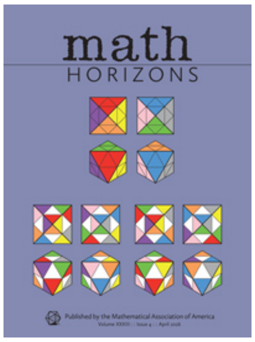

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Patterns on an Octahedron

Rainer Hollerbach

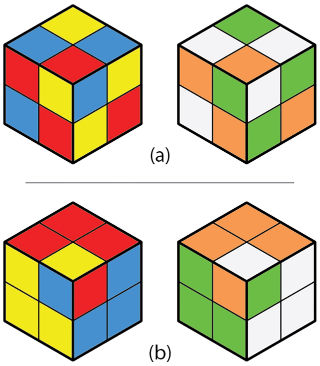
Most readers will be familiar with Rubik's cube and might even be aware that similar puzzles exist with noncubic shapes. The various manufacturers advertise these puzzles as being "educational for children six years and up." I suspect that might be a bit optimistic, but there is no doubt that for university and even high school students there is a great deal of interesting math associated with these puzzles and the patterns one can create on them.

Color Equality and Color Equivalence on a Cube

The patterns we're interested in here are so-called color equality and equivalence patterns, which I had previously explored on Rubik's cubes (see eprints 199822, 211314, and 229635 at *eprints.whiterose.ac.uk*). *Color equality* simply means that all six colors must be distributed exactly equally. That is, if you had two identically arranged cubes, you could pick any one color on the first cube, and any other color on the second cube, and you could then rotate the two cubes with respect to one another in such a way that the two colors coincide in all of their positions. If this holds true for *all* choices of colors on the two cubes, that arrangement satisfies the color equality criterion.

Figure 1(a) shows an example, on the simple $2\times2\times2$ cube. The left panel shows the three "front" faces and in the right panel the cube is turned over to show the three "back" faces. We can easily see how red on the front matches up perfectly with orange on the back; similarly, yellow matches with green, and blue with white. We also notice a threefold rotational symmetry. That is, if you rotate the cube about an axis going through the front and back corners, you can match up red with either yellow or blue on the front, and orange with either green or white on the back. This pattern therefore satisfies the requirement that all six colors are distributed exactly equally.

Figure 1. Two possible arrangements of a $2 \times 2 \times 2$ Rubik's cube: (a) an equality pattern, (b) an equivalence pattern.

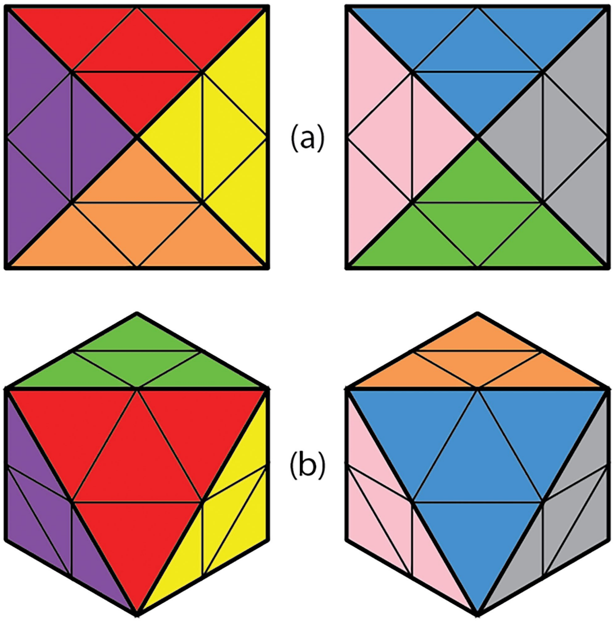


Consider next the arrangement in figure 1(b). Now, red on the front does not match orange on the back, which is already enough to establish that this is not an equality pattern. The ways that red and orange are distributed are nevertheless very closely related: They are mirror images of each other. Furthermore, everything else we did earlier still applies as before. We still have the same threefold rotational symmetry, so red on the front will still match either yellow or blue, and orange on the back will still match either green or white. That is, what we have here is a pattern where equality does not hold among all six colors, but if we split the colors into two sets, here taken as {red, yellow, blue} and {orange, green, white}, then: (1) within each set those three colors are distributed equally, and (2) if you compare a color from the first set with a color from the second set, those two distributions are mirror images of each other. It is arrangements of this type that I call *color equivalence patterns*.

