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Finite-Dimensional Adaptive Observer Design for Euler-Bernouilli Beams with Sensor Delay

H. Rafia(1), F. Giri(2), A. Selivanov(3), E. Fridman(4), A. Benabdelhadi(1), H. Ouadi(1), F.Z. Chaoui(1)

(1) Mohammed V University, ENSAM, Rabat, Morocco.
(2) University of Caen Normandie, ESIX, Caen, France.
(3) Department of Automatic Control and Systems Engineering, The University of Sheffield, UK.
(4) School of Electrical Engineering, Tel Aviv University, Israel.

Abstract. We extend the recent work of (Selivanov and Fridman, 2023) on control design for Partial Differential Equations (PDEs) of Euler-Bernouilli type with viscous and Kelvin–Voigt damping. The extension is focused on adaptive observer design in the presence of sensor delay and parameter uncertainties. The considered formulation of the uncertainties makes it possible to account for the effect of structured disturbances entering the sensor and the state equations. The observer design approach combines the modal decomposition method and the decoupling transformation technique. Making use of these techniques, we design an adaptive observer that includes a (finite-dimensional) state estimator, a parameter estimator of least-squares type, and an auxiliary filter generating the regressor of the parameter estimator. The resulting estimation error system turns to be an interconnected system driven by a signal depending on the neglected PDE modes. Exponential stability of the whole estimation error system is analyzed using suitable Lyapunov and Lyapunov-Krasovskii functional. We show that exponential stability is guaranteed for not too large delay values, under a well-defined Persistent Excitation (PE) condition.

Keywords. Euler-Bernouilli PDE, delay systems, adaptive observer, finite-dimensional design.

1. Introduction

In recent years, observer design for PDEs has gained a substantial interest, but the focus was put on heat and wave PDEs, e.g. (Krstic, Guo, Balogh, and Smyshlyaev, 2008), (Smyshlyaev and Krstic, 2009), (Fridman and Blighovsky, 2012), (Guo, Zhou, and Krstic, 2018), (Terushkin and Fridman, 2019), (Katz, Fridman, and Selivanov, 2021). The various observers can be classified in three main categories:

(i) Natural Luenburger observers in PDE forms. They are copies of the PDE under study plus output injection in the boundary or the domain, see e.g. (Guo and Xu, 2008) for wave PDEs and (Fridman and Blighovsky, 2012) for parabolic semilinear PDEs. In the latter, the spatial domain is split in several subdomains, a dedicated sensor is implemented in each subdomain, and a specific output feedback term is injected; the larger the number of sensors implemented the higher the observer convergence speed.

(ii) Finite-dimensional Luenberger-type observers in ODE forms. They are designed using modal decomposition, see e.g. (Selivanov and Fridman, 2019), (Katz and Fridman, 2020). The fixed observer gain is computed on the basis of a finite number of dominating modes. The remaining modes, not accounted for in the observer design, result in a modeling error that is shown to be exponentially vanishing. Since that modeling error acts as a right-hand side disturbance in the observer estimation error system, it limits the exponential convergence speed of the observer.

(iii) Backstepping observers in PDE forms. They are copies of the PDEs under study with output injection in both domain and boundary equations (Smyshlyaev and Krstic, 2005, 2009). The domain action involves a space-varying gain while the boundary action uses a fixed gain. Both gains are determined using a Volterra-type transformation so that the estimation error system is made similar to some given exponentially stable target system. Interestingly, the observer exponential convergence is guaranteed with a single sensor. A space-varying gain observer in PDE form has also been designed making use of modal-decomposition, e.g. (Katz, Fridman, and Selivanov, 2021)

Parameter uncertainty is a common issue in real-life systems. In observer design for PDEs, parameter uncertainty may come in domain and boundary equations. The issue is generally coped with by incorporating parameter estimators providing online parameter estimates, leading to adaptive observers. Then, besides exponential convergence of the state estimates, the observer is expected to meet the same feature with parameter estimates, which is generally achieved under additional PE conditions. Exponentially convergent adaptive observers for parabolic PDEs have been proposed, e.g., in (Ahmed-Ali, Giri, Krstic, Burlion, Lamnabhi-Lagarrigue, 2016a-b). Those for wave PDEs can be found, e.g., in (Guo and Xu, 2008), (Anfinsen, Diagne, Aamo, Krstic, 2016), (Guo, Zhou, Krstic, 2018), (Benabdelhadi, Giri, Ahmed-Ali, Krstic, El Fadil, Chaoui, 2021).

Time-delay is another important common phenomenon in physical systems. In PDEs, time-delays may come in domain or boundaries, they can be discrete or distributed nature. Most existing works on observer design have been focused on parabolic PDEs with sensor time delays, e.g. (Guo and Xu, 2008), (Ahmed-Ali, Fridman, Giri, Kahelras, Lamnabhi-Lagarrigue, Burlion, 2020), (Selivanov and Fridman, 2019), (Katz, Fridman, Selivanov, 2021). The proposed observers are either Natural Luenburger type, as in (Guo and Xu, 2008), (Ahmed-Ali, Fridman, Giri, Kahelras, Lamnabhi-Lagarrigue, Burlion, 2020), (Katz, Fridman, Selivanov, 2021), or finite-dimensional Luenberger type as in (Selivanov and Fridman, 2019). In (Guo and Yang, 2009), an observer of Natural Luengerger type has been developed for a Euler-Bernouilli PDE with sensor delay, without viscous and Kelvin–Voigt damping.

The problem of observer design for PDEs becomes much more complex in the presence of both parameter-uncertainty and time-delay. So far, quite few existing works have dealt with this problem. In (Ammari, Giri, Krstic, El Majdoub, Chaoui, 2023), an adaptive observer has been designed, using the backstepping approach and decoupling transformation method, for a heat PDEs with time-delays coming in the domain. Interestingly, both discrete and distributed delays have been accounted for and exponential stability of the observer estimation error system was guaranteed with no limitation on the domain length.

The existing adaptive observers for PDEs, with or without delays, are copies of the (infinite-dimensional) systems under study and so are themselves represented by PDEs. It is only recently that finite-dimensional adaptive observers have been designed, using modal decomposition, for some classes of heat PDEs with sensor delays, see (Rafia, Benabdelhadi, Giri, Ouadi, Chaoui, 2024), (Ahmed-Ali, Fridman, Lamnabhi-Lagarrigue, 2025), (Ahmed-Ali, Fridman, Cacace, 2025). The main feature of finite-dimensional observers is that they are much easier to implement. In the present paper, we seek adaptive observer design for a more complex class of PDEs, i.e., Euler-Bernouilli PDEs with viscous and Kelvin–Voigt damping. Control design for this class of systems has recently been focused on (Selivanov and Fridman, 2023) where modal decomposition has been applied. We presently extend the work of (Selivanov and Fridman, 2023) in two directions: (i) we let the system under study be subject to sensor delay and parameter uncertainty; (ii) we develop an observer that provides online state and parameter estimates. The uncertainty formulation allows to account for structured disturbances entering the state equation and the sensor equation. Observer design is dealt with making use of modal decomposition and a decoupling transformation. Modal decomposition consists in letting the observer design be based on a finite number of dominating modes, leading to finite-dimensional state observers. The decoupling transformation is resorted to make the problem of parameter estimation decoupled from the problem of state estimation. Making use of these techniques, we design an adaptive observer that includes a state estimator of Luenberger type, a parameter estimator of least-squares type, and an auxiliary filter. All these components are finite-dimensional making the whole adaptive observer finite-dimensional and, consequently, much easier to implement. The resulting estimation error system turns out to be interconnected and driven by a "disturbing" signal depending on the (infinite number of) neglected modes. The interconnection nature in the error system is coped with thanks to the decoupling transformation. Exponential stability of the error system is analyzed using Lyapunov and Lyapunov-Krasovskii functional. Under a well-defined PE condition, it is shown that both parameter and state estimation errors converge to the origin for not too large sensor delay. A simulation study provides more highlights on the admissible maximal delay and the effect of the number of dominating modes used in the observer design.

The paper is organized as follows: the observer design problem is stated in Section 2; Sections 3 and 4 are respectively devoted to adaptive observer design and analysis; Section 5 provides simulation results and Section 6 presents some concluding remarks. Some notation and useful theoretical tools are recalled first.

Preliminary notation and tools

Norms. Euclidean norms in the spaces $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ are simply denoted $|\ |$. The identity matrix in $\mathbb{R}^n$ is simply denoted I or $I_{n\times n}$, depending on the context. Similarly, the null matrix in $\mathbb{R}^{n\times m}$ is denoted 0 or $0_{n\times m}$. $L^2(0,D)$ denotes the Hilbert space of square integrable functions $f:[0,D]\to\mathbb{R}^n, \varsigma\to f(x)$ and $\|f\|_{L^2(0,D)}$ denotes the associated Euclidean norm and $\langle\ .\ ,.\rangle$ the associated scalar product. Similarly, $H^k(0,D)$ denotes the Sobolev space with index k and the associated norm is

$$\|f\|_{H^k(0,D)} = \sqrt{\sum_{j=0}^{k} \left\|\frac{d^j f}{d\varsigma^j}\right\|^2_{L^2(0,D)}}$$

Sobolev's inequality. For any $f \in H^1[0,D]$, such that $f(0) = f(D) = 0$, the following inequality holds (whose proof can be found in Kang and Fridman, 2019)

$$\max_{0<\varsigma<D} |f(\varsigma)|^2 \leq \frac{D}{2} \|f'\|^2_{L^2(0,D)}.$$

Orthonormal basis. The linear operator $\mathcal{B}\phi = \phi_{xxxx}$, defined in the domain $D(\mathcal{B}) = \{\phi \in H^4 \,/\, \phi(0) = \phi(1) = \phi''(0) = \phi''(1) = 0\}$, has the following eigenfunctions and eigenvalues,

$$\varphi_n(x) = \sqrt{2}\sin(\pi n x), \quad \lambda_n = (\pi n)^4, \quad n \in \mathbb{N} \tag{1}$$

Furthermore, the family of eigenfunctions form a complete orthonormal set on $L^2(0,1)$.

