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Article:

Fomin, F.V., Lokshantov, D., Panolan, F. et al. (2026) Decomposition of Map Graphs with Applications. Discrete and Computational Geometry. ISSN: 0179-5376

<https://doi.org/10.1007/s00454-026-00840-y>

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Decomposition of Map Graphs with Applications

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Received: date / Accepted: date

Abstract Bidimensionality is the most common technique to design subexponential time parameterized algorithms on special classes of graphs, particularly planar graphs. The core engine behind it is a combinatorial lemma of Robertson, Seymour and Thomas that states that every planar graph either has a $\sqrt{k} \times \sqrt{k}$ -grid as a minor, or its treewidth is $\mathcal{O}(\sqrt{k})$. However, bidimensionality theory cannot be extended directly to several well-known classes of geometric graphs. The reason is very simple: a clique on $k - 1$ vertices has no $\sqrt{k} \times \sqrt{k}$ -grid as a minor and its treewidth is $k - 2$, while classes of geometric graphs such as unit disk graphs or map graphs can have arbitrarily large cliques. Thus, the combinatorial lemma of Robertson, Seymour and Thomas is inapplicable

The preliminary version of the paper will appear in the proceedings of ICALP 2019. This work is supported by the European Research Council (ERC) via grant LOPPRE, reference 819416, the Norwegian Research Council via project MULTIVAL, and Israel Science Foundation individual research grant no. 1176/18.

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to these classes of geometric graphs. Nevertheless, a relaxation of this lemma has been proven useful for unit disk graphs. Inspired by this, we prove a new decomposition lemma for map graphs, the intersection graphs of finitely many simply-connected and interior-disjoint regions of the Euclidean plane. Informally, our lemma states the following. For any map graph G , there exists a collection (U_1, \dots, U_t) of cliques of G with the following property: G either contains a $\sqrt{k} \times \sqrt{k}$ -grid as a minor, or it admits a tree decomposition where every bag is the union of $\mathcal{O}(\sqrt{k})$ of the cliques in the above collection.

The new lemma appears to be a handy tool in the design of subexponential parameterized algorithms on map graphs. We demonstrate its usability by designing algorithms on map graphs with running time $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ for CONNECTED PLANAR \mathcal{F} -DELETION (that encompasses problems such as FEEDBACK VERTEX SET and VERTEX COVER). Obtaining subexponential algorithms for LONGEST CYCLE/PATH and CYCLE PACKING is more challenging. We have to construct tree decompositions with more powerful properties and to prove sublinear bounds on the number of ways an optimum solution could “cross” bags in these decompositions.

For LONGEST CYCLE/PATH, these are the first subexponential-time parameterized algorithms on map graphs. For FEEDBACK VERTEX SET and CYCLE PACKING, we improve upon known $2^{\mathcal{O}(k^{0.75} \log k)} \cdot n^{\mathcal{O}(1)}$ -time algorithms on map graphs.

Keywords Longest Path · Longest Cycle · Cycle Packing · Feedback Vertex Set · Map graphs · Parameterized Complexity

Mathematics Subject Classification (2010) MSC 68W01 · MSC 68W40 · MSC 68Q25 · MSC 05C85 · 68R10

1 Introduction

In this paper, we develop new proof techniques to design parameterized subexponential-time algorithms for problems on map graphs, particularly problems that involve hitting or connectivity constraints. The class of map graphs was introduced by Chen, Grigni, and Papadimitriou [12, 13] as a modification of the class of planar graphs. Roughly speaking, map graphs are graphs whose vertices represent countries in a map, where two countries are considered adjacent if and only if their boundaries have at least one point in common; this common point can be a single common point rather than necessarily an edge as standard planarity requires. Formally, a *map* \mathcal{M} is a pair (\mathcal{E}, ω) defined as follows (see Figure 1): \mathcal{E} is a plane graph¹ where each connected component of \mathcal{E} is biconnected, and ω is a function that maps each face f of \mathcal{E} to 0 or 1. A face f of \mathcal{E} is called *nation* if $\omega(f) = 1$ and *lake* otherwise. The graph associated with \mathcal{M} is the simple graph G where $V(G)$ consists of the nations of \mathcal{M} , and $E(G)$ contains $\{f_1, f_2\}$ for every pair of faces f_1 and f_2 that are

¹ That is, a planar graph with a drawing in the plane.

adjacent (that is, share at least one vertex). Accordingly, a graph G is called a map graph if there exists a map \mathcal{M} such that G is the graph associated with \mathcal{M} .

Every planar graph is a map graph [12,13], but the converse does not hold true. Moreover, map graphs can have cliques of any size and thus they can be “highly non-planar”. These two properties of map graphs can be contrasted with those of H -minor free graphs and unit disk graphs: the class of H -minor free graphs generalizes the class of planar graphs, but can only have cliques of constant size (where the constant depends on H), while the class of unit disk graphs does not generalize the class of planar graphs, but can have cliques of any size. At least in this sense, map graphs offer the best of both worlds. Nevertheless, this comes at the cost of substantial difficulties in the design of efficient algorithms on them.

Arguably, the two most natural and central algorithmic questions concerning map graphs are as follows. First, we would like to efficiently recognize map graphs, that is, determine whether a given graph is a map graph. In 1998, Thorup [34] announced the existence of a polynomial-time algorithm for map graph recognition. Although this algorithm is complicated and its running time is about $\mathcal{O}(n^{120})$, where n is the number of vertices of the input graph, no improvement has yet been found; the existence of a simpler or faster algorithm for map graph recognition has so far remained an important open question in the area (see, e.g., [10]).

The second algorithmic question—or rather family of algorithmic questions—concerns the design of efficient algorithms for various optimization problems on map graphs. Most well-known problems that are NP-complete on general graphs remain NP-complete when restricted to planar (and hence on map) graphs. Nevertheless, a large number of these problems can be solved faster or “better” when restricted to planar graphs. For example, nowadays we know of many problems that are APX-hard on general graphs, but which admit polynomial time approximation schemes (PTASes) or even efficient PTASes (EPTASes) on planar graphs (see, e.g., [4,18,19,26]). Similarly, many parameterized problems that on general graphs cannot be solved in time $2^{o(k)} \cdot n^{\mathcal{O}(1)}$ unless the Exponential Time Hypothesis (ETH) of Impagliazzo, Paturi and Zane [29] fails, admit parameterized algorithms with running times subexponential in k on planar graphs (see, e.g., [1,2,18,32]). It is compelling to ask whether the algorithmic results and techniques for planar graphs can be extended to map graphs.

For approximation algorithms, Chen [11] and Demaine et al. [16] developed PTASes for the MAXIMUM INDEPENDENT SET and MINIMUM r -DOMINATING SET problems on map graphs. Moreover, Fomin et al. [25,26] developed an EPTAS for TREEWIDTH- η MODULATOR for any fixed constant $\eta \geq 0$, which encompasses FEEDBACK VERTEX SET and VERTEX COVER. For parameterized subexponential-time algorithms on map graphs, the situation is less explored. While on planar graphs there are general algorithmic methods—in particular, the powerful theory of bidimensionality [19,17]—to design parameterized subexponential-time algorithms, we are not aware of any general algorithmic

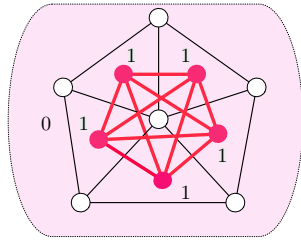


Fig. 1: A map $\mathcal{M} = (\mathcal{E}, \omega)$ where the internal 1-faces are nations and the 0-exterior face is a lake. The corresponding map graph is a complete graph on five vertices.

method that can be easily adapted to map graphs. Demaine et al. [16] gave a parameterized algorithm for DOMINATING SET, and more generally for (k, r) -CENTER, with running time $2^{\mathcal{O}(r \log r \sqrt{k})} n^{\mathcal{O}(1)}$ on map graphs. Moreover, Fomin et al. [25, 26] gave $2^{\mathcal{O}(k^{0.75} \log k)} n^{\mathcal{O}(1)}$ -time parameterized algorithms for FEEDBACK VERTEX SET and CYCLE PACKING on map graphs. Additionally, Fomin et al. [25, 26] noted that the same approach yields $2^{\mathcal{O}(k^{0.75} \log k)} n^{\mathcal{O}(1)}$ -time parameterized algorithms for VERTEX COVER and CONNECTED VERTEX COVER on map graphs. However, the existence of a parameterized subexponential-time algorithm for LONGEST PATH/CYCLE² on map graphs was left open. Furthermore, time complexities of $2^{\mathcal{O}(k^{0.75} \log k)} n^{\mathcal{O}(1)}$, although having subexponential dependency on k , remain far from time complexities of $2^{\mathcal{O}(\sqrt{k} \log k)} n^{\mathcal{O}(1)}$ and $2^{\mathcal{O}(\sqrt{k})} n^{\mathcal{O}(1)}$ that commonly arise for planar graphs [32]. We remark that time complexities of $2^{\mathcal{O}(\sqrt{k} \log k)} n^{\mathcal{O}(1)}$ and $2^{\mathcal{O}(\sqrt{k})} n^{\mathcal{O}(1)}$ are particularly important since they are often known to be essentially optimal under the aforementioned ETH [32].

In the field of Parameterized Complexity, LONGEST PATH/CYCLE, FEEDBACK VERTEX SET and CYCLE PACKING serve as testbeds for development of fundamental algorithmic techniques such as color-coding [3], methods based on polynomial identity testing [30, 31, 35, 6], cut-and-count [15], and methods based on matroids [23]. We refer to [14] for an extensive overview of the literature on parameterized algorithms for these three problems on general graphs. By combining the bidimensionality theory of Demaine et al. [17] with efficient algorithms on graphs of bounded treewidth [21, 14], LONGEST PATH/CYCLE, CYCLE PACKING and FEEDBACK VERTEX SET are solvable in time $2^{\mathcal{O}(\sqrt{k})} n^{\mathcal{O}(1)}$ on planar graphs. Furthermore, the parameterized subexponential-time “tractability” of these problems can be extended to graphs excluding some fixed graph as a minor [19].

² In the LONGEST PATH/CYCLE problem, we ask whether a given graph G contains a path/cycle on at least k vertices. Here, the parameter is k .

Our results and methods

Our results. We design parameterized subexponential-time algorithms with running time $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ for a number of natural and well-studied problems on map graphs.

Let \mathcal{F} be a family of connected graphs that contains at least one planar graph. Then CONNECTED PLANAR \mathcal{F} -DELETION (or just \mathcal{F} -DELETION) is defined as follows.

\mathcal{F} -DELETION

Input: A graph G and a non-negative integer k .

Question: Is there a set S of at most k vertices such that $G - S$ does not contain any of the graphs in \mathcal{F} as a minor?

\mathcal{F} -DELETION is a general problem and several problems such as VERTEX COVER, FEEDBACK VERTEX SET, TREewidth- η VERTEX DELETION, PATHwidth- η VERTEX DELETION, TREEDepth- η VERTEX DELETION, DIAMOND HITTING SET and OUTERPLANAR VERTEX DELETION are its special cases. We give the first parameterized subexponential algorithm for this problem on map graphs, which runs in time $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$. Our approach for \mathcal{F} -DELETION also directly extends to yield $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ -time parameterized algorithms for CONNECTED VERTEX COVER and CONNECTED FEEDBACK VERTEX SET on map graphs. (In this versions we are asked if there is a *connected* vertex cover or a feedback vertex set of size at most k .)

With additional ideas, we derive the first subexponential-time parameterized algorithm on map graphs for LONGEST PATH/CYCLE. Our technique also allows to improve the running time for CYCLE PACKING (does a map graph contains at least k vertex-disjoint cycles) from $2^{\mathcal{O}(k^{0.75} \log k)} \cdot n^{\mathcal{O}(1)}$ to $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$.

Our results are summarized in Table 1.

	Our results	Previous work
VERTEX COVER	$2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ [Thm 7]	$2^{\mathcal{O}(k^{0.75} \log k)} \cdot n^{\mathcal{O}(1)}$ [26]
CONNECTED VERTEX COVER	$2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ [Thm 6]	$2^{\mathcal{O}(k^{0.75} \log k)} \cdot n^{\mathcal{O}(1)}$ [26]
FEEDBACK VERTEX SET	$2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ [Thm 4]	$2^{\mathcal{O}(k^{0.75} \log k)} \cdot n^{\mathcal{O}(1)}$ [26]
CFVS	$2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ [Thm 6]	$2^{\mathcal{O}(k^{0.75} \log k)} \cdot n^{\mathcal{O}(1)}$ [26]
\mathcal{F} -DELETION	$2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ [Thm 7]	$2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ [22]
LONGEST PATH	$2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ [Thm 10]	$2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ [14]
LONGEST CYCLE	$2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ [Thm 8]	$2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ [14]
CYCLE PACKING	$2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ [Thm 11]	$2^{\mathcal{O}(k^{0.75} \log k)} \cdot n^{\mathcal{O}(1)}$ [26]

Table 1: Parameterized complexity of problems on map graphs. For \mathcal{F} -DELETION, LONGEST CYCLE, and LONGEST PATH no faster (than on general graphs) algorithms were known for map graphs. Here, CFVS is a short form of CONNECTED FEEDBACK VERTEX SET.

Methods. The starting point of our study is the technique of bidimensionality [19, 17]. The core engine behind this technique is a combinatorial lemma of Robertson, Seymour and Thomas [33] that states that every planar graph either has a $\sqrt{k} \times \sqrt{k}$ -grid as a minor, or its treewidth is $\mathcal{O}(\sqrt{k})$. Unfortunately, a clique on $k - 1$ vertices has no $\sqrt{k} \times \sqrt{k}$ -grid as a minor and its treewidth is $k - 2$. Notice that a clique on $k - 1$ vertices with an arbitrary long path attached to one vertex in the clique is a graph G of arbitrary size with the property that its treewidth is $k - 2$ and it has no $\sqrt{k} \times \sqrt{k}$ -grid as a minor. It is easy to verify that G is a unit disk graph as well as a map graph. So the above combinatorial lemma is inapplicable to the classes of geometric graphs such as unit disk graphs and map graphs. Nevertheless, a relaxation of this lemma has been proven useful for unit disk graphs. Specifically, every unit disk graph G has a natural partition (U_1, \dots, U_t) of $V(G)$ such that each part induces a clique with “nice” properties—in particular, it has neighbors only in a *constant number* (to be precise, this constant is at most 24) of other parts; it was shown that G either has a $\sqrt{k} \times \sqrt{k}$ -grid as a minor, or it has a tree decomposition where every bag is the union of $\mathcal{O}(\sqrt{k})$ of these cliques [24]. In particular, given a parameterized problem where any two cliques have constant-sized “interaction” in a solution, it is implied that any bag has $\mathcal{O}(\sqrt{k})$ -sized “interaction” with all other bags in a solution. For any map graph G , there also exists a natural collection of subsets of $V(G)$ that induce cliques with “nice” properties. However, not only are these cliques not vertex disjoint, but each of these cliques can have neighbors in *arbitrarily many* other cliques.

In this paper, we first prove that every map graph either has a $\sqrt{k} \times \sqrt{k}$ -grid as a minor, or it has a tree decomposition where every bag is the union of $\mathcal{O}(\sqrt{k})$ of the cliques in the above collection. For \mathcal{F} -DELETION, CONNECTED VERTEX COVER, and CONNECTED FEEDBACK VERTEX SET, this combinatorial lemma alone already suffices to design $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ -time algorithms on map graphs. Indeed, we can choose a fixed constant $c > 0$ so that in case we have a $c\sqrt{k} \times c\sqrt{k}$ -grid as a minor, there does not exist a solution, and otherwise we can solve the problem by using dynamic programming over the given tree decomposition. Specifically, since every bag is the union of $\mathcal{O}(\sqrt{k})$ cliques, and the size of each clique is upper bounded by $\mathcal{O}(k)$ (once we know that no $c\sqrt{k} \times c\sqrt{k}$ -grid exists), only $\mathcal{O}(\sqrt{k})$ vertices in the bag are not to be taken into a solution—there are only $2^{\mathcal{O}(\sqrt{k} \log k)}$ choices to select these vertices, and once they are selected, the information stored about the remaining vertices is the same as in normal dynamic programming over a tree decomposition of $\mathcal{O}(\sqrt{k})$ width.

This approach already substantially improves upon the previously best known algorithms for FEEDBACK VERTEX SET, VERTEX COVER and CONNECTED VERTEX COVER of Fomin et al. [25, 26]. However, $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ -time algorithms for LONGEST PATH/CYCLE and CYCLE PACKING on map graphs require more efforts. The main reason why we cannot apply the same arguments as for unit disk graphs is the following. Recall that for unit disk graphs, given a parameterized problem where any two cliques have constant-

sized “interaction” in a *solution* (in our case, this means a path/cycle on at least k vertices, or a cycle packing of k cycles), it is implied that any bag has $\mathcal{O}(\sqrt{k})$ -sized “interaction” with all other bags in a solution. Here, interaction between two cliques refers to the number of edges in a solution “passing” between these two cliques; similarly, interaction between a bag B and a collection of other bags refers to the number of edges in a solution that have one endpoint in B and the other endpoint in some bag in the collection. In this context, dealing with map graphs is substantially more difficult than dealing with unit disk graphs. In map graphs vertices in a clique can have neighbors in arbitrarily many other cliques in the collection rather than only in a constant number as in unit disk graphs. This is why it is difficult to obtain an $\mathcal{O}(\sqrt{k})$ -sized “interaction” as before.

This is the reason why we are forced to take a different approach for map graphs by bounding “the interaction within a clique across all the bags of a decomposition”. Towards this, we first need to strengthen our tree decomposition. To explain the new properties required, we note that every clique in the aforementioned collection of cliques, say \mathcal{K} , is either a single vertex or the neighborhood of some “special vertex” in an exterior bipartite graph (see Section 2). Further, every vertex of G occurs as a singleton in \mathcal{K} . We construct our decomposition in a way such that every bag is not necessarily a union of $\mathcal{O}(\sqrt{k})$ cliques in \mathcal{K} , but a union of carefully chosen subcliques of $\mathcal{O}(\sqrt{k})$ cliques in \mathcal{K} (with one subclique for each of these $\mathcal{O}(\sqrt{k})$ cliques); subcliques of the same clique chosen in different bags may be different. We then prove properties that roughly state that, if we look at the collection of bags that include some vertex v of G , then this collection induces a subtree and a path as follows: (\clubsuit) *the subtree consists of the bags that correspond to the singleton clique v , and the path goes “upwards” (in the tree decomposition) from the root of this subtree.* We thereby implicitly derive that in every bag B , every subclique of size larger than 1 can only have as neighbors vertices that are (i) in the bag B itself or in one of its descendants, or (ii) in cliques that have a subclique in the bag B . In particular, this means that if we prove that there exists a solution such that for any clique K in \mathcal{K} , the number of edges in $E(K)$ that “cross any bag B ” (i.e., the edges in $E(K)$ with one endpoint in B and the other in the collection of all bags that are not descendants of B) is a constant, then we obtain a bound of $\mathcal{O}(\sqrt{k})$ on the interaction between any bag B and the collection of all bags that are not descendants of B . We prove the mentioned statement using property (\clubsuit). The proof that such a property simultaneously holds for all cliques and all bags is the most challenging part of the proof.

