \vee \psi(v, u, w, z) \vee \psi(w, z, u, v) \vee \psi(z, w, u, v)].\end{aligned}$$

We omit the details, but the idea here is to take account of different possible special branch conditions in D-sets where $S(u, u; w, z)$ holds, or in higher D-sets. The first disjunct in the above formula holds when, in the D-set witnessing $S(u, v; w, z)$ or in a higher D-set, the branch containing w is special at $\mathrm{ram}(u, v, w)$ and the branch containing u is special at $\mathrm{ram}(u; w, z)$. The formula $\chi(u, v, w, z)$ describes quadruples (satisfying $S(u, v; w, z)$) such that the branch containing u at $\mathrm{ram}(u, v, z)$ is special (possibly in a higher D-set), as is the branch containing w at $\mathrm{ram}(w, z, u)$. The formula $\psi(u, v, w, z)$ describes quadruples satisfying $S(u, v; w, z)$ with the branches containing u at $\mathrm{ram}(u, v, z)$ and at $\mathrm{ram}(u, w, z)$ special (possibly in higher D-sets). Note here that

$$L(v; w, z) \wedge L(u; v, w) \wedge L(u; v, z) \wedge L(u; w, z)$$

could also hold for a quadruple satisfying no S-relation in any ordering, but the existential quantifier rules this out. ■

The analogue of Proposition 1.2.3 for $\mathcal{M}_D$ is the following.

Proposition 1.2.5. *Let $G = \mathrm{Aut}(\mathcal{M}_D^1)$. Then the following hold.*

(1) *G is 2-primitive and 3-homogeneous but not 3-transitive on M_D, and is transitive on the set $\{(x, y, z) \in M_D^3 : \mathcal{M}_D^1 \models L(x; y, z)\}$;*

(2) *G is transitive on the set $\{(i, j) \in J^2 : i < j\}$;*

(3) *for each $j \in J$, G_j is transitive on the set $\{(x, y, z) : L(x; y, z)$ holds witnessed in $Z_j\}$ of triples, and induces a 2-transitive Jordan group on Z_j in which branches are Jordan sets;*

(4) *each pre-direction and branch (as a subset of M_D) is a Jordan set for G, as is each set X_j;*

(5) *the stabiliser in G of any 3-set $\{x, y, z\}$ induces C_2 on $\{x, y, z\}$ and fixes x if $L(x; y, z)$ holds (in particular, there is a unique choice of first element such that L holds of the triple).*

To obtain the description of a limit of D-sets in Definition 1.2.1, we replace J by a (not G-invariant) coinitial sub-chain. The Y_i and σ_i of 1.2.1 are then the corresponding Z_i and $\sim_i$.

We comment briefly on the amalgamation constructions of $\mathcal{M}_B^1$ and $\mathcal{M}_D^1$. In the case of $\mathcal{M}_B^1$, a certain class $\mathcal{B}$ of finite $\mathcal{L}_B^1$-structures, namely ‘trees of B-sets’ is

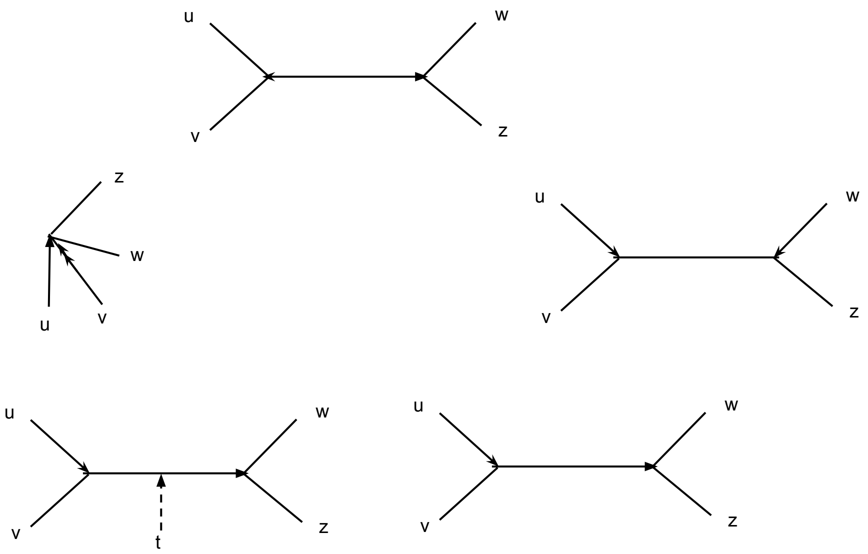

Figure 1.2. Figures relevant for Lemma 1.2.4 Clockwise from the top: the disjunct in the first line of the formula defining S, the formula χ, two figures for the two conjuncts of ψ, and the case where no S-relation holds (with the double arrow indicating the branch is special in a higher D-set).

identified. We refer to [9] for details, but roughly, such a structure is isomorphic to a finite substructure K of $\mathcal{M}_B^1$ such that the finite B-sets have ‘positive type’, that is, if x, y, z are distinct elements of such a B-set then there is $t \in K$ in the B-set lying between any two of x, y, z. It is shown that the class $\mathcal{B}$ has the amalgamation property, and $\mathcal{M}_B^1$ is the Fraïssé limit. The construction of $\mathcal{M}_D^1$ is similar, obtained by amalgamating a class $\mathcal{D}$ of finite $\mathcal{L}_D^1$-structures. The members of $\mathcal{D}$ are isomorphic to finite substructures K of $\mathcal{M}_D^1$ such that any node of any (finite) D-set has a special branch within the structure K. See [7, Section 3] for a detailed description of $\mathcal{D}$. It follows from the construction of $\mathcal{M}_B^1$ that any isomorphism between substructures of $\mathcal{M}_B^1$ which lie in $\mathcal{B}$ extends to an automorphism of $\mathcal{M}_B^1$. The corresponding assertion holds for $\mathcal{M}_D^1$.

1.3 Proofs of Theorem 1.1.1 and Theorem 1.1.3

We first prove the non-homogeneity assertion in Theorem 1.1.1(i), through the following two examples. The essential point is that though the structures are Fraïssé limits of classes of finite structures with the amalgamation property, these classes are not closed under substructure.

Example 1.3.1. In the structure $\mathcal{M}^1_D$, consider the set $C = \{x, y, z, w, u\}$ and $C' = \{x', y', z', w', u'\}$ and consider the two structures A and A' depicted in Figures 1.5 and 1.6 respectively, with $C < A$ and $C' < A'$. We may view A and A' as substructures of $\mathcal{M}^1_D$. It can be checked that the only $\mathcal{L}^1_D$-relations holding on C and C' are L and S, and that the map $h : (x, y, z, w, u) \longrightarrow (x', y', z', w', u')$ is an isomorphism. However, h does not extend to an automorphism of $\mathcal{M}^1_D$, since the relations $S(z, y; u, x)$ and $S(z, w; x, y)$ are witnessed in different D-sets in $\mathcal{M}^1_D$ whereas $S(z', y'; u', x')$ and $(z', w'; x', y')$ are witnessed in the same D-set. In the diagrams, which adopt the same conventions as [7, Figure 2], the arrow indicates which branch is special at each node.

A similar construction shows that $\mathcal{M}^1_B$ is not homogeneous, but we omit the details. Figures 1.3 and 1.4 illustrate two configurations for (a, b, c, d, e, f) in $\mathcal{B}^1$ which are isomorphic but not in the same orbit.