2. Problem Formulation

Class of systems and observer objective

We are considering the problem of observer design for Euler-Bernouilli beam defined by the following equations, for all t:

$$u_{tt}(x,t) = -\mu u_t(x,t) - \nu u_{txxxx}(x,t) - \alpha u_{xxxx}(x,t) + \psi_D(x,t)\theta_D, \text{ for } 0 < x < 1 \tag{2}$$

$$u(0,t) = u(1,t) = 0 \tag{3}$$

$$u_{xx}(0,t) = u_{xx}(1,t) = 0 \tag{4}$$

with initial condition $u(.,0) \triangleq u_0^*(.)$ satisfying:

$$u_0^* \in H^2(0,1), \;\; \dot{u}_0^* \in H^2(0,1), \;\; u_0^*(0) = u_0^*(1) = \frac{d^2u_0^*}{dx^2}(0) = \frac{d^2u_0^*}{dx^2}(1) = 0 \tag{5}$$

In domain equation (2), $u: [0,1] \times [0,\infty) \to \mathbb{R}$ is the beam displacement w.r.t the equilibrium position; $\mu > 0$ is the viscous damping coefficient; $\nu > 0$ is Kelvin-Voigt damping coefficient; $\alpha > 0$ is a beam coefficient (depending on the elasticity modulus, second moment area, and density). The parameter vector $\theta_D \in \mathbb{R}^{m_D}$ is unknown; $\psi_D: [0,1] \times [0,\infty) \to \mathbb{R}^{1\times m_D}$ is a known bounded function of the form:

$$\psi_D(x,t) = \sum_{n=1}^{N_\psi} \psi_{D,n}(t)\varphi_n(x) \tag{6}$$

for some $N_\psi \geq 1$ where the functions $\psi_{D,n}(t) = \int_0^1 \psi_D(x,t)\varphi_n(x)dx \in \mathbb{R}^{1\times m_D}$ are bounded and differentiable almost-everywhere (ae) on $\mathbb{R}_+$ and their derivatives $\psi'_{D,n}(t)$ are bounded on $\mathbb{R}_+$.

The system distributed state $u(x,t)$ $(0 \leq x \leq 1)$ is not accessible to measurements and the only available measurements are those of the delayed output i.e.

$$y(t) = u(x_S, t-h) + \psi_S(t-h)\theta_S, \quad \text{for all } t \geq 0 \tag{7}$$

and $y(t) = 0$ for $t < 0$, where $h > 0$ is a known time delay, $0 < x_S < 1$ is a known sensor position such that x_S is nonrational (i.e., $x_S \neq \frac{n}{p}$ for any integer n, p), $\theta_S \in \mathbb{R}^{m_S}$ is an unknown parameter vector, and $\psi_S: [0,\infty) \to \mathbb{R}^{1\times m_S}$ is a bounded known function and its derivative $\dot{\psi}_S$ is also bounded.

Note that the uncertain quantities $\psi_D(x,t)\theta_D$ and $\psi_S(t)\theta_S$ in (2) and (7) might account for the effect of external disturbances, modeling errors, or component failures.

In this paper, we seek an adaptive observer that is able to provide online estimates $\hat{u}(x,t)$ and $\hat{\theta}_i(t)$, of $u(x,t)$ and θ_i $(i=D,S)$, based only on the measurements of the delayed output signal $y(t)$.

First, the well-posedness of (2)-(4) is analyzed following a similar analysis in (Selivanov and Emilia, 2023). To this end, we define the following boundary-consistent Hilbert spaces:

$$H^2_{BC}(0,1)=\{f\in H^2(0,1)\mid f(0)=f(1)=0\}$$
$$H^4_{BC}(0,1)=\{f\in H^4(0,1)\mid f(0)=f''(0)=0, f(1)=f''(1)=0\}$$

The energy space associated to (2) is $X=H^2_{BC}(0,1)\times L^2(0,1)$ with the scalar product

$$\langle (f_1,g_1),(f_2,g_2)\rangle_X=\langle f_1'',f_2''\rangle_{L^2}+\langle g_1,g_2\rangle_{L^2}.$$

Introducing the operator $A_0 f=-f''$, with $D(A_0)=H^2_{BC}(0,1)$, the system (2) is given the operator form

$$\dot{\bar{u}}=\mathcal{A}\bar{u}+F, \tag{8}$$

with

$$\bar{u}(t)=\begin{bmatrix}u(.,t)\\ u_t(.,t)\end{bmatrix},\quad \mathcal{A}=\begin{bmatrix}0 & I\\ -\alpha A_0^2 & -(\mu I+\nu A_0^2)\end{bmatrix},\quad F(t)=\begin{bmatrix}0\\ \psi_D(.,t)\theta_D\end{bmatrix}$$

Clearly, $D(\mathcal{A})=H^4_{BC}(0,1)\times H^4_{BC}(0,1)\subset X$. It is readily checked that $\mathcal{A}$ and its adjoint $\mathcal{A}^*$ are dissipative and, consequently, $\mathcal{A}$ generates a C_0-semi-group of contractions on X (Corollary 4.4 in Pazy, 1983). Furthermore, from (6) and associated assumptions, F is differentiable a.e. on $\mathbb{R}_+$ and $F'\in L^1(\mathbb{R}_+,\mathrm{X})$. Then, by Corollary 2.10, in Chapter 4 of (Pazy, 1983), the initial value problem (8) has a unique strong solution on $\mathbb{R}_+$, for every $\bar{u}(0)\in D(\mathcal{A})$.

Finite-dimensional reformulation of the observer problem.

The initial observer design problem, involving the infinite-dimensional system (2)-(6), will now be converted into a new problem involving a finite-dimensional system. To this end, $u(x,t)$ is represented by the following development:

$$u(x,t)=\sum_{n=1}^{\infty}u_n(t)\varphi_n(x),\ \text{for } t\geq 0 \tag{9a}$$

with

$$u_n(t)=\int_0^1 u(x,t)\varphi_n(x)dx,\ \text{for } t\geq 0 \tag{9b}$$

Using notation (1) and equations (2)-(4), it is readily shown (see Appendix A) that the functions $u_n(t)$ are governed by the following equation:

$$\ddot{u}_n(t)=-(\mu+\nu\lambda_n)\dot{u}_n(t)-\alpha\lambda_n u_n(t)+\psi_{D,n}(t)\theta_D,\quad \text{for } t\geq 0 \text{ and } n=1,2\ldots \tag{10}$$

Substituting (9a) in (7) gives:

$$y(t)=\sum_{n=1}^{\infty}u_n(t-h)\varphi_n(x_s)+\psi_S(t-h)\theta_S,\quad \text{for } t\geq 0 \tag{11}$$

In the sequel we let $N\geq N_\psi$ be any integer and define the following vectors and matrices, for $t\geq 0$:

$$U(t) = [u_1(t), \ldots, u_N(t), \dot{u}_1(t), \ldots, \dot{u}_N(t)]^T \tag{12}$$

$$\bar{u}_n(t) = [u_n(t), \dot{u}_n(t)]^T \tag{13}$$

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \tag{14}$$

$$A = \begin{bmatrix} 0_{N\times N} & I_{N\times N} \\ -\alpha\Lambda & -(\mu\, I_{N\times N} + \nu\Lambda) \end{bmatrix} \in \mathbb{R}^{2N\times 2N} \tag{15}$$

$$\Psi_D(t) = \begin{bmatrix} 0_{1\times m_D} \\ \cdots \\ 0_{1\times m_D} \\ \psi_{D,1}(t) \\ \cdots \\ \psi_{D,N}(t) \end{bmatrix} \in \mathbb{R}^{2N\times m_D}, \tag{16}$$

$$C = [\varphi_1(x_S), \ldots, \varphi_N(x_S), 0, \ldots 0]$$

$$= \sqrt{2}[\sin(\pi x_S), \ldots, \sin(\pi N x_S), 0, \ldots, 0] \in \mathbb{R}^{1\times 2N}, \tag{17}$$

$$A_n = \begin{bmatrix} 0 & 1 \\ -\alpha\lambda_n & -(\mu + \nu\lambda_{\mathrm{n}}) \end{bmatrix} \in \mathbb{R}^{2\times 2} \tag{18}$$

$$\zeta(x,t) = \textstyle\sum_{n=N+1}^{\infty} u_n(t)\varphi_n(x) \tag{19}$$

where $\psi_{D,n} = 0$, for $n > N_\psi$. Using notation (12)-(18), it follows from (9a) and (10) that $U(t)$ and $\bar{u}_n(t)$ are governed by the following ODEs:

$$\dot{U}(t) = AU(t) + \Psi_D(t)\theta_D, \text{ for } t \geq 0 \tag{20}$$

$$\dot{\bar{u}}_n(t) = A_n\bar{u}_n(t), \quad \text{, for } t \geq 0 \text{ and } n \geq N+1 \tag{21}$$

Using (9a), (12), (17) and (19), one gets the following output equation associated to (20):

$$y(t) = CU(t-h) + \psi_S(t-h)\theta_S + \zeta(x_S, t-h), \text{ for } t \geq 0 \tag{22}$$

From equations (9a), (12) and (19), one immediately gets:

$$u(x,t) = C(x)U(t) + \zeta(x,t), \text{ for } t \geq 0 \tag{23}$$

with

$$C(x) = [\varphi_1(x), \ldots, \varphi_N(x), 0, \ldots 0],$$

$$= \sqrt{2}[\sin(\pi x), \ldots, \sin(\pi N x), 0, \ldots 0] \in \mathbb{R}^{1\times 2N}, \tag{24}$$

Clearly, $C(x)$ is known and bounded. Then, provided that $\zeta(x,t)$ is exponentially vanishing, estimating $u(t)$ amounts to estimating $U(t)$. The latter is focused on in Section 3.

3. Observer Design

Observability analysis

The following analysis is an adaptation from (Selivanov and Fridman, 2023):

Hautus Lemma. A pair (A, C) is observable if there exists no $w \in \mathbb{C}^{2N} - \{0\}$ such that $Aw = \beta w$ and $Cw = 0$, for some $\beta \in \mathbb{C}$.