We remark that we discussed above two types of tree decompositions, first the special one and then its stronger variant which is used for LONGEST PATH/CYCLE and CYCLE PACKING. Since the stronger variant of the decomposition can be used to work with \mathcal{F} -DELETION too, in the technical part of this paper we derive only the stronger variant of the decomposition. In Section 2, we give definitions, notations and some known results which we use throughout the paper. In Section 3, we design a tree decomposition of map graphs which we call as few cliques tree decomposition, and in Section 4 we explain its direct

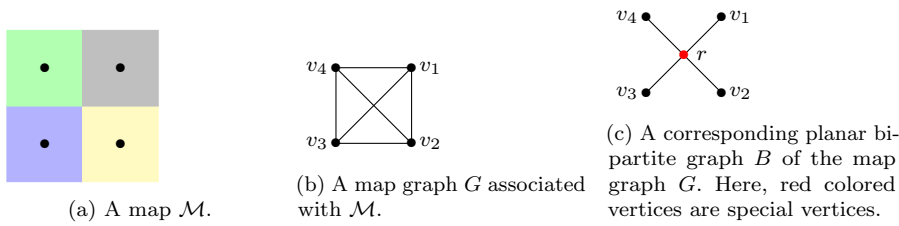


Fig. 2: Example of a map graph G obtained from a corresponding planar bipartite graph B .

applicability for FEEDBACK VERTEX SET and \mathcal{F} -DELETION. In Sections 5 and 6 we design subexponential-time parameterized algorithms for LONGEST PATH/CYCLE and CYCLE PACKING on map graphs, respectively. For these two problems we need additional, somehow technically involved, combinatorial “sublinear crossing” lemmata.

2 Preliminaries

The set of natural numbers is denoted by \mathbb{N} . For any $t \in \mathbb{N}$, we use $[t]$ and $[t]_0$ as shorthands for $\{1, 2, \dots, t\}$ and $\{0, 1, \dots, t\}$, respectively. For a set U , we use 2^U to denote the power set of U . Two disjoint sets A and B , we use $A \uplus B$ to denote the disjoint union of A and B . For a sequence $\sigma = x_1 x_2 \dots x_n$ and any $1 \leq i \leq j \leq n$, the sequence $\sigma' = x_i \dots x_j$ is called a *segment* of σ . For a sequence $\sigma = x_1 x_2 \dots x_n$ and a subset $Z \subseteq \{x_1, \dots, x_n\}$, the restriction of σ on Z , denoted by $\sigma|_Z$, is the sequence obtained from σ by deleting the elements of $\{x_1, \dots, x_n\} \setminus Z$.

Standard graph notations. We use standard notation and terminology from the book of Diestel [20] for graph-related terms that are not explicitly defined here. Given a graph G , let $V(G)$ and $E(G)$ denote its vertex-set and edge-set, respectively. When the graph G is clear from context, we denote $n = |V(G)|$ and $m = |E(G)|$. For a set \mathcal{Q} of graphs we slightly abuse terminology and let $V(\mathcal{Q})$ and $E(\mathcal{Q})$ denote the union of the sets of vertices and edges of the graphs in \mathcal{Q} , respectively. A graph is *simple* if it contains neither loops nor multiple edges between pairs of vertices. Throughout the paper, when we use the term *graph* we refer to a simple graph. Given $U \subseteq V(G)$, let $G[U]$ denotes the subgraph of G induced by U . For an edge subset $E \subseteq E(G)$, let $V(E)$ denotes the set of endpoints of the edges in E , and $G[E]$ denotes the graph with vertex set $V(E)$ and edge set E . Given $X \subseteq V(G)$, let $E(X)$ denotes the edge set $\{\{u, v\} \in E(G) : u, v \in X\}$. Moreover, let $N_G(U)$ denotes the open neighborhood of U in G ; we omit the subscript G when the graph is clear from context. In case $U = \{v\}$, we slightly abuse terminology and use $N_G(v) = N_G(U)$. For a graph G and a vertex $v \in V(G)$, let $d_G(v) = |N_G(v)|$. A graph H is called a *minor* of G if H can be obtained from G by a sequence

of edge deletions, edge contractions, and vertex deletions. For a graph G and a degree-2 vertex $v \in V(G)$, by *contracting* v , we mean deleting v from G and adding an edge between the two neighbors of v in G .

In a graph G , a sequence of vertices $[u_1 u_2 \dots u_\ell]$ is a path in G if for any distinct $i, j \in [\ell]$, $u_i \neq u_j$, and for any $r \in [\ell - 1]$, $\{u_r, u_{r+1}\} \in E(G)$. We also call the path $P = [u_1 u_2 \dots u_\ell]$ as u_1 - u_ℓ *path*, and its internal vertices are $u_2, u_3, \dots, u_{\ell-1}$. For any two paths $P_1 = [u_1 \dots u_i]$ and $P_2 = [u_i \dots u_\ell]$ with $\{u_1, \dots, u_{i-1}\} \cap \{u_{i+1}, \dots, u_\ell\} = \emptyset$, let $P_1 P_2$ denotes the path $[u_1 u_2 \dots u_\ell]$. A sequence of vertices $[u_1 u_2 \dots u_\ell]$ is a cycle in G if $u_1 = u_\ell$, $[u_1 u_2 \dots u_{\ell-1}]$ is a path, and $\{u_{\ell-1}, u_\ell\} \in E(G)$. Since in a multi graph there can be more than one edges between a pair of vertices, we use sequence $[u_0 e_0 u_1 e_1 \dots e_\ell u_1]$ to denote a cycle. In that context, for each $i \in [\ell]_0$, e_i is an edge between u_i and $u_{(i+1) \bmod \ell}$. For a graph G , we say that $U \subseteq V(G)$ is a clique if $G[U]$ is a complete graph. Given $a, b \in \mathbb{N}$, an $a \times b$ grid is a graph on $a \cdot b$ vertices, $v_{i,j}$ for $(i, j) \in [a] \times [b]$, such that for all $i \in [a - 1]$ and $j \in [b]$, it holds that $v_{i,j}$ and $v_{i+1,j}$ are neighbors, and for all $i \in [a]$ and $j \in [b - 1]$, it holds that $v_{i,j}$ and $v_{i,j+1}$ are neighbors.

A binary tree is a rooted tree where each node has at most two children. In a labelled binary tree, for each node with two children one of the children is labelled as “left child” and the other child is labelled as “right child”. A *postorder transversal* of a labelled binary tree T is the sequence σ of $V(T)$ where for each node $t \in V(T)$, t appears after all its descendants, and if t has two children, then the nodes in the subtree rooted at the left child appear before the nodes in the subtree rooted at the right child. For a binary tree T , we say that a sequence σ of $V(T)$ is a postorder transversal if there is a labelling of T such that σ is its postorder transversal.

A tree decomposition of a graph G , which is defined as follows, measures how close the graph G is to a tree like structure.

Definition 1 (Treewidth) A *tree decomposition* of a graph G is a pair $\mathcal{T} = (T_\mathcal{T}, \beta_\mathcal{T})$, where T is a rooted tree and $\beta_\mathcal{T}$ is a function from $V(T_\mathcal{T})$ to $2^{V(G)}$, that satisfies the following three conditions. (We use the term *nodes* to refer to the vertices of $T_\mathcal{T}$.)

- (a) $\bigcup_{x \in V(T_\mathcal{T})} \beta_\mathcal{T}(x) = V(G)$.
- (b) For every edge $\{u, v\} \in E(G)$, there exists $x \in V(T_\mathcal{T})$ such that $\{u, v\} \subseteq \beta_\mathcal{T}(x)$.
- (c) For every vertex $v \in V(G)$, the set of nodes $\{t \in V(T_\mathcal{T}) : v \in \beta_\mathcal{T}(t)\}$ induces a (connected) subtree of $T_\mathcal{T}$.

The *width* of \mathcal{T} is $\max_{x \in V(T_\mathcal{T})} |\beta_\mathcal{T}(x)| - 1$. Each set $\beta_\mathcal{T}(x)$ is called a *bag*. Moreover, $\gamma_\mathcal{T}(x)$ denotes the union of the bags of x and its descendants. The *treewidth* of G is the minimum width among all possible tree decompositions of G , and it is denoted by $\text{tw}(G)$.

A *nice tree decomposition* is a tree decomposition of a form that simplifies the design of dynamic programming (DP) algorithms. Formally,

Definition 2 A tree decomposition $\mathcal{T} = (T_{\mathcal{T}}, \beta_{\mathcal{T}})$ of a graph G is *nice* if for the root r of $T_{\mathcal{T}}$, it holds that $\beta_{\mathcal{T}}(r) = \emptyset$, and each node $v \in V(T_{\mathcal{T}})$ is of one of the following types.

- **Leaf:** v is a leaf in $T_{\mathcal{T}}$ and $\beta_{\mathcal{T}}(v) = \emptyset$. This bag is labelled with **leaf**.
- **Forget vertex:** v has exactly one child u , and there exists a vertex $w \in \beta_{\mathcal{T}}(u)$ such that $\beta_{\mathcal{T}}(v) = \beta_{\mathcal{T}}(u) \setminus \{w\}$. This bag is labelled with **forget**(w).
- **Introduce vertex:** v has exactly one child u , and there exists a vertex $w \in \beta_{\mathcal{T}}(v)$ such that $\beta_{\mathcal{T}}(v) \setminus \{w\} = \beta_{\mathcal{T}}(u)$. This bag is labelled with **introduce**(w).
- **Join:** v has exactly two children, u and w , and $\beta_{\mathcal{T}}(v) = \beta_{\mathcal{T}}(u) = \beta_{\mathcal{T}}(w)$. This bag is labelled with **join**.

We will use the following two folklore observations in Section 3. The correctness of these observations follows from Condition (c) of a tree decomposition.

Observation 1 *Let \mathcal{T} be a nice tree decomposition of a graph G . For any $v \in V(G)$, there is exactly one node $t \in V(T_{\mathcal{T}})$ such that t is labelled with **forget**(v).*

Observation 2 *Let \mathcal{T} be a nice tree decomposition of a graph G , $v \in V(G)$, and $t \in V(T_{\mathcal{T}})$ be the node labelled with **forget**(v). For any node t' in the subtree of $T_{\mathcal{T}}$ rooted at t and $t' \neq t$, either $v \in \beta_{\mathcal{T}}(t')$ or $v \notin \gamma_{\mathcal{T}}(t')$.*

The following proposition concerns the computation of a nice tree decomposition.

Proposition 1 ([7]) *Given a graph G and a tree decomposition \mathcal{T} of G , a nice tree decomposition \mathcal{T}' of the same width as \mathcal{T} can be computed in linear time.*

Planar Graphs and Map Graphs. A graph G is *planar* if there is a mapping of every vertex of G to a point on the Euclidean plane, and of every edge e of G to a curve on the Euclidean plane where the extreme points of the curve are the points mapped to the endpoints of e , and all curves are disjoint except on their extreme points.

Lemma 1 (Theorem 7.23 in [14],[28,33]) *For any $t \in \mathbb{N}$, every planar graph G of treewidth at least $9t/2$ contains a $t \times t$ grid minor. Furthermore, for every $\epsilon > 0$, there exists an $\mathcal{O}(n^2)$ time algorithm that given an n -vertex planar graph G and $t \in \mathbb{N}$, either outputs a tree decomposition of G of width at most $(9/2 + \epsilon)t$, or constructs a $t \times t$ grid minor in G .*

By substituting $\epsilon = 1/3$ in Lemma 1, we get the following corollary.

Corollary 1 *There exists an $\mathcal{O}(n^2)$ time algorithm that given an n -vertex planar graph G and $t \in \mathbb{N}$, either outputs a tree decomposition of G of width less than $5t$, or constructs a $t \times t$ grid minor in G .*

Map graphs are the intersection graphs of finitely many connected and interior-disjoint regions of the Euclidean plane. Any number of regions can meet at a common corner which results (in the map graph) in a clique on the vertices corresponding to these regions. Map graphs can be represented as the *half-squares of planar bipartite graphs*. For a bipartite graph B with bipartition $V(B) = W \uplus U$, the half-square of B is the graph G with vertex set W and edge set is defined as follows: two vertices in W are adjacent in G if they are at distance 2 in B . It is known that the half-square of a planar bipartite graph is a map graph [12,13]. Moreover, for any map graph G , there exists a planar bipartite graph B such that G is a half-square of B [12,13]; we refer to such B as a planar bipartite graph *corresponding* to the map graph G (see Figure 2).

Throughout this paper, we assume that any input map graph G is given with a corresponding planar bipartite graph B ³. We remark that we consider map graphs as simple graphs, that is, there are no multiple edges between two vertices u and v , even if there are two or more internally vertex disjoint paths of length 2 between u and v in the corresponding planar bipartite graph. For a map graph G with a corresponding planar bipartite graph B having bipartition $V(B) = W \uplus U$, we refer to the vertices in $W = V(G)$ simply as *vertices* and the vertices in U as *special vertices*. Moreover, we denote the special vertices by $S(G)$. Notice that for any $s \in S(G)$, $N_B(s)$ forms a clique in G ; we refer to these cliques as *special cliques* of G . We remark that the collection \mathcal{K} of cliques mentioned in Section 1 refers to $\{N_B(s) : s \in S(G)\} \cup \{\{v\} : v \in V(G)\}$.

3 Few Cliques Tree Decomposition of Map Graphs

In this section, we define a special tree decomposition for map graphs. This decomposition will be derived from a tree decomposition of the bipartite planar graph corresponding to the given map graph. Once we have defined our new decomposition, we will gather a few of its structural properties. These properties will be useful in designing fast subexponential time algorithms on map graphs.

Definition 3 Let G be a map graph with a corresponding planar bipartite graph B . Let $\mathcal{D} = (T_{\mathcal{D}}, \beta_{\mathcal{D}})$ be a tree decomposition of B of width less than ℓ . A pair $\mathcal{D}' = (T_{\mathcal{D}'}, \beta_{\mathcal{D}'})$ is called the *ℓ -few cliques tree decomposition derived from \mathcal{D}* , or simply an (ℓ, \mathcal{D}) -FewCliTD, if it is constructed as follows (see Figure 3).

1. The tree $T_{\mathcal{D}'}$ is equal to $T_{\mathcal{D}}$. Whenever \mathcal{D}' and \mathcal{D} are clear from context, we denote both $T_{\mathcal{D}'}$ and $T_{\mathcal{D}}$ by T .
2. For each node $t \in V(T)$, $\beta_{\mathcal{D}'}(t) = (\beta_{\mathcal{D}}(t) \cap V(G)) \cup (\bigcup_{s \in \beta_{\mathcal{D}}(t) \cap S(G)} N_B(s) \cap \gamma_{\mathcal{D}}(t))$. That is, for each node $t \in V(T)$, we derive $\beta_{\mathcal{D}'}(t)$ from $\beta_{\mathcal{D}}(t)$ by replacing every special vertex $s \in \beta_{\mathcal{D}}(t) \cap S(G)$ by $N_B(s) \cap \gamma_{\mathcal{D}}(t)$.

In words, the second item states that for every vertex $v \in V(G)$ and node $t \in V(T)$, we have that $v \in \beta_{\mathcal{D}'}(t)$ if and only if either (i) $v \in \beta_{\mathcal{D}}(t) \cap V(G)$ or

³ This assumption is made without loss of generality in the sense that if G is given with an embedding instead to witness that it is a map graph, then B is easily computable in linear time [12,13].

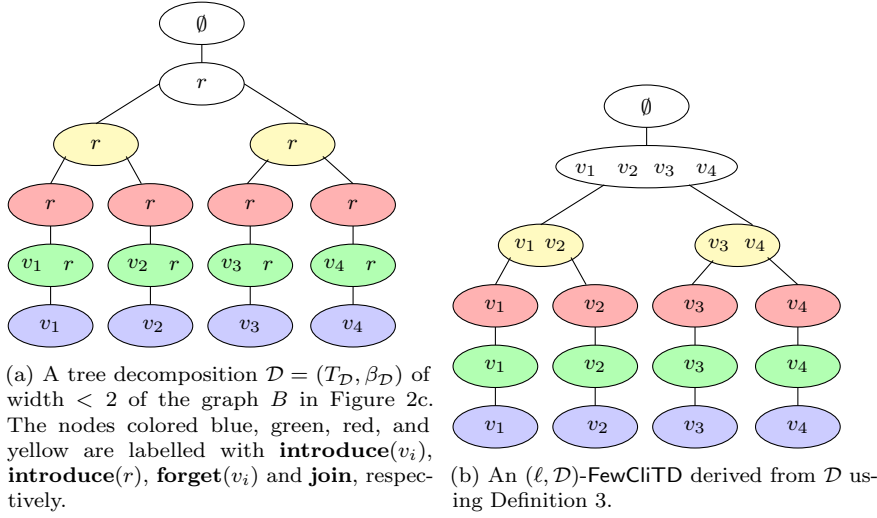


Fig. 3: Figure 3a represents a tree decomposition of B (drawn in Figure 2c). In fact it is a tree decomposition obtained after deleting the leaves of a nice tree decomposition.

(ii) $v \in N_B(s)$ for some $s \in S(G) \cap \beta_{\mathcal{D}}(t)$ and $v \in \beta_{\mathcal{D}}(t')$ for some node t' in the subtree of T rooted at t .

Next, we prove that the (ℓ, \mathcal{D}) -FewCliTD $(T, \beta_{\mathcal{D}'})$ in Definition 3 is a tree decomposition of G . We remark that if we replace the term $N_B(s) \cap \gamma_{\mathcal{D}}(t)$ by the term $N_B(s)$ in the second item of Definition 3, then we still derive a tree decomposition, but then some of the properties proved later do not hold true.

