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Logical convergence laws via stochastic approximation and Markov processes*

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Abstract

Since the paper of Kleinberg and Kleinberg, SODA'05, where it was proven that the preferential attachment random graph with degeneracy at least 3 does not obey the first order 0-1 law, no general methods were developed to study logical limit laws for recursive random graph models with arbitrary degeneracy. Even in the (possibly) simplest case of the uniform attachment, it is still not known whether the first order convergence law holds in this model. We prove that the uniform attachment random graph with bounded degrees obeys the first order convergence law. To prove the law, we describe dynamics of first order equivalence classes of the random graph using Markov chains. The convergence law follows from the existence of a limit distribution of the considered Markov chain. To show the latter convergence, we use stochastic approximations.

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1 Introduction

We consider first-order (FO) sentences about graphs in the language containing the adjacency $\sim$ and the equality $=$ relations. For the sake of readers' convenience, let us recall the definitions of sentences in this language and their quantifier depths (for more details, see surveys [4, 14, 27]). *FO formulas* are words consisting of symbols of several types: variables (lowercase letters with or without integer subscripts $x, y, z, x_1, x_2, \ldots$); relational symbols $\sim, =$; logical connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow, \neg$; quantifiers $\forall, \exists$; and brackets. The formulas are defined recursively as follows:

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- For any two variables x, y, the expressions $(x \sim y)$ and $(x = y)$ are FO formulas with *free* variables x and y and without *bounded* variables. These formulas have *quantifier depth* $\mathrm{qd}(x \sim y) = \mathrm{qd}(x = y) = 0$.
- If expressions ϕ, ϕ_1, ϕ_2 are FO formulas, then $\neg\phi, (\phi_1 \vee \phi_2), (\phi_1 \wedge \phi_2), (\phi_1 \Rightarrow \phi_2), (\phi_1 \Leftrightarrow \phi_2)$ are also FO formulas. For any logical connective L, the set of free variables of the formula $(\phi_1 L \phi_2)$ is the union of the sets of free variables of ϕ_1, ϕ_2; the same applies for sets of bounded variables. The sets of free variables of ϕ and $\neg\phi$ coincide. The quantifier depth of $(\phi_1 L \phi_2)$ equals $\mathrm{qd}(\phi_1 L \phi_2) = \max(\mathrm{qd}(\phi_1), \mathrm{qd}(\phi_2))$; the quantifier depths of ϕ and $\neg\phi$ coincide.
- Finally, if ϕ is a FO formula, then $\exists x \phi$ and $\forall x \phi$ are FO formulas with bounded variable x; the set of free variables of $\exists x \phi$, $\forall x \phi$ excludes the variable x. The quantifier depth of both $\exists x \phi$, $\forall x \phi$ equals $\mathrm{qd}(\phi) + 1$.

A FO *sentence* is a FO formula that does not have free variables. Informally speaking, the quantifier depth of a sentence is the maximum number of "nested" quantifiers. When we say that a graph G satisfies a FO sentence ϕ and write $G \models \phi$, we mean that ϕ evaluates to *true* under G (the process of evaluation of a formula is defined in accordance to its recursive structure introduced above, see details in [4, 14]; for instance, for vertices v, u of G, the sentence $x \sim y$ is true according to G and the variable assignment $x = u$ and $y = v$, if and only if u, v are adjacent in G). For example, the FO sentence

$$\forall x \forall y\, (x = y) \vee (x \sim y) \vee (\exists z\, (z \sim x) \wedge (z \sim y))$$

has quantifier depth 3, three variables, and describes the property of having diameter at most 2. It is worth noting that the number of variables does not necessarily coincide with the quantifier depth of a sentence. Nevertheless, the minimum quantifier depth among all tautologically equivalent reformulations of a given FO sentence is always at least the minimum number of variables (see [14, Chapters 3,6]). For instance, the sentence

$$\forall x \forall y\, (x = y) \vee (x \sim y) \vee (\exists z\, (z \sim x) \wedge (z \sim y)) \vee (\exists z\, (z \sim x) \wedge (\exists x\, (z \sim x) \wedge (x \sim y)))$$

has quantifier depth 4, three variables, and describes the property of having diameter at most 3 (it can be proven that these parameters are minimum possible).