Proposition 1. *Assume the coefficients in the system (2)-(5) satisfy the condition $\alpha \neq \mu\nu$. Then, the pair (A, C) defined by (15) and (17) is observable.*

Proof. Assume there is a $w \in \mathbb{C}^{2N} - \{0\}$ such that $Aw = \beta w$ and $Cw = 0$, for some $\beta \in \mathbb{C}$. Let us show that $w = 0$. Consider the partition $w = [w_1^T,\ w_2^T]^T$ with $w_i \in \mathbb{C}^N$ $(i = 1,2)$. One gets using (14)-(15):

$$w_2 = \beta w_1 \tag{25}$$

$$-\alpha\Lambda w_1 - (\mu\, I_{N\times N} + \nu\Lambda)w_2 = \beta w_2 \tag{26}$$

$$C_1 w_1 = 0 \tag{27}$$

with $C_1 \triangleq \sqrt{2}[\sin(\pi x_S), \ldots, \sin(\pi N x_S)] \in \mathbb{R}^{1\times N}$. First notice that, if $\beta = 0$ then it follows from (25)-(26) that $w = 0$. Let us show that the same result holds with $\beta \neq 0$. From the relation (25) it is seen that if $w_1 = 0$ then $w_2 = 0$ and so $w = 0$. So, let us assume $w_1 \neq 0$. Substituting (25) in (26) gives:

$$\big((\beta^2 + \beta\mu)I_{N\times N} + (\nu\beta + \alpha)\Lambda\big)w_1 = 0 \tag{28}$$

where $I_{N\times N}$ denotes the identity matrix of dimension N. As $\mu\nu \neq 0$, it is easily checked that the numbers $\beta^2 + \beta\mu$ and $\nu\beta + \alpha$ cannot be simultaneously null. Let w_{1i} $(i = 1, \ldots, N)$ denote the components of w_1. If $w_{1i} \neq 0$ for some $i = 1, \ldots, N$, then it follows from (28) that $\lambda_i = -\frac{\beta^2+\beta\mu}{\nu\beta+\alpha}$. Since the λ_i's are different from each other and $w_1 \neq 0$, it follows that one, and only one, component of w_1 is nonzero. Let w_{1i} denote that unique nonzero component. Then, one gets from (27) that $\sin(\pi i x_S)\, w_{1i} = 0$ which implies that $i x_S$ is an integer. But this contradicts the fact that x_S is a nonrational number. Hence, the assumption $w_1 \neq 0$ cannot hold. We have shown that $Aw = \beta w$ and $Cw = 0$ implies that $w = 0$. This proves Proposition 1.

<u>Observer structure</u>

The following observer structure is proposed for the system (20):

$$\dot{\hat{U}}(t) = A\hat{U}(t) + \Psi_D(t)\hat{\theta}_D(t) - L\big(\hat{y}(t) - y(t)\big) + v_1(t),\ \text{ for } t \geq 0 \tag{29}$$

with $\hat{U}(0) = 0 \in \mathbb{R}^{2N}$, where

$$\hat{y}(t) = C\hat{U}(t-h) + \psi_S(t-h)\hat{\theta}_S(t),\ \text{ for } t \geq 0 \tag{30}$$

$v_1(t) \in \mathbb{R}^{2N}$ is extra action yet to be found; $L \in \mathbb{R}^{2N}$ is selected so that $A - LC$ is Hurwitz. This is not an issue because the pair (A, C) is observable.

<u>Error system</u>

Introduce the following errors:

$$\tilde{U}(t) = \hat{U}(t) - U(t), \quad \tilde{y}(t) = \hat{y}(t) - y(t)$$

$$\tilde{\theta}_D(t) = \hat{\theta}_D(t) - \theta_D,\ \tilde{\theta}_S(t) = \hat{\theta}_S(t) - \theta_S$$

Subtracting (22) from (30) give:

$$\tilde{y}(t) = C\tilde{U}(t-h) + \psi_S(t-h)\tilde{\theta}_S(t) - \zeta(x_S, t-h), \text{ for } t \geq 0 \tag{31}$$

which, together with (29), gives, for $t \geq 0$:

$$\dot{\hat{U}}(t) = A\hat{U}(t) + \Psi_D(t)\hat{\theta}_D(t) - LC\tilde{U}(t-h) - L\psi_S(t-h)\tilde{\theta}_S(t) + L\zeta(x_S, t-h) + v_1(t)$$

Subtracting side by side the above equation from (20) gives, for $t \geq 0$:

$$\dot{\widetilde{U}}(t) = (A - LC)\widetilde{U}(t) + \Psi_D(t)\tilde{\theta}_D(t) - L\psi_S(t-h)\tilde{\theta}_S(t) + LC\left(\widetilde{U}(t) - \widetilde{U}(t-h)\right)$$
$$+L\zeta(x_S, t-h) + v_1(t) \tag{32}$$

Design of the extra action $v_1(t)$

We introduce the augmented parameter vectors

$$\Theta = \begin{bmatrix}\theta_D \\ \theta_S\end{bmatrix} \in \mathbb{R}^{m_D+m_S}, \quad \widehat{\Theta} = \begin{bmatrix}\hat{\theta}_D \\ \hat{\theta}_S\end{bmatrix}, \quad \widetilde{\Theta} = \begin{bmatrix}\tilde{\theta}_D \\ \tilde{\theta}_S\end{bmatrix}$$

$$\Omega(t) = \begin{bmatrix}\Psi_D(t) & -L\psi_S(t-h)\end{bmatrix} \in \mathbb{R}^{2N\times(m_D+m_S)}$$

and consider the following decoupling transformation:

$$Z(t) = \widetilde{U}(t) - \lambda(t)\widetilde{\Theta}(t), \text{ for } t \geq -h \tag{33}$$

where the auxiliary state $\lambda(t) \in \mathbb{R}^{2N\times(m_D+m_S)}$ will be selected so that the transformed state $Z(t)$ is made decoupled from $\widetilde{\Theta}(t)$. To this end, we differentiate (33), for $t \geq 0$, and use (32) when necessary:

$$\begin{aligned}
\dot{Z}(t) &= \dot{\widetilde{U}}(t) - \dot{\lambda}(t)\widetilde{\Theta}(t) - \lambda(t)\dot{\widetilde{\Theta}}(t) \\
&= (A - LC)\widetilde{U}(t) + \Psi_D(t)\tilde{\theta}_D(t) - L\psi_S(t-h)\tilde{\theta}_S(t) + LC\left(\widetilde{U}(t) - \widetilde{U}(t-h)\right) \\
&\quad +L\zeta(x_S, t-h) + v_1(t) - \dot{\lambda}(t)\widetilde{\Theta}(t) - \lambda(t)\dot{\widetilde{\Theta}}(t) \\
&= (A - LC)Z(t) + (A - LC)\lambda(t)\widetilde{\Theta}(t) + \Omega(t)\widetilde{\Theta}(t) \\
&\quad +LC\big(Z(t) - Z(t-h)\big) + LC\left(\lambda(t)\widetilde{\Theta}(t) - \lambda(t-h)\widetilde{\Theta}(t-h)\right) \\
&\quad +L\zeta(x_S, t-h) + v_1(t) - \dot{\lambda}(t)\widetilde{\Theta}(t) - \lambda(t)\dot{\widetilde{\Theta}}(t) \\
&= (A - LC)Z(t) + LC\big(Z(t) - Z(t-h)\big) + L\zeta(x_S, t-h) \\
&\quad +\big[(A - LC)\lambda(t) + \Omega(t) + LC\big(\lambda(t) - \lambda(t-h)\big) - \dot{\lambda}(t)\big]\widetilde{\Theta}(t) \\
&\quad +LC\lambda(t-h)\left(\widehat{\Theta}(t) - \widehat{\Theta}(t-h)\right) + v_1(t) - \lambda(t)\dot{\widetilde{\Theta}}(t)
\end{aligned} \tag{34}$$

where we have used the fact that $\widetilde{\Theta}(t-h) - \widetilde{\Theta}(t) = \widehat{\Theta}(t) - \widehat{\Theta}(t-h)$. Furthermore, as $\dot{\widetilde{\Theta}}(t) = \dot{\widehat{\Theta}}$ equation (34) suggests the following definitions of $\lambda(t)$ and $v_1(t)$:

$$\dot{\lambda}(t) = (A - LC)\lambda(t) + LC\big(\lambda(t) - \lambda(t-h)\big) + \Omega(t), \text{ for } t \geq 0 \tag{35}$$

$$v_1(t) = \lambda(t)\dot{\widehat{\Theta}}(t) - LC\lambda(t-h)\left(\widehat{\Theta}(t) - \widehat{\Theta}(t-h)\right), \text{ for } t \geq 0 \tag{36}$$

with initial conditions $\lambda(\tau) = 0$, for $\tau \leq 0$. Doing so, relation (38) boils down to:

$$\dot{Z}(t) = (A - LC)Z(t) + LC\big(Z(t) - Z(t-h)\big) + L\zeta(x_S, t-h), \text{ for } t \geq 0 \tag{37}$$

As $\lambda(\tau) = 0$, for $\tau \leq 0$, il follows from (33) that $Z(\tau) = \widetilde{U}(\tau)$, for $\tau \leq 0$.

Design of parameter estimator

Writing (33) at $t - h$ and pre-multiplying by C gives:

$$C\widetilde{U}(t-h) = C\lambda(t-h)\widetilde{\Theta}(t-h) + CZ(t-h), \text{ for } t \geq 0$$

Adding $C\lambda(t-h)\widetilde{\Theta}(t) - C\lambda(t-h)\widetilde{\Theta}(t)$ to the right-hand side, and rearranging terms, yields:

$$C\widetilde{U}(t-h) = C\lambda(t-h)\widetilde{\Theta}(t) + C\lambda(t-h)\left(\widetilde{\Theta}(t-h) - \widetilde{\Theta}(t)\right) + CZ(t-h) \tag{38}$$

On the other hand, from equation (31) one gets:

$$\begin{aligned} C\widetilde{U}(t-h) &= \tilde{y}(t) - \psi_S(t-h)\tilde{\theta}_S(t) + \zeta(x_S, t-h) \\ &= \tilde{y}(t) - \Psi_S(t-h)\widetilde{\Theta}(t) + \zeta(x_S, t-h) \end{aligned} \tag{39}$$

with $\Psi_S(t) = \left[0_{1\times m_D} \;\; \psi_s(t-h)\right] \in \mathbb{R}^{1\times(m_D+m_S)}$. Replacing $C\widetilde{U}(t-h)$ in (38) by the right-hand side of (39) gives:

$$\begin{aligned} &\tilde{y}(t) - \Psi_S(t-h)\widetilde{\Theta}(t) + \zeta(x_S, t-h) \\ &\qquad = C\lambda(t-h)\widetilde{\Theta}(t) + C\lambda(t-h)\left(\widetilde{\Theta}(t-h) - \widetilde{\Theta}(t)\right) + CZ(t-h) \end{aligned} \tag{40}$$