Lemma 2 *Let G be a map graph with a corresponding planar bipartite graph B . Let $\mathcal{D}' = (T, \beta_{\mathcal{D}'})$ be an (ℓ, \mathcal{D}) -FewCliTD where $\mathcal{D} = (T, \beta_{\mathcal{D}})$ is a tree decomposition of B of width less than ℓ . Then, \mathcal{D}' is a tree decomposition of G .*

Proof We first prove that every vertex of G is present in at least one bag. Towards this, notice that Property (a) of Definition 1 of the tree decomposition \mathcal{D} of B implies that $\bigcup_{t \in V(T)} \beta_{\mathcal{D}}(t) = V(B) = V(G) \cup S(G)$. Therefore, since $\beta_{\mathcal{D}'}(t) \supseteq \beta_{\mathcal{D}}(t) \cap V(G)$ for any $t \in V(T)$, we conclude that $\bigcup_{t \in V(T)} \beta_{\mathcal{D}'}(t) = V(G)$. Now, we prove that for any edge $\{u, v\} \in E(G)$, there exists a bag $\beta_{\mathcal{D}'}(t)$ for some $t \in V(T)$ such that $\{u, v\} \subseteq \beta_{\mathcal{D}'}(t)$. Because $\{u, v\} \in E(G)$, there exists a special vertex $s \in S(G)$ such that $\{u, s\}, \{v, s\} \in E(B)$. By Property (c) of \mathcal{D} , the set of nodes $Q = \{t \in V(T) : s \in \beta_{\mathcal{D}}(t)\}$ induces a (connected) subtree of T . Let $z \in Q$ be a node such that the distance from z to the root of T is minimized. Therefore, the choice of z is unique. Since $\{u, s\}, \{v, s\} \in E(B)$, by Property (b) of \mathcal{D} and the definition of Q , there exist $x, y \in Q$ such that $\{u, s\} \in \beta_{\mathcal{D}}(x)$ and $\{v, s\} \in \beta_{\mathcal{D}}(y)$. Then, because x and y must be descendants of z in T , we have that $\{u, v\} \subseteq \gamma_{\mathcal{D}}(z)$. This implies that $\{u, v\} \subseteq \beta_{\mathcal{D}'}(z)$. So we have proved Properties (a) and (b) of Definition 1.

To prove Property (c) of Definition 1 with respect to \mathcal{D}' , we pick an arbitrary vertex $u \in V(G)$, and prove that the set of nodes $R = \{t \in V(T) : u \in \beta_{\mathcal{D}'}(t)\}$ induces a (connected) subtree of T . Observe that $R = O_u \cup (\bigcup_{s \in N_B(u)} R'_s)$ where $O_u = \{t \in V(T) : u \in \beta_{\mathcal{D}}(t)\}$, and $R'_s = \{t \in V(T) : s \in \beta_{\mathcal{D}}(t) \text{ and } u \in \gamma_{\mathcal{D}}(t)\}$ for each $s \in N_B(u)$. To prove that $T[R]$ is connected, it is enough to prove that (i) $T[O_u]$ is connected, (ii) $R'_s \cap O_u \neq \emptyset$ for all $s \in N_B(u)$, and (iii) $T[R'_s]$ is connected for all $s \in N_B(u)$. Statement (i) follows from Property (c) of the tree decomposition \mathcal{D} of B . For any $s \in N_B(u)$, since $\{u, s\} \in E(B)$ and by Property (b) of \mathcal{D} , we have that $R'_s \cap O_t \neq \emptyset$, and hence Statement (ii) follows.

The proof of the lemma will be complete with the proof of Statement (iii). Towards this, let $R_s = \{t \in V(T) : s \in \beta_{\mathcal{D}}(t)\}$ for all $s \in N_B(u)$. Clearly $R'_s \subseteq R_s$. By Property (c) of \mathcal{D} , we know that for any $s \in N_B(u)$, R_s induces a (connected) subtree T_s of T . We claim that for any $s \in N_B(u)$, $T[R'_s]$ (which is a subgraph of T_s) is a (connected) subtree of T_s . Towards a contradiction, suppose that $T[R'_s]$ is not connected. Then, let C and C' be two connected components of $T[R'_s]$ such that there exists a path P in T_s from a vertex in C to a vertex in C' whose internal vertices all belong to $V(T_s) \setminus R'_s$. Then, there is an internal vertex w of P such that w is an ancestor of one of the end-vertices of P (because in a rooted tree any internal vertex of a path is an ancestor of an end-vertex of the path and P has at least one internal vertex, else C and C' form one connected component). This implies that $u \in \gamma_{\mathcal{D}}(w)$, because $w \in V(T_s) = R_s$ and w is an ancestor a vertex in R'_s (where by the definition of R'_s , u must belong to the bag of that vertex). This is a contradiction to the fact that $w \notin R'_s$. Therefore, we conclude that $T[R'_s]$ is connected. This completes the proof of the lemma. \square

To simplify statements ahead, from now on, we have the following notation.

Throughout the section, we fix a map graph G , a corresponding planar bipartite graph B of G , an integer $\ell \in \mathbb{N}$, a nice tree decomposition \mathcal{D} of B of width less than ℓ and an ℓ -few cliques tree decomposition \mathcal{D}' of G derived from \mathcal{D} using Definition 3

Recall that $T = T_{\mathcal{D}} = T_{\mathcal{D}'}$ and that for each node $t \in V(T)$, $\beta_{\mathcal{D}'}(t)$ was obtained from $\beta_{\mathcal{D}}(t)$ by replacing every special vertex $s \in S(G)$ with $N_B(s) \cap \gamma_{\mathcal{D}}(s)$.

Definition 4 For a node $t \in V(T)$, we use $\text{Original}(t)$ to denote the set $\beta_{\mathcal{D}}(t) \cap \beta_{\mathcal{D}'}(t)$, $\text{Fake}(t)$ to denote the set $\beta_{\mathcal{D}'}(t) \setminus \beta_{\mathcal{D}}(t)$, and $\text{Cliques}(t)$ to denote the set $\{N_B(s) : s \in S(G) \cap \beta_{\mathcal{D}}(t)\}$ of special cliques of G .

Informally, for a node $t \in V(T)$, $\text{Original}(t)$ denotes the set of vertices of $V(G)$ present in the bag $\beta_{\mathcal{D}}(t)$, $\text{Fake}(t)$ denotes the set of “new” vertices added to $\beta_{\mathcal{D}'}(t)$ while replacing special vertices in $\beta_{\mathcal{D}}(t)$, and $\text{Cliques}(t)$ is the set of special cliques in G that consist of one for each special vertex $s \in \beta_{\mathcal{D}}(t)$. For example, let t be the node in Figure 3 that is labelled with **forget**(v_1) by \mathcal{D} . Then, $\text{Original}(t) = \emptyset$, $\text{Fake}(t) = \{v_1\}$ and $\text{Cliques}(t) = \{\{v_1, \dots, v_4\}\}$.

In the remainder of this section we prove properties related to \mathcal{D} and \mathcal{D}' , which we use later in the paper to design some of our subexponential-time parameterized algorithms. Towards the formulation of the first property, consider the tree decomposition \mathcal{D}' in Figure 3 and the set of its nodes whose bags contain the vertex v_1 as a “fake” vertex. This set of nodes forms a path with one end-vertex being the unique node t_{v_1} of T labelled with **forget**(v_1) by \mathcal{D} and the other end-vertex being an ancestor of t_{v_1} . In fact, the set of nodes $Q = \{t \in V(T) : v_1 \in \text{Fake}(t) \text{ and } r \in \beta_{\mathcal{D}}(t)\}$ forms the unique path in T from t_{v_1} to t_r where t_r is the unique child of the node labelled with **forget**(r) by \mathcal{D} . This observation is abstracted and formalized in the following lemma.

Lemma 3 *Let $v \in V(G)$ and $s \in S(G)$ such that $v \in N_B(s)$ and $Q = \{t \in V(T) : v \in \text{Fake}(t) \text{ and } s \in \beta_{\mathcal{D}}(t)\} \neq \emptyset$. Let x be the node in T labelled with **forget**(v) by \mathcal{D} , and y be the unique child of the node labelled with **forget**(s) by \mathcal{D} . Then, y is an ancestor of x , and Q induces a path in T which is the unique path between x and y in T .*

Proof First, we prove that Q induces a (connected) subtree of T . Suppose not. Then, there exist two connected components C_1 and C_2 of $T[Q]$ such that there exists a path P in T from a vertex in C_1 to a vertex in C_2 whose internal vertices all belong to $V(T) \setminus Q$. By Property (c) of the tree decomposition \mathcal{D} , we have that $s \in \beta_{\mathcal{D}}(t)$ for any $t \in V(P)$. Moreover, there is an internal vertex w of P such that w is an ancestor of one of the end-vertices of P . This implies that $v \in \gamma_{\mathcal{D}}(w)$, because v belong to the bags of the endpoints of P (by the definition of Q and **Fake**). As we have also shown that $s \in \beta_{\mathcal{D}}(t)$ for all $t \in V(P)$, this implies that $w \in Q$, which is a contradiction. Hence, we have proved that $T[Q]$ is connected.

Next, we prove that $T[Q]$ is a path such that one of its endpoints is a descendant of the other. Towards this, it is enough to prove that (i) for any distinct $t, t' \in Q$, either t is a descendant of t' or t' is a descendant of t . For the sake of contradiction, assume that there exist $t, t' \in Q$ such that neither t is a descendant of t' nor t' is a descendant of t . By the definition of Q and because $t, t' \in Q$, we have that $v \in \gamma_{\mathcal{D}}(t)$ and $v \in \gamma_{\mathcal{D}}(t')$. Thus by Property (c) of the tree decomposition \mathcal{D} , we have that $v \in \beta_{\mathcal{D}}(t)$ and $v \in \beta_{\mathcal{D}}(t')$. Because $v \in \text{Fake}(t)$ and $v \in \text{Fake}(t')$, this is a contradiction to the definition of **Fake**.

It remains to prove that y is an ancestor of x and that x and y are endpoints of $T[Q]$. Towards this, recall that x is the node in T labelled with **forget**(v) by the tree decomposition \mathcal{D} . First, we prove that x is an end-vertex of the path $T[Q]$. Let x' be the only child of x . To prove x is an end-vertex of the path $T[Q]$, it is enough to show that $x \in Q$ and $x' \notin Q$. Since x is labelled with **forget**(v) by \mathcal{D} , we have that $v \notin \beta_{\mathcal{D}}(x)$, $v \in \beta_{\mathcal{D}}(x')$, and $v \in \gamma_{\mathcal{D}}(x)$. This implies that $v \in \text{Original}(x')$ and hence $x' \notin Q$. Now, we prove that $x \in Q$. For this purpose, let $R = \{t \in V(T) : s \in \beta_{\mathcal{D}}(t)\}$. Clearly, $Q \subseteq R$. By Property (c) of the tree decomposition \mathcal{D} , we have that $T[R]$ is connected. We have already proved that $T[Q]$ is a path and since $Q \subseteq R$, $T[Q]$ is a path in $T[R]$. Since x is labelled with **forget**(v) by \mathcal{D} , for any node x'' in the subtree rooted at x and $x'' \neq x$, by Observation 2, either $v \in \beta_{\mathcal{D}}(x'')$ or $v \notin \gamma_{\mathcal{D}}(x'')$. This implies

that Q contains no node in the subtree of T rooted at x and not equal to x . Moreover, observe that there exists a node x^* in the subtree of T rooted at x such that $\{s, v\} \subseteq \beta_{\mathcal{D}}(x^*)$ and hence $x^* \in R$. Now, since Q is non-empty and $T[Q]$ is connected, we have that $s \in \beta_{\mathcal{D}}(x)$. Since $v \notin \beta_{\mathcal{D}}(x)$, $v \in \gamma_{\mathcal{D}}(x)$ and $s \in \beta_{\mathcal{D}}(x)$, we conclude that $x \in Q$. Thus, we have proved that x is an end-vertex of the the path $T[Q]$.

Next we prove that y is the other end-vertex of the path $T[Q]$ and y is an ancestor of x . Since y is the only child of the node y' labelled with **forget**(s), we have that $s \in \beta_{\mathcal{D}}(y)$ and $s \notin \beta_{\mathcal{D}}(y')$. This implies that $y' \notin Q$. Thus to prove that y is an end-vertex of the path $T[Q]$, it is enough to prove that $y \in Q$. Since $s \in \beta_{\mathcal{D}}(x)$, $s \in \beta_{\mathcal{D}}(y)$, $s \notin \beta_{\mathcal{D}}(y')$ and y' is the parent of y , by Property (c) of \mathcal{D} , we have that y is an ancestor of x . This also implies that $v \in \gamma_{\mathcal{D}}(y)$ and $v \notin \beta_{\mathcal{D}}(y)$. Hence, $y \in Q$. This completes the proof of the lemma. \square

In the next lemma we show that for any special vertex $s \in S(G)$ and any node t in T labelled with **introduce**(s) by \mathcal{D} , it holds that t and its child carry the “same information”.

Lemma 4 *Let $s \in S(G)$ and t be a node in T labelled with **introduce**(s) by \mathcal{D} . Let t' be the only child of t . Then, $\text{Original}(t) = \text{Original}(t')$ and $\text{Fake}(t) = \text{Fake}(t')$.*

Proof We know that $\beta_{\mathcal{D}}(t) \setminus \{s\} = \beta_{\mathcal{D}}(t')$. This implies that $\text{Original}(t) = \text{Original}(t')$. To prove that $\text{Fake}(t) = \text{Fake}(t')$, it is enough to show that $\text{Fake}(t) \cap N_B(s) = \emptyset$ (because any special vertex s' in $\beta_{\mathcal{D}}(t')$ and not equal to s , is also belongs to $\beta_{\mathcal{D}}(t)$). Suppose by way of contradiction that $\text{Fake}(t) \cap N_B(s) \neq \emptyset$ and let $u \in \text{Fake}(t) \cap N_B(s)$. Then, there is a descendent t_1 of t such that $t_1 \neq t$ and $u \in \beta_{\mathcal{D}}(t_1)$. Thus, since $\{u, s\} \in E(B)$ and by Properties (b) and (c) of the tree decomposition \mathcal{D} , we get that $\{u, s\} \subseteq \beta_{\mathcal{D}}(t)$. This implies that $u \in \text{Original}(t)$, which is a contradiction to the assumption that $u \in \text{Fake}(t) \cap N_B(s)$. \square

Next, we see a property of nodes $t \in V(T)$ labelled with **join**.

Lemma 5 *Let t be a node in T labelled with **join** by \mathcal{D} , and t_1 and t_2 are its children. Then, $\text{Original}(t) = \text{Original}(t_1) = \text{Original}(t_2)$, $\text{Cliques}(t) = \text{Cliques}(t_1) = \text{Cliques}(t_2)$, $\text{Fake}(t_1) \cap \text{Fake}(t_2) = \emptyset$, and $\text{Fake}(t) = \text{Fake}(t_1) \cup \text{Fake}(t_2)$.*

Proof Since t is a node in T labelled with **join** by \mathcal{D} and t_1 and t_2 are its children, we have that $\beta_{\mathcal{D}}(t) = \beta_{\mathcal{D}}(t_1) = \beta_{\mathcal{D}}(t_2)$. This implies that $\text{Original}(t) = \text{Original}(t_1) = \text{Original}(t_2)$, $\text{Cliques}(t) = \text{Cliques}(t_1) = \text{Cliques}(t_2)$ and $\text{Fake}(t) = \text{Fake}(t_1) \cup \text{Fake}(t_2)$. For any $v \in \text{Fake}(t_i)$, $i \in \{1, 2\}$, we know that $v \notin \beta_{\mathcal{D}}(t_i)$, but there is a descendent t'_i of t_i such that $v \in \beta_{\mathcal{D}}(t'_i)$. Thus, by Property (c) of the tree decomposition \mathcal{D} , we have that $v \notin \gamma_{\mathcal{D}}(t_j)$ where $j \in \{1, 2\} \setminus \{i\}$. This implies that $\text{Fake}(t_1) \cap \text{Fake}(t_2) = \emptyset$. \square

Now, we define a notion of *nice ℓ -few cliques tree decomposition* of G as the tree decomposition of G derived from a nice tree decomposition \mathcal{D} of B of width less than ℓ (see Definition 3) with additional labeling of nodes. In what

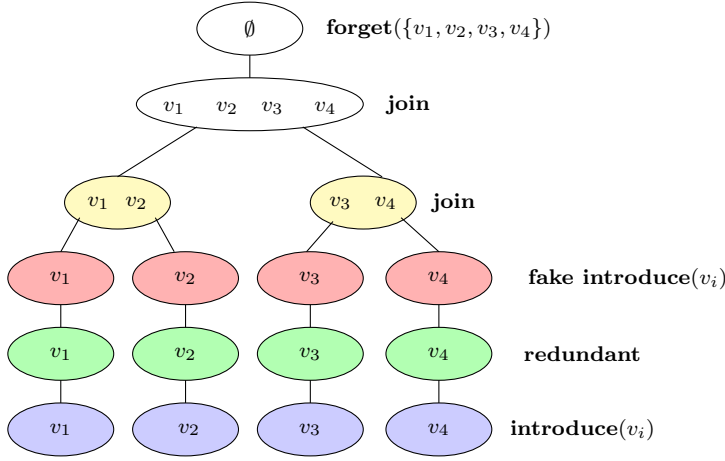


Fig. 4: Labeling the nodes in the nice 2-few cliques tree decomposition \mathcal{D}' derived from the nice tree decomposition \mathcal{D} in Figure 3.

follows, we describe this additional labeling of nodes. Towards this, observe that because of Lemma 4, for any special vertex $s \in S(G)$ and any node $t \in V(T)$ labelled with **introduce**(s) by the nice tree decomposition \mathcal{D} , the bags $\beta_{\mathcal{D}'}(t)$ and $\beta_{\mathcal{D}'}(t')$ carry the “same information” where t' is the only child of t . Informally, one may choose to handle these nodes by contracting them. However, to avoid redundant proofs ahead, instead of getting rid of such nodes, we label them with **redundant** in \mathcal{D}' . Next, we explain how to label other nodes of T in the decomposition \mathcal{D}' (see Figure 4). To this end, let $t \in V(T)$.

- If t is labelled with **leaf** by \mathcal{D} , then we label t with **leaf**. Here, $\beta_{\mathcal{D}'}(t) = \emptyset$.
- If t is labelled with **introduce**(v) by \mathcal{D} for some $v \in V(G)$, then we label t with **introduce**(v). In this case, t has only one child t' in T (because any node labelled with **introduce** by \mathcal{D} has only one child) and $\beta_{\mathcal{D}'}(t) \setminus \{v\} = \beta_{\mathcal{D}'}(t')$.
- If t is labelled with **forget**(v) by \mathcal{D} for some $v \in V(G)$ and $v \in \text{Fake}(t)$, then we label t with **fake introduce**(v). In this case, t has only one child t' and $\beta_{\mathcal{D}'}(t) = \beta_{\mathcal{D}'}(t')$, but $\text{Original}(t) = \text{Original}(t') \setminus \{v\}$ and $\text{Fake}(t) = \text{Fake}(t') \cup \{v\}$.
- If t is labelled with **forget**(v) by \mathcal{D} for some $v \in V(G)$ and $v \notin \text{Fake}(t)$, then we label t with **forget**(v). In this case, t has only one child t' , $\beta_{\mathcal{D}'}(t) = \beta_{\mathcal{D}'}(t') \setminus \{v\}$, $\text{Original}(t) = \text{Original}(t') \setminus \{v\}$ and $\text{Fake}(t) = \text{Fake}(t')$.
- Suppose t is labelled with **forget**(s) by \mathcal{D} for some $s \in S(G)$. Then, t has only one child t' . Here, we label t with **forget**($\beta_{\mathcal{D}'}(t') \setminus \beta_{\mathcal{D}'}(t)$). In this case, $\text{Fake}(t) \subseteq \text{Fake}(t')$ and $\text{Original}(t) = \text{Original}(t')$.
- If t is labelled with **join** by \mathcal{D} , then we label t with **join**. Let t_1 and t_2 be the children of t . Then, $\text{Original}(t) = \text{Original}(t_1) = \text{Original}(t_2)$, $\text{Cliques}(t) = \text{Cliques}(t_1) = \text{Cliques}(t_2)$, $\text{Fake}(t_1) \cap \text{Fake}(t_2) = \emptyset$, and $\text{Fake}(t) = \text{Fake}(t_1) \cup \text{Fake}(t_2)$. (See Lemma 5).