Next, we prove homogenizability of $\mathcal{M}^1_D$. We define a relational language $\mathcal{L}^2_D$ with the relation symbols L and S interpreted on M_D as before, but, in place of the symbols L', S', Q, R of L_D, the language $\mathcal{L}^2_D$ has symbols P of arity 6, $Q^{\leq}$ and $Q^{\geq}$ of arity 7, and T of arity 8, interpreted on M_D as follows: we have $P(x; y, z : p; q, s)$ if $L(x; y, z)$ holds witnessed at i, $L(p; q, s)$ holds witnessed at j, and $i \leq j$. Likewise, $T(x, y; z, w : p, q; s, t)$ holds if $S(x, y; z, w)$ holds witnessed at i, $S(p, q; s, t)$ holds witnessed at j, and $i \leq j$. Similarly $Q^{\leq}(x, y; z, w : p; q, s)$ if $S(x, y, z, w)$ is witnessed at i and $L(p; q, s)$ is witnessed at j with $i \leq j$, and $Q^{\geq}(x, y; z, w : p; q, s)$ if $S(x, y, z, w)$ is witnessed at i and $L(p; q, s)$ is witnessed at j with $i \geq j$.

Lemma 1.3.2. (1) $\mathrm{Aut}(\mathcal{M}^1_D) = \mathrm{Aut}(\mathcal{M}^2_D)$.

(2) *The $\mathcal{L}^2_D$-structure $\mathcal{M}^2_D$ is homogeneous.*

Proof. (1) It is clear from our description above of $\mathcal{M}^1_D$ that $G = \mathrm{Aut}(\mathcal{M}^1_D)$ preserves $Q^{\leq}, Q^{\geq}, P$ and T. For the converse, observe that for distinct $x, y, z, t \in M_D$, $L'(x; y, z; t)$ holds if and only if $L(t; y, z)$ is witnessed in a D-set strictly below that witnessing $L(x; y, z)$, and this information is given by P. Likewise, $S'(x, y; z, w; t)$ holds if and only if $Q^{\geq}$ holds with (x, y, z, w) in the first four entries and (t, z, w) in the last three. Finally, we have $R(x; y, z; p; q, s)$ if and only if $P(x; y, z : p; q, s)$ and $P(p; q, s : x; y, z)$ both hold, and $Q(x, y; z, w : p; q, s)$ holds if and only if $Q^{\geq}$ and $Q^{\leq}$ both hold of (x, y, z, w, p, q, s).

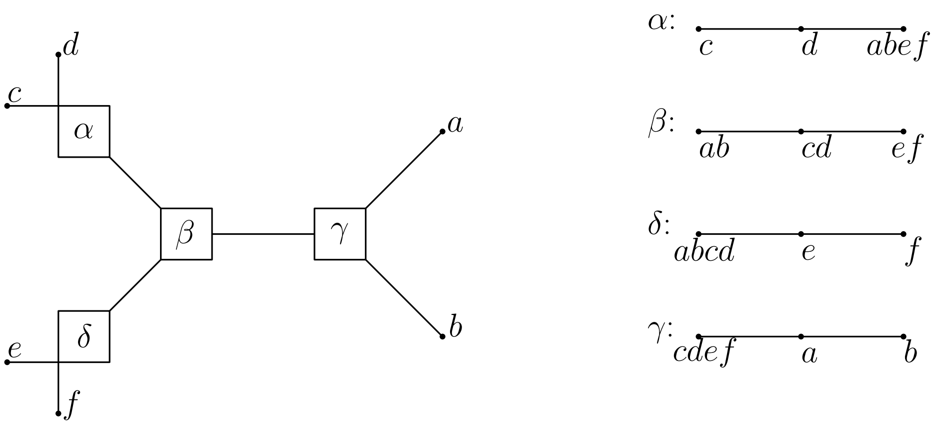

Figure 1.3. The first of two isomorphic substructures of $\mathcal{M}^1_B$ in different orbits.

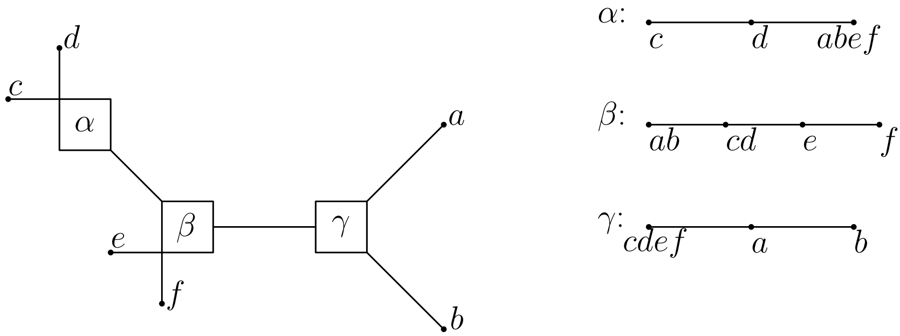

Figure 1.4. The second of two isomorphic substructures of $\mathcal{M}^1_B$ in different orbits.

(2) It suffices by the partial homogeneity of $\mathcal{M}^1_D$ to prove the following assertion. Let A_1, A_2 be finite subsets of M_D and $h : A_1 \to A_2$ an isomorphism of the induced substructures of M^2_D. Then there are finite $A_1' \supseteq A_1$ and $A_2' \supseteq A_2$ such that h extends to an isomorphism between the substructures of $\mathcal{M}^1_D$ induced on A_1' and A_2' respectively, and these $\mathcal{L}^1_D$-structures on A_1' and A_2' lie in $\mathcal{D}$. We build A_1' and A_2' as follows. Using P, $Q^{\leq}$, $Q^{\geq}$, and T, we may identify which tuples satisfying S or L are witnessed in A_1 and A_2 respectively at the lowest level (indexed in J by i_1 and i_2 respectively). These determine a 'lowest' D-set of A_1 and A_2 respectively, and h induces an isomorphism of these D-sets, and hence of the corresponding B-sets interpretable in them. Using $Q^{\leq}, Q^{\geq}$ we may also identify which L-relations are witnessed in these lowest D-sets, and hence identify the special branches at certain nodes. The

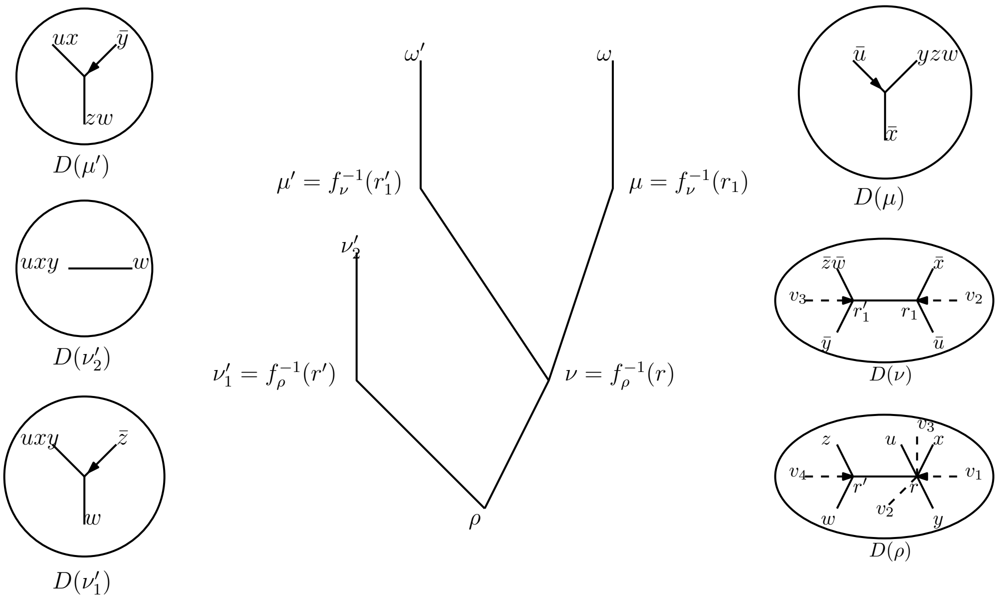

Figure 1.5. The structure $A \subset \mathcal{M}_D^1$.

