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A note on incremental stability of externally positive Lurie systems

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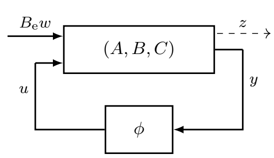
***Abstract*—The Kalman conjecture is shown to be true for Lurie systems in which the linear part is *externally positive* and the nonlinear element satisfies a generalised incremental gain bound. It is shown further that the system satisfies a range of both traditional and incremental stability properties. Multivariable systems are treated in the same framework as single-input-single output systems, giving the results a wider scope than those normally covered by the Kalman conjecture. Some numerical examples illustrate the utility of the results.**

***Index Terms*—Lurie systems, Nonlinear systems, Robust control, Stability of nonlinear systems**

I. Introduction

Consider the Lurie (also Lur'e or Lurye) system in Figure I.1a. Assuming u and y are both scalar-valued signals, the Kalman Conjecture [1] claims that the (nonlinear) system is stable, in some sense, if a corresponding set of linear systems, obtained by setting $\phi(y) = ky$ are stable for all $k \in [k_{\min}, k_{\max}]$. Here, $k_{\min}$ and $k_{\max}$ represent the upper and lower bounds of the *slope* of the nonlinearity $\phi(\cdot)$, respectively, the condition shown in Figure I.1b. Such a $\phi(\cdot)$ is called incrementally bounded with gain δ when $k_{\max} = -k_{\min} = \delta$. In other words, a nonlinear stability analysis problem can be replaced by a linear one. It is known that, in general, the (real) Kalman Conjecture is *false*.



(a)

$$k_{\min} \leqslant \frac{\phi(z_1)-\phi(z_2)}{z_1-z_2} \leqslant k_{\max} \quad \forall\, z_1, z_2,\ z_1 \neq z_2$$

(b)

Figure I.1: (a) Forced Lurie system (b) Slope condition for scalar nonlinearity $\phi(\cdot)$ with bounds $k_{\min}$ and $k_{\max}$.

The Kalman Conjecture was initially proposed to address the shortcomings of a companion conjecture, the Aizerman Conjecture, in which $\phi(\cdot)$ is also replaced by a gain k, but with the gain bounds given by the upper and lower bounds of the *sector* $\phi(\cdot)$ is assumed to inhabit. The Aizerman Conjecture has a long history in automatic control and has been considered in both the former Soviet and the Western literature — see, for example, [2, 3, 4, 5] for a brief overview of the vast literature on this subject. In summary, the real Aizerman Conjecture is also *false*, with 3rd order counter-examples being discovered over 50 years ago [6]. It was hoped that the Kalman Conjecture would be more general, but in fact 4$^{\text{th}}$-order counter-examples exist and it has been recently proved that, in discrete-time, even 2$^{\text{nd}}$ order counter-examples exist [7]. In fact, much of the work on the Kalman Conjecture seeks to prove it is indeed fallacious and the interested reader may consult, for instance, [2, 8, 9, 10, 11] and references therein.

However, when one focuses on positive systems — systems with non-negative impulse responses and/or non-negative state trajectories — the situation with the Aizerman and Kalman conjectures appears much more promising. A feature of positive systems (and particularly their stability radii [12]) is that the "*linear hypothesis*" in the Aizerman/Kalman conjectures, namely that all (real) gains in a certain ball are stabilizing, is equivalent, at least in the single-input single-output case, to a small-gain condition. Over recent years, a variety of Aizerman conjectures have shown to be true for certain classes of positive systems under a variety of conditions — see [4, 13, 14]. Most recently, it was proved in [3] that, for multivariable *internally positive* systems — systems with state evolving on the positive orthant — the Aizermann Conjecture held under quite mild conditions. There only positive exponential stability of the origin $x(0) = 0$ was established, along with some exponential input-to-state stability results.

Here we prove that the Kalman Conjecture is *true* when the linear components of the underlying Lurie system are *externally positive* — a more expansive class than those considered in [3]. The main thrust of this work is captured by the corollary below, which is a special case of our main result.

Result: Consider the feedback connection shown in Figure I.1a in the single-input single-output setting. Let $\mathbf{G}$ and $\mathbf{G}_{\mathrm{e}}$ denote the transfer functions of (A, B, C) and (A, B_{e}, C), respectively. If $\mathbf{G}$ and $\mathbf{G}_{\mathrm{e}}$ are externally positive, ϕ is incrementally bounded with gain δ, and $A + B\gamma C$ is Hurwitz for all $|\gamma| \leqslant \delta$, then the following incremental stability estimate holds

$$\|y_1 - y_2\|_{L^s(\mathbb{R}_+)} \leqslant \frac{1}{1-\delta\mathbf{G}(0)}\Big(\|C\mathrm{e}^{A\,\cdot}\|_{L^s(\mathbb{R}_+)}\|(x_1 - x_2)(0)\| + \mathbf{G}_{\mathrm{e}}(0)\|w_1 - w_2\|_{L^s(\mathbb{R}_+)}\Big)\,, \qquad \text{(I.1)}$$

for all $1 \leqslant s \leqslant \infty$ and $x_1(0), x_2(0) \in \mathbb{R}^n$, whenever $\|w_1 - w_2\|_{L^s(\mathbb{R}_+)} < \infty$. The Lebesgue spaces $L^s(\mathbb{R}_+)$ are recapped in the Notation section below.