- If t is labelled with **introduce**(s) for some $s \in S(G)$, then we label t with **redundant** in \mathcal{D}'

This completes the definition of the nice ℓ -few cliques tree decomposition of G derived from \mathcal{D} , to which we simply call an (ℓ, \mathcal{D}) -NFewCliTD. Notice that for each node t in T , $|\text{Original}(t)| + |\text{Cliques}(t)| \leq \ell$. That is, for any node $t \in V(T)$, there exist $i, j \in \mathbb{N}$ such that $i + j \leq \ell$, the cardinality of $\text{Original}(t)$ is at most i , and the vertices in $\beta_{\mathcal{D}'}(t) \setminus \text{Original}(t)$ were obtained from at most j special cliques.

Since the number of nodes with label **forget**(v) in the tree decomposition \mathcal{D} is exactly one for any $v \in V(B)$ (see Observation 1), *at most one* node in T is labelled with **fake introduce**(v) in \mathcal{D}' . This is formally stated in the following observation.

Observation 3 *Let $\mathcal{D}' = (T, \beta_{\mathcal{D}'})$ be an (ℓ, \mathcal{D}) -NFewCliTD of a map graph G , for some $\ell \in \mathbb{N}$, derived from a nice tree decomposition \mathcal{D} of a corresponding planar bipartite graph of G . Let $t \in V(T)$ and $v \in \text{Fake}(t)$. Then,*

- (i) *there is a unique node $t' \in V(T)$ such that t' is labelled with **fake introduce**(v) in \mathcal{D}' ,*
- (ii) *t is an ancestor of t' or $t = t'$, and*
- (iii) *for any node t'' in the unique path between t and t' , we have that $v \in \text{Fake}(t'')$.*

The correctness of Observation 3 follows from Observation 1 and Lemma 3. The discussion above, along with Corollary 1 and Proposition 1, implies the following lemma.

Lemma 6 *Given a map graph G with a corresponding planar bipartite graph B and an integer $\ell \in \mathbb{N}$, in time $\mathcal{O}(n^2)$, one can either correctly conclude that B contains an $\ell \times \ell$ grid as a minor, or compute a nice tree decomposition \mathcal{D} of B of width less than 5ℓ and a $(5\ell, \mathcal{D})$ -NFewCliTD of G .*

Lastly, we prove an important property of a tree decomposition \mathcal{D}' of a map graph G that is derived from a tree decomposition \mathcal{D} of a corresponding bipartite planar graph of G . In particular, the edges considered in the following lemma are precisely those that connect the vertices “already seen” (when we use dynamic programming (DP)) with vertices to “see in the future”.

Lemma 7 *For any node $t \in V(T)$, the edges with one endpoint in $\gamma_{\mathcal{D}'}(t)$ and other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ are of two kinds.*

- *Edges incident with vertices in $\text{Original}(t)$.*
- *Edges belonging to some special clique in $\text{Cliques}(t)$ (these edges are incident to vertices in $\text{Fake}(t)$).*

Proof Fix $t \in V(T)$. Since \mathcal{D}' is a tree decomposition of G , for any edge $e \in E(G)$ with one endpoint in $\gamma_{\mathcal{D}'}(t)$ and other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$, the endpoint of e in $\gamma_{\mathcal{D}'}(t)$ should belong to $\beta_{\mathcal{D}'}(t)$. Let u be the endpoint of e that belongs

to $\beta_{\mathcal{D}'}(t)$, and v be the other endpoint of e . Notice that the set $\beta_{\mathcal{D}'}(t)$ is partitioned into $\text{Original}(t)$ and $\text{Fake}(t)$, so u belongs to either $\text{Original}(t)$ or $\text{Fake}(t)$, and in the former case we are done. We now assume that $u \in \text{Fake}(t)$.

Since $\{u, v\} = e \in E(G)$, there is a special vertex $s \in S(G)$ such that $\{u, s\}, \{v, s\} \in E(B)$. If $s \in \beta_{\mathcal{D}}(t)$, then the edge $\{u, v\}$ belongs to the special clique $K = N_B(s)$ in G and $K \in \text{Cliques}(t)$. We claim that indeed $s \in \beta_{\mathcal{D}}(t)$. Towards this, notice that $u \in \text{Fake}(t)$. This implies that $u \notin \beta_{\mathcal{D}}(t)$, but u is present in a bag $\beta_{\mathcal{D}}(t')$ of some descendant t' of t . Moreover, since $\{u, s\} \in E(B)$, we further know that $\{u, s\} \subseteq \beta_{\mathcal{D}}(t_1)$ for some descendent t_1 of t . Since $\{v, s\} \in E(B)$ and $v \notin \gamma_{\mathcal{D}'}(t)$, there is a bag $\beta_{\mathcal{D}}(t_2)$ such that $\{v, s\} \subseteq \beta_{\mathcal{D}}(t_2)$ and t_2 is not a descendent of t . Thus, since $s \in \beta_{\mathcal{D}}(t_1) \cap \beta_{\mathcal{D}}(t_2)$, by Property (c) of the tree decomposition \mathcal{D} , we have that $s \in \beta_{\mathcal{D}}(t)$. This completes the proof of the lemma. \square

4 Feedback Vertex Set

In this section, we give some simple applications of the computation of an (ℓ, \mathcal{D}) -FewCliTD in designing subexponential-time parameterized algorithms on map graphs. We exemplify our approach by developing a subexponential-time parameterized algorithm for FEEDBACK VERTEX SET. This approach can be used to design subexponential-time parameterized algorithms for the more general CONNECTED PLANAR \mathcal{F} -DELETION problem (discussed below) as well as CONNECTED VERTEX COVER and CONNECTED FEEDBACK VERTEX SET. These simple applications already substantially improve upon the known algorithms for these problems on map graphs [25,26]. We first prove the following theorem, and then show how the idea of the proof can be generalized to CONNECTED PLANAR \mathcal{F} -DELETION.

Theorem 4 FEEDBACK VERTEX SET *on map graphs can be solved in time $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$.*

The starting point of the algorithm is that the existence of a large grid as a minor in the corresponding planar bipartite graph B or a large clique implies that the given instance is a NO-instance. Indeed we now observe that if we find a large grid as a minor in B , then we can find many vertex disjoint cycles in the map graph, and hence we can answer NO.

Observation 5 *Let (G, B, k) be an instance of FEEDBACK VERTEX SET. If B contains a $3(\sqrt{k} + 1) \times 3(\sqrt{k} + 1)$ grid as a minor, then (G, B, k) is a NO-instance.*

Proof From the existence of a $3\sqrt{k+1} \times 3(\sqrt{k} + 1)$ grid as a minor in B , we can conclude that B contains $k + 1$ vertex-disjoint cycles of length at least 8 each. For any cycle C of length $\ell \geq 6$ in B , there is a cycle of length $\ell/2$ in G whose set of vertices is a subset of $V(C)$. This implies that if B has $k + 1$ vertex-disjoint cycles of length at least 8 each, then there are $k + 1$ vertex-disjoint cycles in G . \square

This observation leads us to the following lemma.

Lemma 8 *There is an algorithm that given an instance (G, B, k) of FEEDBACK VERTEX SET, runs in time $\mathcal{O}(n^2)$, and either correctly concludes that the minimum size of a feedback vertex set of G is more than k , or outputs a nice tree decomposition \mathcal{D} of B and a $(15(\sqrt{k}+1), \mathcal{D})$ -NFewCliTD \mathcal{D}' of G such that for each $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq 15(k+2)(\sqrt{k}+1)$ and $\beta_{\mathcal{D}'}(t)$ is a union of $15(\sqrt{k}+1)$ many cliques of size at most $k+2$ each.*

Proof If $|N_B(s)| \geq k+3$ for some special vertex $s \in S(G)$, then G has a clique of size $k+3$, and hence (G, B, k) is a NO instance. Thus, we now suppose that this is not the case. Now, we apply Lemma 6 with $\ell = 3(\sqrt{k}+1)$. If the output is a $3(\sqrt{k}+1) \times 3(\sqrt{k}+1)$ grid minor of B , then by Observation 5, (G, B, k) is a NO-instance. Otherwise, we have a nice tree decomposition \mathcal{D} of B of width less than $15(\sqrt{k}+1)$ and a nice $(15(\sqrt{k}+1))$ -few cliques tree decomposition \mathcal{D}' of G . In this case, since $|N_B(s)| \leq k+2$ for every $s \in S(G)$, we have that for any $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq 15(k+2)(\sqrt{k}+1)$. The bound on the number of cliques follows from the width of \mathcal{D} . \square

Because of Lemma 8, to prove Theorem 4, we can focus on FEEDBACK VERTEX SET on map graphs where the input is accompanied with a nice $(15(\sqrt{k}+1))$ -few cliques tree decomposition \mathcal{D}' of G such that for each $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq 15(k+2)(\sqrt{k}+1)$ and $\beta_{\mathcal{D}'}(t)$ is a union of $15(\sqrt{k}+1)$ many cliques of size at most $k+2$ each. The proof of Theorem 4 is by a dynamic programming (DP) algorithm using the fact that for any $t \in V(T)$, $\beta_{\mathcal{D}'}(t)$ is a union of $15(\sqrt{k}+1)$ many cliques of size at most $k+2$. Observe that for any $t \in V(T)$, any feedback vertex set must contain all but two vertices in each clique. Thus, for each clique we have at most $\mathcal{O}(k^2)$ choices of which vertices of the clique belong to a solution. Briefly, for any node $t \in V(T)$, a subset $S \subseteq \beta_{\mathcal{D}'}(t)$ such that S contains all but at most 2 vertices from each clique in the bag $\beta_{\mathcal{D}'}(t)$, a partition \mathcal{P} of $\beta_{\mathcal{D}'}(t) \setminus S$ and $k' \leq k$, we have DP table entry $\mathcal{A}[t, S, \mathcal{P}, k']$ which stores a Boolean value. The table entry $\mathcal{A}[t, S, \mathcal{P}, k']$ is set to 1 if and only if there is a feedback vertex set F of $G[\gamma_{\mathcal{D}'}(t)]$ of size k' such that $F \cap \beta_{\mathcal{D}'}(t) = S$ and for any block P of \mathcal{P} , all the vertices of P belong to a connected component of $G[\gamma_{\mathcal{D}'}(t)] - F$. Notice that the cardinality of $\beta_{\mathcal{D}'}(t) \setminus S$ is upper bounded by $\mathcal{O}(\sqrt{k})$. This allows us to bound the number of states by $2^{\mathcal{O}(\sqrt{k} \log k)}$. After this observation the dynamic programming is identical to the one made for FEEDBACK VERTEX SET on graphs of bounded treewidth. See the book [14] for further details on the dynamic programming algorithm for FEEDBACK VERTEX SET on graphs of bounded treewidth.

By following similar lines as in the case of the above algorithm for FEEDBACK VERTEX SET, we can design subexponential-time parameterized algorithms for CONNECTED FEEDBACK VERTEX SET and CONNECTED VERTEX COVER on map graphs.

Theorem 6 *CONNECTED FEEDBACK VERTEX SET and CONNECTED VERTEX COVER on map graphs can be solved in time $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$.*

Our algorithm for FEEDBACK VERTEX SET can be generalized to a large class of problems, namely, the class of CONNECTED PLANAR \mathcal{F} -DELETION problems. In this class, each problem is defined by family \mathcal{F} of connected graphs that contains at least one planar graph. Here, the input is a graph G and an integer parameter k . The goal is to find a set S of size at most k such that $G - S$ does not contain any of the graphs in \mathcal{F} as a minor. This definition captures problems such as VERTEX COVER, FEEDBACK VERTEX SET, TREewidth- η VERTEX DELETION, PATHwidth- η VERTEX DELETION, TREEDePTH- η VERTEX DELETION, DIAMOND HITTING SET and OUTERPLANAR VERTEX DELETION. Theorem 4 can be generalized to the following general theorem.

Theorem 7 *Every CONNECTED PLANAR \mathcal{F} -VERTEX DELETION problem on map graphs can be solved in time $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$.*

Similarly to FEEDBACK VERTEX SET, we can prove that there is a constant c (depending only on \mathcal{F}) such that if there is a $c\sqrt{k} \times c\sqrt{k}$ -grid minor in B , then the given instance is a NO-instance. Moreover, if there is a clique of size at least $k + d + 1$ in G , where d is the size of the smallest graph in \mathcal{F} , then also the given instance is a NO-instance. These two arguments imply that there is an algorithm which given an instance (G, B, k) of CONNECTED PLANAR \mathcal{F} -VERTEX DELETION, runs in time $\mathcal{O}(n^2)$, and either correctly concludes that (G, B, k) is a NO-instance or outputs a nice $\mathcal{O}(\sqrt{k})$ -few cliques tree decomposition \mathcal{D}' of G such that for each $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq \mathcal{O}(k^2)$ and $\beta_{\mathcal{D}'}(t)$ is a union of $\mathcal{O}(\sqrt{k})$ many cliques of size at most $k + d$ each. As in the case of FEEDBACK VERTEX SET, any solution of CONNECTED PLANAR \mathcal{F} -DELETION contains all but at most $d - 1$ vertices from any clique. Thus, for each clique of size k' in $\beta_{\mathcal{D}'}(t)$ we have at most $\mathcal{O}\binom{k'}{k'-d-1} = \mathcal{O}(k^d)$ (because $k' \leq k + d + 1$) choices of which vertices of the clique belong to a solution. This allows us to bound the number of “states” by $(k^d)^{\mathcal{O}(\sqrt{k})} = 2^{\mathcal{O}(\sqrt{k} \log k)}$. After this observation the dynamic programming is identical to the one made for CONNECTED PLANAR \mathcal{F} -VERTEX DELETION on graphs of bounded treewidth. That is, given a tree decomposition of width w , there is an algorithm solving CONNECTED PLANAR \mathcal{F} -VERTEX DELETION in time $2^{\mathcal{O}(w \log w)} n^{\mathcal{O}(1)}$ [5]. Following this algorithm with our observation results in an algorithm with time complexity $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ for CONNECTED PLANAR \mathcal{F} -VERTEX DELETION problem on map graphs.

5 Longest Cycle

In the last section we saw simple applications of the computation of an (ℓ, \mathcal{D}) -FewCliTD on map graphs. In this section as well as Section 6, we will see more involved applications of (ℓ, \mathcal{D}) -FewCliTD. Specially, in this section we prove that LONGEST CYCLE admits a subexponential-time parameterized algorithm on map graphs.

Theorem 8 *LONGEST CYCLE on map graphs can be solved in $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ time.*

Towards the proof of Theorem 8, we prove that if there is a solution (i.e., a cycle of length at least k), then there is one for which a “sublinear crossing lemma” holds. Informally, the sublinear crossing lemma asserts the existence of a solution such that at any separator (bag) of a given (ℓ, \mathcal{D}) -NFewCliTD, the number of edges crossing the separation is $\mathcal{O}(\sqrt{k})$. This lemma lies at the heart of the proof and is one of the main technical contributions of the paper.

Towards proving Theorem 8, we design an algorithm that given a map graph G along with a corresponding bipartite planar graph B and $k \in \mathbb{N}$, runs in time $2^{\mathcal{O}(\sqrt{k} \log k)} n^{\mathcal{O}(1)}$ and decides whether G has a cycle of length at least k . Notice that if there is a special vertex $s \in S(G)$ such that $|N_B(s)| \geq k$, then G has a cycle of length at least k , because $N_B(s)$ forms a clique in G . Moreover, observe that if there is a “large enough” grid in B , then we can answer YES. These observations lead to the following lemma.

Lemma 9 *There is an algorithm that given an instance (G, B, k) of LONGEST CYCLE, runs in time $\mathcal{O}(n^2)$, and either correctly concludes that G has a cycle of length at least k , or outputs a nice tree decomposition \mathcal{D} of B of width $< 5\sqrt{2k}$ and a $(5\sqrt{2k}, \mathcal{D})$ -NFewCliTD \mathcal{D}' of G such that for each node $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq 5\sqrt{2} \cdot k^{1.5}$.*

Proof As mentioned before, if $|N_B(s)| \geq k$ for some special vertex $s \in S(G)$, then G has a cycle of length k . Thus, we now suppose that this is not the case. Now, we apply Lemma 6 with $\ell = \sqrt{2k}$. If the output is a $\sqrt{2k} \times \sqrt{2k}$ grid minor of B , then B has a cycle of length at least $2k$, and this implies that G has a cycle of length at least k . Otherwise, we have a nice tree decomposition \mathcal{D} of B of width less than $5\sqrt{2k}$ and a nice $(5\sqrt{2k}, \mathcal{D})$ -NFewCliTD \mathcal{D}' of G . In this case, since $|N_B(s)| < k$ for every $s \in S(G)$, we have that for each node $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq 5\sqrt{2} \cdot k^{1.5}$. \square

Because of Lemma 9, to prove Theorem 8, it is enough to prove the following lemma.

Lemma 10 *There is an algorithm that given an instance (G, B, k) of LONGEST CYCLE, and a $(5\sqrt{2k}, \mathcal{D})$ -NFewCliTD \mathcal{D}' of G (derived from a nice tree decomposition \mathcal{D} of B) such that for each node $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq 5\sqrt{2} \cdot k^{1.5}$, runs in time $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$, and outputs a longest cycle in G .*

Towards proving Lemma 10, the main ingredient is to prove the following claim: if G has a cycle of length ℓ , then there is a cycle C of length ℓ , with the following property.

For each node $t \in V(T)$, the number of edges of $E(C)$ with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is upper bounded by $\mathcal{O}(\sqrt{k})$.

The above mentioned property is encapsulated in the following sublinear crossing lemma.

Lemma 11 (Sublinear Crossing Lemma) *Let (G, B, k) be an instance of LONGEST CYCLE. Let \mathcal{D} be a nice tree decomposition of B and \mathcal{D}' be a $(5\sqrt{2k}, \mathcal{D})$ -NFewCliTD of G . Let C be a cycle in G . Then there is a cycle C' of the same length as C such that for any node $t \in V(T)$, the number of edges in $E(C')$ with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $20\sqrt{2k}$.*

Towards proving Lemma 11, we first prove the following lemma.