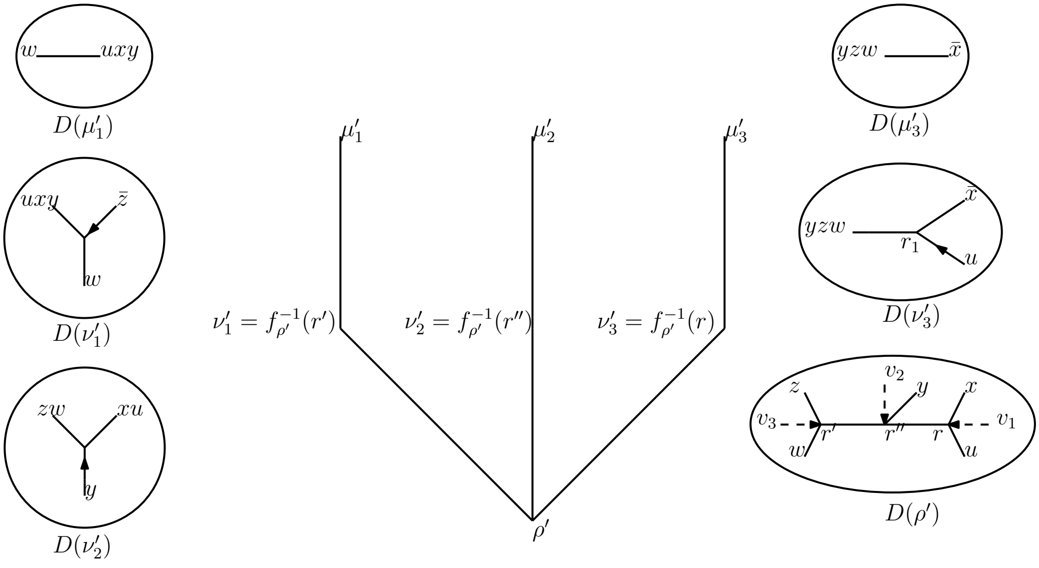

Figure 1.6. The structure $A' \subset \mathcal{M}_D^1$.

isomorphism h induces a bijection between the nodes of A with no special branches (in this lowest D-set) and the corresponding nodes of A_2. We add to A_1 and A_2 a direction (in the special branch) at each of these nodes. The map h extends to these expanded structures. We now continue inductively. For each point a of the B-set induced by A_1 at level i_1, h induces a map from the cone at i_1 corresponding to a to the cone at $h(i_1)$ corresponding to $h(a)$. We may identify those tuples of A_1 which satisfy S or L witnessed at the lowest level within this cone at i_1, thereby identifying a D-set of A_1 directly above the lowest level, and h maps this isomorphically to the corresponding D-set of A_2. Again, we may add elements to A_1 and A_2 to ensure that each node of the corresponding B-sets has a special branch. Repeating this argument, we eventually obtain an isomorphism $h : A_1' \to A_2'$, as claimed.

■

We give without details a similar result for $\mathcal{M}_B$, namely the following.

Lemma 1.3.3. *The structure $\mathcal{M}_B^1$ is homogenizable.*

Proof. We just give a sketch. We replace the original language $\mathcal{L}_B^1$ by a language $\mathcal{L}_B^2$ which has the relation symbols, L (arity 3), N and S (both arity 4) but not L', and also has relation symbols which express that if the B-set indexed by i witnesses L, N or S for some tuple, and the B-set indexed by j witnesses L, N or S for some other tuple, then $i \leq j$ (these all have arity at most 8). It can be checked that these relations are all 0-definable in $\mathcal{M}_B^1$ and that L' is 0-definable from them. Furthermore, given a finite subset A of M, we can identify the lowest B-set of $\mathcal{M}_B^1$ in which a tuple from A satisfies one of L, N or S, and which tuples satisfy such relations in this B-set. Using these relations, we may uniquely reconstruct this lowest B-set restricted to A. It may not have positive type, i.e. may miss certain ramification points, but we can add these in a canonical manner. We proceed inductively to embed A canonically in a substructure of $\mathcal{M}_B^1$ lying in $\mathcal{B}$. Now if $f : A_1 \to A_2$ is an isomorphism of finite substructures of $\mathcal{M}_B^2$, then f extends to an isomorphism $f' : A_1' \to A_2'$ of finite substructures of $\mathcal{M}_B^1$ lying in $\mathcal{B}$, and the homogeneity properties given by the amalgamation construction of $\mathcal{M}_B^1$ ensure that f' extends to an automorphism of $\mathcal{M}_B^1$. ■

Before proving that $\mathcal{M}_B$ and $\mathcal{M}_D$ are NIP, we shall obtain a description of indiscernible sequences of singletons, useful also for later results.

Lemma 1.3.4. *Let $I = (a_i : i \in \omega)$ be an indiscernible sequence of singletons in $\mathcal{M}_D$. Then one of the following holds.*

(i) *I is a D-*set indiscernible*; that is, there is $j \in J$ indexing the D-set (Z_j, D_j) such that if $a_i^* := a_i/{\sim_j}$ for each $i \in \omega$, then the a_i^* are in Z_j as depicted in Figure 1.7. That is, for any $p < q < r < s$ we have $S(a_p, a_q; a_r, a_s)$ witnessed in Z_j.*

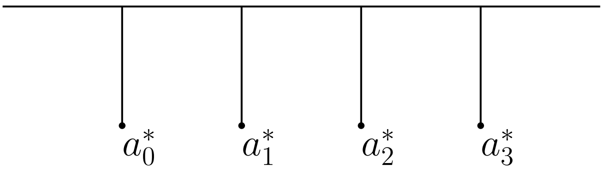

Figure 1.7. A D-set indiscernible sequence.

(ii) *I is an S-free* indiscernible, *meaning that no quadruple from I satisfies S. Now one of:*
(a) I is upwards S-free*; that is, there is an increasing sequence $(n_i : i \in \omega)$ in J such that for any $i_1, i_2, i_3 \in \omega$ with $i_1 < i_2 < i_3$, the relation $L(a_{i_1}; a_{i_2}, a_{i_3})$ holds and is witnessed at n_{i_1}. We have $a_j \notin X_{n_i}$ for $j < i$.*
(b) I is downwards S-free*; that is, there is a decreasing sequence $(n_i : i \in \omega, i \geq 3)$ in J such that for any $i_1, i_2, i_3 \in \omega$ with $i_1 < i_2 < i_3$, the relation $L(a_{i_3}; a_{i_1}, a_{i_2})$ holds and is witnessed at n_{i_3-2}. We have $a_j \notin X_{n_i}$ for $j > n_i + 2$.*

(iii) *I is a* mixed indiscernible*; that is, there is a decreasing sequence $n_1 > n_2 > \ldots$ in J such that for each $i \geq 1$ we have: $a_0, \ldots, a_{i+1}$ all lie in distinct branches at the vertex v_i of Z_{n_i}, $a_j \sim_{n_i} a_{i+2}$ for $j > i+2$, and $S(a_p, a_q; a_{i+1}, a_r)$ is witnessed in Z_{n_i} for all $r > i+1$ and distinct $p, q \leq i$.*

Note that in the above cases some information is omitted – e.g. in Case (i), for D-set-indiscernibles, we do not specify the L-relations. Also, (ii)(a) and (b) are very similar: if we consider such an indiscernible sequence indexed by $\mathbb{Q}$, then for each $i \in \mathbb{Q}$ there is a corresponding D-set Z_i with vertex v_i such that one of the following holds: either (in (a)) we have $a_j \notin X_i$ for all $j < i$ and for $k \geq i$ the a_k all lie in distinct branches at v_i with the branch of a_i special; or (in (b)) we have $a_j \notin X_i$ for $j > i$ and the a_k for $k \leq i$ all lie in distinct branches at v_i, again with the branch of a_i special.

