Article:

Brodrick, J. P., Del Sorbo, D. and Ridgers, C. P. (2023) An alternative justification for the stationary assumption made by many reduced models for nonlocal electron heat flow in plasmas. Physics of Plasmas. 054501. ISSN: 1089-7674

https://doi.org/10.1063/5.0134966

Physics of Plasmas

BRIEF REPORT | MAY 17 2023

An alternative justification for the stationary assumption made by many reduced models for nonlocal electron heat flow in plasmas

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Phys. Plasmas 30, 054501 (2023)
https://doi.org/10.1063/5.0134966

Physics of Plasmas BRIEF COMMUNICATION pubs.aip.org/aip/pop

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Cite as: Phys. Plasmas **30**, 054501 (2023); doi: 10.1063/5.0134966
Submitted: 15 November 2022 · Accepted: 30 April 2023 ·
Published Online: 17 May 2023

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ABSTRACT

Nonlocal models are widely used for approximating kinetic effects on electron heat flow in fusion-relevant plasmas. Almost universally, such models have no explicit time dependence and are designed to make heat flow predictions based directly on instantaneous profiles of macroscopic plasma parameters. While this is usually justified by the claim that transient effects fade before temperature profiles evolve appreciably, a more rigorous justification of the stationarity assumption in terms of kinetic theory is desirable. In this Letter, such a justification is provided by demonstrating that nonstationary effects related to the time dependence of the isotropic part of the electron distribution function vanish up to third order in Chapman–Enskog theory (irrespective of ion charge state or presence of magnetic fields). However, it is found that the electron inertia term (whose appearance in Ohm's law stems from the time derivative of the anisotropic part of the electron distribution function) does have a small but finite third order effect that is most prominent for plasmas with low average ion charges. This Letter additionally provides a convenient analytic inverse for the isotropic part of the Landau electron–electron collision operator.

Braginskii's local theory of plasma heat transport[1] has long been known to be inaccurate for scenarios relevant to attaining net energy gain through thermonuclear fusion due to the presence of steep temperature gradients with respect to the electron mean free path.[2–5] However, full-scale integrated simulations that account for fine-scale effects on the electron energy distribution function (EDF) using the Vlasov–Fokker–Planck (VFP) equation are prohibitively computationally intensive; examples of nanosecond-scale VFP simulations[6–11] are typically limited in complexity and scale and do not fully incorporate important phenomena such as laser-plasma instabilities, atomic physics and equations of state critical to fully understanding indirect drive inertial fusion (ICF).

Deviations from locality are mostly caused by the rapid diffusion of suprathermal heat-carrying electrons across the temperature gradient (sometimes referred to as "multigroup diffusion"[5]) with the most noticeable effect being a reduction of the peak heat flux from that predicted by Braginskii's theory. A common approach (especially in the laser-plasma field) for approximating this "nonlocal" reduction in heat flow is the use of *flux-limiters*, which simply restrict the heat flow to a user-specified fraction of the free-streaming limit. However, such an approach fails to capture other nonlocal effects, such as preheat at the foot of the gradient enabled by the long collisional mean free path of the aforementioned suprathermal electrons, and importantly the "correct" value of the flux-limiter (strictly speaking this should generally be time-dependent,[12] but in practice, a constant value is often used) is not known *a priori* and must be tuned to VFP simulation or experiment,[13] and getting its value right has been critical in accurately predicting the results of ICF experiments.[4,14] Therefore, more sophisticated "reduced nonlocal models," typically based on simplifications of kinetic equations, are required for enhanced predictive capability.

In devising reduced models that can account for these nonlocal effects, it is usually required that these models be able to provide predictions for the electron heat flow directly from the instantaneous macroscopic plasma parameters and not depend explicitly on time derivatives of the EDF. The main reason for this is to enable coupling to traditional hydrodynamic codes by simply replacing the local heat flow or thermal conductivity with the model's nonlocal prediction thereby avoiding simulation of the entire EDF over the course of the simulation and neglecting possible kinetic effects of more complex phenomena (such as atomic physics and laser-plasma interactions).

For models based directly on the Vlasov–Fokker–Planck (VFP) equation, such as the widely used SNB multigroup diffusion model,[5,15] the M1 model[16–18] and the NRL's velocity-dependent Krook approach (VDK),[19] this is often achieved by simply neglecting the time derivative of the EDF.