The hypotheses of the above result are recognised as those

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of the real Kalman conjecture. The conclusions are *linear* incremental L^s-input-output stability, with L^s-input-output gain which is independent of s. Since incrementally-bounded nonlinearities are globally Lipschitz, the conjunction of the estimate (I.1) with $s = 2$ and [15, Theorem 2] yields that the Lurie system depicted in Figure I.1a is exponentially input-to-state stable [16] if additionally $\phi(0) = 0$. Consequently, it is also, *a forteriori*, globally exponentially stable when unforced.

Contributions: The main objective of this paper is to prove a formal and multivariable (MIMO) version of the above result and, in particular to establish that, for Lurie systems with *externally positive* linear part, the Kalman Conjecture is true. As well as supplementing the literature on the Aizerman and Kalman conjectures, the current work is relevant to other recent papers which recognise the importance of incremental stability properties, such as [17, 18, 19, 20]. Incremental stability properties are known to afford a toolbox for nonlinear observer design, the study of synchronization-type problems, and the analysis of convergent or (almost) periodic inputs in forced differential equations [21], a property sometimes called entrainment. Recent applications of incremental stability concepts include to infer robustness properties of certain neural network architectures in [22, 23, 24]. In many ways, the presented results can be understood as an extension of [4] to the incremental setting, whilst also generalising the norms of the stability bounds and providing explicit expressions for the bound's constants. Moreover, the results highlight some of the technical intricacies of the MIMO version of the Kalman Conjecture, and introduce a class of system able to satisfy the question raised in [25], that is: "*whether the Kalman conjecture can be extended to a MIMO setting?*". The present results show that externally positive systems satisfy this MIMO extension in the case that the nonlinearity satisfies a non-standard, but quite natural, incremental bound.

The presence of $1-\delta\mathbf{G}(0) > 0$ in (I.1) above indicates a small-gain approach (see, for example [26, 27]) which, as mentioned earlier, positive Lurie systems are amenable to. As such, the estimate (I.1) also follows from classical incremental small-gain results such as [28, Corollary 4.3], which even extend to certain Lurie systems with infinite dimensional linear components. However, as we demonstrate in Section III, our main results in the multivariable case impose less restrictive assumptions than classical small-gain and are, to the best of our knowledge, novel.

The note is organised as follows. Sections II and III contain a description of the system under study and our main results, respectively. Examples and a short Conclusion appear in Sections IV and V, respectively.

Notation We use standard mathematical notation, and mention here only a few items. We recall that $\rho(M)$ denotes the spectral radius of the square matrix M. Throughout, unless otherwise indicated, the magnitude operation and inequalities are taken to be component-wise, that is, $|M| < \ (\leqslant)\ |N|$ implies that $|M_{ij}| < \ (\leqslant)\ |N_{ij}|$ for all i, j of compatibly-dimensioned matrices or vectors, and similarly for $> (\geqslant)$. We call vectors v with $v > 0$ strictly positive. Elsewhere in the literature, the notation $\ll$ or $\leqslant$ is used for component-wise $<$. For clarity, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^n_+$ is the usual nonnegative orthant in n-dimensional Euclidean space, $\mathbb{R}^n$. We denote by $|\xi|$ for $\xi \in \mathbb{R}^n$ the vector of absolute values of ξ, that is, $|\xi|_i = |\xi_i|$ for each i, and analogously for $|M|$ corresponding to matrix M.

Externally positive systems are defined, for example, in [29, Definition 1, p. 8] or [30, p. 323].

Definition I.1. The linear control system specified by (A, B, C) is called *externally positive* if every (componentwise) nonnegative input subject to zero initial state produces a (componentwise) nonnegative output.

Externally positive systems are characterised by possessing (componentwise) nonnegative impulse response, that is $g(t) = Ce^{At}B \geqslant 0 \quad \forall t \geqslant 0$, and also go by the name of *positive impulse response* systems in the literature, cf. [31, 32, 33].

As usual, for $1 \leqslant s \leqslant \infty$, interval $J \subset \mathbb{R}$ and normed space X, we let $L^s(J, X)$ denote the normed space of Lebesgue measurable, s-integrable ($s < \infty$) or essentially bounded ($s = \infty$) functions $J \to X$. We let $L^s_{\mathrm{loc}}(J, X)$ denote the vector space of locally s-integrable or essentially bounded functions $J \to X$. We write $L^s(0, t, X)$ for the more cumbersome $L^s([0, t], X)$, and shall often omit the symbol X for brevity.

For locally integrable function $h : \mathbb{R}_+ \to \mathbb{R}^n$, $v \in \mathbb{R}^n_+$, $v > 0$, $1 \leqslant s \leqslant \infty$ and $t > 0$ we write

$$\|h\|_{L^s_v(0,t)} = \|v^\top |h(\cdot)|\|_{L^s(0,t)} = \Big(\int_0^t (v^\top |h(\tau)|)^s \,\mathrm{d}\tau\Big)^{\frac{1}{s}},$$

which, note, is a norm on $L^s(0, t, \mathbb{R}^n)$, with the last equality being valid for $s < \infty$.