Lemma 12 *Let (G, B, k) be an instance of LONGEST CYCLE. Let \mathcal{D} be a nice tree decomposition of B and \mathcal{D}' be a $(5\sqrt{2k}, \mathcal{D})$ -NFewCliTD of G . Let C be a cycle in G and K be a special clique in G . Then, there is a cycle C' of the same length as C such that $E(C') \setminus E(K) = E(C) \setminus E(K)$ and for any node $t \in V(T)$, the number of edges of $E(C') \cap E(K)$ with one endpoint in $\text{Fake}(t) \cap K$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most 4.*

Before formally proving Lemma 12, we give a high level overview of the proof and an auxiliary lemma which we use in the proof of Lemma 12. The proof idea is to change the edges of $E(K) \cap E(C)$ in C (because in Lemma 12 our objective is to bound the “crossing edges” from a subset of $E(K)$ for each node $t \in V(T)$) to obtain a new cycle C' of the same length as C that satisfies the following property: (i) for any node $t \in V(T)$, the number of edges of $E(C') \cap E(K)$ with one endpoint in $\text{Fake}(t) \cap K$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most 4. For the ease of presentation, assume that $K \subseteq V(C)$. Now, consider the graph \mathcal{P} obtained from the cycle C after deleting the edges in $E(K)$. Without loss of generality assume that $E(C) \cap E(K) \neq \emptyset$. Otherwise, Lemma 12 is true where $C' = C$. We consider \mathcal{P} as a collection of vertex-disjoint paths where the end-vertices of the paths are in K . Some paths in \mathcal{P} may be of length 0. Let Z be the set of end-vertices of the paths in \mathcal{P} . Clearly, $Z \subseteq K$. We will “complete” the collection of paths \mathcal{P} to a cycle by adding edges from $E(Z)$ satisfying Statement (i). Any cycle C' with $V(C') = V(\mathcal{P})$ has the same length as C . So, all the work that is required for us is to complete the collection of paths \mathcal{P} to a cycle by adding edges from $E(Z)$ satisfying Statement (i). Towards that, let $\hat{\sigma} = v_1, \dots, v_{k'}$ be an arbitrary sequence of vertices in Z . We show (in Claim 1) that (ii) there is a subset of edges $F \subseteq E(Z)$ such that $E(\mathcal{P}) \cup F$ forms a cycle C' with vertex set $V(\mathcal{P})$ and for any $j \in [k']$, the number of edges in F with one endpoint in $\{v_1, \dots, v_j\}$ and the other in $\{v_{j+1}, \dots, v_{k'}\}$ is at most 2. This implies that for any $1 \leq i \leq j \leq k'$, the number of edges in F with one endpoint in $\{v_i, \dots, v_j\}$ and the other in $Z \setminus \{v_i, \dots, v_j\}$ is at most 4. In the light of Statement (ii), our aim will be to prove that (iii) for any node $t \in V(T)$, there exist $1 \leq i \leq j \leq k'$ such that $\text{Fake}(t) \cap Z \subseteq \{v_i, \dots, v_j\}$ and no vertex in $Z \setminus \gamma_{\mathcal{D}'}(t)$ belongs to $\{v_i, \dots, v_j\}$. Then, Statement (i) will follow (because edges of C' incident with vertices in $K \setminus Z$ are from $E(G) \setminus E(K)$ and will not be counted in Statement (i)). In fact, we will prove that there a sequence σ on Z (derived from a postorder transversal of T) such that Statement (iii) is true (see Claim 2). The proof of Statement (ii) is encapsulated in the following lemma (which we will use in the proof of Lemma 12).

Lemma 13 *Let $\ell \geq 3$ be an integer. Let u_1, \dots, u_ℓ be a sequence of vertices in a graph H where $X = \{u_1, \dots, u_\ell\}$ is a clique in H . Let \mathcal{Q} be a family of vertex disjoint paths in H (which possibly contains paths of length 0) such that each $v \in X$ is an end-vertex of a path in \mathcal{Q} and $E(\mathcal{Q}) \cap E(X) = \emptyset$. Then, there is a set $F \subseteq E(X)$ such that the following conditions hold.*

- (a) $E(\mathcal{Q}) \cup F$ forms a cycle containing all the vertices of $V(\mathcal{Q})$,
- (b) For any $j \in [\ell]$, the number of edges in F with one endpoint in $\{u_1, \dots, u_j\}$ and the other in $\{u_{j+1}, \dots, u_\ell\}$ is at most 2.
- (c) If the degree of u_1 is one in \mathcal{Q} (i.e., u_1 is an end-vertex of a path in \mathcal{Q}), then the number of edges in F with u_1 as an endpoint is exactly 1.

Proof We prove the lemma using induction on the length of the sequence ℓ . By slightly abusing the notation, we also use \mathcal{Q} as a subgraph of H where each connected component is a path in the family \mathcal{Q} . Notice that for any vertex $u \in X$, $d_{\mathcal{Q}}(u) \in \{0, 1\}$. Additionally, notice that for any set $F \subseteq E(H)$ such that $E(\mathcal{Q}) \cup F$ forms a cycle containing all the vertices of $V(\mathcal{Q})$, if $d_{\mathcal{Q}}(u_1) = 1$, then the number of edges in F with u_1 as an endpoint is exactly 1 (because the degree of each vertex in a cycle is 2). So to prove the lemma, it is enough to prove conditions (a) and (b) of the lemma.

The base case is when $\ell = 3$. Towards the proof of the base case, suppose that $\mathcal{Q} = \{[u_1, u_2], [u_2, u_3], [u_3, u_1]\}$. Then, the set of edges $F = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_1\}\}$ is a set as required to satisfy the lemma. Otherwise, $\mathcal{Q} = \{[xy], [z]\}$, where $\{x, y, z\} = \{u_1, u_2, u_3\}$. Then, $F = \{\{x, z\}, \{y, z\}\}$ is a set as required to satisfy the lemma.

Now, we consider the induction step. For this purpose, we assume the lemma for any sequence of length at most $\ell - 1$, and consider a sequence of length $\ell > 3$. The proof consist of three cases as follows, depending on the degrees of u_1 and u_2 in \mathcal{Q} .

Case 1: $d_{\mathcal{Q}}(u_1) = 0$ and $d_{\mathcal{Q}}(u_2) = 1$. Let $A = \{\{u_1, u_2\}\}$. Let $\mathcal{Q}' = (V(\mathcal{Q}), E(\mathcal{Q}) \cup A)$ and $X' = X \setminus \{u_2\}$. The subgraph \mathcal{Q}' is a collection of vertex disjoint paths in H such that each $v \in X'$ is an end-vertex of a path in \mathcal{Q}' and $d_{\mathcal{Q}'}(u_1) = 1$. Thus, by induction hypothesis, there is a set $F' \subseteq E(X')$ such that (i) $E(\mathcal{Q}') \cup F'$ forms a cycle containing all the vertices of $V(\mathcal{Q}')$, (ii) for any $i \in [\ell] \setminus \{1\}$, the number of edges in F' with one endpoint in $\{u_1, \dots, u_i\} \setminus \{u_2\}$ and the other in $\{u_{i+1}, \dots, u_\ell\} \setminus \{u_2\}$ is at most 2, and (iii) the number of edges in F' with u_1 as an endpoint is exactly 1. We claim that $F = F' \cup \{\{u_1, u_2\}\}$ is the required set of edges. Since $E(\mathcal{Q}) \cup F = E(\mathcal{Q}') \cup F'$, by (i), $E(\mathcal{Q}) \cup F$ forms a cycle and it uses all the vertices in $V(\mathcal{Q})$ because $\{u_1, u_2\} \in F$. For any $j \in [\ell] \setminus \{1\}$, the edges in F with one endpoint in $\{u_1, \dots, u_j\}$ and other in $\{u_{j+1}, \dots, u_\ell\}$ are also edges in F' , and $\{u_2, u_r\} \notin F'$ for all $r \in [\ell]$. Thus, by (ii), condition (b) of the statement holds for $j \in [\ell] \setminus \{1\}$. Lastly, notice that the number of edges in F with u_1 as one endpoint is exactly 2.

Case 2: $d_{\mathcal{Q}}(u_1) = d_{\mathcal{Q}}(u_2) = 0$. Let $A = \{\{u_1, u_2\}, \{u_1, u_3\}\}$. Let $\mathcal{Q}' = (V(\mathcal{Q}), E(\mathcal{Q}) \cup A)$. The subgraph \mathcal{Q}' is a collection of vertex disjoint paths in H . Let X' is the set of end-vertices of paths in \mathcal{Q}' .

Clearly, $X' \subseteq X$. Since the degree of u_1 in \mathcal{Q}' is 2, we have that $u_1 \notin X'$ and hence $|X'| < |X| = \ell$. Since $d_{\mathcal{Q}}(u_2) = 0$ and only one edge in A is incident with u_2 , we have that $d_{\mathcal{Q}'}(u_2) = 1$ and hence $u_2 \in X'$. More precisely, $X' = X \setminus \{u_1\}$ if $d_{\mathcal{Q}'}(u_3) = 1$ (equivalently $d_{\mathcal{Q}}(u_3) = 0$) and $X' = X \setminus \{u_1, u_3\}$ otherwise. Suppose $|X'| \geq 3$. Then, by induction hypothesis, there is a set $F' \subseteq E(X')$ such that (i) $E(\mathcal{Q}') \cup F'$ forms a cycle containing all the vertices of $V(\mathcal{Q}')$, (ii) for any $i \in [\ell] \setminus \{1\}$, the number of edges in F' with one endpoint in $\{u_2, \dots, u_i\}$ and the other in $\{u_{i+1}, \dots, u_\ell\}$ is at most 2, and (iii) the number of edges in F' with u_2 as an endpoint is exactly 1. We claim that $F = F' \cup \{\{u_1, u_2\}, \{u_1, u_3\}\}$ is the required set of edges. Since $E(\mathcal{Q}) \cup F = E(\mathcal{Q}') \cup F'$, by (i), $E(\mathcal{Q}) \cup F$ forms a cycle and it uses all the vertices in $V(\mathcal{Q})$ because $\{u_1, u_2\} \in F$. For any $j \in [\ell] \setminus \{1, 2\}$, the edges in F with one endpoint in $\{u_1, \dots, u_j\}$ and the other in $\{u_{j+1}, \dots, u_\ell\}$ are also edges in F' , and hence by (ii), condition (b) of the statement holds for $j \in [\ell] \setminus \{1, 2\}$. By (iii) and the fact that $F = F' \cup \{\{u_1, u_2\}, \{u_1, u_3\}\}$, we have that the number of edges in F with one endpoint in $\{u_1, u_2\}$ and the other in $\{u_3, \dots, u_\ell\}$ is exactly 2. Lastly, notice that the number of edges in F with one endpoint u_1 is exactly 2 (these edges are $\{u_1, u_2\}$ and $\{u_1, u_3\}$).

Now consider the case $|X'| = 2$. Then, $|X| = \ell = 4$ and $\mathcal{Q} = \{u_1, u_2, P\}$, where P is a path with endpoints u_3 and u_4 . Here, $F = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_1, u_4\}\}$ is a required set of edges satisfying the conditions of the lemma.

Case 3: $d_{\mathcal{Q}}(u_1) = 1$. Let P be the path in \mathcal{Q} such that u_1 is its end-vertex and let z be the other end-vertex of P . Let x be the first vertex in u_2, \dots, u_ℓ that is not equal to z . That is, $x = u_2$ if $z \neq u_2$ and $x = u_3$ if $z = u_2$. Let $A = \{\{u_1, x\}\}$ and $\mathcal{Q}' = (V(\mathcal{Q}), E(\mathcal{Q}) \cup A)$. Notice that $d_{\mathcal{Q}'}(u_1) = 2$ and $d_{\mathcal{Q}'}(x) \in \{1, 2\}$. If $d_{\mathcal{Q}'}(x) = 1$, then denote $X' = X \setminus \{u_1\}$, and otherwise denote $X' = X \setminus \{u_1, x\}$. The subgraph \mathcal{Q}' is a collection of vertex disjoint paths in H such that each $v \in X'$ is an end-vertex of a path in \mathcal{Q}' . Thus, by induction hypothesis, there is a set $F' \subseteq E(X')$ such that (i) $E(\mathcal{Q}') \cup F'$ forms a cycle containing all the vertices of $V(\mathcal{Q}')$, (ii) for any $i \in [\ell]$, the number of edges in F' with one endpoint in $X' \cap \{u_1, \dots, u_i\}$ and the other in $X' \cap \{u_{i+1}, \dots, u_\ell\}$ is at most 2, and (iii) if $u_2 \in X'$, then $d_{\mathcal{Q}'}(u_2) = 1$ and the number of edges in F' with u_2 as an endpoint is exactly 1. We claim that $F = F' \cup \{\{u_1, x\}\}$ is the required set of edges. Since $E(\mathcal{Q}) \cup F = E(\mathcal{Q}') \cup F'$, by (i), $E(\mathcal{Q}) \cup F$ forms a cycle and it uses all the vertices in $V(\mathcal{Q})$ because $\{u_1, x\} \in F$. Since $x \in \{u_2, u_3\}$ and $F = F' \cup \{\{u_1, x\}\}$, we have that for any $j \in [\ell] \setminus \{1, 2\}$, the edges in F with one endpoint in $\{u_1, \dots, u_j\}$ and the other in $\{u_{j+1}, \dots, u_\ell\}$ are also edges in F' and hence by (ii), condition (b) holds for $j \in [\ell] \setminus \{1, 2\}$. If $x = u_2$, then the number of edge in F with one endpoint in $\{u_1, u_2\}$ and the other in $\{u_3, \dots, u_\ell\}$ is at most 1 (because $d_{\mathcal{Q}}(u_1) = 1$ and $\{u_1, u_2\} \in F$). If $x \neq u_2$, then $z = u_2$ and hence the number of edge in F with one endpoint in $\{u_1, u_2\}$ and the other in $\{u_3, \dots, u_\ell\}$ is exactly 2. So, condition (b) holds for $j = 2$. Lastly, notice that the number of edges in F with u_1 as an endpoint is exactly 1 (because $d_{\mathcal{Q}}(u_1) = 1$).

This completes the proof of the lemma. \square

Next, we move to a formal proof of Lemma 12. For the convenience of the reader we restate the lemma.

Lemma 12 *Let (G, B, k) be an instance of LONGEST CYCLE. Let \mathcal{D} be a nice tree decomposition of B and \mathcal{D}' be a $(5\sqrt{2k}, \mathcal{D})$ -NFewCliTD of G . Let C be a cycle in G and K be a special clique in G . Then, there is a cycle C' of the same length as C such that $E(C') \setminus E(K) = E(C) \setminus E(K)$ and for any node $t \in V(T)$, the number of edges of $E(C') \cap E(K)$ with one endpoint in $\text{Fake}(t) \cap K$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most 4.*

Proof Without loss of generality, assume that $K \subseteq V(C)$. Otherwise, we can consider the statement of the lemma for cycle C in the graph $G' = G - (K \setminus V(C))$ and special clique $K \cap V(C)$ of G' . We also assume that $E(C) \cap E(K) \neq \emptyset$, else the correctness is trivial because we can take C' as C .

Recall that $T = T_{\mathcal{D}} = T_{\mathcal{D}'}$. Let π' be a postorder transversal of the nodes in the rooted binary tree T , and let π be the restriction of π' where we only keep the nodes that are labeled with **fake introduce** (v) for some $v \in K$. Denote $\pi = t_1, \dots, t_{k''}$ such that each $t_i, i \in [k'']$, is labelled with **fake introduce** (x_i) where $x_i \in K$. Notice that $\bigcup_{t \in V(T)} \text{Fake}(t) \cap K = \{x_1, \dots, x_{k''}\}$ (by Observation 3). Let σ_1 be the sequence $x_1, \dots, x_{k''}$ and $U = \{x_1, \dots, x_{k''}\}$. Let σ_2 be a fixed arbitrary sequence of $K \setminus U$, i.e., all the vertices of K that are never “fakely introduced”. Let σ be the sequence which is a concatenation of σ_1 and σ_2 .

Let $\mathcal{P} = (V(C), E(C) \setminus E(K))$. That is, \mathcal{P} is the graph obtained by deleting edges of $E(K)$ from the cycle C . Notice that each connected component of \mathcal{P} is a path (may be of length 0) with end-vertices in K . Let Z be the set of end-vertices of the paths in \mathcal{P} . Notice that for any vertex $u \in K \setminus Z$, both edges of C incident with u are from $E(C) \setminus E(K)$ (see the left part of Figure 5). That is, $E(C) \setminus E(K) = E(C) \setminus E(Z) = E(\mathcal{P})$. Since we seek a cycle C' in which $E(C') \setminus E(K) = E(C) \setminus E(K)$, no edge of C' incident with u for any vertex $u \in K \setminus Z$, is in $E(K)$. That is, all the edges of $E(C') \cap E(K)$ will belong to $E(Z)$. This leads to the following simple observation.

Observation 9 *Let C' be a cycle in G such that $E(C') \setminus E(K) = E(C) \setminus E(K)$ and $t \in V(T)$. The number of edges of $E(C') \cap E(K)$ with one endpoint in $\text{Fake}(t) \cap K$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is equal to the number of edges of $E(C') \cap E(Z)$ with one endpoint in $\text{Fake}(t) \cap Z$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$.*

Let $Z = \{v_1, \dots, v_{k'}\}$ and $\sigma' = \sigma|_Z = v_1, \dots, v_{k'}$. The main ingredients of the proof are the following two claims.