The necessity of neglecting the time derivative is often justified by the observation that fine-scale transient effects on the EDF that might affect the heat flow take place (and die out) on a quicker timescale than the evolution of macroscopic plasma parameters. However, in our studies of the relation between nonlocal models and the kinetic equations, we have never been completely comfortable with this reasoning. First, a simplistic linear analysis (intended to address the damping of linearized temperature sinusoids studied in Ref. 20) using a velocity-dependent Krook approximation for electron–electron collisions suggests that the lowest-order corrections to the EDF as a result of multigroup diffusion and nonstationarity are of equal order in wavenumber—This suggests that while some transient effects may indeed dissipate more rapidly (and further insight on the exact timescales over, which such dissipation may occur could be gleaned through a spectral analysis of the electron–electron collision operator itself), there would still remain a longer lived non-zero contribution of nonstationarity to the heat flow. Additionally, in our own kinetic simulations of temperature ramp relaxation, we have observed that initial transient effects take several thermal collision times until they are no longer clearly distinguishable from the bulk; at which point the temperature profile is found to have evolved considerably from initial conditions,[20–22] suggesting that the timescales over, which transient effects act are not fully separated from those at which thermal transport occurs.

Finally, *if* we were to construct a "nonlocal model" by simply striking off the time derivative from the VFP equation while retaining the full Landau expression for C_{ee0} along with a self-consistent electric field and solving for the EDF, it can be shown that this would always (for the case of an unmagnetised plasma) predict a heat flux with zero divergence everywhere by computing the energy moment of the relevant equation:

$$\vec{\nabla} \cdot \vec{Q} + \vec{E} \cdot \vec{j} = 0. \tag{1}$$

This would mean the "model" would predict that there would be no evolution of the temperature profile whatsoever, which is clearly incorrect as otherwise hot spots would never be allowed to dissipate. Thus, how then are successful reduced nonlocal models that assume stationarity able to predict heat fluxes that are not divergence-free, but are in fact found to closely approximate the true kinetic heat flow?[10,20]

For a long time, we believed that the only resolution to the above contradiction was through the use of model collision operators that do not conserve energy (such as the VDK or AWBS[23] operators) and, thus, allow for a non-zero term on the right-hand side of Eq. (1). However, this explanation does not shed any light on whether the assumption of quasistationarity made by other models, such as the SNB should be considered significant (for its potential effect on the predicted nonlocal heat flux) or not. However, by performing a detailed analysis of the VFP equation in the limit of "marginal nonlocality," we can demonstrate that the time derivative of the isotropic part of the distribution function has no direct impact on the lowest-order nonlocal deviations of the EDF and heat flow; and we consider this exposition to be a more satisfactory justification for assuming stationarity in popular reduced nonlocal models such as the SNB model. This work is complementary to a linearized analysis performed by Brantov *et al.*[24] While this analysis can be used to demonstrate the validity of the stationarity assumption, the linearized case is of limited use in scenarios relevant to experiments.[20] Our work, being based on the Chapman–Enskog expansion, is not valid in the strongly nonlocal case;[25,26] hence, we limit ourselves to the aforementioned "marginal nonocality" where the thermal mean free path is not much less than 100 times the mean free path of the thermal electrons. It has been found that many experimentally relevant cases lie in this regime and indeed most nonlocal models break down beyond it.[20]

The analysis is based on a second-order Chapman–Enskog expansion of the VFP equations in the Cartesian tensor formulation.[27,28] Pressure anisotropy shall largely be neglected in this Letter and truncation will be performed above the first degree of anisotropy (often referred to as the P1 expansion). Under this truncation, the EDF is expressed in terms of its isotropic part and first-order anisotropy $f(\vec{v}) = f_0(v) + \vec{v} \cdot \vec{f}_1(v)/v$, respectively, where $\vec{v}$ is the electron velocity and v its magnitude. In the Lorentz limit, as the (effective root mean square) ion charge number $Z \to \infty$, the effect of electron–electron collisions on the anisotropic part of the EDF is negligible in comparison to electron-ion collisions allowing the reduced VFP equations to be expressed as