Proof. The argument is elementary but there are many cases to consider, so we only sketch the idea.

Exactly one of $L(a_0; a_1, a_2)$, $L(a_2; a_0, a_1)$ or $L(a_1; a_0, a_2)$ holds (see part (5) of Proposition 1.2.5). We shall suppose this relation is witnessed in the D-set Z_{n_0}, with $\mathrm{ram}(a_0, a_1, a_2) = v_0$. Thus, by indiscernibility exactly one of the following three cases holds.

Case 1. Whenever $i, j, k \in \omega$ are distinct with $i < \mathrm{Min}\{j, k\}$ we have $L(a_i; a_j, a_k)$.

Case 2. Whenever $i, j, k \in \omega$ are distinct with $k > \mathrm{Max}\{i, j\}$ we have $L(a_k; a_i, a_j)$.

Case 3. Whenever $i < j < k$ or $k < j < i$ we have $L(a_j; a_i, a_k)$.

Case 1a. Suppose $L(a_0; a_1, a_2)$ is witnessed in Z_{n_0} and a_0, a_1, a_2, a_3 lie in distinct branches at v_0. Then automorphisms inducing $(a_0, a_1, a_2, a_3) \mapsto (a_0, a_1, a_2, a_i)$ for $i >$

3 show that $L(a_0; a_1, a_i)$ is witnessed in Z_{j_0}, and maps $(a_0, a_1, a_2, a_3) \mapsto (a_0, a_1, a_i, a_j)$ (for $3 \leq i < j$) show that all the a_i are $\sim_{n_0}$ inequivalent and lie in distinct branches at the same vertex v_0 of Z_{n_0}. Using indiscernibility, there is $n_1 > n_0$ such that the same configuration holds in Z_{n_1} (but with a_0 omitted, and with a_1 in the special branch at the central vertex v_1). Now iterate this argument, to obtain that I is an upwards S-free indiscernible.

Case 1b. $a_3 \notin X_{n_0}$. In this case, $L(a_3; a_0, a_1)$ is witnessed in a D-set below Z_{n_0}, contrary to the assumption that $L(a_0; a_1, a_2)$ holds (see Proposition 1.2.5(5)).

Case 1c. a_3 lies in the same branch at v_0 in Z_{n_0} as a_0. In this case $L(a_3; a_1, a_2)$ holds witnessed in Z_{n_0}, contradicting $L(a_1; a_2, a_3)$.

Case 1d. $a_1 \nsim_{n_0} a_3$, and a_3 lies in the same branch at v_0 in Z_{n_0} as a_1. In this case the D-set which witnesses $L(a_1; a_2, a_3)$ must also witness $L(a_1; a_0, a_3)$ contradicting that $L(a_0; a_1, a_3)$ holds by indiscernibility.

Case 1e. $a_1 \sim_{n_0} a_3$. Now a D-set below Z_{n_0} will have the D-set configuration of Case 1d and is eliminated for the same reason (the D-set which witnesses $L(a_1; a_2, a_3)$ must also witness $L(a_1; a_2, a_0)$, contrary to $L(a_0; a_1, a_2)$).

Case 1f. $a_3 \nsim_{n_0} a_2$ and a_3 lies in the same Z_{n_0} branch as v_0 as a_2. Now Z_{n_0} witnesses both $S(a_0, a_1; a_2, a_3)$ and $L(a_0; a_1, a_2)$ and $L(a_0; a_1, a_3)$. Hence an automorphism extending $(a_0, a_1, a_2, a_3) \mapsto (a_0, a_1, a_3, a_4)$ ensures that for distinct $p, q \in \{0, 1, 2\}$, $S(a_p, a_q; a_3, a_4)$ holds, witnessed in Z_{n_0}, and iteration of the argument shows that I is a D-set indiscernible.

Case 1g. $a_3 \sim_{n_0} a_2$. Now for $i > 3$ an automorphism inducing $(a_0, a_1, a_2, a_3) \mapsto (a_0, a_1, a_2, a_i)$ fixes the D-set witnessing $L(a_0; a_1, a_2)$, so ensures that $a_2 \sim_{n_0} a_i$ holds. Now there is $n_1 < n_0$ in J and vertex v_1 in Z_{n_1} such that a_0, a_1, a_2 are in distinct branches at v_1, and a_3 in the same branch as a_2 at v_1, with $a_3 \nsim_{n_1} a_2$. Now if $L(a_0; a_2, a_3)$ is witnessed in this D-set, then the map $(a_0, a_1, a_2, a_i) \mapsto (a_1, a_2, a_3, a_i)$ (for $i > 3$) shows that $a_i \sim_{n_1} a_3$ for $i > 3$ and I is a mixed indiscernible. On the other hand, if $L(a_0; a_2, a_3)$ is not witnessed in Z_{n_1}, then if $a_3 \sim_{n_1} a_4$ we again get a mixed indiscernible; the key point is that $S(a_0, a_1; a_2, a_3)$ is witnessed in a D-set Z_{n_1} such that $a_3 \sim_{n_1} a_4$ holds, and as the same D-set witnesses $S(a_0, a_1; a_2, a_4)$, any automorphism inducing $(a_0, a_1, a_2, a_3, a_4) \mapsto (a_0, a_1, a_2, a_4, a_5)$ fixes this D-set, ensuring $a_4 \sim_{n_1} a_5$. Otherwise (that is if $a_3 \nsim_{n_1} a_4$) we obtain a D-set indiscernible (with the relevant S-relations witnessed in Z_{n_1}).

Case 2a. a_0, a_1, a_2, a_3 lie in distinct branches at v_0. In this case $L(a_2; a_3, a_0)$ holds, contradicting that $L(a_3; a_0, a_2)$ holds by indiscernibility.

Case 2b. $a_3 \notin X_{n_0}$. Now there is $n_1 < n_0$ in J so that a_0, a_1, a_2, a_3 lie in distinct branches at vertex v_1 in Z_{n_1} with a_3 in the special branch. By indiscernibility $a_4 \notin X_{n_1}$ and we iterate the argument and obtain that I is a downwards S-free indiscernible.

Case 2c. a_3 lies in the same branch at v_0 as a_0 or a_1. This would imply $L(a_2; a_3, a_1)$ or $L(a_2; a_3, a_0)$ respectively, both of which are false by indiscernibility.

Case 2d. $a_3 \not\sim_{n_0} a_2$ but a_2, a_3 lie in the same branch at v_0. Arguments similar to Case 1f yield that I is a D-set indiscernible.