Finally, for locally-integrable and compatibly-sized functions h_1 and h_2, defined on $\mathbb{R}_+$, we let $h_1 * h_2$ denote the convolution of h_1 and h_2, viz,

$$(h_1 * h_2)(t) = \int_0^t h_1(t-s)h_2(s)\,\mathrm{d}s \quad \text{almost all } t \geqslant 0.$$

II. Preliminaries

Consider the forced Lurie system

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + B\phi(y(t)) + B_{\mathrm{e}}w(t), \\ y(t) &= Cx(t)\,, \end{aligned}\right\} \qquad \text{(II.1)}$$

so that the underlying linear system containing the feedback element has transfer function $\mathbf{G}(s) := C(sI - A)^{-1}B$, and impulse response denoted $g(t) := C\mathrm{e}^{At}B$. Furthermore, let $\mathbf{G}_{\mathrm{e}}(s) := C(sI - A)^{-1}B_{\mathrm{e}}$ and $g_{\mathrm{e}}(t) := C\mathrm{e}^{At}B_{\mathrm{e}}$ denote the transfer function and impulse response of the linear system from external input to output, respectively. Here A, B, B_{e}, and C are $n \times n$, $n \times m$, $n \times q$, $p \times n$ real matrices, respectively, for $m, n, p, q \in \mathbb{N}$. The single-input single-output (SISO) case refers to $m = p = 1$.

We formulate the following hypotheses:

(H1) $\mathbf{G}$ is externally positive;

(H2) $\phi : \mathbb{R}^p \to \mathbb{R}^m$ satisfies the incremental bound

$$|\phi(y_1) - \phi(y_2)| \leqslant \Delta |y_1 - y_2| \quad \forall\, y_1, y_2 \in \mathbb{R}^p\,,$$

for some fixed $\Delta \in \mathbb{R}^{m\times p}_+$.

We shall use throughout that, under hypothesis (H1), if A is Hurwitz, then

$$\mathbf{G}(0) = \int_0^\infty g(t)\,\mathrm{d}t \geqslant 0\,. \tag{II.2}$$

Observe that the inequality in (H2) ensures that ϕ is globally Lipschitz. Therefore, it follows routinely from the theory of ordinary differential equations that for each forcing term $w \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^q)$ and $\xi \in \mathbb{R}^n$, there exists a unique solution $x : \mathbb{R}_+ \to \mathbb{R}^n$ of (II.1) with $x(0) = \xi$. We let $y = y(\cdot\,; x(0), w)$ denote the corresponding output of (II.1).

Remark II.1. In the typical absolute stability literature, the corresponding Assumption (H2) is often expressed a little differently. Indeed, often Δ would be assumed diagonal and ϕ would be assumed to be incrementally sector bounded. In the SISO case, this would usually be expressed as $\tilde{\phi} \in \mathrm{Sector}[0, 2\gamma]$ for $\gamma > 0$. However, note that by appropriate loop-shifting, this can be equivalently expressed as $\phi \in \mathrm{Sector}[-\gamma, \gamma]$, which is equivalent to Assumption (H2). ▫

One class of systems that can be shown to satisfy (H1) are the so-called "*symmetric systems*": SISO systems with $A^\top = A$, $B = b$ and $C = c^\top$ with $c = b$ are externally positive (see, for example, [34, Fact 2]). If A is not Metzler, then symmetric systems are not *internally* positive.

The following linear stability hypothesis for the Aizerman conjecture is key:

(AC1) $A + B\Sigma C$ is Hurwitz for all $\Sigma \in \mathbb{R}^{m\times p}$ with $|\Sigma| \leqslant \Delta$.

An immediate consequence of (AC1) is that A itself is Hurwitz (take $\Sigma = 0$). Note that since we deal with externally positive systems, no Metzler assumption is made. The next lemma contains a characterisation of the hypothesis (AC1) when $\mathbf{G}$ is externally positive.

Lemma II.2. *Assume that* $\mathbf{G}$ *satisfies assumption* (H1) *and let* $\Delta \in \mathbb{R}^{m\times p}_+$ *be fixed.*

(i) Hypothesis (AC1) *is equivalent to the conditions that* A *is Hurwitz and* $\rho(\mathbf{G}(0)\Delta) < 1$.

(ii) Further, if hypothesis (AC1) *holds, then there exist* $\zeta \in (0, 1)$ *and strictly positive* $v \in \mathbb{R}^p_+$ *such that*

$$v^\top \mathbf{G}(0)\Delta \leqslant \zeta v^\top\,. \tag{II.3}$$

Of course, in the single-input single-output setting, the condition $\rho(\mathbf{G}(0)\Delta) < 1$ reduces to the small-gain inequality $\mathbf{G}(0)\Delta < 1$. Although Lemma II.2 essentially replaces one eigenvalue condition, namely on the eigenvalues of $A + B\Sigma C$, with another eigenvalue condition $\rho(\Delta\mathbf{G}(0)) = \rho(\mathbf{G}(0)\Delta) < 1$, in usual applications $m, p \ll n$ so that at least one of the second eigenvalue problems is smaller than the first.

Proof of Lemma II.2. Statement (i) is a small-gain lemma for externally positive linear control systems. It may be derived analogously to the equivalence of statements (ii) and (iii) of [35, Lemma 3.1]. For brevity, we do not give the details.

For statement (ii), observe that $\mathbf{G}(0)\Delta$ is componentwise nonnegative by (II.2) and choice of Δ. Choose nonnegative $P \in \mathbb{R}^{p\times p}_+$ such that $\mathbf{G}(0)\Delta + P$ is strictly positive, yet also satisfies $\zeta := \rho(\mathbf{G}(0)\Delta + P) < 1$, which is possible by statement (i). The Perron-Frobenius Theorem (see, for example, [36, Theorem 1.4, p. 27]) yields strictly positive $v \in \mathbb{R}^p_+$ such that

$$v^\top(\mathbf{G}(0)\Delta + P) = \rho(\mathbf{G}(0)\Delta + P)v^\top = \zeta v^\top\,,$$

from which inequality (II.3) follows as $v^\top P \geqslant 0$. □

III. Incremental stability for externally positive Lurie systems

The following theorem is our main result.