Logical limit laws A *random graph* G_n on the vertex set $[n] := \{1, \ldots, n\}$ is a random element of the set of all (simple) graphs on $[n]$ with an arbitrary distribution over this set. It was proven by Glebskii, Kogan, Liogon'kii and Talanov [7] and independently by Fagin [5] that, for every FO sentence ϕ, either asymptotically almost all graphs on $[n]$ satisfy ϕ, or asymptotically almost all graphs on $[n]$ do not satisfy ϕ. In other words, letting G_n be uniformly distributed, we get that either $\mathbb{P}(G_n \models \phi) \to 1$, or $\mathbb{P}(G_n \models \phi) \to 0$ as $n \to \infty$. This means that the descriptive power of FO logic is weak in the sense that it does not express properties that are not trivial on typical, sufficiently large graphs: if there exist arbitrarily large n_1 and n_2 such that a (non-vanishing) fraction of graphs on $[n_1]$ has the property and a (non-vanishing) fraction of graphs on $[n_2]$ does not have the property, then this property cannot be described in FO logic. This phenomenon is known as the *FO zero-one law*, or, for brevity, *FO 0-1 law*. More generally, a sequence of random graphs G_n, $n \in \mathbb{N}$, *obeys the FO 0-1 law*, if, for any FO sentence ϕ, $\lim_{n\to\infty} \mathbb{P}(G_n \models \phi) \in \{0, 1\}$. FO limit laws are known to be helpful in the comparison of descriptive powers of different logics. For example, the failure of the FO 0-1 law in sparse binomial random graphs was used in [29] to prove that the minimum quantifier depth of a sentence that expresses the property of containing an induced subgraph isomorphic to a given graph F is at least $\frac{|E(F)|}{|V(F)|}+2$. The latter fact implies some limitations of algorithms that solve the *induced subgraph isomorphism problem*[1] by evaluating first order sentences, since the direct evaluation of a FO sentence of quantifier depth q on an n-vertex graph runs in time $\Theta(n^q)$, see [14, Proposition 6.6].

The most studied model in the context of FO 0-1 laws is the *binomial random graph* $G(n,p)$ (see, e.g., [12, 24, 26]), where every edge is drawn independently with probability p. In particular, $G(n,1/2)$ is just a graph chosen uniformly at random. The above mentioned classical FO 0-1 law (for $p=1/2$) is generalised to all $p=p(n)$ such that $\min\{p,1-p\}n^\alpha\to\infty$ for every $\alpha>0$ in [26] (in particular, this is true for all constant $p\in(0,1)$). On the other hand, the FO 0-1 law fails for $G(n,p(n))$, where $p(n)=n^{-\alpha}$ and $\alpha\in(0,1)$ is rational [25]. Moreover, even the FO convergence law fails for this random graph (a sequence of random graphs G_n, $n\in\mathbb{N}$, obeys *the FO convergence law*, if, for every FO sentence ϕ, $\lim_{n\to\infty}\mathbb{P}(G_n\models\phi)$ exists). Many other models are studied in the context of logical laws: random regular graphs [9], random geometric graphs [20], uniform random trees [21], etc. (see, e.g., [11, 27, 28, 30]). However, the usual combinatorial tools that are applied to prove logical laws seem to be insufficient to study the logical behaviour of attachment models that are, in particular, used to model real networks (see, e.g., [10]).

Attachment models The attachment models are built recursively. Fix a positive integer m, which is the *degeneracy parameter* of the model. At each step, one new vertex is added to the graph, from which m new edges are drawn randomly to the old vertices. The most studied attachment models are uniform and preferential attachment. In the *uniform attachment model* [6, 15], probabilities to draw an edge to a newly added vertex are the same for all existing vertices, while in the *Bollobás–Riordan preferential attachment model* probabilities are proportional to the degrees of the respective existing vertices. Let us recall that the preferential attachment graphs were introduced by Barabási and Albert [1] and later were formalised by Bollobás and Riordan [2].