Case 2e. $a_2 \sim_{n_0} a_3$. Now I is a mixed indiscernible, as in Case 1g.

Case 3a. a_0, a_1, a_2, a_3 lie in distinct branches at v_0. This yields that $L(a_1; a_2, a_3)$ holds, contradicting that $L(a_2; a_1, a_3)$ holds by indiscernibility.

Case 3b. $a_3 \notin X_{n_0}$. In this case, $L(a_3; a_0, a_1)$ is witnessed in a D-set below X_{n_0}, contradicting $L(a_1; a_0, a_3)$.

Case 3c. a_3 lies in the same branch at v_0 as a_1. Then $L(a_3; a_0, a_2)$, contradicting $L(a_2; a_0, a_3)$.

Case 3d. a_3 lies in the same branch at v_0 as a_0. Then $L(a_1; a_2, a_3)$ holds, contradicting $L(a_2; a_1, a_3)$.

Case 3e. $a_3 \sim_{n_0} a_2$. Now I is a mixed indiscernible, much as in Case 1g.

Case 3f. $a_3 \not\sim_{n_0}$ and a_3, a_2 lie in the same branch at v_0. Arguments similar to Case 1f show that I is a D-set indiscernible.

■

The following lemma gives a similar description of indiscernible sequences of singletons in $\mathcal{M}_B$.

Lemma 1.3.5. *Let* $I = (a_i : i \in \omega)$ *be an indiscernible sequence of singletons in* $\mathcal{M}_B$. *Then one of the following holds.*

(i) *I is a B-*set indiscernible*; that is, there is a B-set* (Z_j, B_j) *such that one of the following holds.*
   (a) Z_j *witnesses that* $L(a_q; a_p, a_r)$ *whenever* $p, q, r \in \omega$ *with* $p < q < r$.
   (b) *There are* $(b_i : i \in \omega)$ *such that* Z_j *witnesses that* $L(b_q; b_p, b_r)$ *whenever* $p, q, r \in \omega$ *with* $p < q < r$, *and furthermore,* Z_j *witnesses that* $L(b_p; a_p, b_q)$ *for any* $p \neq q$ *in* ω.

(ii) *One of the following holds.*
   (a) *I is an* upwards star-sequence*; that is, there is an increasing sequence* $(n_i : i \in \omega)$ *in J such that in* Z_{n_i} *the* a_j *for* $j > i$ *lie in distinct branches at* a_i, *and* $a_k \notin X_{n_i}$ *for* $k < i$.
   (b) *I is a* downwards star-sequence*; this means that there is an decreasing sequence* $(n_i : i \in \omega)$ *in J such that in* Z_{n_i} *the elements* $a_0, \ldots a_{i+1}$ *lie in distinct branches at* a_{i+2}, *and* $a_k \notin X_{n_i}$ *for* $k > i + 2$.

(iii) *One of the following holds.*
   (a) $L(a_2; a_0, a_1)$ *and I is a* downwards mixed indiscernible*; that is, there is a decreasing sequence* $n_1 > n_2 > \ldots$ *in J such that in* Z_{n_i} *the elements* $a_0, \ldots, a_{i+1}$ *are* $\sim_{n_i}$*-inequivalent, the* a_j *are pairwise* $\sim_{n_i}$*-equivalent for* $j \geq i + 1$, *and there is a vertex* v_i *of* Z_{n_i} *such that* $a_0, \ldots a_i$ *lie in different branches at* v_i *and* a_i, a_{i+1} *lie in the same branch at* v_i, *with* a_{i+1} *between* v_i *and* a_i.

(b) I is an upwards mixed indiscernible*; that is, there is an increasing sequence* $n_1 < \ldots$ *in J, and* $a_0, \ldots, a_{i-1}$ *are* $\sim_{n_i}$*-equivalent and the* a_j *are* $\sim_{n_i}$*-inequivalent for* $j \geq i-1$*, and one of: (I)* $L(a_0; a_1, a_2)$*, there is one branch at* a_0 *containing* a_i*, and another branch at* a_0 *containing a vertex* v_i *and* a_p *for all* $p > i$*, with* $a_0, a_{i+1}, a_{i+2}, \ldots$ *all in different branches at* v_i*; or (II)* $L(a_1; a_0, a_2)$*, there is one branch at* a_i *containing* a_0*, and another branch at* a_i *containing a vertex* v_i *and* a_p *for all* $p > i$*, with* $a_0, a_{i+1}, a_{i+2}, \ldots$ *all in different branches at* v_i*; in* Z_{n_i}*.*

Again, in (ii), (a) and (b) are very similar. In the case of an indiscernible star-sequence indexed by $\mathbb{Q}$, we have $(a_i)_{i \in \mathbb{Q}}$, and for each $i \in \mathbb{Q}$ the following holds in Z_i: in (a), we have $a_j \notin X_i$ for $j < i$, and the a_k all lie in distinct branches at a_i for $k \geq i$. In (b), this is reversed.

Likewise in (iii), suppose $(a_i : i \in \mathbb{Q})$ is a mixed-indiscernible sequence. Then for each $i \in \mathbb{Q}$, one of the following holds at the vertex v_i, according to the three sub-cases. In (iii)(a) the a_j are all $\sim_i$ equivalent for $j < i$, and for $k \geq i$ the a_k all lie in distinct branches at v_i. Furthermore, for $j < i$ we have a_j between v_i and a_i. In (b), all the a_j are $\sim_i$-equivalent for $j > i$, and for $k \leq i$ the a_k all lie in distinct branches at v_i. In (b)(I) for $j > i$ we have a_j between v_i and a_i, and in (b)(II) we have a_i between v_i and v_j.

Proof. This is very similar to the proof of Lemma 1.3.4, and we omit the details. ■

Lemma 1.3.6. *The structures* $\mathcal{M}_B$ *and* $\mathcal{M}_D$ *have NIP theory.*

Proof. Again, we focus on $\mathcal{M}_D$, just mentioning how to adjust the argument for $\mathcal{M}_B$. Since we use quantifier-elimination, we work with $\mathcal{M}_D^2$, i.e. in the language with quantifier-elimination. Let $(a_i : i \in \omega)$ be a sequence of indiscernibles of singletons. Given that any finite Boolean combination of NIP formulas is NIP, it suffices to show that the atomic formulas of $\mathcal{L}_D^2$ are NIP. For example, we must show that a formula of the form $L(x; b_1, b_2)$ cannot define the set $\{a_{2i} : i \in \omega\}$, with a similar statement for the formula $L(b_1; x, b_2)$, and for the other atomic formulas of $\mathcal{L}_D^2$. All such statements are clear by inspection of the three types of indiscernible sequences. We omit the details.