Theorem III.1. *Consider the forced Lurie system* (II.1) *and assume that hypotheses* (H1), (H2), *and* (AC1) *hold. Then the following incremental stability estimate for trajectories* (w_i, x_i) *of* (II.1) *is valid:*

$$\begin{aligned}\|y_1 - y_2\|_{L^1_v(0,T)} \leqslant &\frac{1}{1-\zeta}\Big(\|C\mathrm{e}^{A\,\cdot}\|_{L^1_v(0,T)}\|(x_1 - x_2)(0)\| \\ &+ \big\||v^\top g_{\mathrm{e}}| * \big(|w_1 - w_2|\big)\big\|_{L^1(0,T)}\Big)\end{aligned} \tag{III.1}$$

for all $T > 0$*, all* $x_1(0), x_2(0) \in \mathbb{R}^n$ *and all* $w_1, w_2 \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^q)$*, where* v *and* ζ *are as in* (II.3)*.*

In words, Theorem III.1 provides sufficient conditions under which the Lurie system (II.1) is incrementally L^1-input-output stable in the sense of (III.1), with *linear* incremental L^1-gains. Moreover, if $m = p = 1$, then v and ζ in (III.1) may be replaced by 1 and $\Delta\mathbf{G}(0)$, respectively. If, additionally $q = 1$ and $\mathbf{G}_{\mathrm{e}}$ is externally positive, then, for all $T > 0$,

$$\big\||v^\top g_{\mathrm{e}}| * \big(|w_1 - w_2|\big)\big\|_{L^1(0,T)} \leqslant \mathbf{G}_{\mathrm{e}}(0)\|w_1 - w_2\|_{L^1(0,T)}\,.$$

Proof of Theorem III.1. Let y_i for $i = 1, 2$, denote the outputs of (II.1) subject to initial state $x_i(0)$ and forcing term w_i. From the variation of parameters formula, we have

$$\begin{aligned}y_i(t) = &\,C\mathrm{e}^{At}x_i(0) \\ &+ \int_0^t C\mathrm{e}^{A(t-\tau)}\big(B\phi(y_i(\tau)) + B_{\mathrm{e}}w_i(\tau)\big)\,\mathrm{d}\tau\,,\end{aligned}$$

for all $t \geqslant 0$, $i = 1, 2$. In convolution notation, this reads

$$y_i = C\mathrm{e}^{A\,\cdot}x_i(0) + g * \phi(y_i) + g_{\mathrm{e}} * w_i, \quad i = 1, 2\,.$$

Thus, the output-error $y_1 - y_2$ is given by

$$\begin{aligned}y_1 - y_2 = &\,C\mathrm{e}^{A\,\cdot}\big(x_1(0) - x_2(0)\big) + g * \big(\phi(y_1) - \phi(y_2)\big) \\ &+ g_{\mathrm{e}} * (w_1 - w_2)\,.\end{aligned} \tag{III.2}$$

Taking componentwise absolute values and then applying $v^\top$ in (III.2) we obtain, after application of (H2) with $\xi := |y_1 - y_2|$

$$\begin{aligned}v^\top\xi \leqslant &\,v^\top|C\mathrm{e}^{A\,\cdot}|\cdot|x_1(0) - x_2(0)| + v^\top g\Delta * \xi \\ &+ v^\top|g_{\mathrm{e}}| * \big(|w_1 - w_2|\big)\,.\end{aligned} \tag{III.3}$$

Here we have used the external positivity (H1) to give $g(t) \geqslant 0$, and that $|Mz| \leqslant |M|\cdot|z| = M|z|$ for all compatibly sized nonnegative matrices M and vectors z. Let $T > 0$. We invoke Fubini's Theorem (see, for example [37, Theorem 2.16.2]) to rewrite the integral of the key, nonnegative term $v^\top g\Delta * \xi$ as follows:

$$\|v^\top g\Delta * \xi\|_{L^1(0,T)} = \int_{t=0}^T \int_{\tau=0}^t v^\top g(t-\tau)\Delta\xi(\tau)\,\mathrm{d}\tau\,\mathrm{d}t = \int_{\tau=0}^T \int_{t=\tau}^T v^\top g(t-\tau)\Delta\,\mathrm{d}t\,\xi(\tau)\mathrm{d}\tau\,.$$

From the external positivity again, we now majorise as follows:

$$\|v^\top g\Delta * \xi\|_{L^1(0,T)} \leqslant \int_{\tau=0}^T v^\top \mathbf{G}(0)\Delta\xi(\tau)\,\mathrm{d}\tau \leqslant \zeta\big\|v^\top\xi\big\|_{L^1(0,T)}\,, \tag{III.4}$$

invoking (II.3). Therefore, integrating both sides of (III.3) from 0 to T, and invoking (III.4), we have that

$$\big\|\xi\big\|_{L^1_v(0,T)} \leqslant \big\|C\mathrm{e}^{A\,\cdot}\big\|_{L^1_v(0,T)}\|x_1(0)-x_2(0)\| + \zeta\big\|\xi\big\|_{L^1_v(0,T)} + \big\||v^\top g_\mathrm{e}| * \big(|w_1-w_2|\big)\big\|_{L^1(0,T)}\,.$$