In [13] Kleinberg and Kleinberg observed that the classical Bollobás–Riordan preferential attachment random graph with degeneracy at least 3 does not obey the FO 0-1 law. Since then, there was no significant progress in the study of logical limit laws for attachment models with an arbitrary degeneracy parameter m — we summarise all known results below. In particular, it is still unknown whether the classical preferential attachment random graph obeys the FO convergence law. Though the study of random graphs is dominated by the binomial random graph (and a similar uniform model), properties of preferential attachment models better resemble those of real-world networks such as the graph of the Web, social networks, and citation networks.

Let FO^γ be the fragment of the FO logic comprising all sentences with at most γ variables. In the context of FO limit laws, the following are known:

- the FO 0-1 law holds for the tree models (when only one edge is drawn at each step, i.e. $m=1$), for both preferential and uniform attachment [17],
- for the non-tree uniform model (when we draw $m\geq 2$ edges at each step) and the preferential attachment model with the degeneracy m at least 3 there is no FO 0-1 law [13, 17],

[1] Induced subgraph isomorphism problem is the problem of deciding whether a given input host graph contains a subgraph isomorphic to a given input pattern graph.

- the FO^{m-2} convergence law is known to be true for the uniform attachment [16], the FO^{m-3} convergence law holds true for some variations of the preferential attachment [18].

Thus, for the entire FO logic, we only know that the FO 0-1 law fails if m is large enough ($m \geq 2$ for the uniform attachment and $m \geq 3$ for the preferential attachment), while it is still unclear whether the FO convergence law fails at least for some m. Constructions of sentences with non-trivial limit probabilities are quite straightforward: since, for $m \geq 3$, the expected number of cliques of size $m+1$ converges to a finite limit, a sentence saying that there exist at least K cliques of size $m+1$ is satisfied with probability which is bounded away both from 0 and 1, for K large enough (see details in [13, 17]). Though for the existential fragment of the FO logic, the convergence law clearly holds for any attachment model (it immediately follows from the definition of the model), no approach to study the validity of the convergence law for the entire FO logic has been developed. In this paper, we develop a method to prove FO convergence laws for attachment models, and apply it to the uniform attachment with bounded degrees.

Ehrenfeucht-Fraïssé game The main tool for proving logical laws is *the Ehrenfeucht-Fraïssé game* (see, e.g., [14, Chapter 11.2]). Let us recall the rules of the game. The board consists of two vertex–disjoint graphs G and H. There are two players, *Spoiler* and *Duplicator*. The number of rounds R is fixed. In each round, Spoiler chooses a vertex either in G, or in H; then Duplicator chooses a vertex in the *other* graph. When R rounds are played, vertices $x_1, \ldots, x_R$ are chosen in G and vertices $y_1, \ldots, y_R$ are chosen in H. Duplicator wins if and only if the bijection that maps each x_i to y_i, $i \in [R]$, is an isomorphism of graphs $G[\{x_1, \ldots, x_R\}]$ and $H[\{y_1, \ldots, y_R\}]$.[2] The Ehrenfeucht-Fraïssé game provides a connection between the existence of a winning strategy for Duplicator in the game in R rounds on two graphs and their indistinguishability in terms of FO sentences with quantifier depth at most R. This connection could be formulated in the following way. Let us say that two graphs are FO_R*-equivalent* if, for every FO sentence ϕ with quantifier depth at most R, either ϕ is true on both graphs or it is false on both graphs.

Theorem 1.1. *Duplicator has a winning strategy on graphs G and H in R rounds if and only if G and H are FO_R-equivalent.*

We will need the following direct consequence of Theorem 1.1.