In the case of $\mathcal{M}_B^2$, the statement of the Claim is unchanged, and the rest of the argument is essentially as for $\mathcal{M}_D^2$. ■

Lemma 1.3.7. *The structures* $\mathcal{M}_B$ *and* $\mathcal{M}_D$ *have distal theory.*

Proof. We apply Theorem 2.28 and Corollary 2.9 of [30], which both have a hidden assumption NIP which holds due to Lemma 1.3.6. The key point here is that Theorem 2.28 reduces us to verifying distality for indiscernible sequences of singletons, and Corollary 2.9 to working over the empty set. Thus, we must show the following, for any indiscernible sequence of singletons $(a_q : q \in \mathbb{Q})$, distinct $r, r' \in \mathbb{R} \setminus \mathbb{Q}$, and elements $a_r, a_{r'}$: if $(a_q : q \in \mathbb{Q})$ remains indiscernible (over $\emptyset$) when a_r is inserted in the appropriate place in the sequence, and also remains indiscernible when $a_{r'}$ is inserted in the appropriate place, then it is still indiscernible over $\emptyset$ when both $a_r, a_{r'}$ are inserted. It is routine to check that this holds for all the types of indiscernible sequence listed in Lemmas 1.3.4 and 1.3.5. We use the quantifier elimination which follows from the homogenizability results in Lemmas 1.3.2 and 1.3.3. We omit the details. ■

Lemma 1.3.8. *The structures $\mathcal{M}_D$ and $\mathcal{M}_B$ have trivial algebraic closure, that is, for any subset A of the domain we have* $\mathrm{acl}(A) = A$.

Proof. We work in $\mathcal{M}_D$ but the argument is the same in $\mathcal{M}_B$. We may suppose that A is finite. Suppose $a \in M_D \setminus A$ and let $A' = A \cup \{a\}$. There is a largest $j \in J$ such that the elements $x/{\sim_j}$ for $x \in A'$ are all distinct. For any k, we choose $a_1 = a, \dots, a_k \in a/{\sim_j}$ all realising $\mathrm{tp}(a/A)$. Using that pre-nodes are Jordan sets, it follows that $a \notin \mathrm{acl}(A)$.

■

Proof of Theorem 1.1.1. This follows immediately from Example 1.3.1 and Lemmas 1.3.2, 1.3.3, 1.3.6, 1.3.7 and 1.3.8. ■

Conjecture 1.3.9. *The structures $\mathcal{M}_B$ and $\mathcal{M}_D$ are dp-minimal.*

By [22, Corollary 1], to prove dp-minimality, it suffices to show that for any finite parameter set A and any two mutually indiscernible (over A) sequences I and J of singletons, there is no singleton a such that neither I nor J remains indiscernible over Aa. This seems feasible to verify given the description of indiscernibles in Lemmas 1.3.4 and 1.3.5, but there are many configurations to analyse.

We turn next to the proof of Theorem 1.1.3. We consider first a structure $M_{C,\mathrm{lev}}$ which arises under the name $\partial T(\prec)$ in [16, Section 6]. We shall prove statements similar to those in Theorem 1.1.3 for $M_{C,\mathrm{lev}}$, and then prove the theorem by showing that $M_{C,\mathrm{lev}}$ is interpretable with parameters in $\mathcal{M}_B$ and $\mathcal{M}_D$.

Let X be a countably infinite set with a distinguished element e, and $L_{C,\mathrm{lev}}$ be a language with a ternary relation C and arity 4 relation $V(x, y; z, w)$. The domain of $M_{C,\mathrm{lev}}$ is the set of sequences $a = (a(i))_{i\in\mathbb{Q}}$ where the $a(i)$ lie in X, and a has finite support in the sense that all but finitely many $a(i)$ are equal to e. We put $C(a; b, c)$ if either $b = c$ and $a \neq b$, or a, b, c are distinct and $\mathrm{Min}\{i : a(i) \neq b(i)\} < \mathrm{Min}\{i : b(i) \neq c(i)\}$. Then C is a C-relation on $M_{C,\mathrm{lev}}$, and is in fact the universal homogeneous C-relation, as described in Section 4 of [16] where it is denoted by ∂T. If $a, b, c, d \in M_{C,\mathrm{lev}}$, we put $M_{C,\mathrm{lev}} \models V(a, b; c, d)$ if $a \neq b$, $c \neq d$, and $\mathrm{Min}\{i : a(i) \neq b(i)\} \leq \mathrm{Min}\{i : c(i) \neq d(i)\}$. This structure $M_{C,\mathrm{lev}}$ admits an iterated wreath product (in the sense of [16, Section 6]) as an oligomorphic group of automorphisms which is a Jordan group.

Below, we sketch a proof of homogeneity of $M_{C,\mathrm{lev}}$. This appears not to be in the literature, but was shown for a richer structure, in which there is a total order on the domain compatible with C, in [19, Theorem 6.1, Theorem 6.2]. The latter theorem shows that (in the setting of [19]) one can put extra structure on the set of levels, and iterate the process. For convenience, for $a, b \in M_{C,\mathrm{lev}}$ we write $a \wedge b = q$ if $a \neq b$ and $q = \mathrm{Min}\{i : a(i) \neq b(i)\}$.

Lemma 1.3.10. *The following hold for the structure* $M_{C,\mathrm{lev}}$.

(1) *The structure* $M_{C,\mathrm{lev}}$ *is homogeneous and so has quantifier elimination.*

(2) *If* (K, v) *is any countable algebraically closed non-trivially-valued field, then there is a copy of* $M_{C,\mathrm{lev}}$ *with domain* K *definable without parameters in the valued field* (K, v).

(3) $M_{C,\mathrm{lev}}$ *is dp-minimal and so NIP.*

Proof. (1) Let $\bar{a} = (a_1, \ldots, a_n)$, $\bar{b} = (b_1, \ldots, b_n)$ be tuples from $M_{C,\mathrm{lev}}$ such that the map f with $f(a_i) = b_i$ for each i is an isomorphism, and let $a \in M_{C,\mathrm{lev}} \setminus \{a_1, \ldots, a_n\}$. By back-and-forth, it suffices to find $b \in M_{C,\mathrm{lev}}$ such that we may extend f to a partial isomorphism with domain $\{a_1, \ldots, a_n, a\}$ by putting $f(a) = b$. Choose $j_0 \in \{1, \ldots, n\}$ so as to maximise $a \wedge a_{j_0}$.

Case 1. There is $k_0 \in \{1, \ldots, n\} \setminus \{j_0\}$ with $a \wedge a_{j_0} = a_{j_0} \wedge a_{k_0}$. Now let

$$I := \{i \in \{1, \ldots, n\} : a \wedge a_{j_0} = a_{j_0} \wedge a_i\}.$$

Put $q := b_{j_0} \wedge b_{k_0}$. Choose $b \in M_{C,\mathrm{lev}}$ so that $b(r) = b_{j_0}(r)$ for $r < q$, and $b(q) \neq b_i(q)$ for each $i \in I$.

Case 2. Not Case 1, but there are distinct $k_0, l_0 \in \{1, \ldots, n\}$ with $a \wedge a_{j_0} = a_{k_0} \wedge a_{l_0}$. Now let $q = b_{j_0} \wedge b_{l_0}$. Choose b so that $b(r) = b_{j_0}(r)$ for $r < q$, and $b(q) \neq b_{j_0}(q)$.

Case 3. Not Cases 1 or 2. Now choose $q \in \mathbb{Q}$ so that $q < b_k \wedge b_l$ whenever $a \wedge a_{j_0} < a_k \wedge a_l$, and $q > b_k \wedge b_l$ whenever $a \wedge a_{j_0} > a_k \wedge a_l$. As in Case 2, choose b so that $b(r) = b_{j_0}(r)$ for $r < q$, and $b(q) \neq b_{j_0}(q)$.

(2) Define the relation C on K by putting $C(x; y, z)$ if and only if $v(x - y) \leq v(y - z)$, and $V(x, y; z, w)$ if and only if $x \neq y$ and $z \neq w$ and $v(x - y) \leq v(z - w)$.

(3) This follows from (ii); the dp-minimality of any model of the theory ACVF of algebraically closed valued fields is noted in [31, Theorem A.11].