Rearranging the above yields (III.1), as required. □

Our next result shows that incremental input-output stability in any L^s-norm is possible if (AC1) is strengthened as follows:

(AC1)' There exists strictly positive $v \in \mathbb{R}^p_+$ such that

$$\zeta(v) := \int_0^\infty \max_j \frac{(v^\top g(t)\Delta)_j}{v_j}\,\mathrm{d}t < 1\,.$$

In the SISO case, assumption (AC1)' is equivalent to (AC1) by Lemma II.2 and (II.2), as, in that case, ζ is independent of the scalar v. In the MIMO case, hypothesis (AC1)' implies (AC1), but not conversely in general. Observe that $\zeta(v) = \zeta(\lambda v)$ for all scalars $\lambda > 0$. Thus, hypothesis (AC1)' holds if the optimisation problem of minimising ζ over $p-1$ variables in v (fixing one component of v equal to one) subject to the constraint $\zeta(v) < 1$ is feasible, which provides a numerical test for the property. As an illustrative example, consider the MIMO impulse response

$$g(t) = \begin{pmatrix} \mathrm{e}^{-t} - \mathrm{e}^{-2t} & 0.5\mathrm{e}^{-t} \\ 0.5\mathrm{e}^{-10t} & \mathrm{e}^{-t} - t\mathrm{e}^{-2t} \end{pmatrix} \quad t \geqslant 0\,.$$

We seek to determine the maximal $\delta > 0$ such that (AC1) or (AC1)' holds for $\Delta = \delta I$. Noting that

$$\mathbf{G}(0) = \begin{pmatrix} 0.5 & 0.5 \\ 0.05 & 0.75 \end{pmatrix} \quad \text{has} \quad \rho(\mathbf{G}(0)) = 0.8266\,,$$

assumption (AC1) holds for all $\delta \in (0, 1/\rho(\mathbf{G}(0))) = (0, 1.21)$. Writing $v^\top = \begin{pmatrix} w & 1 \end{pmatrix}$ and $\zeta = \zeta(w)$ for $w > 0$, minimisation of ζ using MATLAB's `fminsearch` command gives the minimiser $w' = 0.254$ and $\zeta(w') = 0.905 < 1$. Consequently, hypothesis (AC1)' holds for all $\delta \in (0, 1/0.905) = (0, 1.104)$; that is the conservatism of (AC1)' over (AC1) is relatively small in this case.

Proposition III.2. *Consider the forced Lurie system* (II.1) *and assume that hypotheses* (H1), (H2) *and* (AC1)' *are satisfied. Let* $T > 0$ *and* $1 \leqslant s \leqslant \infty$. *Then the estimate* (III.1) *holds with* L^1_v*-norms replaced by* L^s_v*-norms throughout, where now* v *and* ζ *are as in* (AC1)'.

The commentary after Theorem III.1 applies here as well and, in particular, the simplifications in the case that $m = p = q = 1$. Indeed, the estimate (I.1) is a special case of (III.1) with 1 replaced by $1 \leqslant s \leqslant \infty$.

Proof of Proposition III.2. The proof shares the same start with that of Theorem III.1 up to inequality (III.3). Again fix $T > 0$. We take L^s-norms in (III.3) to give

$$\big\|\xi\big\|_{L^s_v} \leqslant \big\|C\mathrm{e}^{A\,\cdot}\big\|_{L^s_v}\|(x_1-x_2)(0)\| + \|v^\top g\Delta * \xi\|_{L^s} + \big\||v^\top g_\mathrm{e}| * \big(|w_1-w_2|\big)\big\|_{L^s}\,, \tag{III.5}$$

writing here L^s_v for $L^s_v(0,T)$. We record a consequence of hypothesis (AC1)'. Setting $\theta(t) := \max_j (v^\top g(t)\Delta)_j/v_j$, then

$$(v^\top g(t)\Delta)_j \leqslant \theta(t)v_j, \ \text{ that is, } \ v^\top g(t)\Delta \leqslant \theta(t)v^\top\,, \tag{III.6a}$$

for all $t \geqslant 0$ and, furthermore,

$$\zeta = \int_0^\infty \theta(\tau)\,\mathrm{d}\tau < 1\,. \tag{III.6b}$$

From (III.6a) it follows that

$$v^\top g\Delta * \xi \leqslant \theta * v^\top \xi\,,$$

and we invoke Young's convolution inequality; see, for example [38, Proposition A.3.14, p. 755]:

$$\|\theta * v^\top\xi\|_{L^s(0,T)} \leqslant \|\theta\|_{L^1(0,T)}\|v^\top\xi\|_{L^s(0,T)} \leqslant \zeta\|\xi\|_{L^s_v(0,T)}\,.$$

Substituting the above into (III.5) and rearranging the resulting inequality as in the proof of Theorem III.1 completes the proof. □

We provide further commentary in the form of a remark.

Remark III.3. (a) Under the hypotheses of Theorem III.1, the incremental L^1-input-to-state estimate

$$\big\|x_1 - x_2\big\|_{L^1(0,T)} = c_{1,1}\|x_1(0)-x_2(0)\| + c_{2,1}\big\|w_1-w_2\big\|_{L^1(0,T)}$$

holds for the Lurie system (II.1), for some positive constants $c_{i,1}$. This bound is established by taking $L^1(0,T)$-norms of the difference $x_1 - x_2$ in convolution form, and majorising the term involving $\phi(y_1) - \phi(y_2)$ by (III.1). Here, we need to use norm equivalence to bound $\|y_1-y_2\|_{L^1_v(0,T)}$ by $\|y_1-y_2\|_{L^1(0,T)}$, leading to somewhat complicated expressions for the constants $c_{i,1}$. The general $1 \leqslant s \leqslant \infty$ case holds under the hypotheses of Proposition III.2.