Corollary 1.2. *If for every $\varepsilon > 0$ and $R \in \mathbb{N}$ there exist a positive integer M and graph families $\mathcal{A}_i$, $i \in [M]$, such that, for any two representatives of one family, Duplicator wins the game in R rounds (which is equivalent to indistinguishability in the FO logic with quantifier depth at most R) and*

$$\mathbb{P}\left(G_n \in \mathcal{A}_i\right) \to p_i, \;\; i \in [M], \qquad \sum_{i=1}^{M} p_i > 1 - \varepsilon,$$

then G_n satisfies the FO convergence law.

Indeed, assume that the requirements of Corollary 1.2 hold and let ϕ be a first order sentence of quantifier depth R. Due to Theorem 1.1, any two graphs from $\mathcal{A}_i$ are not distinguishable by ϕ. Therefore, for n large enough and every $i \in [M]$ such that graphs from $\mathcal{A}_i$ satisfy ϕ,

$$\mathbb{P}(G_n \models \phi, G_n \in \mathcal{A}_i) = \mathbb{P}(G_n \in \mathcal{A}_i) \in (p_i - \varepsilon/M, p_i + \varepsilon/M),$$

[2] In the usual way, we denote by $G[A]$ the induced subgraph of G induced on the set of vertices $A \subset V(G)$.

implying

$$\left|\mathbb{P}(G_n \models \phi) - \sum_{i:\forall G\in\mathcal{A}_i, G\models\phi} p_i\right| \leq \varepsilon + \left|\mathbb{P}(G_n \models \phi) - \sum_{i:\forall G\in\mathcal{A}_i, G\models\phi} \mathbb{P}(G_n \in \mathcal{A}_i)\right|$$
$$\leq \varepsilon + \mathbb{P}(G_n \notin \cup_i \mathcal{A}_i) \leq 2\varepsilon + \left(1 - \sum_{i=1}^{M} p_i\right) \leq 3\varepsilon,$$

as required.

It is worth noting that the FO_R-equivalence relation partitions the set of all graphs into finitely many equivalence classes, see [14, Corollary 3.16].

Our plan is to use Corollary 1.2 to prove the convergence laws for uniform attachment random graphs with bounded degree. Assume that the two players play the R-rounds game on two sufficiently large uniform attachment random graphs $G_1 \subset G_2$ on vertex sets $[n_1]$ and $[n_2]$ respectively. Let r be large enough (depending on R). For an induced subgraph F of a graph G, we call the induced subgraph of G containing all vertices that are at distance at most r from some vertex of F *the r-neighbourhood of F*. In particular, the induced subgraph spanned by all vertices that are at distance at most r from a given vertex v is *the r-neighbourhood of v*. It can be shown that there exists n_0 such that, with probability at least $1-\varepsilon$, $G_1' := G_1 \setminus [n_0]$ and $G_2' := G_2 \setminus [n_0]$ are almost trees — the r-neighbourhood of every vertex contains at most one cycle (we call a connected graph with exactly one cycle *unicyclic*), see, e.g., [16, Lemma 3]. Moreover, with the same probability bound, for every *admissible*[3] rooted tree T of depth r, there are many vertices such that their rooted r-neighbourhoods are isomorphic to T in both graphs G_1', G_2' (cf. Lemma 6.1). Then, for Duplicator to win it is enough to guarantee that, for each FO_R*-equivalence class* $\mathcal{C}$ and for every $a \in \{3, \ldots, r\}$, the numbers of r-neighbourhoods of a-cycles C_a that have isomorphic representatives in $\mathcal{C}$ are either equal in G_1', G_2' or large in both graphs. On the one hand, it is not difficult to give a structural description of logical equivalence classes of unicyclic graphs and distinguish between unicyclic graphs that appear as r-neighbourhoods with probability arbitrarily close to 1 and unicyclic graphs such that the probabilities of their appearance as r-neighbourhoods have non-trivial limits. On the other hand, there are equivalence classes with infinitely many admissible unicyclic graphs, and this makes it hard to study the limit behaviour of the number of r-neighbourhoods that belong to such a class.[4] Indeed, note that Corollary 1.2 requires a finite decomposition into graph families, so that a further refinement of logical equivalence classes into, say, isomorphism classes does not help. However, if we bound the degrees of G_n, then this is no longer the case — the number of representatives in each of the equivalence classes becomes bounded as well.