■

Proposition 1.3.11. *The structure* $M_{C,\mathrm{lev}}$ *satisfies the conclusions of Theorem 1.1.3, that is,*

(1) $(f_k(\mathrm{Aut}(M_{C,\mathrm{lev}}))) \geq \lfloor k/3 \rfloor !$ *for sufficiently large* k, *so in particular* $(f_k(\mathrm{Aut}(M_{C,\mathrm{lev}})))$ *has super-exponential growth rate.*

(2) $Age(M_{C,\mathrm{lev}})$ *is not wqo.*

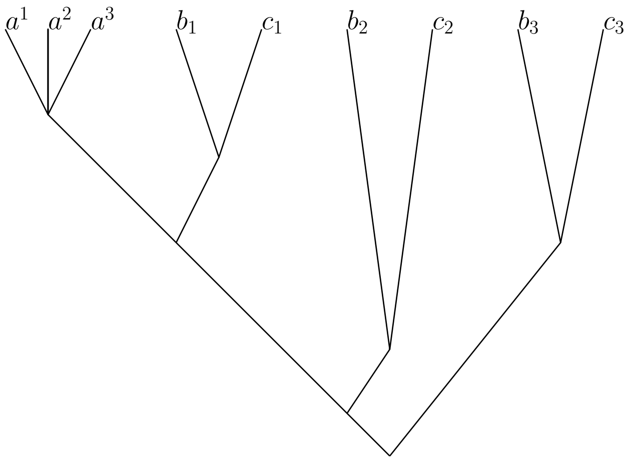

Figure 1.8. The structure M_σ, where σ is the permutation of $\{1, 2, 3\}$ with $1 <_1 2 <_1 3$ and $2 <_2 3 <_2 1$. The order in which $\{b_i, c_i\}$ meets the leftmost branch defines $<_1$, while the height of $b_i \wedge c_i$ defines $<_2$.

(3) *$M_{C,\mathrm{lev}}$ is not monadically NIP.*

Proof. The basic idea is to encode permutations (viewed as sets equipped with two linear orders $<_1, <_2$) of $\{1, \dots, n\}$ into sets of size $2n + 3$. Suppose $n \geq 2$ and let $\sigma \in \mathrm{Sym}_n$. Pick distinct $a^1, a^2, a^3, b_1, \dots, b_n, c_1, \dots, c_n \in M_{C,\mathrm{lev}}$ such that

(a) $\neg C(a^i; a^j, a^k)$ for distinct $i, j, k \in \{1, 2, 3\}$,

(b) $C(a^j; b_i, c_i)$ for every $j \in \{1, 2, 3\}$ and i with $1 \leq i \leq n$,

(c) $a^k \wedge b_i < a^k \wedge b_j$ for every k whenever $i <_1 j$,

(d) $b_i \wedge c_i < b_j \wedge c_j$ whenever $i <_2 j$.

An example of the resulting structure M_σ is depicted in Figure 1.8. There is an equivalence relation E on $\{b_i, c_i | i \in [n]\}$ whose classes are $\{\{b_i, c_i\} | i \in [n]\}$, which is definable from any a_i by $E(x, y)$ if $x \wedge a_i = y \wedge a_i$. Given E-classes E_j, E_k we may define $E_j <_1 E_k$ if $a_i \wedge E_j > a_i \wedge E_k$, where $a_i \wedge E_j$ is the element given by the meet of any a_i with either element of E_j. We may define a second order $E_j <_2 E_k$ by $b_j \wedge c_j < b_k \wedge c_k$.

We claim that if $\sigma \not\hookrightarrow \tau$ then $M_\sigma \not\hookrightarrow M_\tau$. To see this, recall that embeddings must preserve the fact that an existential formula holds of a tuple. Now suppose $f \colon M_\sigma \hookrightarrow M_\tau$ is an embedding. First observe that for any permutation π, the set $\{a^1, a^2, a^3\} \in M_\pi$ is existentially definable by the formula $\phi(x)$ saying there exist y, z such that $x \wedge y = x \wedge z = y \wedge z$. Note that fixing a^i, the definition of E given above is quantifier-free, and so both E and $\neg E$ are existentially definable. Similarly for the definitions of $<_1$ and $<_2$.

Thus f sends $\{a^1, a^2, a^3\} \subset M_\sigma$ to the corresponding 3-subset of M_τ, and preserves E-classes and both linear orders on them. Thus $f(M_\sigma)$ encodes the permutation σ as a substructure of τ.

In particular, if σ and τ are finite, then $\sigma \not\cong \tau$ implies $M_\sigma \not\cong M_\tau$. Part (1) of the Proposition now follows immediately, since $f_{2k+3}(\mathrm{Aut}(M_{C,\mathrm{lev}})) \geq k!$.

Part (2) follows immediately from the existence of an infinite antichain of finite permutations under order-preserving embeddings. For this, see e.g. [33, Figure 17].

Finally, we show (3). Recall from [14] that a theory T has *endless indiscernible triviality* if for every $M \models T$, every endpointless indiscernible sequence I in M, and every tuple B of parameters, if I is indiscernible over each $b \in B$ then I is indiscernible over B. It is shown in [14, Theorem 1.2] that if T is monadically NIP then T has endless indiscernible triviality. In $M_{C,\mathrm{lev}}$ consider the infinite substructure M_σ as above where $\sigma = (\mathbb{Z}, <_1, <_2)$ where both $<_1$ and $<_2$ agree with the standard order. Using quantifier-elimination in $M_{C,\mathrm{lev}}$, we see the sequence $I = (b_i c_i : i \in \mathbb{Z})$ is indiscernible over $A = \{a^1, a^2, a^3\}$. Consider a further pair of elements $b, c \in M_{C,\mathrm{lev}}$ corresponding to an E-class coming $<_1$-before every E-class of I but defining a non-trivial $<_2$-cut in I. Then I is still indiscernible over Ab and over Ac but not over Abc. ■

Proof of Theorem 1.1.3. We give the argument for $\mathcal{M} = \mathcal{M}_D^2$. The proof is essentially the same for $\mathcal{M} = \mathcal{M}_B^2$.

We claim that the structure $M_{C,\mathrm{lev}}$ is definable on singletons in $\mathcal{M}$. For this, recall the paragraph after the statement of Proposition 1.2.3. It was noted that if $[a]$ is a prenode of $\mathcal{M}_D^2$ at level j_0, then there is a $G_{a/\sim_{j_0}}$-invariant sequence of equivalence relations E_i and F_i on $[a]$ indexed by $\{i : i < j_0\}$ (which is order-isomorphic to $(\mathbb{Q}, <)$) and for $i < k < j_0$ we have $E_i \subset F_i \subset E_k \subset F_k$. This determines a C-relation on $[a]$ with $C(x; y, z)$ holding if for some i, some E_i-class contains y, z and omits x. By [7, Lemma 5.4], if $x, y \in [a]$ are distinct, then there is a greatest $i < j_0$ such that $\neg E_i xy$, and for such i we have $F_i xy$. We shall put $\mathrm{lev}(x, y) = i$, and define V on $[a]^4$ by putting $V(x, y; z, w)$ if $x \neq y$, $z \neq w$, and $\mathrm{lev}(x, y) \geq \mathrm{lev}(z, w)$.

Easily, C and V are $a/\sim_j$-definable. Also, the structure $([a], C, V)$ is isomorphic to the structure $M_{C,\mathrm{lev}}$ of Proposition 1.3.11. We do not give full details of this, but for example, in the proof of [7, Lemma 5.7] a bijection is constructed from $[a]$ (there denoted $[n]$) to a set Ω which is pretty clearly identifiable with the domain of $M_{C,\mathrm{lev}}$, and this bijection respects C and V.