$$\frac{\partial f_0}{\partial t} + \frac{v}{3}\vec{\nabla} \cdot \vec{f}_1 - \frac{e}{3m_e v^2}\frac{\partial}{\partial v}\left(v^2 \vec{E} \cdot \vec{f}_1\right) = C_{ee0}[f_0], \tag{2}$$

$$\frac{\partial \vec{f}_1}{\partial t} + v\vec{\nabla} f_0 - \frac{e\vec{E}}{m_e}\frac{\partial f_0}{\partial v} - \frac{e}{m_e}\vec{B} \times \vec{f}_1 = -\nu_{ei}\vec{f}_1, \tag{3}$$

along with an appropriate closure for the electric field. The velocity-dependent electron-ion collision frequency is defined as

$$\nu_{ei} = \frac{4\pi Z n_e e^4 \log \Lambda_{ei}}{v^3}, \tag{4}$$

where n_e is the electron density, $\log \Lambda_{ei}$ is the Coulomb logarithm, and e is the magnitude of the electron charge in Gaussian units. For our purposes, it is sufficient to use the linearized form for isotropic electron–electron collisions provided by Bychenkov *et al.*,[29]

$$C^{(L)}_{ee0} = \frac{\nu^{(0)}_{ei}}{ZV^2}\frac{\partial}{\partial V}\left(\frac{f^{(mb)}}{2}\hat{L}\left[\frac{\partial \psi}{\partial V}\right]\right), \tag{5}$$

where

$$\hat{L}\left[\frac{\partial \psi}{\partial V}\right] := \frac{2}{\sqrt{\pi}}\frac{\gamma\left(\frac{3}{2}, V^2\right)}{V}\frac{\partial \psi}{\partial V} - \frac{8}{\sqrt{\pi}}\int_0^V U^2 \int_U^\infty e^{-W^2}\frac{\partial \psi}{\partial W}\,dW\,dU, \tag{6}$$

$\nu^{(0)}_{ei} = \nu_{ei}\left(v = \sqrt{2k_B T_e/m_e}\right)$ is the thermal electron-ion frequency and $\psi = \delta f_0/f^{(mb)}$.

The classical derivation of the local heat flow (according to Epperlein's methodology[30]) commences with the substitution of a Maxwellian $f^{(mb)} = n_e e^{-v^2/v_T^2}/\pi^{3/2} v_T^3$ into Eq. (3) and adopts the ambipolar electric field (which is appropriate over timescales longer than that of a few plasma oscillations and is often referred to as the zero current constraint and specified in terms of a moment equation: $\int_0^\infty V^3 \vec{f}_1 dV = 0$). In the absence of magnetic fields, Eq. (3) finds the expression for the first-order anisotropy of the EDF to be

$$\vec{f}_1^{\,(\mathrm{mb})} = -V^4(V^2-4)f^{(\mathrm{mb})}\lambda_{\mathrm{ei}}^{(0)}\vec{\nabla}\log(T_{\mathrm{e}}), \tag{7}$$

where $V = v/v_{\mathrm{T}}$ is the relative velocity compared to the thermal velocity $v_{\mathrm{T}} = \sqrt{2k_{\mathrm{B}}T_{\mathrm{e}}/m_{\mathrm{e}}}$ and $\lambda_{\mathrm{ei}}^{(0)} = v_{\mathrm{T}}/\nu_{\mathrm{ei}}^{(0)}$ is the thermal electron-ion mean free path. Despite Eq. (7) only being accurate for unmagnetised high-Z plasmas, it is this expression that will be used to derive nonlocal perturbations to the EDF and heat flow. While this will have a bearing on the universality of some later expressions in this Letter, such as Eq. (19), which quantifies the relative role of multigroup diffusion, electron inertia and pressure anisotropy on the nonlocal heat flow, it does not undermine the central finding concerning the stationary assumption.