Patterns on an Octahedron

In my previous work I explored all possible equality and equivalence patterns on $2\times 2\times 2$ and $3\times 3\times 3$ Rubik's cubes, studied their various symmetries, looked for interesting comparisons between different patterns, and so on. In this article, I explore the same concepts on an octahedron. Figure 2 shows an octahedral puzzle called the Skewb Diamond, consisting of six corner pieces that have four colors each, together with eight central pieces having a single color.

Figure 2. Four views of the Skewb Diamond puzzle. (a) "Top" and "bottom" views along a corner-to-corner axis, as in figure 1. (b) The (a) positions tilted forward to lie flat on one face. Note the *fourfold symmetry* from the (a) perspective, and *the threefold symmetry* from the (b) perspective. Cubes similarly have a threefold symmetry when viewed as in figure 1, and a fourfold symmetry when viewed straight down on one of the faces.



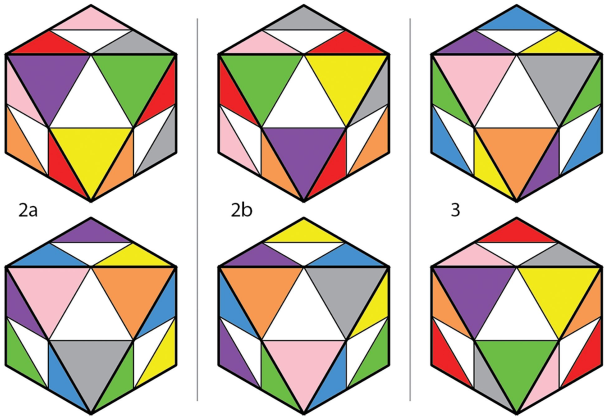
Now, as a puzzle, the Skewb Diamond is somewhat limited; too many arrangements that one thinks ought to be accessible cannot actually be achieved by the allowed rotations. The mathematical analysis of which arrangements of various puzzles are achievable is itself a fascinating topic, but not what we want to consider here. Instead, let's just take this puzzle as our inspiration, but imagine that we are allowed to disassemble and reassemble it at will, so that all permutations and rotations of the six corners can be achieved, and similarly we allow all permutations of the eight centers. What are the possible equality and equivalence patterns, and what interesting symmetries and other properties might they have?

To be sure we really have all the patterns we want, it is necessary to write a computer program that scans through all possible arrangements and picks out the right ones. There is an online supplement available at https://www.tandfonline.com/doi/suppl/10.1080/10724117.2026.2618769 that outlines how to formulate such a code, but it is also possible to understand the patterns without considering how they were obtained. The only thing necessary to understand all the results to follow is a willingness to take the various templates (also available at the previous URL), cut them out, and tape them together to form octahedra, and make sure you understand the various symmetries of the resulting three-dimensional patterns. There will also be a few challenges, which I urge you to spend some time trying to work out yourself before turning to the supplement for the solutions.

Equality Patterns

Including the original configuration in figure 2, which I refer to as pattern 1, there are 16 equality patterns. Figure 3 shows the next three. Patterns 2a and 2b constitute a mirror image pair. All six corners remain in their original positions and rotate by either 90 degrees (2a) or -90 degrees (2b), where we will follow

Figure 3. Patterns 2a, 2b, and 3, as seen from the threefold symmetry view (figure 2(b)). The orientations shown here make it particularly easy to see that red and blue are distributed equally, but the other six colors are also the same. Also, we are considering only the six corners so far, so the centers are left blank.



the usual convention that positive angles are measured in a counterclockwise direction.