Now let $b, c \in X_j$ with a, b, c inequivalent modulo $\sim_j$, so j and hence $[a]$ are abc-definable. Then the structure $([a], C, V)$ is abc-definable, and we identify it with $M_{C,\mathrm{lev}}$ as above. Note that by the homogeneity of $\mathcal{M}$, these definitions can be taken quantifier-free, and so both they and their negations are preserved by passing to substructures.

Part (i) of the theorem follows immediately from these observations and Proposition 1.3.11, since we may name the parameters a, b, c by unary predicates.

Parts (ii) and (iii) follow by taking substructures of $([a], C, V)$ as in Proposition 1.3.11.

■

We remark that while the first two points of Theorem 1.1.3 depend only on the definable relations of $\mathcal{M}$, and so are not dependent on the choice of language, the proof of the last point does use that we are working in the homogeneous language, because it is concerned with substructures of $\mathcal{M}$ and because embeddings need only preserve existential formulas. Nevertheless, we expect the third point to continue to hold regardless of the choice of language, in particular for $\mathcal{M} \in \{\mathcal{M}_B^0, \mathcal{M}_D^0\}$.

1.4 Proof of Theorem 1.1.4

We show first that if $\mathcal{M} = \mathcal{M}_B^0$ and $G = \mathrm{Aut}(\mathcal{M})$, then G is a maximal-closed subgroup of $\mathrm{Sym}(M)$. We omit the proof in the case when $G = \mathrm{Aut}(\mathcal{M}_D)$ – the details are very similar. Note that the latter uses Lemma 1.2.4, which ensures that $\mathcal{M}_D$ can be viewed as a structure with just the ternary relation L.

First observe that G is a 3-homogeneous 2-primitive Jordan group. It follows from the description of primitive Jordan groups in [3] that if $G < H < \mathrm{Sym}(M)$ with H closed, then one of the following holds.

(a) H preserves a linear separation relation.

(b) H preserves a D-relation.

(c) H preserves a Steiner system.

(d) H preserves a limit of Steiner systems.

(e) H preserves a limit of betweenness relations or D-relations.

To prove the theorem, we must eliminate each of these cases. Cases (a), (b), (c) are eliminated respectively in Lemmas 6.4, 6.6 and 6.5 of [9].

To eliminate (e), suppose that H preserves a limit of betweenness relations or D-relations. Then (by definition) H is not 3-transitive. However, G, the automorphism group of a ternary relation on M, is 3-homogeneous and induces C_2 on each 3-set. Since C_2 is maximal in Sym_3, there is no such H.

It remains to show that the group H cannot preserve a limit of Steiner systems, and for this we argue as in [10, Section 3]. We suppose for a contradiction that for some $n \geq 3$, H preserves a limit of Steiner $(n-1)$-systems, and follow the notation of Definition 1.2.2. As in [10], we may suppose that the S_k are all infinite.

We argue as in [10], but with the notion of *branch* (as subset of M_B) in place of cone. We work in the case of $\mathcal{M}_B^0$. The main point is that branches are a special class of Jordan sets for $G = \mathrm{Aut}(\mathcal{M}_B^0)$ with the property that if U is a branch, $A \subset U$ is finite, and $a \in U \setminus A$, then there is a branch $U' \subset U$ with $a \in U'$ and $A \cap U' = \emptyset$.

We claim that no set S_k can contain a branch. To see this, suppose that U is a branch lying in S_k. Pick distinct $a_1, \ldots, a_{n-1} \in S_k$ and let l be the unique Steiner block (of the Steiner system on S_k) containing these points. Let b_{n-1} be a point of S_k not on l, and m be the Steiner block of S_k containing $a_1, \ldots a_{n-2}, b_{n-1}$. As Steiner blocks have more than $n-1$ points, there are $a_n \in l \setminus \{a_1, \ldots, a_{n-1}\}$ and $b_n \in m \setminus \{a_1, \ldots, a_{n-2}, b_{n-1}\}$. So l is the unique block containing $a_1, \ldots, a_{n-3}, a_{n-1}, a_n$ and m is the unique block containing $a_1, \ldots, a_{n-3}, b_{n-1}, b_n$. Let $A = \{a_1, \ldots, a_{n-3}, a_{n-1}, a_n, b_{n-1}, b_n\}$. Since A contains $n-1$ points of each of l and m, any automorphism of the Steiner system on S_k fixing A fixes l and m setwise, so fixes the remaining point $a = a_{n-2}$ of their intersection. Choose a branch $U' \subset U$ with $a \in U'$ and $U' \cap A = \emptyset$. Now $H_{\{S_k\},(A)}$ fixes a, contradicting the fact that $G_{(M \setminus U')} \leq H_{\{S_k\},(A)}$ and that U' is a Jordan set for G containing a.

Claim. Any infinite subset V of M meets infinitely many disjoint branches.

Proof of Claim. Case 1. There is $j \in J$ such that $V/{\sim_j}$ is infinite. Now let $j' < j$. There is $a \in Z_{j'}$ such that V meets the branches corresponding to infinitely many branches of $Z_{j'}$ at a. These branches are disjoint and satisfy the Claim.

Case 2. For each $j \in J$, $V/{\sim_j}$ is finite. We may choose $j_0 \in J$ small enough so that there are $v_1, v_2 \in V$, inequivalent under $\sim_{j_0}$, with $V_1 := V \setminus (v_1/{\sim_{j_0}})$ infinite. Then if $j_1 < j_0$ there is a branch W_1 of Z_{j_1} contained in $v_1/{\sim_{j_1}}$ and meeting V. Indeed, there is a ramification point w_1 of Z_{j_1} such that $v_1/{\sim_{j_0}}$ is a union of branches at w_1, and we may choose W_1 to be one of these branches meeting $V \cap (v_1/{\sim_{j_0}})$. By choosing j_1 small enough we may also ensure that V_1 meets at least two $\sim_{j_1}$-classes, say $v_1'/{\sim_{j_1}}$ and $v_2'/{\sim_{j_1}}$, with $V_2 := V_1 \setminus (v_1'/{\sim_{j_1}})$ infinite. Now we may repeat the argument to V_1 in place of V to find $j_2 < j_1$ and a suitable branch W_2 of Z_{j_2} meeting V_1 and contained in $v_1'/{\sim_{j_1}}$. Iterating this argument yields the Claim.

Finally, pick a cofinal subset $\{k_n : n \in \omega\}$ of the index set K of the Steiner systems. By the Claim, S_{k_0} meets infinitely many disjoint branches $\{U_n : n \in \omega\}$, so for each $n \in \omega$ there is $x_n \in S_{k_0} \cap U_n$. By the paragraph before the Claim there is also $y_n \in U_n \setminus S_{k_n}$, for each n. Using that branches are Jordan sets and that G is the full automorphism group of $\mathcal{M}$ (*i.e.* is topologically closed in $\mathrm{Sym}(M)$), there is $g \in G$ with $x_n^g = y_n$ for each n. Thus, for each n we have $S_{k_0}^g \cap (M \setminus S_{k_n}) \neq \emptyset$, so for each $j \geq k_0$ and each $k \in K$ we have $S_j^g \cap (M \setminus S_k) \neq \emptyset$. This contradicts condition (iv) in Definition 1.2.2. ∎

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