(b) A consequence of the incremental gain bound from part (a) is that any other outputs of the form $z = C_z x + D_{zw}w$ will satisfy a similar incremental gain bound. Of course, since no conditions on the model data C_z, D_{zw} are imposed in Theorem III.1 or Proposition III.2, it is less clear how tight such bounds are.

(c) Consider forced Lurie systems with biases, viz:

$$\dot{x}(t) = Ax(t) + B\phi(y(t)) + B_\mathrm{e}w(t) + b, \quad y(t) = Cx(t)\,,$$

where $b \in \mathbb{R}^n$ is a bias term. Many dynamic systems involving neural networks can be expressed precisely in this form —

see [22, 23] for examples. Versions of Theorem III.1 and Proposition III.2 can be proved for such systems with little additional effort.

(d) Finally, we comment that discrete-time analogues of Theorem III.1 and Proposition III.2 are valid, and may be derived along similar lines, *mutatis mutandis*. For brevity, we do not give formal statements. ▫

A. Comparison to the small-gain approach

In this short section, we compare and contrast the hypotheses of Theorem III.1 to those of the classical small-gain approach. To summarise, these hypotheses coincide in the SISO case, and not in the MIMO case, presented in Lemma III.4 below.

We record that the usual Euclidean 2-norm on $\mathbb{R}^n$ satisfies

$$\|y\| = \||y|\| \quad \forall\, y \in \mathbb{R}^n ,$$

and is monotonic, so that $0 \leqslant x \leqslant y \quad \Rightarrow \quad \|x\| \leqslant \|y\|$. Consequently hypothesis (H2) ensures that

$$\begin{aligned}\|\phi(y_1) - \phi(y_2)\| &= \||\phi(y_1) - \phi(y_2)|\| \leqslant \|\Delta|y_1 - y_2|\| \\ &\leqslant \|\Delta\| \cdot \|y_1 - y_2\| \quad \forall\, y_1, y_2 \in \mathbb{R}^p ,\end{aligned}$$

that is, ϕ satisfies a classical incremental gain (norm) condition with gain $\|\Delta\|$.

The classical small-gain condition for incremental stability is

$$\|\mathbf{G}\|_{H^\infty} \|\Delta\| < 1 . \tag{III.7}$$

In the SISO case, so that $\delta = \Delta$, this reduces to $\mathbf{G}(0)\delta = \rho(\mathbf{G}(0)\Delta) < 1$ which coincides with (AC1) by Lemma II.2.

Lemma III.4. *Assume that* (H1) *and* (H2) *hold, and that A is Hurwitz. If* (III.7) *holds, then so does hypothesis* (AC1). *The converse is false in the MIMO case.*

Lemma III.4 ensures that the hypotheses of Theorem III.1, at least for nonlinearities ϕ satisfying (H2), are weaker than classical small-gain assumptions in the MIMO case.

Proof. Let $\Sigma \in \mathbb{R}^{m\times p}$ with $|\Sigma| \leqslant \Delta$. Then, by monotonicity of the Euclidean norm and (III.7),

$$\|\Sigma\| \leqslant \||\Sigma|\| \leqslant \|\Delta\| < \frac{1}{\|\mathbf{G}\|_{H^\infty}} .$$

Therefore, by usual stability radius arguments, Σ is a stabilising feedback for A, that is, $A + B\Sigma C$ is Hurwitz. We conclude that (AC1) holds. For the converse, it suffices to obtain a counterexample. For which purpose, consider the 2×2 example

$$\mathbf{G}(0) = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \quad \Delta = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \quad a \in (0,1), \quad c > 0 .$$

Here

$$\mathbf{G}(0)\Delta = \begin{pmatrix} a & ac+1 \\ 0 & a \end{pmatrix} \quad \text{so that} \quad \rho(\mathbf{G}(0)\Delta) = a < 1 ,$$

whence (AC1) holds for all $c > 0$. However,

$$\frac{\|\Delta\|_2}{c} = \frac{1}{c}\sqrt{\lambda_{\max}(\Delta^\top\Delta)} \to 1 \quad \text{as } c \to \infty ,$$

so that the small-gain condition (III.7) fails for large c. □

We highlight that the $\mathbf{G}$ considered above is the transfer function of, for example,

$$A = \begin{pmatrix} -\frac{1}{a} & \frac{1}{a^2} \\ 0 & -\frac{1}{a} \end{pmatrix}, \quad B = C = I .$$

B. Context of work

The main novelty of this work over related work on Lurie systems is our focus in *incremental stability*. The vast majority of work on such systems focuses either on asymptotic stability of the origin, or on regular L^s-stability. For instance, the paper [39] focusses on asymptotic (not incremental) state-space stability properties of internally positive Lurie inclusions, and the assumed internal positivity structure is crucial. The result [39, Corollary 2.13], for internally positive Lurie differential equations, is somewhat related to a non-incremental version of Theorem III.1. Moreover, our earlier paper on the Aizerman Conjecture [3] establishes only asymptotic stability and associated integral-to-state stability properties. Similarly, paper [40], does not consider incremental stability properties, and is closer in terms of approach and results to [3] than the present work; a comparison between those two works is contained in [3, Section III. C]. It is emphasized the absolute stability conditions for Lurie systems which are based on integral-quadratic constraints [41], in general, do not guarantee incremental stability and as such the papers [34, 42] do not provide as strong guarantees as those here.