In order to find the lowest-order deviation from the local heat flow, the nonlocal correction to the isotropic part of the EDF, δf_0, must first be found by substituting $\vec{f}_1^{\,(\mathrm{mb})}$ into Eq. (2). Assuming that δf_0 does not vary in time much faster than $f^{(\mathrm{mb})}$ (in accordance with the Chapman–Enskog ansatz that the EDF can be expressed directly in terms of macroscopic plasma parameters and their spatial gradients) only the Maxwellian part of the EDF needs to be considered in the time-derivative term

$$\frac{\partial f^{(\mathrm{mb})}}{\partial t} = \left(V^2 - \frac{3}{2}\right)f^{(\mathrm{mb})}\frac{\partial \log(T_{\mathrm{e}})}{\partial t}. \tag{8}$$

The resulting equation

$$\left(V^2-\frac{3}{2}\right)f^{(\mathrm{mb})}\frac{\partial\log(T_{\mathrm{e}})}{\partial t} - \vec{\nabla}\cdot\left(\frac{V^7-4V^5}{3}f^{(\mathrm{mb})}\lambda_{\mathrm{ei}}^{(0)}\vec{\nabla}\log(T_{\mathrm{e}})\right) = C_{\mathrm{ee0}}^{(\mathrm{L})}(\delta f_0) \tag{9}$$

must then be solved for δf_0 by inversion of the electron–electron collision operator. After multiplying Eq. (9) by v^2, integrating from ∞ to v (to take advantage of the EDF vanishing at ∞), this simplifies to

$$\lambda_{\mathrm{ei}}^{(0)}V^3 f^{(\mathrm{mb})}\frac{\partial\log(T_{\mathrm{e}})}{\partial t} + \lambda_{\mathrm{ei}}^{(0)}\vec{\nabla}\cdot\left(\frac{V^8}{3}f^{(\mathrm{mb})}\lambda_{\mathrm{ei}}^{(0)}\vec{\nabla}\log(T_{\mathrm{e}})\right) = \frac{\hat{L}}{Z}\left[\frac{\partial\psi}{\partial v}\right], \tag{10}$$

where the first term on the left-hand side is still the one relating to nonstationarity (and the source of the second could be labeled "multigroup diffusion"). Finally, by identifying that one solution to the homogeneous equation $\hat{L}[\frac{\partial\psi}{\partial v}] = 0$ is given by $\frac{\partial\psi}{\partial v} = V$ and invoking the Wronskian, an exact expression for the inverse of the second-order integrodifferential operator $\hat{L}$ applied to an arbitrary function S_V of normalized velocity V can be derived,

$$\hat{L}_*^{-1}[S_V] = \frac{\sqrt{\pi}}{2}\frac{VS_V}{\gamma\left(\frac{3}{2},V^2\right)} - 2\sqrt{\pi}V\int_*^V \frac{Y^2\int_0^Y S_T T e^{-T^2}\mathrm{d}T}{\gamma\left(\frac{3}{2},Y^2\right)^2}\mathrm{d}Y, \tag{11}$$

where γ is a lower incomplete gamma function the constant of integration $*$ is determined by adhering to energy conservation principles. For completeness, a proof of this is provided in the appendix.

This inverse operator can then be used to consider the perturbation owing to the nonstationarity source term proportional to V^3

$$\begin{aligned}\hat{L}_*^{-1}[V^3] - \frac{\sqrt{\pi}}{2}\frac{V^4}{\gamma\left(\frac{3}{2},V^2\right)} &= -2\sqrt{\pi}V\int_*^V \frac{Y^2\int_0^Y T^4 e^{-T^2}\mathrm{d}T}{\gamma\left(\frac{3}{2},Y^2\right)^2}\mathrm{d}Y \\ &= -\sqrt{\pi}V\int_*^V \frac{3}{2}\frac{Y^2}{\gamma\left(\frac{3}{2},Y^2\right)} - \frac{Y^5 e^{-Y^2}}{\gamma\left(\frac{3}{2},Y^2\right)^2}\mathrm{d}Y \\ &= -\frac{\sqrt{\pi}}{2}V\left[\frac{Y^3}{\gamma\left(\frac{3}{2},Y^2\right)}\right]_*^V,\end{aligned} \tag{12}$$

$$\Rightarrow \hat{L}_*^{-1}[V^3] = 0. \tag{13}$$

Hence, inclusion of the time derivative of f_0 has no effect on δf_0 at second order in Chapman–Enskog theory. Note that this finding holds irrespective of magnetization or value of the ion charge number, as the source term will always be proportional to V^3.