Again, noting that $\widetilde{\Theta}(t-h) - \widetilde{\Theta}(t) = \widehat{\Theta}(t) - \widehat{\Theta}(t-h)$ is known at time t and rearranging terms in the above equality, one gets:

$$\begin{aligned} &\tilde{y}(t) + C\lambda(t-h)\left(\widehat{\Theta}(t) - \widehat{\Theta}(t-h)\right) \\ &\qquad = (C\lambda(t-h) + \Psi_S(t-h))\widetilde{\Theta}(t) - \zeta(x_S, t-h) + CZ(t-h) \end{aligned} \tag{41}$$

where we have moved to the left-hand side all terms that are known at time t. Finally, dividing both sides of (41) by $|C|$ gives:

$$Y(t) = \Lambda(t)\widetilde{\Theta}(t) + \bar{C}Z(t-h) - \bar{\zeta}(x_S, t-h) \tag{42}$$

with

$$\Lambda(t) = (\bar{C}\lambda(t-h) + \overline{\Psi}_S(t-h)) \in \mathbb{R}^{1\times(m_D+m_S)} \tag{43}$$

$$Y(t) = \frac{\tilde{y}(t)}{|C|} + \bar{C}\lambda(t-h)\left(\widehat{\Theta}(t) - \widehat{\Theta}(t-h)\right) \in \mathbb{R} \tag{44}$$

where $\bar{C} = \frac{C}{|C|}$, $\bar{\zeta} = \frac{\zeta}{|C|}$ and $\overline{\Psi}_S = \frac{\Psi_S}{|C|}$. Doing so one has $|\bar{C}| = 1$ a property that will prove to be useful. The quantities $\bar{C}Z(t-h)$ and $\bar{\zeta}(x_S, t-h)$ will be shown to be exponentially vanishing. It turns out that relation (42) is (asymptotically) affine in $\widetilde{\Theta}(t)$, suggesting the following least-squares type estimator:

$$\dot{\widehat{\Theta}}(t) = \dot{\widetilde{\Theta}}(t) = -R(t)\Lambda^{\mathrm{T}}(t)Y(t), \quad \text{for} \quad t \geq 0 \tag{45}$$

$$\dot{R}(t) = R(t) - R(t)\Lambda^{\mathrm{T}}(t)\Lambda(t)R(t), \quad \text{for} \quad t \geq 0 \tag{46}$$

with arbitrary $\widehat{\Theta}(\tau) \in \mathbb{R}^{(m_D+m_S)}$ and arbitrary positive definite matrix $R(\tau) = R^T(\tau) \in \mathbb{R}^{(m_D+m_S)\times(m_D+m_S)}$, for $-h \leq \tau \leq 0$. Note that, in view of (43)-(44), the estimator (45)-(46) makes use of $\lambda(t-h)$ and $\widehat{\Theta}(t) - \widehat{\Theta}(t-h)$ (for all $t \geq 0$) which are available at time t. This also explains why we initialized the ODEs (35) and (45)-(46) over the interval $-h \leq s \leq 0$. In particular, the initialization of $\lambda(s)$ at zero will prove to be useful in the observer analysis (especially, the proof of Theorem 1). For convenience, the whole adaptive observer equations are recapitulated in Table 1.

Table 1. Adaptive Observer

State Estimator

$$\dot{\hat{U}}(t) = A\hat{U}(t) + \Psi_D(t)\hat{\theta}_D(t) - L\big(\hat{y}(t) - y(t)\big) + v_1(t), \text{ for } t \geq 0 \tag{47}$$

$$\hat{u}(x,t) = C(x)\hat{U}(t) = \sum_{n=1}^{N} \hat{u}_n(t)\varphi_n(x), \text{ for } 0 \leq x \leq 1,\ t \geq 0 \tag{48}$$

with

$$\hat{y}(t) = C\hat{U}(t-h) + \psi_S(t-h)\hat{\theta}_S(t) \tag{49}$$

$$v_1(t) = \lambda(t)\dot{\hat{\Theta}}(t) - LC\lambda(t-h)\left(\hat{\Theta}(t) - \hat{\Theta}(t-h)\right) \tag{50}$$

and $\hat{U}(\tau) = 0 \in \mathbb{R}^{2N}$, for $-h \leq \tau \leq 0$, where $L \in \mathbb{R}^{2N}$ is selected such that $A - LC$ is Hurwitz.

Parameter estimator

$$\dot{\hat{\Theta}}(t) = -R(t)\Lambda^{\mathrm{T}}(t)Y(t), \text{ for } t \geq 0 \tag{51}$$

$$\dot{R}(t) = R(t) - R(t)\Lambda^{\mathrm{T}}(t)\Lambda(t)R(t), \text{ for } t \geq 0 \tag{52}$$

with

$$Y(t) = \frac{\tilde{y}(t)}{|C|} + \bar{C}\lambda(t-h)\left(\hat{\Theta}(t) - \hat{\Theta}(t-h)\right) \tag{53}$$

$$\Lambda(t) = (\bar{C}\lambda(t-h) + \bar{\Psi}_S(t-h)) \tag{54}$$

where $\bar{C} = \frac{C}{|C|}$ and $\bar{\Psi}_S(t) = \frac{\Psi_S}{|C|}, \Psi_S(t) = \begin{bmatrix} 0_{1\times m_D} & \psi_S(t) \end{bmatrix}$ with arbitrary initial conditions $\hat{\Theta}(\tau) \in \mathbb{R}^{(m_D+m_S)}$ and arbitrary positive definite matrix $R(\tau) = R^T(\tau) \in \mathbb{R}^{(m_D+m_S)\times(m_D+m_S)}$, for $-h \leq \tau \leq 0$. Then, one gets estimates, $\hat{\theta}_D$ and $\hat{\theta}_S$, of the initial system parameter vectors using the partition $\hat{\Theta} = \begin{bmatrix} \hat{\theta}_D \\ \hat{\theta}_S \end{bmatrix}$.

Auxiliary filter

$$\dot{\lambda}(t) = (A - LC)\lambda(t) + LC\big(\lambda(t) - \lambda(t-h)\big) + \Omega(t), \quad \text{for } t \geq 0 \tag{55}$$

with $\lambda(\tau) = 0 \in \mathbb{R}^{N\times(m_D+m_S)}$ $(-h \leq \tau \leq 0)$, where $\Omega(t) = \begin{bmatrix} \Psi_D(t) & -L\psi_S(t) \end{bmatrix} \in \mathbb{R}^{2N\times(m_D+m_S)}$.

Remark 1. 1) Table 1 shows that the adaptive observer is constituted of three components: (i) the (finite-dimensional) state estimator, described by (47)-(48), providing the state estimates $\hat{U}(t)$; (ii) the parameter estimator, described by (51)-(54), providing the parameter estimates $\hat{\Theta}(t)$; (iii) the auxiliary filter (55) generating the matrix $\lambda(t)$ which, together with $\bar{\Psi}_S$, determines the matrix signal $\Lambda(t)$, as shown by (54). The signal $\Lambda(t)$ determines the trajectory of the parameter vector estimate $\hat{\Theta}(t)$, according to (51)-(52).

2) The observer state trajectory, defined by equation (47), is not only driven by the innovation term $\big(\hat{y}(t) - y(t)\big)$, but also by the parameter estimate rate involved in the extra term $v_1(t) = \lambda(t)\dot{\hat{\Theta}}(t) - LC\lambda(t-h)\left(\tilde{\Theta}(t) - \tilde{\Theta}(t-h)\right)$.

3) To reinforce learning capability of the parameter estimator (51)-(52), we make use of all information that are available at a given time t. Accordingly, the signal $Y(t)$ that is injected in the parameter estimator (51) is augmented by the term in $(\hat{\Theta}(t) - \hat{\Theta}(t-h))$.

4) As mentioned in the Introduction, Euler-Bernouilli PDEs are much more complex than heat PDEs. The higher complexity results in $2N$-dimensional observers (see Table 1), while N-dimensional observers proved to be sufficient in the case of heat PDEs (Rafia, Benabdelhadi, Giri, Ouadi, Chaoui, 2024), (Ahmed-Ali, Fridman, Lamnabhi-Lagarrigue, 2025), (Ahmed-Ali, Fridman, Cacace, 2025). Besides, to ensure observability of Euler-Bernouilli PDEs the sensor positioning requirement $x_S \neq \frac{n}{p}$ must be

satisfied (for any integer n, p). There was no such requirement in heat PDEs for which observability is ensured with the standard positioning $x_S = 0$.

4. Observer Analysis

From (48) and (23), it follows that the distributed state estimation error $\tilde{u}(x,t) = \hat{u}(x,t) - u(x,t)$ and the finite-dimensional estimation error $\widetilde{U}(t)$ are related as follows:

$$\tilde{u}(x,t) = C(x)\widetilde{U}(t) - \zeta(x,t) \tag{56}$$

Showing that $\tilde{u}(x,t)$ is exponentially vanishing, as $t \to \infty$, amounts to showing that $\zeta(x,t)$ and $\widetilde{U}(t)$ are so. First, $\zeta(x,t)$ is focused on in the next proposition using the fact that A_n has the eigenvalues,

$\varrho_1(n) = -\frac{1}{2}\left[(\mu + \nu\lambda_n) + \sqrt{(\mu + \nu\lambda_n)^2 - 4\alpha\lambda_n}\right]$, $\varrho_2(n) = -\frac{1}{2}\left[(\mu + \nu\lambda_n) - \sqrt{(\mu + \nu\lambda_n)^2 - 4\alpha\lambda_n}\right]$,

and the eigenvectors $\begin{bmatrix} 1 \\ \varrho_1(n) \end{bmatrix}$, $\begin{bmatrix} 1 \\ \varrho_2(n) \end{bmatrix}$. It is readily seen that

$$\varrho_1(n) \xrightarrow[n\to\infty]{} -\infty; \quad \varrho_2(n) \xrightarrow[n\to\infty]{} -\frac{\alpha}{\nu}. \tag{57}$$

Proposition 2. *Let the integer $N \geq N_\psi$ be such that $\varrho_2(n) < -\rho_0 := -\frac{\alpha}{2\nu}$, for $n > N$. Then, there exists $\kappa > 1$ such that $\zeta(x,t)$ defined in (19) satisfies*

$$\max_{0<x<1} |\zeta(x,t)| \leq \kappa \|[u_0^*(.), \dot{u}_0^*(.)]\|_{H^1(0,1)} e^{-\rho_0 t}, \textit{ for } t \geq 0 \tag{58}$$

Proof. From (21) one has $\bar{u}_n(t) = e^{A_n t}\bar{u}_n(0)$, for all t. Since $\varrho_1(n) \leq \varrho_2(n) \leq -\rho_0$, there exists $M > 1$ such that,

$$|\bar{u}_n(t)| = |e^{A_n t}\bar{u}_n(0)| \leq M e^{-\rho_0 t}|\bar{u}_n(0)|, \text{ for } t \geq 0,\ n > N$$

Then, applying Sobolev's inequality, it follows using (19) that

$$\begin{aligned}\max_{0\leq x\leq 1} |\zeta(x,t)|^2 &\leq \tfrac{1}{2}\|\zeta_x(x,t)\|^2_{L^2(0,1)} = \tfrac{1}{2}\textstyle\sum_{n=N+1}^{\infty}(\pi n)^2 u_n^2(t) \\ &\leq \tfrac{M^2}{2} e^{-2\rho_0 t} \textstyle\sum_{n=N+1}^{\infty}(\pi n)^2|\bar{u}_n(0)|^2 \\ &\leq \tfrac{M^2}{2} e^{-2\rho_0 t}\|[u_0^*(.), \dot{u}_0^*(.)]\|^2_{H^1(0,1)} e^{-\rho_0 t}, \quad \text{for } t \geq 0\end{aligned} \tag{59}$$

where the last inequality is obtained by applying Parseval's inequality to the equality $[u_0^*(.), \dot{u}_0^*(.)]^T = \sum_{n=1}^{\infty} \bar{u}_n(0)\, \varphi_n(x)$. Inequality (59) proves Proposition 2 with $\kappa = M/\sqrt{2}$.