It is straightforward to observe that the convergence of probabilities for isomorphism classes does not necessarily imply the convergence for logical equivalence classes. Consider the following example: Define a deterministic sequence of nested rooted trees of the same depth $T_1 \subset T_2 \subset \ldots$, where T_i is a perfect tree of a large enough arity a_i when i is odd, while, for even i, the tree T_i is obtained from T_{i-1} by attaching to the root a path of length equal to the depth of the tree. Although each isomorphism class contains at most one tree from the sequence, the convergence law fails. Indeed, for every i, T_i

[3]A rooted graph H is *admissible*, if, for n large enough, with positive probability, for some vertex v of G_n, its r-neighbourhood rooted in v is isomorphic to H.

[4]There is also a similar obstacle with subtrees "growing from $[n_0]$" (i.e. rooted in $[n_0]$ and having all the other vertices outside of $[n_0]$) in G_1, G_2 since there are trees such that the probabilities of their appearance as r-neighbourhoods of vertices from $[n_0]$ have non-trivial limits. See Section 9 for the description of the winning strategy of Duplicator where, in particular, it is explained why we should take care of trees rooted in $[n_0]$.

and T_{i-1} are distinguished by a sentence that asserts the existence of a vertex that has degree 2. In particular, Spoiler has a winning strategy on T_i and T_{i-1} in 4 rounds. Note that in this example there are two equivalence classes, each containing infinitely many trees: one consists of all T_i with odd i and the other of all T_i with even i.

The formal statement of the new result Let us introduce the model of graphs $G_n = G_n(m, d)$ that we consider in the paper. We start with a complete graph G_m on m vertices. Then, at each step, we construct a graph G_n by adding to G_{n-1} a new vertex and drawing m edges from it to different vertices, chosen uniformly at random out of existing vertices each of whom has degree less than d. Note that, for such a procedure to be possible, we need the condition $d \geq 2m$. The case $d = 2m$ is easier since in this case all but a constant number of vertices have degree d. It has been already considered separately in [19] and requires a different approach that cannot be generalised to other d.

Note that G_n, for $m \geq 2$, still does not obey the FO 0-1 law — the reason is the same as for the original uniform attachment model, see [17]. Indeed, if we consider the number of *diamond graphs* (for $m = 2$) or the number of complete graphs on $m + 1$ vertices (for $m \geq 3$), it could be proven (similar to the way it was done in Section 2 of [17], but with modifications based on the arguments that appear in Sections 4, 5 of the present paper) that the probability to have a certain number of such graphs is bounded away from both 0 and 1.

Let us formulate our main result.

Theorem 1.3. *For every $m \geq 2$ and $d > 2m$, $G_n(m, d)$ obeys the* FO *convergence law.*

Proof outline We derive Theorem 1.3 from Corollary 1.2. Specifically, we partition the set of asymptotically almost all graphs (with respect to the measure induced by $G_n(m, d)$) into finitely many disjoint families $\mathcal{A}_i$, so that each family $\mathcal{A}_i$ lies within a single FO_R-equivalence class, and the probability $\mathbb{P}(G_n(m, d) \in \mathcal{A}_i)$ converges as $n \to \infty$. As previously noted, it suffices to characterise the distribution of FO_R-equivalence classes of r-neighbourhoods of vertices in the random graph, that are typically either trees or unicyclic graphs, where $r = r(R)$ does not depend on n. Since, in bounded-degree graphs, there are only finitely many isomorphism classes of subtrees and unicyclic graphs with a given fixed diameter, it suffices to analyse the limit behaviour of these isomorphism classes (which is done in Sections 5–8, as described below). We now outline the proof strategy in more detail and refer to the sections where each part is developed.