Consequently, the anisotropic part of the EDF $\vec{f}1$ is also not affected by the time derivative of f_0, and therefore, the assumption of isotropic stationarity is valid in what could be described as the limit of "marginal nonlocality." This provides adequate resolution of the potential issues with the stationary assumption raised above: While the time derivative of f_0 appears at the same order as the lowest-order deviation to the isotropic part of the EDF δf_0 (and indeed represents a non-zero rate of change of the temperature profile) it lies in the kernel of the inverse electron–electron collision operator and therefore has no actual effect on either δf_0 itself or consequently the lowest-order nonlocal perturbation to the heat flow. This reasoning justifies the neglection of the f_0 time derivative for models designed for approximating electron thermal conduction in plasmas exhibiting moderate to low degrees of nonlocality (such as near the expanding hohlraum bubble in indirect inertial fusion experiments or even along quiescent tokamak scrape-off layers).

Of course, it is not only the time derivative of the isotropic part of the EDF that could potentially affect heat flow but also that of the first-order anisotropy, i.e., $\frac{\partial \vec{f}_1}{\partial t}$. This term is commonly known as "electron inertia" in reference to the delayed response of electrons to changing macroscopic conditions. It turns out that this term does indeed contribute at third order in Chapman–Enskog theory; this can be shown by first explicitly calculating the rate of change of the Maxwellian form of $\vec{f}1$ with time

$$\frac{\partial\vec{f}_1^{\,(\mathrm{mb})}}{\partial t} = -V^4(V^2-4)f^{(\mathrm{mb})}\lambda_{\mathrm{ei}}^{(0)}\vec{\nabla}\frac{\partial\log(T_{\mathrm{e}})}{\partial t} - V^4(V^4-13V^2/2+6)\frac{\partial\log(T_{\mathrm{e}})}{\partial t}\lambda_{\mathrm{ei}}^{(0)}\vec{\nabla}\log(T_{\mathrm{e}}), \tag{14}$$

where the term on the second line is a purely nonlinear contribution (i.e., negligible when relative temperature perturbations are small).

The contribution of electron inertia to $\vec{f}1$ itself at third order can be assessed by substituting Eq. (14) into Eq. (3) while ignoring terms involving f_0,

$$\delta\vec{f}1^{(\mathrm{EI})} = -\frac{V^3}{\nu_{\mathrm{ei}}^{(0)}}\frac{\partial\vec{f}1^{(\mathrm{mb})}}{\partial t} - \lambda_{\mathrm{ei}}^{(0)}\frac{e\delta\vec{E}^{(\mathrm{EI})}}{k_B T_0}V^4 f^{(\mathrm{mb})}, \tag{15}$$

where $\delta\vec{E}^{(\mathrm{EI})}$ is the perturbation to the electric field that balances the current driven by the electron inertia term,

$$\int_0^\infty V^3\delta\vec{f}1^{(\mathrm{EI})}\,dV = 0, \tag{16}$$

$$\Rightarrow \frac{e\delta\vec{E}^{(\mathrm{EI})}}{k_B T_0} = \frac{315\sqrt{\pi}}{64}\vec{\nabla}\left(\frac{\vec{\nabla}\cdot\vec{Q}^{\mathrm{B}}}{\nu_{\mathrm{ei}}^{(0)}P_e}\right) + \frac{315\sqrt{\pi}}{16}\frac{\vec{\nabla}\cdot\vec{Q}^{\mathrm{B}}}{\nu_{\mathrm{ei}}^{(0)}P_e}\vec{\nabla}\log(T_e), \tag{17}$$

where the time derivatives have been re-expressed in terms of the *local* Braginskii heat flow $\vec{Q}^{\mathrm{B}}$ (which is justified at this order in Chapman–Enskog theory) and $P_e = n_e k_B T_e$ is the electron pressure. By substituting Eq. (17) into Eq. (15), we find that the effect of electron inertia on the heat flow at second order in Chapman–Enskog theory is given by