For pattern 3 the corners again remain in their original positions, but now rotate by 180 degrees. Because ± 180 degrees denote the same rotation, this pattern has no handedness and is its own mirror image. From the way that these patterns are constructed it is clear also that they have no preferred axis; that is, the patterns look essentially the same when viewed along any of the axes.

Figure 4 shows the next two patterns and compares them with pattern 3. Both patterns 4 and 5 yet again leave all six corners in their original positions; pattern 4 then rotates the top and bottom corners by 180 degrees, whereas pattern 5 rotates the other four corners by 180 degrees. That is, we could also think of pattern 3 as being a superposition of patterns 4 and 5, which is indeed reflected in figure 4. Unlike pattern 3, though, which has no preferred axis, patterns 4 and 5 clearly do have a preferred axis, namely which two opposite corners you choose to constitute the top and bottom. There would, therefore, be three possibilities for these patterns, depending on which axis we choose. These are all basically the same patterns though, so we will consistently only choose one of the three axes.

Figure 4. Patterns 4 and 5, also compared with pattern 3, all seen from the fourfold symmetry view (figure 2(a)).

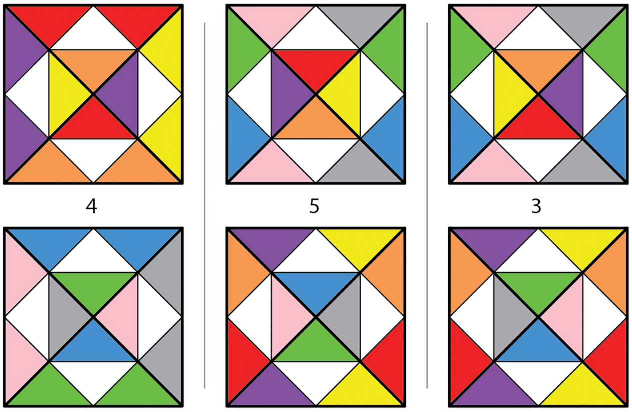


Figure 5. Patterns 6a and 6b, seen from both the fourfold symmetry and threefold symmetry views.

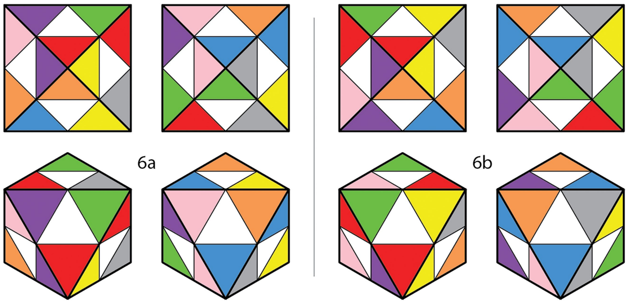


Figure 5 shows another mirror image pair, patterns 6a and 6b. This pattern has both a preferred axis and handedness. That is, it forms a mirror image pair. In this case it is left up to you to print out the provided templates and work out for yourself how the pieces move to create these patterns.

Overall, there are then four possible symmetries: either a preferred axis or no preferred axis, and either handedness or no handedness. In addition to the eight patterns shown here, there are four other mirror image pairs, with all four of them also having a preferred axis. By studying the patterns given here, can you perhaps guess what these other four mirror image pairs are? The complete set of all 16 patterns follows remarkably simple rules, but still yields quite a variety of patterns, including all four of these symmetry types.

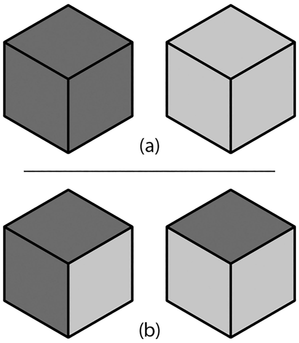
Equivalence Patterns

Based on the results so far, and comparing also with figure 1(b), can you guess a few equivalence patterns? The full set is presented in the supplement. Before we even get to any of the details though, equivalence patterns raise a question that fundamentally does not arise for equality patterns: How do you divide the colors up into the two mirror image sets?