Claim 1 *There is a cycle C' of the same length as C such that (i) $E(C') \setminus E(Z) = E(C) \setminus E(Z)$, and (ii) for any $j \in [k']$, the number of edges of $E(C') \cap E(Z)$ with one endpoint in $\{v_1, \dots, v_j\}$ and the other in $\{v_{j+1}, \dots, v_{k'}\}$ is at most 2.*

Proof Clearly when $k' \leq 2$, $|E(Z)| \leq 1$ and $C' = C$ satisfies the conditions of the claim. To prove the claim for $k' \geq 3$, we apply Lemma 13. Recall

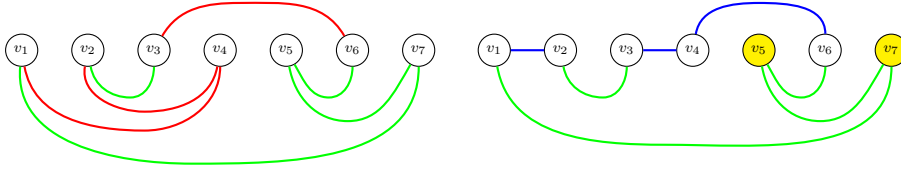


Fig. 5: Left part illustrates a cycle C interacting with a special clique $K = \{v_1, \dots, v_7\}$. The red curves represent edges in $E(C) \cap E(K)$ and green curves represent paths in C with endpoints in K and (at least one) internal vertices in $V(G) \setminus K$. Thus, \mathcal{P} is the collection of paths that is a union of the set of two “green” paths $((v_2 - v_3)$ and $(v_1 - v_7 - v_5 - v_6))$ and $\{[v_4]\}$. Here $Z = \{v_1, v_2, v_3, v_4, v_6\}$. Any edge of $E(C)$ incident with v_5 and v_7 (i.e., vertices in $K \setminus Z$) are from $E(C) \setminus E(K)$. The right part illustrates the proof of Claim 1. The edges of $E(C') \setminus E(C)$ mentioned in the proof of Claim 1 are colored blue.

that $\{v_1, \dots, v_{k'}\} = Z \subseteq K$ and hence Z forms a clique in G . Additionally, recall that \mathcal{P} is a collection of paths such that Z is the set of end-vertices of the paths in \mathcal{P} . Thus, we apply Lemma 13 for the sequence $v_1, \dots, v_{k'}$ of vertices in G and family of paths \mathcal{P} . Then, by Lemma 13, there is a subset $F \subseteq E(Z)$ such that (a) $E(\mathcal{P}) \cup F$ forms a cycle C' containing all the vertices of $V(\mathcal{Q})$ and (b) for any $j \in [k']$, the number of edges in F with one endpoint in $\{v_1, \dots, v_j\}$ and the other in $\{v_{j+1}, \dots, v_{k'}\}$ is at most 2. Since $V(C) = V(\mathcal{P}) = V(C')$, the lengths of cycles C' and C are same. Because of statement (a) and $E(\mathcal{P}) = E(C) \setminus E(K) = E(C) \setminus E(Z)$, we have that $E(C') \setminus E(Z) = E(C) \setminus E(Z)$. Finally, condition (ii) in the claim follows from statement (b). This completes the proof of the claim.

Recall that for a sequence $\sigma' = u_1 u_2 \dots u_\ell$ and any $1 \leq i \leq j \leq \ell$, the sequence $\sigma'' = u_i \dots u_j$ is called a segment of σ' .

Claim 2 *For any node $t \in V(T)$, there is a segment σ'' of σ' such that each vertex in $\text{Fake}(t) \cap Z$ appears in σ'' , and each vertex in $Z \setminus \gamma_{\mathcal{D}'}(t)$ does not appear in σ'' .*

Proof Fix a node $t \in V(T)$. Recall that $\sigma = \sigma_1 \sigma_2$ and $\sigma' = \sigma|_Z$. Here, the set of vertices present in σ_1 is $U = (\bigcup_{t \in V(T)} \text{Fake}(t) \cap K) \supseteq (\bigcup_{t \in V(T)} \text{Fake}(t) \cap Z)$ (because $Z \subseteq K$), and no vertex in σ_2 is from U . This implies that all the vertices of $\text{Fake}(t) \cap Z$ are in the sequence σ_1 . That is, the sequence σ'' we seek is also a sequence of $\sigma_1|_Z$ and this is the reason we defined σ to be $\sigma_1 \sigma_2$. Thus, to prove the claim it is enough to prove that there is a segment σ'_1 of $\sigma_1|_Z$ such that each vertex in $\text{Fake}(t) \cap Z$ appears in σ'_1 and each vertex in $Z \setminus \gamma_{\mathcal{D}'}(t)$ does not appear in σ'_1 .

Recall that $\sigma_1 = x_1 \dots x_{k''}$ is obtained from the sequence $\pi = t_1, \dots, t_{k''}$. In turn, recall that π is the restriction of the postorder transversal π' of T , where for each $i \in [k'']$, t_i is labelled with **fake introduce**(x_i) for $x_i \in K$. Let

W_t be the nodes of the subtree of T rooted at t , and

$$V_t = \{v \in K : \text{there is } t' \in W_t \text{ such that } t' \text{ is labelled with } \mathbf{fake\ introduce}(v)\}.$$

The vertices in W_t appear consecutively in π . Thus, we can let π_t be the minimal segment of π that contains all the nodes in V_t . Let $i, j \in [k'']$ be such that $\pi_t = t_i, \dots, t_j$. Now, we define σ_t be the segment x_i, \dots, x_j of σ_1 . Now we prove the claim. By conditions (i) and (ii) in Observation 3, $\mathbf{Fake}(t) \cap Z \subseteq V_t$. Clearly, no vertex in $Z \setminus \gamma_{\mathcal{D}'}(t)$ is in V_t . This implies that each vertex in $\mathbf{Fake}(t) \cap Z$ appears in σ_t and no vertex from $Z \setminus \gamma_{\mathcal{D}'}(t)$ appears in σ_t . In turn, this implies that $\sigma_t|_Z$ is the required segment σ'' of $\sigma' = \sigma|_Z$. \square

Now, having the above two claims, we are ready to prove the lemma. By Claim 1, we have that there is a cycle C' such that (i) $E(C') \setminus E(Z) = E(C) \setminus E(Z)$, and (ii) for any $j \in [k']$, the number of edges of $E(C) \cap E(Z)$ with one endpoint in $\{v_1, \dots, v_j\}$ and other in $\{v_{j+1}, \dots, v_{k'}\}$ is at most 2. By Claim 2, we know that for any $t \in V(T)$, there is a segment σ'' of σ' such that all vertices in $\mathbf{Fake}(t) \cap Z$ appear in a segment σ'' and no vertex from $Z \setminus \gamma_{\mathcal{D}'}(t)$ appears in σ'' . That is, there exist $i, j \in [k']$ such that $\mathbf{Fake}(t) \cap Z \subseteq \{v_i, \dots, v_j\}$ and $(Z \setminus \gamma_{\mathcal{D}'}(t)) \cap \{v_i, \dots, v_j\} = \emptyset$. Therefore, by (ii), the number of edges of $E(C') \cap E(Z)$ with one endpoint in $\mathbf{Fake}(t) \cap Z$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most 4. Then, by Observation 9, the proof of the lemma is complete. \square

Now we are ready to prove Lemma 11. We restate the lemma below.

Lemma 11 (Sublinear Crossing Lemma) *Let (G, B, k) be an instance of LONGEST CYCLE. Let \mathcal{D} be a nice tree decomposition of B and \mathcal{D}' be a $(5\sqrt{2k}, \mathcal{D})$ -NFewCliTD of G . Let C be a cycle in G . Then there is a cycle C' of the same length as C such that for any node $t \in V(T)$, the number of edges in $E(C')$ with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $20\sqrt{2k}$.*

Proof Let $S(G) = \{s_1, \dots, s_\ell\}$, and let $K_i = N_B(s_i)$ for all $i \in [\ell]$. For any $i \in [\ell]$, let $Z_i = \bigcup_{j \in [i]} K_j$ and $F_i = \bigcup_{j \in [i]} E(K_j)$. We remind that $T = T_{\mathcal{D}} = T_{\mathcal{D}'}$. Towards the proof of the lemma we first prove the following claim using induction on i .

Claim 3 *Let S be a cycle in G . Then, for any $i \in [\ell]$, there is a cycle S_i of the same length as S such that $E(S_i) \setminus F_i = E(S) \setminus F_i$, and for any $t \in V(T)$, the number of edges in $E(S_i) \cap F_i$ with one endpoint in $\mathbf{Fake}(t) \cap Z_i$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $4r$ where $r = |\beta_{\mathcal{D}}(t) \cap \{s_1, \dots, s_i\}|$.*

Proof The base case is when $i = 1$, and it follows from Lemma 12 (by substituting $C = S$ and $K = K_1$). Now, we consider the induction step for $i > 1$. By induction hypothesis, we have that the claim is true for $i - 1$. That is, there is a cycle S_{i-1} of the same length as S such that (i) $E(S_{i-1}) \setminus F_{i-1} = E(S) \setminus F_{i-1}$, and (ii) for any $t \in V(T)$, the number of edges in $E(S_{i-1}) \cap F_{i-1}$ with one endpoint in $\mathbf{Fake}(t) \cap Z_{i-1}$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $4r'$

where $r' = |\beta_{\mathcal{D}}(t) \cap \{s_1, \dots, s_{i-1}\}|$. Now we apply Lemma 12 (by substituting $C = S_{i-1}$ and $K = K_i$). Then, there is a cycle S_i of the same length as S_{i-1} such that (a) $E(S_i) \setminus E(K_i) = E(S_{i-1}) \setminus E(K_i)$, and (b) for any $t \in V(T)$, the number of edges in $E(S_i) \cap E(K_i)$ with one endpoint in $\text{Fake}(t) \cap K_i$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most 4.

Now, we prove that S_i satisfies the conditions in the claim. We begin by proving that $E(S_i) \setminus F_i = E(S) \setminus F_i$.

$$\begin{aligned}
E(S) \setminus F_i &= E(S) \setminus (F_{i-1} \cup E(K_i)) \\
&= (E(S) \setminus F_{i-1}) \setminus E(K_i) \\
&= (E(S_{i-1}) \setminus F_{i-1}) \setminus E(K_i) && \text{(By (i))} \\
&= (E(S_{i-1}) \setminus E(K_i)) \setminus F_{i-1} \\
&= (E(S_i) \setminus E(K_i)) \setminus F_{i-1} && \text{(By (a))} \\
&= E(S_i) \setminus F_i && \text{(Because } F_i = F_{i-1} \cup E(K_i)\text{)}
\end{aligned}$$

Next, we prove that for any $t \in V(T)$, the number of edges in $E(S_i) \cap F_i$ with one endpoint in $\text{Fake}(t) \cap Z_i$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $4r$ where $r = |\beta_{\mathcal{D}}(t) \cap \{s_1, \dots, s_i\}|$. Fix a node $t \in V(T)$. First, suppose $s_i \notin \beta_{\mathcal{D}}(t)$. Then, by (ii), we have that the number of edges in $E(S_{i-1})$ with one endpoint in $\text{Fake}(t) \cap Z_{i-1} = \text{Fake}(t) \cap Z_i$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $4r'$ where $r' = |\beta_{\mathcal{D}}(t) \cap \{s_1, \dots, s_{i-1}\}| = |\beta_{\mathcal{D}}(t) \cap \{s_1, \dots, s_i\}| = r$. Moreover, since $E(S_i) \setminus E(K_i) = E(S_{i-1}) \setminus E(K_i)$, we have that the number of edges in $E(S_i)$ with one endpoint in $\text{Fake}(t) \cap Z_i$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $4r$.

Second, suppose $s_i \in \beta_{\mathcal{D}}(t)$. Then, by (ii), we have that the number of edges in $E(S_{i-1}) \cap F_{i-1}$ with one endpoint in $\text{Fake}(t) \cap Z_{i-1}$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $4r'$ where $r' = |\beta_{\mathcal{D}}(t) \cap \{s_1, \dots, s_{i-1}\}|$. By (a), we have that $E(S_i) \setminus E(K_i) = E(S_{i-1}) \setminus E(K_i)$, and by (b), we have that the number of edges of $E(S_i) \cap E(K_i)$, with one endpoint in $\text{Fake}(t) \cap K_i$ and other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most 4. Thus, we have that the number of edges in $E(S_i) \cap (F_{i-1} \cup E(K_i)) = E(S_i) \cap F_i$ with one endpoint in $\text{Fake}(t) \cap Z_i$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $4(r' + 1)$, and $r' + 1 = |\beta_{\mathcal{D}}(t) \cap \{s_1, \dots, s_i\}|$. This completes the proof of the claim. \square

By applying Claim 3 with $S = C$, we get that there is a cycle $C' = S_\ell$ of the same length as C such that (iii) for any $t \in V(T)$, the number of edges in $E(C') \cap F_\ell = E(C')$ with one endpoint in $\text{Fake}(t) \cap Z_\ell = \text{Fake}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $4r$ where $r = |\beta_{\mathcal{D}}(t) \cap \{s_1, \dots, s_\ell\}| = |\beta_{\mathcal{D}}(t) \cap S(G)|$.

We claim that C' has the required property. Towards the proof, fix a node $t \in V(T)$. By Lemma 7, we know that for any edge e with one endpoint u in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$, we have that either $u \in \text{Original}(t)$ or e belongs to some special clique $K \in \text{Cliques}(t)$. By (iii), the number of edges of C' with one endpoint in $\text{Fake}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $4r$ where $r = |\beta_{\mathcal{D}}(t) \cap S(G)|$. Notice that, since C' is a cycle, the number of edges of C' with one endpoint in $\text{Original}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $2 \cdot |\text{Original}(t)|$. That is, the number of edges of C' with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most

$2 \cdot |\text{Original}(t)| + 4r \leq 4(|\text{Original}(t)| + r) = 4|\beta_{\mathcal{D}}(t)| \leq 20\sqrt{2k}$, because \mathcal{D} is a tree decomposition of width $< 5\sqrt{2k}$. This completes the proof of the lemma. \square

Now we ready to give a proof sketch of Lemma 10. For the convenience of the reader we restate the lemma.

Lemma 10 *There is an algorithm that given an instance (G, B, k) of LONGEST CYCLE, and a $(5\sqrt{2k}, \mathcal{D})$ -NFewCliTD \mathcal{D}' of G (derived from a nice tree decomposition \mathcal{D} of B) such that for each node $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq 5\sqrt{2} \cdot k^{1.5}$, runs in time $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$, and outputs a longest cycle in G .*

Proof Recall that we are given an instance (G, B, k) of LONGEST CYCLE, a nice tree decomposition \mathcal{D} of B , and a $(5\sqrt{2k}, \mathcal{D})$ -NFewCliTD \mathcal{D}' of G , such that for each $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq 5\sqrt{2} \cdot k^{1.5}$. Lemma 11 ensures that if (G, B, k) is a YES instance of LONGEST CYCLE, then there is a cycle C of length at least k such that for any $t \in V(T)$, the number of edges with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $20\sqrt{2k}$. We give a dynamic programming (DP) algorithm, called \mathcal{A} , to find a cycle satisfying properties described in Lemma 11.

Algorithm \mathcal{A} is a DP algorithm over the given $(5\sqrt{2k}, \mathcal{D})$ -NFewCliTD \mathcal{D}' of G . For any node $t \in V(T)$, we define G_t as the induced subgraph $G[\gamma_{\mathcal{D}'}(t)]$ of G . Let \mathcal{C} be the set of maximum length cycles in G such that for any node $t \in V(T)$, the number of edges with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $20\sqrt{2k}$. This allows us to keep only $2^{\mathcal{O}(\sqrt{k} \log k)}$ states for any node in our DP algorithm. Algorithm \mathcal{A} will construct a cycle $C \in \mathcal{C}$. For a set Q of paths (of length 0 or more) and cycles, define $\widehat{Q} = \{\{u, v\} : \text{there is a } u\text{-}v \text{ path } P \text{ in } Q\}$. Let $C \in \mathcal{C}$. For any $t \in V(T)$, define C_t to be the set of connected components when we restrict C to G_t . That is, each element in C_t is a path (maybe of length 0) or C itself (in that case $C_t = \{C\}$). We also use C_t to denote the subgraph $G_t[E(C)]$ of G_t . Notice that $\bigcup_{Y \in \widehat{C}_t} Y$ is the set of vertices of degree 0 or 1 in C_t and $\bigcup_{Y \in \widehat{C}_t} Y \subseteq \beta_{\mathcal{D}'}(t)$ (recall that $\widehat{C}_t = \{\{u, v\} : \text{there is a } u\text{-}v \text{ path } P \text{ in } C_t\}$). We know that the number of edges with one endpoint in $\beta_{\mathcal{D}'}(t)$ and other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $20\sqrt{2k}$. This implies that the cardinality of $\bigcup_{P \in \widehat{C}_t} P$ is at most $20\sqrt{2k}$. In our DP algorithm, we will have state indexed by $(t, \widehat{C}_t, |E(C_t)|)$, which will be set to 1. Formally, for any $t \in V(T)$, $\ell \in [n]$ and a family \mathcal{Z} of vertex disjoint sets of size at most 2 of $\beta_{\mathcal{D}'}(t)$ with the property that the cardinality of $\bigcup_{Z \in \mathcal{Z}} Z$ is at most $20\sqrt{2k}$, we will have a table entry $\mathcal{A}[t, \mathcal{Z}, \ell]$. For each $t \in V(T)$, we maintain the following correctness invariant.

Correctness Invariant: (i) For every $C \in \mathcal{C}$, $\mathcal{A}[t, \widehat{C}_t, |E(C_t)|] = 1$, (ii) for any family \mathcal{Z} of vertex-disjoint sets of size at most 2 of $\beta_{\mathcal{D}'}(t)$ with $0 < |\bigcup_{Z \in \mathcal{Z}} Z| \leq 20\sqrt{2k}$, $\ell \in [n]$, and $\mathcal{A}[t, \mathcal{Z}, \ell] = 1$, there is a set \mathcal{Q} of $|\mathcal{Z}|$ vertex-disjoint paths in G_t where the endpoints of each path are specified by a set in \mathcal{Z} and $|E(\mathcal{Q})| = \ell$, and (iii) if $\mathcal{A}[t, \emptyset, \ell] = 1$, then there is a cycle of length ℓ in G_t .

The correctness of the our algorithm will follow from the correctness invariant. The way we fill the table entries is similar to the way it is done for DP algorithms over graphs of bounded treewidth. That is, we fill the table entries by considering various cases for bags (introduce, forget and join) and using the previously computed table entries. This part of our algorithm is similar to the algorithm for LONGEST CYCLE in [24] on a so called *special path decomposition*. \square

Theorem 8 follows from Lemmata 9 and 10. The algorithm for LONGEST PATH goes along the same lines as LONGEST CYCLE. Let (G, B, k) be an instance of LONGEST PATH. We first apply Lemma 6 with $\ell = \sqrt{2k}$ and if we get a $\sqrt{2k} \times \sqrt{2k}$ grid minor of B , then we conclude that G has a path of length k . Otherwise, we construct a $(5\sqrt{2k}, \mathcal{D})$ -NFewCliTD \mathcal{D}' of G where \mathcal{D} is a nice tree decomposition of B , and guess two end-vertices u and v of a path of length k in G (assuming it exists). Then, we add u and v to all the bags of \mathcal{D}' as *original* vertices and let $G' = (V(G), E(G) \cup \{\{u, v\}\})$. Next, to prove the existence of a path of length k in G , it is enough to check the existence of a cycle of length at least k in G' using the tree decomposition \mathcal{D}' (where we added $\{u, v\}$ to all bags). This can be done by using Lemma 10.

Theorem 10 LONGEST PATH on map graphs can be solved in $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ time.

6 Cycle Packing

In this section, we prove that CYCLE PACKING admits a subexponential-time parameterized algorithm on map graphs. That is, we prove the following.

Theorem 11 CYCLE PACKING on map graphs can be solved in time $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$.

Let (G, B, k) be an instance of CYCLE PACKING. Our first observation is that if B has a large grid minor, then (G, B, k) is YES instance.