$$\delta\vec{Q}^{(\mathrm{EI})} = \frac{\int_0^\infty V^5\delta\vec{f}1^{(\mathrm{EI})}\,dV}{\int_0^\infty V^5\vec{f}1^{(\mathrm{mb})}\,dV} = \frac{3255}{32}\lambda_{\mathrm{ei}}^{(0)}\vec{\nabla}\left(\frac{\vec{\nabla}\cdot\vec{Q}^{\mathrm{B}}}{\nu_{\mathrm{ei}}^{(0)}P_e}\right)Q_{\mathrm{fs}} + \frac{18795}{32}\lambda_{\mathrm{ei}}^{(0)2}\vec{\nabla}\cdot\vec{Q}^{\mathrm{B}}\frac{\vec{\nabla}T_e}{T_e} \approx 101.7\lambda_{\mathrm{ei}}^{(0)}\vec{\nabla}\left(\frac{\vec{\nabla}\cdot\vec{Q}^{\mathrm{B}}}{\nu_{\mathrm{ei}}^{(0)}P_e}\right)Q_{\mathrm{fs}} + 587.3\lambda_{\mathrm{ei}}^{(0)2}\vec{\nabla}\cdot\vec{Q}^{\mathrm{B}}\frac{\vec{\nabla}T_e}{T_e}, \tag{18}$$

where $Q_{\mathrm{fs}} = P_e v_T$ is the free-streaming heat flux that would arise if all electrons moved in a single direction with the thermal velocity $v_T = \sqrt{2k_B T_e/m_e}$. Note that this perturbation is positive, representing the inertia of electrons against the nonlocal flux limitation caused by multigroup diffusion. The effect of electron inertia in counteracting nonlocal effects is most noticeable at low ion charge numbers (e.g., $Z = 1$), where the electron-ion mean free path is comparable to the electron–electron mean free path. Additionally, experience gleaned throughout our previous extensive VFP studies[10,20–22] has shown that the effect of including electron inertia on the bulk heat flow profile is fairly transitory and almost indistinguishable from simulations that neglect electron inertia after several collision times.

In a previous work,[31] we directly compared the effect of electron inertia on the heat flow at this order in Chapman–Enskog theory to the more commonly considered flux limitation due to multigroup diffusion [classified as the contribution from the $v\vec{\nabla}\cdot\vec{f}_1/3$ term in Eq. (2)] as well as that of second order or "pressure" anisotropy for the specific case of low-amplitude temperature sinusoids. (For the sake of conciseness, the derivation has not been replicated in this Letter.) The localizing effect of electron inertia was found to be more than compensated by the contribution of the pressure anisotropy term, which also played less of a role at higher values of *Z*. The relative perturbation to the local heat flow was expressed as

$$\frac{\delta Q}{Q^{\mathrm{B}}} = (\underbrace{263.9\xi Z}_{\text{Multigroup Diffusion}} + \underbrace{507\xi^2}_{\text{Pressure Anisotropy}} - \underbrace{115.1\xi^2}_{\text{Electron Inertia}})k^2\lambda_{\mathrm{ei}}^{(B)2}, \tag{19}$$

where $\lambda_{\mathrm{ei}}^{(B)2} = 9\pi\lambda_{\mathrm{ei}}^{(0)2}/32$ is the Braginskii definition of the electron-ion mean free path (squared), k is the wavenumber, and $\xi = (Z+0.24)/(Z+4.2)$ is the *ad hoc* Epperlein Short correction used to approximate the increasing relative importance of the neglected electron–electron collisions on the right-hand side of Eq. (3) with decreasing *Z*. (Here, we have made good an erratum noticed in Eq. (4.3.45) of Ref. 31 where the pressure anisotropy and electron inertia terms were calculated based on the thermal rather than the Braginskii mean free path.)

We have not gone further by providing a derivation of the more general case with large relative temperature or density profile gradients [which would require additional consideration of the nonlinear electric field term in Eq. (2)] as this would be cumbersome and distract from the main message of this Letter: the non-contribution of the nonstationary term and a general prescription of how to further pursue the Chapman–Enskog analysis by provision of an explicit form for the inverse electron–electron collision operator.

In summary, the lowest-order nonlocal correction to the heat flow is of third order in Chapman–Enskog theory and is dominated by multigroup diffusion. The time derivative of the isotropic part of the EDF f_0 has been shown to completely vanish in this limit; this result holds for all plasmas where the Landau collision operator is appropriate (irrespective of magnetization or value of *Z* but potentially not for high energy density plasmas). Therefore, the neglection of this term (the "stationarity assumption") is justified in nonlocal models designed for situations that are not extremely nonlocal (such as hohlraum transport and potentially steady state scrape-off layer profiles). There are however contributions from both the electron inertia term, which can counteract the nonlocal flux limitation caused by multigroup diffusion, and from pressure anisotropy, which enhances the nonlocal heat flow reduction (at least for the case of temperature sinusoids). Both electron inertia and pressure anisotropy only contribute noticeably for plasmas with low average ion charge numbers [where admittedly the numerical multipliers in Eq. (19) will likely require modification due to the increased impact of electron–electron collisions on the shape of the anisotropic part of the EDF] and become less important as *Z* increases above approximately 10.