Incremental stability of Lurie systems has, in fact, been examined in relatively few papers. The paper [21] explores the utility of incremental stability properties in establishing convergence in response to (almost) periodic forcing terms ("entrainment") of forced Lurie systems. The overlap with the present paper of both of these works is minimal owing to the different Lurie systems considered, the assumptions imposed, and stability properties sought. On the other hand, the paper [35] also considers incremental stability properties, but now for internally positive Lurie systems via linear dissipativity theory. As such the approach is different, with a greater emphasis on state-space properties, and the results are not directly comparable.

IV. Examples

We illustrate our main results through two multivariable academic examples for which the linear element has externally, but not internally, positive dynamics. In both examples the linear component of Lurie system (II.1) is given by $\mathbf{G}(s) = \mathbf{G}^j(s)$ where the $\mathbf{G}^j(s)$ has the following form

$$\mathbf{G}^j(s) = \begin{bmatrix} G_{11}^j(s) & G_{12}(s) \\ G_{21}(s) & G_{22}^j(s) \end{bmatrix}, \quad j = a, b$$

In the above systems, the off-diagonal transfer functions are given by $G_{12}(s) = 0.5/(s+1)$, $G_{21}(s) = 0.5/(s+10)$; both transfer functions are clearly externally positive. The diagonal components $G_{11}^j(s)$ and $G_{22}^j(s)$ are different for $j = a$ and $j = b$ and are constructed so that they are also externally positive.

Transfer Function	Estimate of δ			Time to compute estimate [sec]		
	Theorem III.1	Circle	BRL	Theorem III.1	Circle	BRL
$\mathbf{G}^a(s)$	1.40	1.40	1.21	0.006	4.27	0.877
$\mathbf{G}^b(s)$	1.87	1.87	1.44	0.006	2566.53	491.79
Transfer Function	Estimate of α			Time to compute estimate [sec]		
	Theorem III.1	Circle	BRL	Theorem III.1	Circle	BRL
$\mathbf{G}^a(s)$	0.937	0.937	0.806	0.007	5.36	0.896
$\mathbf{G}^b(s)$	1.092	1.092	0.961	0.007	2769.91	460.27

Table IV.1: Comparison of approaches: (i) diagonal Δ, first two rows; (ii) non-diagonal Δ, final two rows

A. Example Construction

It was shown in [43] that an externally positive system, $\mathbf{H}^E(s)$, can be constructed using the state-space matrices of a given transfer function $\mathbf{H}(s) \sim (A_H, B_H, C_H)$. In particular, the state-space matrices of $\mathbf{H}^E(s)$ are given by

$$\mathbf{H}^E(s) \sim (A_H \oplus A_H, B_H \otimes B_H, C_H \otimes C_H)$$

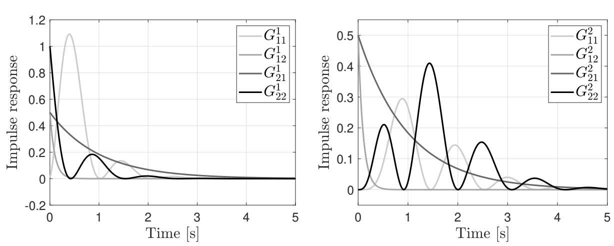
where $\otimes$ and $\oplus$ represent the Kronecker product and sum respectively ([44]). We use this technique to construct the elements of $G_j(s)$ to ensure they are externally positive. Therefore, take

$$G^a_{01}(s) = \frac{0.2s + 10}{2s^2 + 4s + 20} \quad \text{and} \quad G^a_{02}(s) = \frac{s - 0.1}{s^2 + 2s + 9}\,.$$

Let the state-space matrices of these systems be given by $(A^a_{01}, B^a_{01}, C^a_{01})$ and $(A^a_{02}, B^a_{02}, C^a_{02})$, respectively. Then, applying Ebihara's approach, $G^a_{11}(s)$ and $G^a_{22}(s)$ are constructed using the state-space matrices

$$G^a_{11}(s) \sim (A^a_{01} \oplus A^a_{01}, B^a_{01} \otimes B^a_{01}, C^a_{01} \otimes C^a_{01}),$$
$$G^a_{22}(s) \sim (A^a_{02} \oplus A^a_{02}, B^a_{02} \otimes B^a_{02}, C^a_{02} \otimes C^a_{02})\,.$$

Neither $G^a_{11}(s)$ nor $G^a_{22}(s)$ is internally positive but, by construction, both are externally positive. $\mathbf{G}^a(s)$ has 10 states, 2 inputs and 2 outputs and is externally positive, but not internally positive; its impulse response is shown in Figure IV.1a.



(a) (b)

Figure IV.1: Impulse responses of (a) $\mathbf{G}^a(s)$ (b) $\mathbf{G}^b(s)$.