Observation 12 Let (G, B, k) be an instance of CYCLE PACKING on map graphs. If B contains a $3\sqrt{k} \times 3\sqrt{k}$ grid as a minor, then G has k vertex-disjoint cycles.

Proof From a $3\sqrt{k} \times 3\sqrt{k}$ grid minor of B , we can conclude that B contains k vertex-disjoint cycles of length at least 8 each. For any cycle of length $\ell \geq 6$ in B , there is a cycle of length $\ell/2$ in G . This implies that if B has k vertex-disjoint cycles of length at least 8 each, then there are k vertex-disjoint cycles in G . \square

Additionally, notice that if $|N_B(s)| \geq 3k$ for some $s \in S(G)$, then G has k vertex-disjoint cycles of length 3 each, because $N_B(s)$ forms a clique in G . This fact along with Observation 12 leads to the following lemma.

Lemma 14 *There is an algorithm that given an instance (G, B, k) of CYCLE PACKING on map graphs, runs in time $\mathcal{O}(n^2)$, and either correctly concludes that (G, B, k) is a YES instance, or outputs a nice tree decomposition \mathcal{D} of B of width less than $15\sqrt{k}$ and a $(15\sqrt{k}, \mathcal{D})$ -NFewCliTD \mathcal{D}' of G , such that for each $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq 45 \cdot k^{1.5}$.*

Proof For any $s \in S(G)$, if $|N_B(s)| \geq 3k$, then (G, B, k) is a YES instance. Now we apply Lemma 6 with $\ell = 3\sqrt{k}$. If the output is a $3\sqrt{k} \times 3\sqrt{k}$ grid minor of B , then by Observation 12, (G, B, k) is a YES instance. Otherwise, we have a nice tree decomposition \mathcal{D} of B of width less than $15\sqrt{k}$ and a $(15\sqrt{k}, \mathcal{D})$ -NFewCliTD \mathcal{D}' of G . In this case, since $|N_B(s)| < 3k$ for any $s \in S(G)$, we have that for each $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq 45 \cdot k^{1.5}$. \square

Due to Lemma 14, to prove Theorem 11, it is enough to prove the following lemma.

Lemma 15 *There is an algorithm that given an instance (G, B, k) of CYCLE PACKING on map graphs, a nice tree decomposition \mathcal{D} of B of width less than $15\sqrt{k}$, and a $(15\sqrt{k}, \mathcal{D})$ -NFewCliTD \mathcal{D}' of G such that for each $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq 45 \cdot k^{1.5}$, runs in time $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$, and correctly concludes whether (G, B, k) is a YES instance or not.*

As in the case of LONGEST CYCLE, we want to bound the ‘‘interaction’’ of a solution (i.e., a cycle packing) across every bag of \mathcal{D}' to be $\mathcal{O}(\sqrt{k})$. That is, if (G, B, k) is a YES instance of CYCLE PACKING, then there is a solution \mathcal{C} with the following property.

For any node $t \in V(T)$, the number of edges in $E(\mathcal{C})$ with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is upper bounded by $\mathcal{O}(\sqrt{k})$.

The above mentioned property is encapsulated in Lemma 16. Moreover, notice that given a set \mathcal{C} of pairwise vertex-disjoint cycles in a graph G and a cycle $C \in \mathcal{C}$ that is not an induced cycle in G , by replacing C in \mathcal{C} by an induced cycle in $G[V(C)]$, we obtain another set of pairwise vertex-disjoint cycles. So from now onwards we assume that our objective is to look for k vertex-disjoint induced cycles in G .

Lemma 16 (Sublinear Crossing Lemma) *Let (G, B, k) be a YES instance of CYCLE PACKING on map graphs, and let \mathcal{D}' be a $(15\sqrt{k}, \mathcal{D})$ -NFewCliTD of G where \mathcal{D} is a nice tree decomposition of B . Then, there is a solution \mathcal{C} such that each cycle in \mathcal{C} is an induced cycle in G and for any $t \in V(T)$, the number of edges with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $360\sqrt{k}$.*

Towards the proof of Lemma 16, recall that for any $s \in S(G)$, $N_B(s)$ forms a clique in G and we call it a special clique of G . If a solution \mathcal{C} of (G, B, k) contains a cycle with at least three vertices from $N_B(s)$, then it should be

a triangle because we seek induced cycles. Towards finding a solution for (G, B, k) , we consider a solution that maximizes the number of triangles it selects from the special cliques of G . Let \mathcal{S} be the set of solutions of (G, B, k) with maximum number of triangles from the special cliques of G and which consist of induced cycles in G . Before proceeding to the proof of Lemma 16, we prove an analogous result for packing triangles from special cliques.

Lemma 17 *Let (G, B, k) be an instance of CYCLE PACKING on map graphs, and let \mathcal{D}' be a $(15\sqrt{k}, \mathcal{D})$ -NFewCliTD of G where \mathcal{D} is a nice tree decomposition of B . Let \mathcal{C} be a set of vertex-disjoint triangles from special cliques of G . Then, there is a set \mathcal{C}' of vertex-disjoint triangles from special cliques of G such that $|\mathcal{C}| = |\mathcal{C}'|$, $V(\mathcal{C}) = V(\mathcal{C}')$, and for any $t \in V(T)$, the number of edges of $E(\mathcal{C}')$ with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $60\sqrt{k}$.*

Proof Let $S(G) = \{s_1, \dots, s_\ell\}$, and let K_i be the special clique $N_B(s_i)$ in G for any $i \in [\ell]$. Let $\mathcal{C} = \bigsqcup_{i \in [\ell]} \mathcal{C}_i$ be a set of vertex-disjoint triangles such that \mathcal{C}_i is a set of triangles in K_i for any $i \in [\ell]$, and $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for any distinct $i, j \in [\ell]$. First, we prove the following claim.

Claim 4 *Let $i \in [\ell]$. Then, there is a set \mathcal{C}'_i of vertex-disjoint triangles from the special clique K_i of G such that $|\mathcal{C}_i| = |\mathcal{C}'_i|$, $V(\mathcal{C}_i) = V(\mathcal{C}'_i)$, and for any node $t \in V(T)$, the number of edges of $E(\mathcal{C}'_i)$ with one endpoint in $\text{Fake}(t) \cap K_i$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most 4.*

Proof Let π' be a postorder transversal of the nodes in the binary tree T . Let π be the restriction of π' where we keep only the nodes labelled with **fake introduce**(v) for some $v \in K_i$. Recall that, for any vertex $v \in V(G)$, there is at most one node in T which is labelled with **fake introduce**(v) by \mathcal{D}' (see Observation 3). Accordingly, denote $\pi = t_1, \dots, t_r$ where each t_j , $j \in [r]$, is a node labelled with **fake introduce**(v_j) for some $v_j \in K_i$. Let σ_1 be the sequence v_1, \dots, v_r , and $U = \{v_1, \dots, v_r\}$. Let σ_2 be a fixed arbitrary sequence on $K_i \setminus U$. Let σ be the sequence $(\sigma_1 \sigma_2)|_{V(\mathcal{C}_i)}$. That is, σ is the sequence obtained by restricting $\sigma_1 \sigma_2$ (the concatenation of σ_1 and σ_2) to $V(\mathcal{C}_i)$. Since \mathcal{C}_i is a set of vertex-disjoint triangles from K_i , we have that the length of σ is a multiple 3. Thus, we can denote $\sigma = z_1 z_2 \dots z_{3q}$ for some positive integer q . Notice that $z_j = v_j$ for any $j \in [r]$, and $|\mathcal{C}_i| = q$. Now, we define the “required” set of triangles to be $\mathcal{C}'_i = \{[z_{3c-2} z_{3c-1} z_{3c}] : c \in [q]\}$. Clearly, \mathcal{C}'_i is a set of vertex-disjoint triangles, $|\mathcal{C}'_i| = |\mathcal{C}_i|$ and $V(\mathcal{C}_i) = V(\mathcal{C}'_i)$. The proof of the following statement easily follows from the definition \mathcal{C}'_i .

- (a) For any $j \in [3q]$, the number of edges of $E(\mathcal{C}'_i)$ with one endpoint in $\{z_1, \dots, z_j\}$ and the other in $\{z_{j+1}, \dots, z_{3q}\}$ is at most 2.

The proof of the following statement is similar in arguments to that of Claim 2.

- (b) For any $t \in V(T)$, there is a segment σ' of σ such that each vertex in $\text{Fake}(t) \cap V(\mathcal{C}'_i)$ appears in σ' , and each vertex in $V(\mathcal{C}'_i) \setminus \gamma_{\mathcal{D}'}(t)$ does not appear in σ' .

Now, we are ready to complete the proof of the claim. Towards this, let us fix a node $t \in V(T)$. By statement (b), there exist $j_1, j_2 \in [3q]$ such that $\text{Fake}(t) \cap V(\mathcal{C}'_i) \subseteq \{z_{j_1}, \dots, z_{j_2}\}$ and $(V(\mathcal{C}'_i) \setminus \gamma_{\mathcal{D}'}(t)) \cap \{z_{j_1}, \dots, z_{j_2}\} = \emptyset$. Therefore, by statement (a), we conclude that the number of edges of $E(\mathcal{C}'_i)$ with one endpoint in $\text{Fake}(t) \cap K_i$ and other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most 4. \square

To prove the lemma, we apply Claim 4 for all $i \in [\ell]$ to obtain \mathcal{C}'_i from \mathcal{C}_i , and then let $\mathcal{C}' = \bigcup_{i \in [\ell]} \mathcal{C}'_i$. Clearly, by Claim 4, \mathcal{C}' is a set of vertex-disjoint triangles, $|\mathcal{C}| = |\mathcal{C}'|$, and $V(\mathcal{C}) = V(\mathcal{C}')$. Now, fix any $t \in V(T)$. By Claim 4, the number of edges of $E(\mathcal{C}')$ with one endpoint in $\text{Fake}(t) \cap \beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $4r_1$ where $r_1 = |\text{Cliques}(t)|$. Since the degree of each vertex in a graph consisting of only vertex-disjoint cycles is 2, the number of edges of $E(\mathcal{C}')$ with one endpoint in $\text{Original}(t) \cap \beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $2r_2$ where $r_2 = |\text{Original}(t)|$. Therefore, the number of edges of $E(\mathcal{C}')$ with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $4r_1 + 2r_2 \leq 4(r_1 + r_2) = 4|\beta_{\mathcal{D}'}(t)| \leq 60\sqrt{k}$. \square

Recall that \mathcal{S} is the set of solutions of (G, B, k) with maximum number of triangles from special cliques of G which consists only of induced cycles. Next, we state another lemma needed for the proof of Lemma 16. The proof of this lemma will be the focus of most of the rest of this section.

Lemma 18 *Let (G, B, k) be an instance of CYCLE PACKING on map graphs, and let \mathcal{D}' be a $(15\sqrt{k}, \mathcal{D})$ -NFewCliTD of G where \mathcal{D} is a nice tree decomposition of B . Let $\mathcal{C} \in \mathcal{S}$. Let $\mathcal{C}_2 \subseteq \mathcal{C}$ be such that $C \in \mathcal{C}_2$ if and only if C is not a triangle in a special clique of G . Then, for any node $t \in V(T)$, the number of edges of \mathcal{C}_2 with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $300\sqrt{k}$.*

Lemma 16 follows immediately from Lemmata 17 and 18. Our proof of Lemma 18 requires the arguments of the following lemma (see Figure 6).

Lemma 19 *Let (G, B, k) be an instance of CYCLE PACKING on map graphs. Let $\mathcal{C} \in \mathcal{S}$, and denote $\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2$ where $C \in \mathcal{C}_1$ if and only if C is a triangle in a special clique. Let K_1 and K_2 be two special cliques. Then, there does not exist two vertices $u, v \in V(\mathcal{C}_2) \cap K_1 \cap K_2$ and four edges e_1, e_2, e_3, e_4 such that (a) $e_1, e_2 \in E(\mathcal{C}_2) \cap E(K_1)$, (b) $e_3, e_4 \in E(\mathcal{C}_2) \cap E(K_2)$, (c) u is incident with e_1 and e_3 , and (d) v is incident with e_2 and e_4 .*

Proof For the sake of contradiction, we assume that there exist two vertices $u, v \in V(\mathcal{C}_2) \cap K_1 \cap K_2$ and four edges e_1, e_2, e_3, e_4 such that (a) $e_1, e_2 \in E(\mathcal{C}_2) \cap E(K_1)$, (b) $e_3, e_4 \in E(\mathcal{C}_2) \cap E(K_2)$, (c) u is incident with e_1 and e_3 , and (d) v is incident with e_2 and e_4 . Let $e_1 = \{u, w_1\}$, $e_2 = \{v, w_2\}$, $e_3 = \{u, z_1\}$ and $e_4 = \{v, z_2\}$. We claim that all the four vertices w_1, w_2, z_1 and z_2 are distinct. We only prove that $w_1 \notin \{w_2, z_1, z_2\}$. (All other cases are symmetric). Targeting a contradiction, suppose $w_1 \in \{w_2, z_1, z_2\}$. If $w_1 = w_2$, then all the four edges e_1, e_2, e_3 and e_4 are part of a single cycle $C \in \mathcal{C}_2$. Then, $\mathcal{C}' = (\mathcal{C} \setminus \{C\}) \cup \{[uvw_1u]\}$ is a solution to (G, B, k) and \mathcal{C}' contains

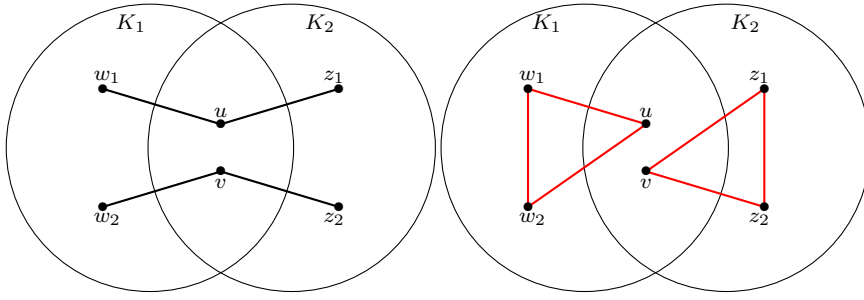


Fig. 6: Illustration for the proof of Lemma 19. Relevant subpaths of C_1 and C_2 are given in the left figure.

strictly more triangles from special cliques than \mathcal{C} . This is a contradiction to the assumption that $\mathcal{C} \in \mathcal{S}$. The proof of the statement $w_1 \neq z_2$ is the same in arguments to that of the statement $w_1 \neq w_2$. Since G is a simple graph and e_1 and e_3 are distinct edges, we have that $w_1 \neq z_1$.

So, now we have that w_1, w_2, z_1 and z_2 are distinct vertices. That is, w_1uz_1 is a subpath of a cycle C_1 in \mathcal{C}_2 and w_2vz_2 is a subpath of a cycle C_2 in \mathcal{C}_2 . This implies that $\mathcal{C}' = (\mathcal{C} \setminus \{C_1, C_2\}) \cup \{[ww_1w_2u], [vz_1z_2v]\}$ is a solution to (G, B, k) and \mathcal{C}' contains strictly more triangles from special cliques than \mathcal{C} . This is a contradiction to the assumption that $\mathcal{C} \in \mathcal{S}$ (see Figure 6 for an illustration). \square

We also require the following lemma in the proof of Lemma 18. To state the lemma we need the following definition. We say that a collection of sets $\{A_1, \dots, A_q\}$ has a system of distinct representatives if there exist distinct vertices a_1, a_2, \dots, a_q such that $a_i \in A_i$ for all $i \in [q]$.

Lemma 20 *Let $\{A_1, \dots, A_q\}$ be a collection of sets of size at least one such that all but at most one set have size 2, and each element appears in at most two sets. Then, $\{A_1, \dots, A_q\}$ has a system of distinct representatives.*

Proof For the base case, $q = 1$, the statement holds trivially. Consider the induction step $q > 1$. There exists at most one set $A \in \{A_1, \dots, A_q\}$ such that $|A| = 1$. If all sets have size 2, then choose A to be an arbitrary set. Select an element z_A from A as its representative. By the definition of $\{A_1, \dots, A_q\}$, there is at most one set B in $\{A_1, \dots, A_q\}$ such that $B \neq A$ and $z_A \in B$. Moreover $|B| = 2$. Then, by induction hypothesis we have that $(\{A_1, \dots, A_q\} \setminus \{A, B\}) \cup \{B'\}$, where $B' = B \setminus \{z_A\}$, has a system of distinct representatives. This system of representatives along with $z_A \in A$ forms a system of distinct representatives for $\{A_1, \dots, A_q\}$. \square

Now, we are ready to give a proof for Lemma 18. For the convenience of the reader we restate the lemma.

Lemma 18 *Let (G, B, k) be an instance of CYCLE PACKING on map graphs, and let \mathcal{D}' be a $(15\sqrt{k}, \mathcal{D})$ -NFewCliTD of G where \mathcal{D} is a nice tree decomposition*

of B . Let $\mathcal{C} \in \mathcal{S}$. Let $\mathcal{C}_2 \subseteq \mathcal{C}$ be such that $C \in \mathcal{C}_2$ if and only if C is not a triangle in a special clique of G . Then, for any node $t \in V(T)$, the number of edges of \mathcal{C}_2 with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $300\sqrt{k}$.

Proof Fix a node $t \in V(T)$. Let $S(G) \cap \beta_{\mathcal{D}}(t) = \{s_1, \dots, s_\ell\}$, and for each $i \in [\ell]$, let K_i be the special clique $N_B(s_i)$ in G . Then, $\text{Cliques}(t) = \{K_1, \dots, K_\ell\}$. Let $G_t = G[\gamma_{\mathcal{D}'}(t)]$. Let \mathcal{P} be the restriction of \mathcal{C}_2 to G_t . That is, $\mathcal{P} = (V(G_t) \cap V(\mathcal{C}_2), E(G_t) \cap E(\mathcal{C}_2))$. Notice that \mathcal{P} is a collection of pairwise vertex-disjoint cycles and paths (some paths could be of length 0) such that the end-vertices of each path belong to $\beta_{\mathcal{D}'}(t)$. Observe that cycles in \mathcal{P} are fully contained inside G_t and hence the edges of these cycles do not contribute to the number of crossing edges we want to bound. Hence, we assume without loss of generality that \mathcal{P} only contains pairwise vertex-disjoint paths such that the end-vertices of each path belong to $\beta_{\mathcal{D}'}(t)$. Let $\mathcal{O} \subseteq \mathcal{P}$ be such that $P \in \mathcal{O}$ if and only if at least one end-vertex of P is in $\text{Original}(t)$. Let $\mathcal{F} = \mathcal{P} \setminus \mathcal{O}$. Since $|\text{Original}(t)| \leq 15\sqrt{k}$, $|\mathcal{O}| \leq 15\sqrt{k}$. By the definition of \mathcal{F} , for any path $P \in \mathcal{F}$, both the end-vertices of P belong $\text{Fake}(t)$. We further classify $\mathcal{F} = \mathcal{F}_0 \uplus \mathcal{F}_1$ where \mathcal{F}_0 contains the paths of length 0 in \mathcal{F} , and \mathcal{F}_1 contains the paths of length at least 1 in \mathcal{F} . Notice that $\mathcal{P} = \mathcal{O} \uplus \mathcal{F}_0 \uplus \mathcal{F}_1$. Moreover, notice that the set X of vertices in $\beta_{\mathcal{D}'}(t)$ that are endpoints of edges in $E(\mathcal{C}_2)$ whose other endpoints in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$, is the set of end-vertices of the paths in \mathcal{P} . If $|\mathcal{P}| \leq 75\sqrt{k}$, then $|X| \leq 150\sqrt{k}$, and hence the number of edges of \mathcal{C}_2 with one endpoint in $\beta_{\mathcal{D}'}(t)$ and the other in $V(G) \setminus \gamma_{\mathcal{D}'}(t)$ is at most $2|X| \leq 300\sqrt{k}$ (because \mathcal{C}_2 is a set of vertex-disjoint cycles). Thus, to complete the proof of the lemma, it is suffice to prove that indeed $|\mathcal{P}| \leq 75\sqrt{k}$. Recall that $|\mathcal{O}| \leq 15\sqrt{k}$. Thus, to prove that $|\mathcal{P}| \leq 75\sqrt{k}$, it is enough to prove that $|\mathcal{F}_0| \leq 45\sqrt{k}$ and $|\mathcal{F}_1| \leq 15\sqrt{k}$.