Proposition 3 *(Boundedness of $\lambda(t)$). Consider the signal $\lambda(t)$ generated by the filter (55). There exists a real $h^* > 0$ such that, if $0 \leq h < h^*$ then $\lambda(t)$ is bounded.*

See Appendix B for the proof of Proposition 3.

By assumption, $\overline{\Psi}_S$ is bounded. Then, it follows from Proposition 3 that $\Lambda(t) = \bar{C}\lambda(t-h) + \overline{\Psi}_S(t-h)$ is also bounded.

In the sequel, the bounds on $|\lambda(t)|$ and $|\Lambda(t)|$ will be denoted λ_{M} and Λ_{M}.

Theorem 1. *(Convergence of $\tilde{U}(t)$, $\tilde{\Theta}(t)$ and $\tilde{u}(t)$). We let the integer N be selected as in Proposition 2 and the delay be such that $0 \leq h < h^*$ (with h^* as in Proposition 3). Assume the signal $\Lambda(t)$ defined by (54) be persistently exciting (PE), in the sense that,*

$$\int_t^{t+T_0} (\Lambda(t+s))^T \Lambda(t+s) ds > \varepsilon_0 I, \ \forall t \geq 0 \tag{60}$$

for some real $\varepsilon_0 > 0, T_0 > 0$. Then, the following properties hold if:

1) *The auxiliary state $Z(t)$, governed by the differential equation (37), exponentially converges to the origin (as $t \to \infty$).*
2) *The parameter estimation error $\tilde{\Theta}(t) = \hat{\Theta}(t) - \Theta$, with $\Theta = [\theta_D^T \ \theta_S^T]$, exponentially converges to the origin.*
3) *The (finite-dimensional) state estimation error $\tilde{U}(t) = \hat{U}(t) - U(t)$, with $U(t)$ defined by (12), exponentially converges to the origin.*
4) *The (infinite-dimensional) state estimation error $\tilde{u}(x,t) = \hat{u}(x,t) - u(x,t)$, with $u(x,t)$ the state of the PDE (2)-(4), exponentially converges to zero.*

Proof. **Proof of Part 1**. Recall the Z −system is described by equation (37) where the state matrix $A - LC$ is Hurwitz. Then, there exists a unique matrix $P = P^T > 0$ satisfying the following Lyapunov equation:

$$(A - LC)^T P + P(A - LC) = -I \tag{61}$$

where I is the identity matrix in $\mathbb{R}^{2N}$. To analyze (37), we consider the Lyapunov-Krasovskii functional:

$$V_z(t) = Z^T(t) P Z(t) + \beta_1 \int_{t-h}^{t} (h + s - t) \dot{Z}^T(s) \dot{Z}(s) ds \tag{62}$$

with $\beta_1 > 0$ arbitrary. Using (61) and (37) and applying Young's and Schwartz inequalities, (62) gives:

$$\begin{aligned}
\dot{V}_z(t) &= -|Z(t)|^2 + 2L\left(C \int_{t-h}^{t} \dot{Z}(s) ds + \zeta(x_s, t-h)\right) P Z(t) - \beta_1 \int_{t-h}^{t} \left|\dot{Z}(s)\right|^2 ds + h\beta_1 \left|\dot{Z}(t)\right|^2 \\
&\leq -\frac{1}{p_M} Z^T(t) P Z(t) + \varsigma_0 Z^T(t) P Z(t) + \frac{4|P||L|^2}{\varsigma_0}\left(|C|^2 \left(\int_{t-h}^{t} \dot{Z}(s) ds\right)^2 + \zeta^2(x_s, t-h)\right) \\
&\quad -\beta_1 \int_{t-h}^{t} \left|\dot{Z}(s)\right|^2 ds + h\beta_1 \left|\dot{Z}(t)\right|^2
\end{aligned} \tag{63}$$

Letting $\sigma_2 = \frac{1}{2p_M}$, $\varsigma_0 = \frac{1}{2p_M}$ and p_M the maximal singular value of P, and using the inequality, $-\frac{\beta_1}{2} \int_{t-h}^{t} \left|\dot{Z}(s)\right|^2 ds \leq -\frac{\beta_1}{2h} \int_{t-h}^{t} (h + s - t) \left|\dot{Z}(s)\right|^2 ds$. Inequality (63) implies:

$$\begin{aligned}
\dot{V}_z(t) &\leq -\sigma_2 Z^T(t) P Z(t) + 8p_M h |P||L|^2 |C|^2 \int_{t-h}^{t} \left|\dot{Z}(s)\right|^2 ds + 8p_M |P||L|^2 \zeta^2(x_s, t-h) \\
&\quad -\frac{\beta_1}{2} \int_{t-h}^{t} (h + s - t) \left|\dot{Z}(s)\right|^2 ds - \frac{\beta_1}{2} \int_{t-h}^{t} \left|\dot{Z}(s)\right|^2 ds + h\beta_1 \left|\dot{Z}(t)\right|^2 \\
&\leq -\sigma_2 Z^T(t) P Z(t) - \frac{\beta_1}{2h} \int_{t-h}^{t} (h + s - t) \left|\dot{Z}(s)\right|^2 ds \\
&\quad -\left(\frac{\beta_1}{2} - 8p_M h |P||L|^2 |C|^2\right) \int_{t-h}^{t} \left|\dot{Z}(s)\right|^2 ds + h\beta_1 \left|\dot{Z}(t)\right|^2 \\
&\quad +8p_M |P||L|^2 \zeta^2(x_s, t-h), \quad \text{for } t \geq 0
\end{aligned} \tag{64}$$

The penultimate term on the right side of (64) is bounded from above as follows, using (37) and applying Young's and Schwartz inequalities:

$$h\beta_1|\dot{Z}(t)|^2 \le 3h\beta_1|A-LC|^2|Z(t)|^2 + 3h^2\beta_1|LC|^2\int_{t-h}^{t}|\dot{Z}(s)|^2 ds + 3h\beta_1|L|^2\zeta^2(x_S, t-h)$$
$$\le \frac{3h\beta_1|A-LC|^2}{p_M} Z^T(t)PZ(t) + 3h^2\beta_1|LC|^2\int_{t-h}^{t}|\dot{Z}(s)|^2 ds$$
$$+3h\beta_1|L|^2\zeta^2(x_S, t-h), \qquad \text{for } t \ge 0 \tag{65}$$

Substituting (65) in (64) gives:

$$\dot{V}_z(t) \le -\left(\sigma_2 - \frac{3h\beta_1|A-LC|^2}{p_M}\right)Z^T(t)PZ(t) - \frac{\beta_1}{2h}\int_{t-h}^{t}(h+s-t)|\dot{Z}(s)|^2 ds$$
$$-\left(\frac{\beta_1}{2} - 8p_M h|P||L|^2|C|^2 - 3h^2\beta_1|LC|^2\right)\int_{t-h}^{t}|\dot{Z}(s)|^2 ds$$
$$+(3h\beta_1|L|^2 + 8p_M|P||L|^2)\zeta^2(x_s, t-h) \tag{66}$$

Let h be small so that $\sigma_2 - \frac{3h\beta_1|A-LC|^2}{p_M} > 0$ and $\frac{\beta_1}{2} - 8p_M h|P||L|^2|C|^2 - 3h^2\beta_1|LC|^2 > 0$. Then (66) implies, using (62):

$$\dot{V}_z(t) \le -\sigma_z V_z(t) + (3h\beta_1|L|^2 + 8p_M|P||L|^2)\zeta^2(x_s, t-h)\,, \quad \text{for } t \ge 0: \tag{67}$$

with $\sigma_z = \min\left(\sigma_2 - \frac{3h\beta_1|A-LC|^2}{p_M}, \frac{1}{2h}\right) > 0$. Recall that by, Proposition 2, $\zeta^2(x_s, t-h)$ exponentially converges to zero, as $t \to \infty$. Then, it follows from (67) that so does $V_z(t)$. This proves Part 1 of Theorem 1, using (62) and the fact that P is positive definite.

Proof of Part 2 of Theorem 1.