Imagine a cube without any colors attached to the faces at all yet. In how many different ways can you divide the faces into two sets of three? From a combinatorics point of view, the answer would be this: Choose three out of six, so $6!/3!^2 = 20$. However, most of these are just rotated versions of one another. From a geometric point of view, there are only two different ways of dividing the faces up into two sets of three.

First, as shown in figure 6(a), you can split every pair of opposite faces between

Figure 6. The two different ways of dividing the six faces of a cube into two sets of three faces each.



the two sets. Alternatively, as in figure 6(b), you can split only one pair of opposite faces between the two sets and have the other two pairs of opposite faces each entirely within a given set. Any of the 20 possible choices will always be just some suitably rotated version of one of these two ways of creating two sets of three faces.

Returning to figure 1(b), it clearly divides the six colors as in figure 6(a). In fact, all equivalence patterns on a cube split the colors up in this way; there are no equivalence patterns where the colors are split up as in figure 6(b). Why should that be the case? At one level we could just say because the computer code checked every possibility, and there simply aren't any patterns that follow the figure 6(b) way of dividing things. At a deeper level, though, we could at least conjecture that it is due to the fact that the 6(a) split is more symmetric than the 6(b) split, with every pair of opposite faces being treated equally, rather than one pair being treated differently from the other two.

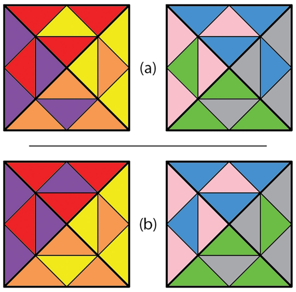
This raises two obvious points. First, how many different ways are there of dividing the eight faces on an octahedron into two sets of four each? That is, what is the equivalent of figure 6 for an octahedron? See if you can work out the possibilities here. There are definitely more than just two! Next, once you think you have them all, apply symmetry considerations as given earlier to try to decide which splitting options are most likely to yield equivalence patterns. The results are presented in the supplement, but these fundamental questions of what is the equivalent of figure 6 for an octahedron and which options are most likely to yield equivalence patterns are something you should definitely think about yourself first.

Adding the Centers

We are rapidly running out of space, and have not considered the centers at all yet, so let's keep this section short.

Figure 7 shows two possible arrangements of the centers, together with the corners in their pattern 1 original configuration. Both arrangements yield color equality patterns. If you focus on the distribution of just a single color at a time, the two arrangements even look the same: In both cases each color has its center moved to one of the immediately adjacent faces. Nevertheless, these arrangements are different. This becomes especially clear if we think of them as actions being applied to the centers and ask how often the actions would have to be repeated before everything returns to its original configuration. In figure 7(a) you have to repeat the action four times to get back to where you started, whereas in figure 7(b) centers are exchanged pairwise, so repeating that action twice gets you back where you started. Next, apply these same center arrangements to some of the other equality patterns. Are the results still equality patterns? If not, why does it work for pattern 1 but not for some of the others?

Figure 7. Two different arrangements of the centers, with the corners in their original positions.



Future Work

You might also enjoy Roice Nelson's "Abstracting the Rubik's Cube" (*Math Horizons*, April 2018), where he discussed how various puzzles such as these can be extended into higher dimensions. The concepts of color equality and equivalence patterns would still apply but actually computing these patterns for even just the simplest such extension, the four-dimensional $2\times2\times2\times2$ hypercube, would be far more challenging than anything we considered here. ●

Rainer Hollerbach is professor of applied math at the University of Leeds, England. He first encountered Rubik's cube as a teenager and is delighted to be working on related patterns now.

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Supplemental materials for this article can be accessed at https://www.tandfonline.com/doi/suppl/10.1080/10724117.2026.2618769.

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