The case for $\mathbf{G}^b(s)$ is similar to $\mathbf{G}^a(s)$, but now we let

$$G^b_{01}(s) = \frac{(0.2s + 10)^2}{(2s^2 + 4s + 20)^2} \quad \text{and}$$
$$G^b_{02}(s) = \frac{(s - 0.1)(0.2s + 10)}{(s^2 + 2s + 9)(2s^2 + 4s + 20)}\,.$$

Similar to before, the state-space matrices of these systems are given by $(A^b_{01}, B^b_{01}, C^b_{01})$ and $(A^b_{02}, B^b_{02}, C^b_{02})$, respectively. Then, again using the technique of [43] $G^b_{11}(s)$ and $G^b_{22}(s)$ are constructed using the state-space matrices

$$G^b_{11}(s) \sim (A^b_{01} \oplus A^b_{01}, B^b_{01} \otimes B^b_{01}, C^b_{01} \otimes C^b_{01}),$$
$$G^b_{22}(s) \sim (A^b_{02} \oplus A^b_{02}, B^b_{02} \otimes B^b_{02}, 5(C^b_{02} \otimes C^b_{02}))\,.$$

Again, $\mathbf{G}^b(s)$ is externally positive, but not internally positive; it has 34 states, 2 inputs and 2 outputs; and the impulse response is shown in Figure IV.1b.

B. Incremental Stability

Incremental stability of the Lurie system, with linear part $\mathbf{G}^j(s)$ and nonlinear term $\phi(\cdot)$ satisfying (H2) is to be investigated in two scenarios: (i) for the diagonally structured $\Delta = \delta I_2$ for $\delta > 0$; (ii) and for the non-diagonal structure

$$\Delta = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}. \tag{IV.1}$$

We compare the incremental stability guarantees computed in a number of different ways:

1) Using Theorem III.1 if Assumption (AC1) is satisfied. This is an easy eigenvalue check and a bisection algorithm can be used to compute the maximum δ for which (AC1) holds.
2) Using the Circle Criterion which is equivalent to the LMI feasibility problem

$$\begin{bmatrix} A_j^\top P + PA_j + C_j^\top \Delta^\top U \Delta C_j & PB_j \\ B_j^\top P & -U \end{bmatrix} \prec 0, \; P \succ 0,$$

where $U \succ 0$ is diagonal, and where (A_j, B_j, C_j) realise $\mathbf{G}^j(s)$. When $\Delta = \delta I$, a bisection algorithm can again be used to solve this problem but note that at every iteration a semi-definite programming problem needs to be solved with a large number of decision variables when n is large. Furthermore, without *a priori* information, it is difficult to select the upper bound to be used in the bisection algorithm.
3) The Bounded Real Lemma. Since A is assumed to be Hurwitz, the linear element of the system is bounded real. and satisfies the following LMI condition

$$\begin{bmatrix} A_j^\top P + PA_j & PB_j & C_j^\top \\ B_j^\top P & -\gamma I & 0 \\ C_j & 0 & -\gamma I \end{bmatrix} \prec 0, \; P \succ 0,$$

for some scalar $\gamma > 0$. By the small gain theorem, the system is therefore incrementally stable for all $\|\Delta\| = 1/\gamma$. This is an LMI minimisation problem. Note however, this approach *bounds* Δ by its norm, introducing conservatism.

When the bound on the incremental sector is diagonal, that is $\Delta = \delta I_2$, the above three conditions can be evaluated efficiently and the maximum δ can be computed. When Δ is given by equation (IV.1), Theorem III.1 is used to compute the maximum series gain which can be inserted into the system before it becomes unstable i.e. we seek the maximum α such that when B is replaced by αB, the system is stable. When the Circle Criterion is applied to this problem, Δ and $-\Delta$ are used as upper and lower bounds of the sector and the maximum α is sought. The bounded real lemma uses the fact that $||\Delta|| = \bar{\sigma}(\Delta) = 1.5$ and again the maximum α is computed.

Results for cases of the diagonal and non-diagonal bounds appear in Table IV.1, and were generated on an Intel i5 laptop computer with 16GB memory, running Linux and MATLAB 2015a. It can be seen that, in all cases, Theorem III.1 provides the largest estimates of δ for diagonal Δ, and the largest estimates of α for non-diagonal Δ given in equation (IV.1). These estimates are matched by those computed from the Circle Criterion but the time to compute the estimates is *orders of magnitude faster* using Theorem III.1. This is not surprising since Theorem III.1 simply requires an eigenvalue problem to be solved. The complexity of the QR (QL) algorithm, which underpins many modern eigenvalue solvers, typically scales with $\mathcal{O}(p^3)$ for $p \times p$ matrices at each step of the iteration, with improvements for certain matrix structures; see [45, Chapter 11]. Here, p is the dimension of the output vector, which typically satisfies $p \ll n$, for state dimension n. Indeed, the hypotheses of Theorem III.1 are *independent of the size of the state vector*. This needs to be contrasted with the LMI solvers used for the Circle and BRL solutions. The complexity of interior point algorithms, such as those used in the Robust Control Toolbox, is typically $\mathcal{O}(n^6)$ for Lyapunov-type problems; see [46, Section 11.3], making them scale far more poorly than Theorem III.1. It is emphasized that the estimates provided by the Circle Criterion are not completely comparable in the non-diagonal case, because the sector bounds used in this are different in form to those given by (H2) when Δ is not diagonal.

V. Conclusions

The Kalman conjecture was shown to hold for forced multivariable Lurie systems with externally positive linear component, and ensure incremental L^1- or L^s-input-output stability with linear gains. Our results give an intuitive, computationally efficient and non-conservative way of verifying incremental absolute stability of such systems.

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