From (42) and (45)-(46) it follows that the parameter estimation error $\widetilde{\Theta}(t) = \widehat{\Theta}(t) - \Theta$ is governed by the following equation, for all $t > 0$:

$$\dot{\widetilde{\Theta}}(t) = -R(t)\Lambda^{\mathrm{T}}(t)\Lambda(t)\widetilde{\Theta}(t) - R(t)\Lambda^{\mathrm{T}}(t)\bar{C}Z(t-h) + R(t)\Lambda^{\mathrm{T}}(t)\bar{\zeta}(x_S, t-h), \tag{68}$$

with $R(t)$ defined by (46). It is shown in many places (e.g., Ioannou and Sun, 1995) that condition (60) ensures that $R^{-1}(t)$ remains positive definite, i.e.

$$R^{-1}(t) \ge r_m I, \quad \text{for all } t \ge 0 \tag{69}$$

for some real $0 < r_m < \infty$. Then, it follows from (52) that $R^{-1}(t)$ undergoes the following equation:

$$\dot{R}^{-1}(t) = -R^{-1}(t) + \Lambda^{\mathrm{T}}(t)\Lambda(\mathrm{t})\,, \quad \text{for } t \ge 0 \tag{70}$$

To analyze (68), consider the following Lyapunov function candidate:

$$V_{\widetilde{\Theta}}(t) = \widetilde{\Theta}^T(t)R^{-1}(t)\widetilde{\Theta}(t), \quad \text{for } t \ge 0 \tag{71}$$

Differentiating $V_{\widetilde{\Theta}}(t)$ along (68) gives, using (70) and applying Young's inequality:

$$\dot{V}_{\widetilde{\Theta}}(t) = \widetilde{\Theta}^T(t)\dot{R}^{-1}(t)\widetilde{\Theta}(t) + 2\widetilde{\Theta}^T(t)R^{-1}(t)\dot{\widetilde{\Theta}}(t)$$
$$= -\widetilde{\Theta}^T(t)R^{-1}(t)\widetilde{\Theta}(t) - |\Lambda(\mathrm{t})\widetilde{\Theta}(t)|^2 - 2\widetilde{\Theta}^T(t)\Lambda^{\mathrm{T}}(t)\Lambda(t)\widetilde{\Theta}(t)$$
$$-2\widetilde{\Theta}^T(t)\Lambda^{\mathrm{T}}(t)\bar{C}Z(t-h) + 2\widetilde{\Theta}^T(t)\Lambda^{\mathrm{T}}(t)\bar{\zeta}(x_S, t-h)$$
$$\le -V_{\widetilde{\Theta}}(t) + |Z(t-h)|^2 + |\bar{\zeta}(x_S, t-h)|^2, \quad \text{for } t \ge 0 \tag{72}$$

using the fact that $|\bar{C}| = 1$. As $\zeta^2(x_s, t-h)$ exponentially converges to zero as $t \to \infty$, by Proposition 2, and $|Z(t-h)|^2$ does so by Part 1 of this theorem, it follows from (72) that $V_{\tilde{\Theta}}(t)$ also exponentially converges to zero. Then, using (69), it follows from (71) that $\tilde{\Theta}(t)$ exponentially converges to the origin.

Proof of Parts 3 and 4 of Theorem 1.

By Parts 1 and 2, $|Z(t)|$ and $|\tilde{\Theta}(t)|$ are exponentially vanishing and, by Proposition 3, $|\lambda(t)|$ is bounded. Then, it follows from (33) that $|\tilde{U}(t)|$ exponentially converges to zero, proving Part 3. Then, using (56) and Proposition 2, one gets that $\tilde{u}(x,t)$ exponentially converges to zero. This ends the proof of Theorem 1.

Remark 2. 1) One complexity of the estimation error system, with state $\tilde{\Theta}(t)$ and $\tilde{U}(t)$, defined by equations (68) and (32), was the interactions between these equations. Indeed, it is seen that $\tilde{\Theta}(t) = \left[\tilde{\Theta}_D^T(t) \;\; \tilde{\Theta}_S^T(t)\right]^T$ acts as external input in equation (32) (governing $\tilde{U}(t)$), while $\tilde{U}(t)$ comes (through $Z(t)$) as an input in equation (68) governing $\tilde{\Theta}(t)$. This interaction has partly been reduced thanks to the decoupling transformation (33). Indeed, the latter led to the Z −system defined by (37) not depending on $\tilde{\Theta}$, making its analysis possible independently of $\tilde{\Theta}$ which is done in Theorem 1 (Part 1).

2) PE condition (60) is a usual requirement ensuring exponential convergence of the parameter estimator (51)-(52) which is a least-squares algorithm with unit forgetting factor (Ioannou and Sun, 2012). Property (60) simply means that the vector $\Lambda^T(t)$ spans the whole parameter space $\mathbb{R}^{(m_D+m_S)}$ in each time interval $[t, t+T_0]$ for all $t > 0$. On the other hand, it is easily seen (see Table 1) that the trajectory of $\Lambda^T(t)$ is determined by the known external signals $\psi_D(t)$ and $\psi_S(t)$. Roughly speaking, the power spectra of these signals should contain a number of (different) frequencies at least equal to the number of unknown parameters to be estimated, see e.g. (Ljung, 1987). Therefore, the PE condition is a priori checkable.

3) Sufficient conditions on the admissible values of delay h are established in the proof of Proposition 3. Accordingly, the delay should satisfy the following inequalities

$$h < \frac{1}{6\beta_1|A-LC|^2}, \quad 8hp_M|P||L|^2|C|^2 + 3h^2\beta_1|LC|^2 < \frac{\beta_1}{2} \tag{73}$$

for some $\beta_1 > 0$, where P is the solution of (61) and p_M its larger singular value. From the second inequality, we immediately get the two stronger conditions, $h < \frac{1}{|LC|\sqrt{6}}$ and $h < \frac{\beta_1}{8p_M|P||L|^2|C|^2}$. Multiplying side-by-side the last inequality and the first inequality in (73), we get a β_1-free inequality, i.e., $h^2 < \frac{1}{48p_M|P||L|^2|C|^2|A-LC|^2}$. In summary, we have the following sufficient condition:

$$h < \min\left(\frac{1}{|LC|\sqrt{6}}, \frac{1}{4\sqrt{3}\sqrt{p_M|P|}|L||C||A-LC|}\right) \tag{74}$$

This new sufficient condition is certainly stronger than (73) but is much simpler to check. It entails some conservative that will be illustrated by simulation in Subsection 5. The effect of the design parameter N, representing the number of dominating modes, will also be highlighted in that Section.

5. Simulation

Throughout this Section, we consider a system of the form (2)-(6) where the various parameters are given the numerical values $\alpha = 2$, $\mu = \nu = 0.02$, so that the condition $\mu\nu < \alpha$ is satisfied. It is checked that the condition $\lambda_{N+1}\nu^2 \leq 2\alpha - \mu\nu$ is satisfied. With regard to (6), we let $\theta_D = 1 \in \mathbb{R}$, $N_\psi = 2$ and

$$\psi_D(x,t) = 15 \times (10 + \cos(23\pi t)) \times \varphi_1(x) + 5 \times (10 + \cos(46\pi t)) \times \varphi_2(x)$$

With regard to (22), we let $\theta_S = 2$ and $\psi_S(t) = 0.4 \times \cos(1.2\pi t) + \cos(0.1\pi t)$. The sensor is placed at $x_S = \frac{\sqrt{2}}{7}$ and its delay value is $h = 3$ seconds. The system initial conditions are set to $u(x,0) \triangleq u_0^*(x) = 10 \times \sin(2\pi x)$. It turns out that $u_0^*(0) = u_0^*(1) = \frac{d^2 u_0^*}{dx^2}(0) = \frac{d^2 u_0^*}{dx^2}(1) = 0$. Also, as $u_n(t) = \int_0^1 u(x,t)\varphi_n(x)dx$, one has $u_n(0) = \int_0^1 u(x,0)\varphi_n(x)dx = \int_0^1 10\sin(2\pi x)\varphi_n(x)dx$.

The observer of Table 1 is successively applied with dimensions $N = 2$ and $N = 3$. The observer of dimension $N = 2$ is given the gain $L \in \mathbb{R}^4$ such that $A - LC$ is Hurwitz with eigenvalues $(-13 \ \ -11 \ \ -9 \ \ -8)$. The parameter estimators are initialized with $\widehat{\Theta}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $R(0) = R^T(0) = \text{diag}[2.3\ 10^{-4} \quad 1.26\ 10^{-3}]$. The gain $L \in \mathbb{R}^6$ of the observer with dimension $N = 3$ is selected such that $A - LC$ has eigenvalues $(-13 \ -11 \ -9 \ -8 \ -15 - 15)$. The parameter estimators are initialized with $\widehat{\Theta}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $R(0) = R^T(0) = \text{diag}[1.74\ 10^{-3} \quad 1.34\ 10^{-2}]$.

The parameter estimates $\hat{\theta}_D$ and $\hat{\theta}_S$ generated by both observers are shown in Figs 2 and 3. The state estimation errors $\tilde{u}(x,t) = \hat{u}(x,t) - u(x,t)$ obtained with both observers are separately plotted in Fig. 4. To make the difference between the two observers more visible we have plotted on Fig. 5 the error norm $\|\tilde{u}(t)\|_{L_2[0,1]} = \int_0^1 \tilde{u}^2(x,t)dx$ in the case $N = 2$ and $N = 3$. Figures 2 to 5 show that the observer with higher dimension $N = 3$ performs better than the observer of smaller dimension $N = 2$. Finally, it is checked that the requirement (74) leads to the delay limitation 2×10^{-4} seconds which is much smaller than the delay value $h = 3$ seconds used in simulation. This shows the maximal admissible delay h^* goes beyond (74) which is a sufficient, not necessary, condition.

6. Conclusion

We have considered the problem of observer design for Euler-Bernouilli PDE described by equations (2)-(7). The novelty of this class lies in the presence of time delay in sensor equation (7) and parameter uncertainty in equations (2) and (7) making possible to account for structured external disturbances in domain and sensor equations. The adaptive observer of Table 1 includes the state observer (47)-(48) designed using modal distribution based on the first N main modes. The effect of parameter uncertainty is compensated for using the least-squares estimator (50)-(51) and the auxiliary filter (52). The parameter and state estimation errors are described by equations (68) and (32) which highlight the existence of an interaction in the estimation error system. Exponential stability of this system is stated in Theorem 1 for not too large delays, under the PE condition (60). Proposition 3 (stating the boundedness of the auxiliary state $\lambda(t)$) and that of Theorem 1 are established using Lyapunov and Lyapunov-Krasovskii functional. Those proofs highlight a sufficient condition, expressed by inequalities (73), on the maximal admissible value of delay. Coping with arbitrary large delays can be performed using the cascade observer principle. This has recently been illustrated for heat PDEs in (Ahmed-Ali, Cacace, Fridman, 2025). Another research direction is one that consists in extending the observer of Table 1 so that it applies to the case of both discrete and distributed delays in state equation.