This appendix exhibits the proof that the generalized expression $\hat{L}_*^{-1}$ given in Eq. (11) inverts the electron–electron collision operator for all non-negative values of "$*$" (although $*$ should be chosen such that the desired correction to f_0 does not affect temperature)—recall that $*$ is the lower limit of the integral in Eq. (11). The proof involves applying $\hat{L}$ to both sides of the equation and integrating by parts multiple times. Recalling $\Gamma\left(\frac{3}{2}\right) = \sqrt{\pi}/2$ and substituting $G_V = S_V V e^{-V^2}$ for compactness it is demonstrated that $\hat{L}_*^{-1}$ successfully inverts $\hat{L}$ as long as the definite integral of G vanishes. This is guaranteed to be the case as it is equivalent to requiring that the v^4 moment of the (left-hand side of the) f_0 equation is zero, which must be true due to energy conservation. For a more detailed version of this derivation see.[32]

$$
\begin{aligned}
\hat{L}\left[\hat{L}_*^{-1}[S_V]\right] - S = {} & -4\Bigg(\gamma\left(\frac{3}{2},V^2\right)\int_*^V \frac{Y^2\int_0^Y G_T\mathrm{d}T}{\gamma\left(\frac{3}{2},V^2\right)^2}\mathrm{d}Y \\
& + \int_0^V U^2\int_U^\infty \frac{G_W}{\gamma\left(\frac{3}{2},W^2\right)} \\
& - 4We^{-W^2}\int_*^W \frac{Y^2\int_0^Y G_T\mathrm{d}T}{\gamma\left(\frac{3}{2},Y^2\right)^2}\mathrm{d}Y\mathrm{d}W\mathrm{d}U\Bigg) \\
= {} & -4\Bigg(\gamma\left(\frac{3}{2},V^2\right)\int_*^V \frac{Y^2\displaystyle\int_0^Y G_T\mathrm{d}T}{\gamma\left(\frac{3}{2},Y^2\right)^2}\mathrm{d}Y \\
& + \int_0^V U^2\int_U^\infty \frac{G_W}{\gamma\left(\frac{3}{2},W^2\right)} - \frac{2W^2e^{-W^2}\displaystyle\int_0^W G_T\mathrm{d}T}{\gamma\left(\frac{3}{2},W^2\right)^2}\mathrm{d}W \\
& - 2U^2e^{-U^2}\int_*^U \frac{Y^2\int_0^Y G_T\mathrm{d}T}{\gamma\left(\frac{3}{2},Y^2\right)^2}\mathrm{d}Y\mathrm{d}U\Bigg) \\
= {} & -4\int_0^V \frac{U^2\int_0^\infty G_T\mathrm{d}T}{\Gamma\left(\frac{3}{2}\right)}\mathrm{d}U.
\end{aligned}
$$

The authors would like to gratefully acknowledge feedback on this manuscript and previous thought-provoking discussions with Mark Sherlock of Lawrence Livermore National Laboratory.

This work was funded by EPSRC Grant Nos. EP/K504178/1 and EP/M011372/1. This work has been carried out within the framework of the EUROfusion Consortium and has received funding from the Euratom research and training programme 2014–2018 under Grant Agreement No. 633053 (Project reference CfP-AWP17-IFE-CCFE-01). The views and opinions expressed herein do not necessarily reflect those of the European Commission.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Jonathan Peter Brodrick: Conceptualization (lead); Investigation (lead); Methodology (lead); Writing – original draft (lead); Writing – review & editing (lead). **Dario Del Sorbo:** Conceptualization (supporting); Methodology (supporting); Supervision (supporting); Writing – original draft (supporting); Writing – review & editing (supporting). **Christopher Ridgers:** Conceptualization (supporting); Funding acquisition (lead); Investigation (supporting); Methodology (supporting); Supervision (equal); Writing – original draft (supporting); Writing – review & editing (supporting).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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