Before formally proving upper bounds on the cardinalities of \mathcal{F}_0 and \mathcal{F}_1 , we give a high level overview of the proof. Towards bounding \mathcal{F}_0 , we construct a planar bipartite subgraph H of B with bipartition $\mathcal{F}_0 \uplus \{s_1, \dots, s_\ell\}$ and the following property: for any $v \in \mathcal{F}_0$, $d_H(v) = 2$. Therefore, $|\mathcal{F}_0| = \frac{|E(H)|}{2}$. To upper bound $|\mathcal{F}_0|$, we construct a minor H' of H on ℓ vertices and $\frac{|E(H)|}{2}$ edges and prove that (a) H' is a graph without self-loops and parallel edges. Since H' is a minor of a planar graph H , H' is also a planar graph. Thus, H' is a planar graph without self-loops and parallel edges, and $|V(H')| = \ell$. This implies that the number of edges in H' is at most $3\ell - 6$ (because the number of edges in a simple planar graph on N vertices is at most $3N - 6$). This, in turn, will imply that $|\mathcal{F}_0| \leq 3\ell - 6 \leq 45\sqrt{k}$. Towards upper bounding $|\mathcal{F}_1|$ by $15\sqrt{k}$, we construct a graph H_1 on the vertex set $\{s_1, \dots, s_\ell\}$. To construct the edge set of H_1 , we add one edge for each path in \mathcal{F}_1 . Thus, $|\mathcal{F}_1| = |E(H_1)|$. To upper bound $|E(H_1)|$, we prove that (b) H_1 is a forest. This implies that $|\mathcal{F}_1| = |E(H_1)| \leq \ell - 1 < 15\sqrt{k}$. The proofs of both the Statements (a) and (b) use the assumption that $\mathcal{C} \in \mathcal{S}$.

Now, we move towards the formal proof of $|\mathcal{F}_0| \leq 45\sqrt{k}$. Notice that \mathcal{F}_0 is a set of paths of length 0 and each $v \in \mathcal{F}_0$ belongs to $\text{Fake}(t)$. Any edge

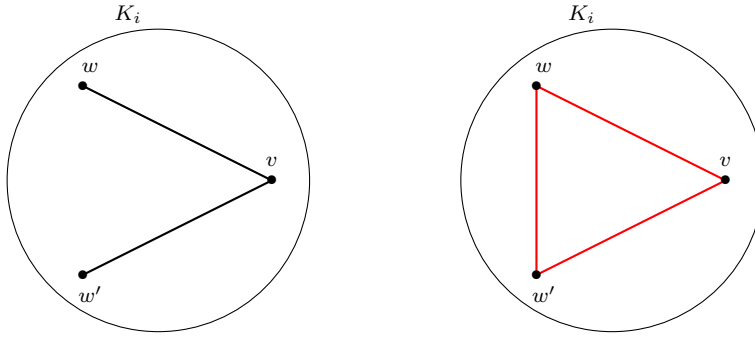


Fig. 7: Illustration of Proof of Claim 5. $C' = [wvw'w]$ is a triangle in the special clique K_i .

in $E(\mathcal{C}_2)$ incident with a vertex $v \in \mathcal{F}_0$ belongs to $\bigcup_{i \in [\ell]} E(K_i)$ (because of Lemma 7). For any $v \in \mathcal{F}_0$, let $e_v = \{v, x_v\}$ and $e'_v = \{v, y_v\}$ be the edges of $E(\mathcal{C}_2)$ incident with v .

Claim 5 *For any $v \in \mathcal{F}_0$, there does not exist $i \in [\ell]$ such that both $e_v, e'_v \in E(K_i)$. Moreover, $x_v \neq y_v$ and $x_v, y_v \notin V(G_i)$.*

Proof Let $v \in \mathcal{F}_0$. Since G is a simple graph, we have that $x_v \neq y_v$. The condition $x_v, y_v \notin V(G_i)$ follows from the definition of \mathcal{F}_0 . Towards a contradiction, suppose there exists $i \in [\ell]$ such that $e_v, e'_v \in E(K_i)$. That is, $v, x_v, y_v \in K_i$. Let $C \in \mathcal{C}_2$ be such that $v, x_v, y_v \in V(C)$. Since $v, x_v, y_v \in K_i$, $C' = [vx_vy_vv]$ forms a triangle in the special clique K_i . Then, $C' = (C \setminus \{C\}) \cup \{C'\}$ is a cycle packing such that $|C'| = |C|$ and C' contain strictly more triangles from special cliques than C . This is a contradiction to the fact that $C \in \mathcal{S}$ (see Figure 7 for an illustration). \square

We proceed to define a subgraph H of B on the vertex set $\{s_1, \dots, s_\ell\} \cup \mathcal{F}_0$, and whose edge set will be defined immediately. Towards this, note that by Lemma 7, we have that for any $v \in \mathcal{F}_0$, $e_v \in E(K_i)$ and $e'_v \in E(K_j)$ for some special cliques K_i and K_j in $\text{Cliques}(t)$. Moreover, by Claim 5, $e'_v \notin E(K_i)$ and $e_v \notin E(K_j)$. For any $v \in \mathcal{F}_0$, we choose $i, j \in [\ell]$ such that $e_v \in E(K_i)$ and $e'_v \in E(K_j)$ (notice that $K_i \neq K_j$). Then, we add the edges $\{v, s_i\}$ and $\{v, s_j\}$ to H . Notice that since $e_v \in E(K_i)$ and $e'_v \in E(K_j)$, we have that $\{v, s_i\}$ and $\{v, s_j\}$ are edges in B . Therefore, H is a subgraph of B . That is, H is a planar bipartite graph with bipartition $\{s_1, \dots, s_\ell\} \uplus \mathcal{F}_0$, and the degree of each $v \in \mathcal{F}_0$ is exactly 2.

Next, we construct a minor of H by arbitrarily choosing, for each $v \in \mathcal{F}_0$, exactly one edge incident to $v \in \mathcal{F}_0$ and contracting it into its other endpoint (so we refer to the resulting vertex using the identity of the other endpoint). Let the resulting graph be H_1 . Notice that H_1 is a planar graph on the vertex set $\{s_1, \dots, s_\ell\}$. As each vertex in $v \in \mathcal{F}_0$ has degree exactly 2 in H , and we contracted exactly one edge incident with v to obtain H_1 , we have that $|\mathcal{F}_0|$ is

equal to the number of edges in H_1 . Since H_1 is a planar graph with vertex set $\{s_1, \dots, s_\ell\}$, if there are no self-loops and parallel edges in H_1 , then the number of edges in H_1 (and hence $|\mathcal{F}_0|$) is at most $3\ell - 6$ (because the number of edges in a simple planar graph on N vertices is at most $3N - 6$). Next, we prove that indeed H_1 is a graph without self-loops and parallel edges.

Claim 6 H_1 is a graph without parallel edges and self-loops.

Proof Targeting a contradiction, suppose H_1 has a self-loop. Then, H has two vertices $v \in \mathcal{F}_0$ and $s_i \in \{s_1, \dots, s_\ell\}$ such that there are two edges between v and s_i in H . That is, both $e_v = \{v, x_v\}$ and $e'_v = \{v, y_v\}$ are edges in $E(K_i)$. However, this is not possible because of Claim 5.

Now, we prove that H_1 does not contain two parallel edges. For the sake of contradiction, assume that there exist s_i and s_j in $V(H_1)$ such that there are two parallel edges between s_i and s_j in H_1 . This implies that there exist two distinct vertices $v, u \in \mathcal{F}_0$ such that $e_u = \{u, x_u\}$ and $e_v = \{v, x_v\}$ are edges in K_i and $e'_u = \{u, y_u\}$, and $e'_v = \{v, y_v\}$ are edges in K_j . However, this contradicts Lemma 19. This completes the proof of the claim. \square

Thus, we have proved that $|\mathcal{F}_0| \leq 3\ell - 6 \leq 3 \cdot 15\sqrt{k} - 6 \leq 45\sqrt{k}$ (because $\ell \leq |\beta_{\mathcal{D}}(t)|$).

Next, we prove that $|\mathcal{F}_1| \leq 15\sqrt{k}$. Let $\ell' = |\mathcal{F}_1|$ and $\mathcal{F}_1 = \{P_1, \dots, P_{\ell'}\}$. Notice that \mathcal{F}_1 is a set of paths of length at least 1 and each end-vertex of any path in \mathcal{F}_1 belongs to $\text{Fake}(t)$. For any $i \in [\ell']$, let u_i and v_i be the end-vertices of P_i (since P_i is a path of length at least 1, we have that $u_i \neq v_i$). Moreover, for any $i \in [\ell']$, let $\{u_i, w_i\}$ and $\{v_i, z_i\}$ be the edges of $E(\mathcal{C}_2) \setminus E(\mathcal{F}_1)$ incident with u_i and v_i , respectively.

Now, we create an auxiliary graph H' on the vertex set $\{s_1, \dots, s_\ell\}$. By Lemma 7, we have that for any $i \in [\ell']$, $u_i, w_i \in K$ and $v_i, z_i \in K'$ for some special cliques K and K' in $\text{Cliques}(t)$. Moreover, it is easy to see that (i) there is no special clique K'' such that $K'' \cap \{u_i, v_i, w_i, z_i\} \geq 3$. Otherwise, we replace the cycle containing $\{u_i, v_i, w_i, z_i\}$ in \mathcal{C}_2 with a triangle in $K'' \cap \{u_i, v_i, w_i, z_i\}$ and thereby contradict the assumption that $\mathcal{C} \in \mathcal{S}$. For any $i \in [\ell']$, we choose $j, j' \in [\ell]$ such that $u_i, w_i \in K_j$ and $v_i, z_i \in K_{j'}$ (notice that $j \neq j'$). Then, we add an edge between s_j and $s_{j'}$ in H' , and we denote this edge with g_i . Clearly, H' is a graph without self-loops.

Claim 7 H' is a forest.

Proof For the sake of contradiction, assume (without loss of generality) that there is a cycle $L = [s_{i_1} g_1 s_{i_2} g_2 \dots s_{i_r} g_r s_{i_1}]$ in H' . Recall that $K_{i_j} = N_B(s_{i_j})$ is a special clique in $\text{Cliques}(t)$ for any $j \in [r]$. From the definition of L , we have that (a) for any $j \in [r]$, $u_j, w_j \in K_{i_j}$, (b) for any $j \in [r - 1]$, $v_j, z_j \in K_{i_{j+1}}$, and (c) $v_r, z_r \in K_{i_1}$. By the definition of \mathcal{F}_1 , we have that $u_1, \dots, u_r, v_1, \dots, v_r$ are distinct vertices. See Figure 8 for an illustration. Let $\mathcal{C}'_2 \subseteq \mathcal{C}_2$ be such that $C \in \mathcal{C}'_2$ if and only if $V(C) \cap \{u_1, \dots, u_r\} \neq \emptyset$. Notice that each cycle in \mathcal{C}'_2 contains at least one vertex in $\{u_1, \dots, u_r\}$. Therefore, $|\mathcal{C}'_2| \leq r$. Moreover, any

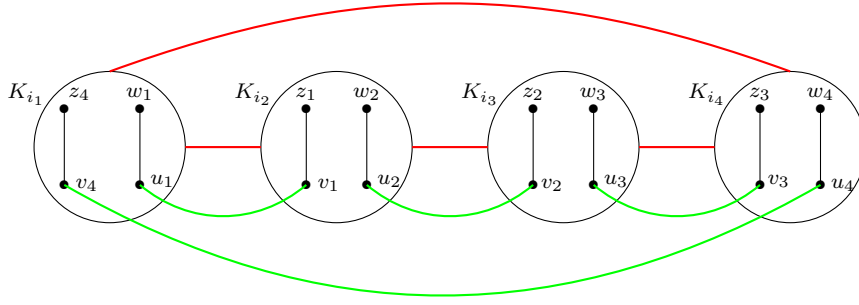


Fig. 8: The red curves are edges in the cycle L . The green curves are some paths in \mathcal{F}_1 that give rise to the edges (drawn as red curves) in L . The vertices $u_1, \dots, u_4, v_1, \dots, v_4$ are distinct. $Z = \{w_1, \dots, w_4, z_1, \dots, z_4\}$, $S_1 = \{w_1, z_4\}$, $S_2 = \{w_2, z_1\}$, $S_3 = \{w_3, z_2\}$ and $S_4 = \{w_4, z_3\}$. Even though the circles representing special cliques are disjoint in the illustration, they may not be disjoint in general.

cycle in \mathcal{C}_2 that has non-empty intersection with $\{u_i, w_i, v_i, z_i \mid i \in [r]\}$ also belongs to \mathcal{C}'_2 .

We prove the claim using a proof by contradiction. Towards that, we will prove that there are r triangles in the special cliques K_{i_1}, \dots, K_{i_r} using vertices from $\{u_i, v_i, w_i, z_i \mid i \in [r]\}$; then, by replacing \mathcal{C}'_2 with these triangles, we will reach a contradiction to the assumption that $\mathcal{C} \in \mathcal{S}$. Let $Z = \{w_j, z_j \mid j \in [r]\}$. Let $S_j = \{w_j, z_{j-1 \bmod r} \mid j \in [r]\}$ for all $j \in [r]$ (here, $0 \bmod r$ is defined to be r). Now, we prove that $\{S_1, \dots, S_r\}$ has a system of distinct representatives using Lemma 20. Towards that, we first prove that $|S_j| = 2$ for any $j \in [r]$. Suppose not. Then, $|S_j| = 1$ for some $j \in [r]$. This implies that $w_j = z_s$, where $s = (j-1) \bmod r$. Then, $[v_s w_j u_j]$ is a subpath of a cycle C in \mathcal{C}_2 . Moreover, $v_s, w_j, u_j \in K_{i_j}$. Therefore, by replacing $C \in \mathcal{C}$ with the triangle $[v_s w_j u_j v_s]$, we contradict the assumption that $\mathcal{C} \in \mathcal{S}$. Notice that since the degree of each vertex in a set of vertex-disjoint cycles is exactly 2, we have that each vertex $z \in Z$ appears in at most two sets in $\{S_1, \dots, S_r\}$. Thus, by Lemma 20, $\{S_1, \dots, S_r\}$ has a system of distinct representatives.

Now, we construct a set of r triangles as follows. For each S_j , let a_j be its representative. Notice that $a_j \in K_{i_j}$. Let $\widehat{\mathcal{C}} = \{[u_j a_j v_{(j-1) \bmod r} u_j] : j \in [r]\}$. Notice that $\widehat{\mathcal{C}}$ is a set of vertex-disjoint triangles in special cliques of G . Therefore, by replacing cycles in \mathcal{C}'_2 with cycles in $\widehat{\mathcal{C}}$, we get a solution of (G, B, k) with more triangles from special cliques than that of \mathcal{C} . This is a contradiction to the assumption that $\mathcal{C} \in \mathcal{S}$. This completes the proof of the claim. \square

Since H' is a forest (by Claim 7), we have that $|E(H')| < |V(H')|$. Notice that for each path in \mathcal{F}_1 , we added exactly one edge to H' and hence $|E(H')| =$

$|\mathcal{F}_1|$. Therefore, we have that $|\mathcal{F}_1| = |E(H')| < |V(H')| \leq \ell \leq 15\sqrt{k}$. This completes the proof of the lemma. \square

Using Lemma 16, we can design an algorithm for CYCLE PACKING and prove Lemma 15. This algorithm is a dynamic programming algorithm over the $(15\sqrt{k}, \mathcal{D})$ -NFewCliTD \mathcal{D}' of G , such that for each $t \in V(T)$, $|\beta_{\mathcal{D}'}(t)| \leq 45 \cdot k^{1.5}$. The number of states at any node $t \in V(T_{\mathcal{D}'})$ is upper bounded by $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$, and thus resulting in an algorithm with running time $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$. For further details, we refer to the proof sketch of Lemma 9.3 in [24] for a similar algorithm on a path decomposition of the input graph with similar properties.

7 Conclusion

In this paper, we gave subexponential algorithms of running time $2^{\mathcal{O}(\sqrt{k} \log k)} \cdot n^{\mathcal{O}(1)}$ for LONGEST CYCLE, LONGEST PATH, FEEDBACK VERTEX SET and CYCLE PACKING on map graphs. The following are some open questions along the direction of our work.

- Is there a parameterized subexponential time algorithm for EXACT k -CYCLE on map graphs? Here, we want to test whether the input graph has a cycle of length exactly k .
- Can we get a better running time (by shaving off the $\log k$ factor in the exponent) for LONGEST CYCLE, LONGEST PATH, FEEDBACK VERTEX SET and CYCLE PACKING on map graphs? The rank and matroid based techniques [8, 23] used to develop single exponential time algorithms parameterized by treewidth uses the size of the bag crucially. Since the treewidth of our decomposition is large, these methods are not directly applicable here.
- It is noteworthy to remark that a simple disjoint union trick [9, 27] implies that LONGEST CYCLE and LONGEST PATH do not admit a polynomial kernel on map graphs. Can we get a Turing polynomial kernel for these problems on map graphs? That is, can we give a polynomial time algorithm that given an instance of LONGEST CYCLE or LONGEST PATH on map graphs, produces polynomially many reduced instances of size polynomial in k such that the input instance is a YES instance if and only if one of the reduced instances is?
- Can we get a general characterization of parameterized problems admitting subexponential algorithms on map graphs like the bidimensionality theory for planar graphs?

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