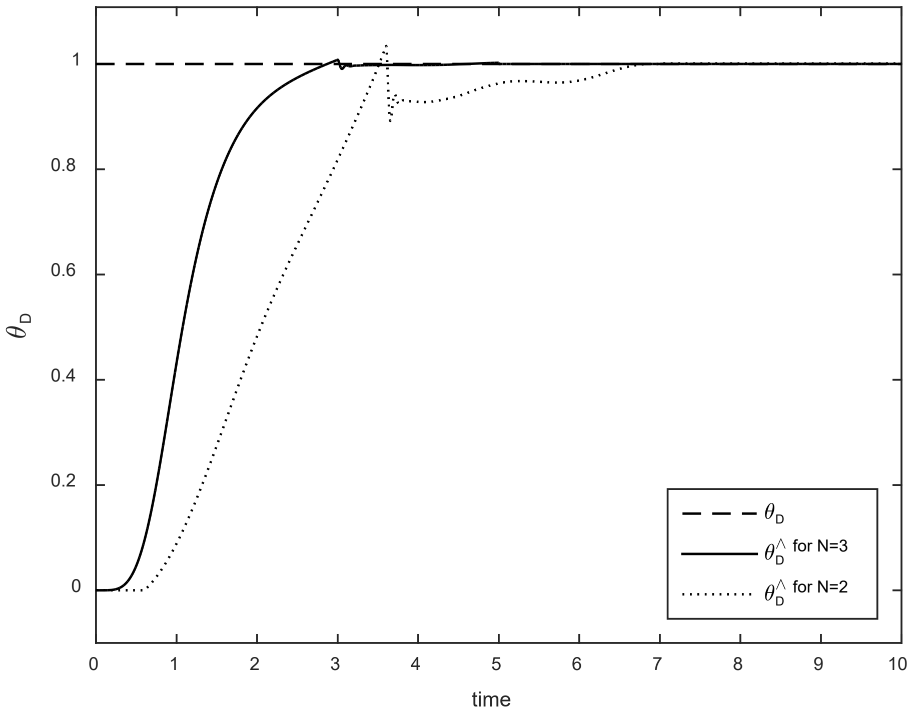


Fig. 2. Parameter θ_D (dashed) and its estimate provided by the observer of Table 1: case $N = 2$ (dotted), case $N = 3$ (solid)

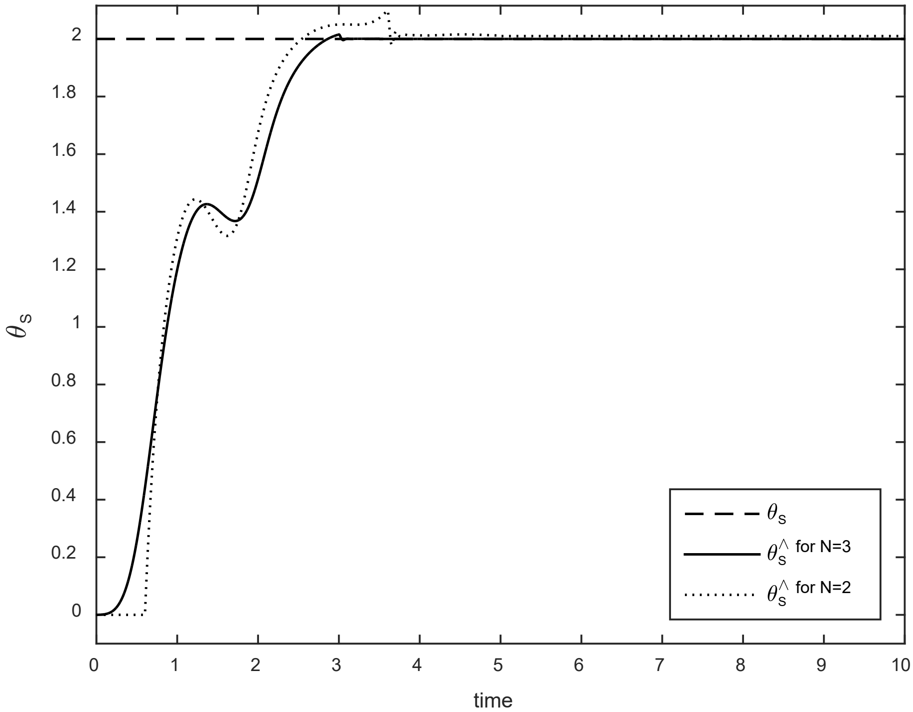
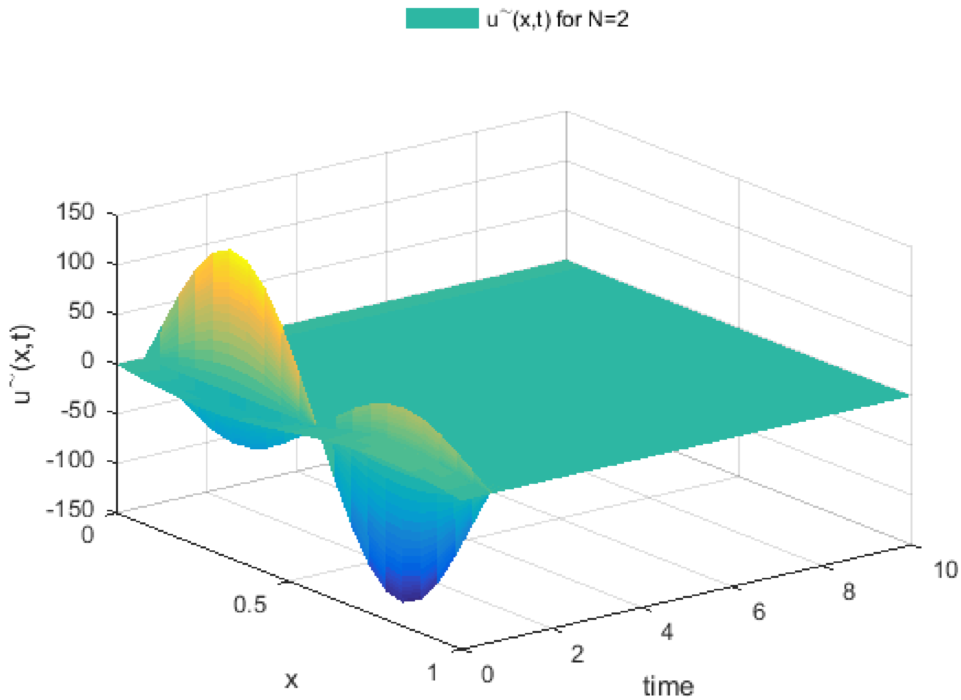


Fig. 3. Parameter θ_S (dashed) and its estimate provided by the observer of Table 1: case $N = 2$ (dotted), case $N = 3$ (solid)



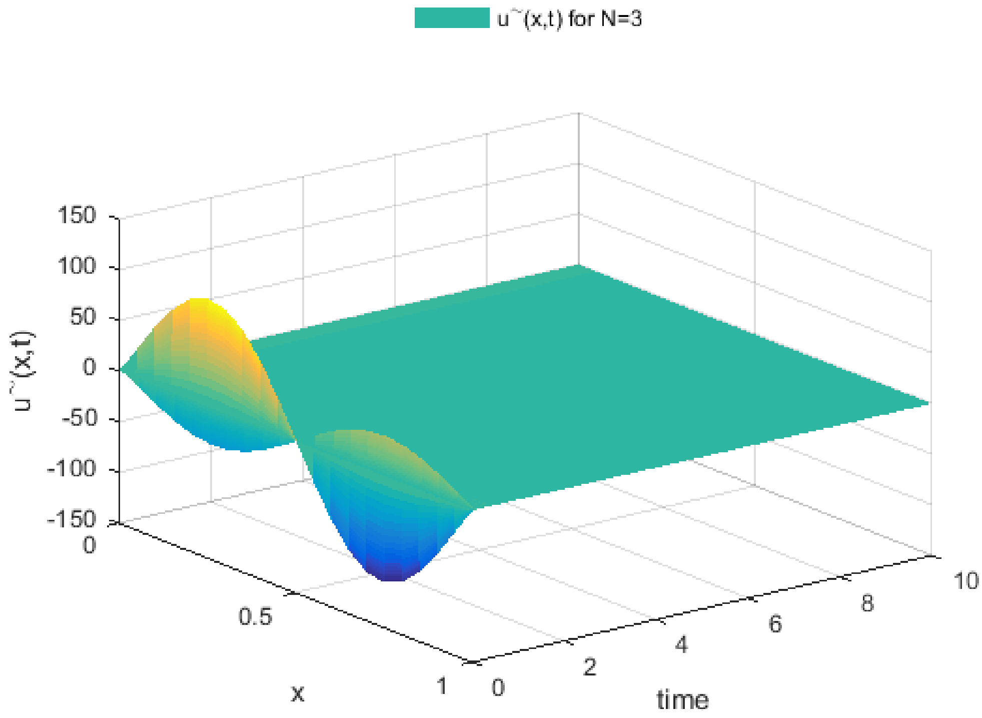


Fig. 4. State estimation error $\tilde{u}(x,t)$ with observer of Table 1: case $N=2$ (top) case $N=3$ (bottom)

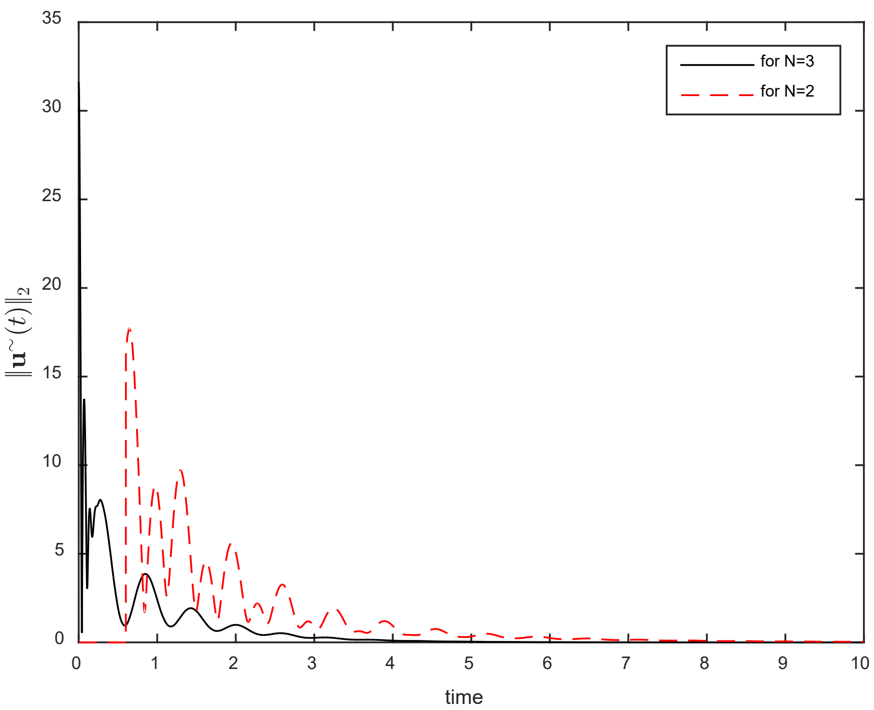


Fig. 5. Norm $\|\tilde{u}(t)\|_{L_2[0,1]}$ with observer of Table 1: case $N=2$ (dashed), case $N=3$ (solid)

Appendix A. Computation of (10)

Differentiating (9b) immediately gives, using (2)

$$\ddot{u}_n(t) = \int_0^1 u_{tt}(x,t)\varphi_n(x)dx$$
$$= -\int_0^1 \mu u_t(x,t)\varphi_n(x)dx - \int_0^1 \nu u_{txxxx}(x,t)\varphi_n(x)dx$$
$$-\int_0^1 \alpha u_{xxxx}(x,t)\varphi_n(x)dx + \int_0^1 \psi_D(x,t)\theta_D\varphi_n(x)dx$$
$$= -\mu\dot{u}_n(t) - \nu\frac{d}{dt}\int_0^1 u_{xxxx}(x,t)\varphi_n(x)dx$$
$$-\alpha\int_0^1 u_{xxxx}(x,t)\varphi_n(x)dx + \left(\int_0^1 \psi_D(x,t)\varphi_n(x)dx\right)\theta_D \tag{A1}$$

The second and third terms on the right side of (A1) are focused on in the sequel. Using (1) and (3)-(4), the second term on the right side develops as follows, applying integration by parts four times:

$$\frac{d}{dt}\int_0^1 u_{xxxx}(x,t)\varphi_n(x)dx = -\frac{d}{dt}\int_0^1 u_{xxx}(x,t)\varphi_n'(x)dx$$
$$= \frac{d}{dt}\int_0^1 u_{xx}(x,t)\varphi_n''(x)dx = -\frac{d}{dt}\int_0^1 u_x(x,t)\varphi_n'''(x)dx$$
$$= \frac{d}{dt}\int_0^1 u(x,t)\varphi_n''''(x)dx$$
$$= \lambda_n\frac{d}{dt}\int_0^1 u(x,t)\varphi_n(x)dx = \lambda_n\dot{u}_n(t) \tag{A2}$$

Similarly, the third term on the right side writes as follows:

$$\int_0^1 u_{xxxx}(x,t)\varphi_n(x)dx = \lambda_n u_n(t) \tag{A3}$$

Using (A2)-(A3), equation (A1) give:

$$\ddot{u}_n(t) = -\mu\dot{u}_n(t) - \alpha\lambda_n u_n(t) + \psi_{D,n}(t)\theta_D - \nu\lambda_n\dot{u}_n(t)$$
$$= -(\mu + \nu\lambda_n)\dot{u}_n(t) - \alpha\lambda_n u_n(t) + \psi_{D,n}(t)\theta_D$$

This establishes equation (10).

Appendix B. Proof of Proposition 3.

We consider the following partition of the vector $\lambda(t) \in \mathbb{R}^{N\times(m_D+m_S)}$:

$$\lambda(t) = \left[\lambda_1(t) \ldots \lambda_{m_D+m_S}(t)\right] \tag{B1}$$

where $\lambda_i(t) \in \mathbb{R}^N$ designates the i^{th} column of $\lambda(t)$. Then, it follows from (35) that, for each $i = 1 \ldots m_D + m_S$, the vector function $\lambda_i(t)$ is governed by the following differential equation:

$$\dot{\lambda}_i(t) = (A - LC)\lambda_i(t) + LC\int_{t-h}^t \dot{\lambda}_i(s)ds + \Omega_i(t), \text{ for } t \geq 0 \tag{B2}$$

where $\Omega_i(t) \in \mathbb{R}^N$ designates the i^{th} column of $\Omega(t) \in \mathbb{R}^{N\times(m_D+m_S)}$. To analyze (B2), we consider the following Lyapunov-Krasovskii functional:

$$V_i(t) = \lambda_i^T(t)P\lambda_i(t) + \beta_1\int_{t-h}^t (h + s - t)\dot{\lambda}_i^T(t)\dot{\lambda}_i(s)ds \tag{B3}$$

with $\beta_1 > 0$ and P as in the proof is very much similar to the proof of Part 1 of Theorem 1. The rest of the present proof is very much similar to that of Part 1 of Theorem 1. Differentiating (B3) gives, using (B2) and (69) and applying Young's and Schwartz inequalities:

$$\dot{V}_i(t) = \dot{\lambda}_i^T(t)P\lambda_i(t) + \dot{\lambda}_i^T(t)P\lambda_i(t) - \beta_1\int_{t-h}^t \left|\dot{\lambda}_i(s)\right|^2 ds + h\beta_1\left|\dot{\lambda}_i(t)\right|^2$$

$$= -|\lambda_i(t)|^2 + 2L\left(C\int_{t-h}^{t}\dot{\lambda}_i(s)ds + \Omega_{\mathrm{i}}(t)\right)P\lambda_i(t) - \beta_1\int_{t-h}^{t}\left|\dot{\lambda}_i(s)\right|^2 ds + h\beta_1\left|\dot{\lambda}_i(t)\right|^2$$

$$\leq -\frac{1}{p_M}\lambda_i^T(t)P\lambda_i(t) + \varsigma_0\lambda_i^T(t)P\lambda_i(t) + \frac{2|P||L|^2}{\varsigma_0}\left(|C|^2\left(\int_{t-h}^{t}\dot{\lambda}_i(s)ds\right)^2 + |\Omega_{\mathrm{i}}(t)|^2\right)$$

$$-\beta_1\int_{t-h}^{t}\left|\dot{\lambda}_i(s)\right|^2 ds + h\beta_1\left|\dot{\lambda}_i(t)\right|^2$$

Letting $\varsigma_0 = \frac{1}{2p_M}$, just as in the proof of Theorem 1, where p_M denotes the maximal singular value of P, the above inequality implies,

$$\dot{V}_i(t) \leq -\varsigma_0\lambda_i^T(t)P\lambda_i(t) + \frac{2h|P||L|^2|C|^2}{\varsigma_0}\int_{t-h}^{t}\left|\dot{\lambda}_i(s)\right|^2 ds + \frac{2|P||L|^2}{\varsigma_0}|\Omega_{\mathrm{i}}(t)|^2$$

$$-\frac{\beta_1}{2}\int_{t-h}^{t}(h+s-t)\left|\dot{\lambda}_i(s)\right|^2 ds - \frac{\beta_1}{2}\int_{t-h}^{t}\left|\dot{\lambda}_i(s)\right|^2 ds + h\beta_1\left|\dot{\lambda}_i(t)\right|^2$$

$$\leq -\varsigma_0\lambda_i^T(t)P\lambda_i(t) - \frac{\beta_1}{2h}\int_{t-h}^{t}(h+s-t)\left|\dot{\lambda}_i(s)\right|^2 ds - \left(\frac{\beta_1}{2} - \frac{2h|P||L|^2|C|^2}{\varsigma_0}\right)\int_{t-h}^{t}\left|\dot{\lambda}_i(s)\right|^2 ds$$

$$+h\beta_1\left|\dot{\lambda}_i(t)\right|^2 + \frac{4|P||L|^2}{\varsigma_0}|\Omega_{\mathrm{i}}(t)|^2, \text{ for } t \geq 0 \tag{B4}$$

where the last inequality is obtained using the inequality:

$$-\frac{\beta_2}{2}\int_{t-h}^{t}\left|\dot{\lambda}_i(s)\right|^2 ds \leq -\frac{\beta_2}{2h}\int_{t-h}^{t}(h+s-t)\left|\dot{\lambda}_i(s)\right|^2 ds \tag{B5}$$

The penultimate term on the right side of (B4) is bounded from above as follows, using (B2) and applying Young's and Schwartz inequalities:

$$h\beta_1\left|\dot{\lambda}_i(t)\right|^2 \leq 3h\beta_1|A-LC|^2|\lambda_i(t)|^2 + 3h^2\beta_1|LC|^2\int_{t-h}^{t}\left|\dot{\lambda}_i(s)\right|^2 ds$$

$$+3h\beta_1|L|^2\zeta^2(x_S, t-h)$$

$$\leq \frac{3h\beta_1|A-LC|^2}{p_M}\lambda_i^T(t)P\lambda_i(t) + 3h^2\beta_1|LC|^2\int_{t-h}^{t}\left|\dot{\lambda}_i(s)\right|^2 ds$$

$$+3h\beta_1|L|^2\zeta^2(x_S, t-h), \text{ for } t \geq 0 \tag{B6}$$

Substituting (B6) in (B2) gives:

$$\dot{V}_i(t) \leq -\left(\varsigma_0 - \frac{3h\beta_1|A-LC|^2}{p_M}\right)\lambda_i^T(t)P\lambda_i(t) - \frac{\beta_1}{2h}\int_{t-h}^{t}(h+s-t)\left|\dot{\lambda}_i(s)\right|^2 ds$$

$$-\left(\frac{\beta_1}{2} - \frac{4h|P||L|^2|C|^2}{\varsigma_0} - 3h^2\beta_1|LC|^2\right)\int_{t-h}^{t}\left|\dot{\lambda}_i(s)\right|^2 ds$$

$$+\frac{4|P||L|^2}{\varsigma_0}|\Omega_{\mathrm{i}}(t)|^2 + 3h\beta_1|L|^2\zeta^2(x_S, t-h) \tag{B7}$$

Let h be small so that $\varsigma_0 - \frac{3h\beta_1|A-LC|^2}{p_M} = \frac{1}{2p_M} - \frac{3h\beta_1|A-LC|^2}{p_M} > 0$ and $\frac{\beta_1}{2} - \frac{4h|P||L|^2|C|^2}{\varsigma_0} - 3h^2\beta_1|LC|^2 > 0$. Then (B7) implies, using (B3):

$$\dot{V}_i(t) \leq -\sigma_z V_i(t) + \left(3h\beta_1|L|^2 + \frac{4|P||L|^2}{\varsigma_0}\right)|\Omega_{\mathrm{i}}(t)|^2, \quad \text{for } t \geq 0 \tag{B8}$$

with $\sigma_z = \min\left(\frac{1-6h\beta_1|A-LC|^2}{2p_M}, \frac{1}{2h}\right) > 0$ just as in the proof of Theorem 1 (Part 1). By assumption, $|\Omega_{\mathrm{i}}(t)|^2$ is bounded and, by Proposition 2, $\zeta^2(x_S, t-h)$ is asymptotically vanishing. Then, it follows from (B8) that

$V_i(t)$ is bounded and, as P is positive definite, it follows that $\lambda_i(t)$ $i = 1 \dots m_D + m_S$) are bounded. This ends the proof of Proposition 3.

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