The existence of a suitable decomposition of the set of all graphs into families $\mathcal{A}_i$, $i \in [M]$, is established in Section 9 via Lemmas 9.1 and 9.2. We introduce two graph properties that we call Q1 and Q2. Lemma 9.1 is deterministic and it guarantees that these two properties are sufficient conditions for the existence of a winning strategy for Duplicator. Lemma 9.2 then shows the existence of families $\mathcal{A}_i$, $i \in [M]$, satisfying the assumptions of Corollary 1.2, such that any pair of graphs from the same family $\mathcal{A}_i$ has properties Q1 and Q2, which in turn impies Theorem 1.3.

The key properties Q1 and Q2 of pairs of graphs $G_1 \subset G_2$ characterise the distribution of small cycles and subtrees in G_1, G_2. Specifically, they say that in both graphs G_1, G_2 there are two small sets of vertices $V_0 \subset V_0'$, where V_0 contains only vertices with degree exactly d and V_0' includes vertices that are close to V_0, such that

(1) any two small cycles not intersecting V_0 are far from each other,
(2) for any *admissible*[5] rooted tree T of small depth r, there exist sufficiently many

[5]With positive probability, there exists a vertex in the random graph such that the r-ball around it is isomorphic to this tree — see the definition in Section 6.

vertices, that are far from V_0' and from each other, whose r-neighbourhoods are isomorphic to T, and

(3) if $G_1[V_0'] = G_2[V_0']$, then for any rooted *unicyclic*[6] graph C of small diameter, there is r such that either in both G_1, G_2 there are sufficiently many vertices, that are far from V_0' and from each other, whose r-neighbourhoods are isomorphic to C, or the numbers of such r-neighbourhoods in the two graphs are equal.

Lemma 9.1 is a stand-alone lemma and it is relatively straightforward to prove. In contrast, Lemma 9.2 is technically much more challenging. The proof makes up for the majority of the paper and relies on results from all Sections 2–8. In particular, the fact that small cycles are typically far from each other, that is required by (1), is exactly Lemma 5.2 given in Section 5. Property (2) follows directly from a law of large numbers for a count of admissible trees given by Lemma 6.1 in Section 6. The proof of the latter lemma uses the so-called stochastic approximation method (see Section 2 and, e.g., [3, 23] for more details). To the best of our knowledge, an application of stochastic approximations to prove logical limit laws is novel. We hope that it may be used to prove FO convergence for the original uniform attachment model and for some other recursive models as well. As discussed earlier, the main difficulty in the derivation of convergence laws for recursive models lies in the treatment of unicyclic graphs — corresponding here to property (3). This property is ensured by Lemma 8.1 that is presented in Section 8. Its proof relies on approximating counts of unicyclic graphs using Markov chains, applying results concerning the existence of limiting distributions for those chains, and leveraging the fact that $G_n(m,d)$ typically has unbounded number of cycles of any fixed length — see Lemma 7.1 in Section 7.

Organisation of the paper In Section 2 we state the stochastic approximation theorem that we use in Sections 3 and 6. In Section 3 we prove auxiliary results about the asymptotic behaviour of the number of vertices of a given degree, which is a particular case of a more general result that asserts the law of large numbers for trees counts, presented in Section 6. In Section 4 we describe the random graph structure induced by $[n_0]$ — that plays the role of the set V_0 from our overview of the proof strategy — where n_0 is a sufficiently large constant. In Sections 5 and 7 we prove an upper bound and a lower bound on the number of small cycles in the random graph, respectively. The limit distribution of unicyclic subgraphs is investigated in Section 8. Finally, Section 9 proves Lemma 9.1 and Lemma 9.2 and therefore completes the proof of Theorem 1.3 by describing the winning strategy of Duplicator.

2 Stochastic approximation

Let us consider an r-dimensional process $Z(n)$, with filtration $\mathcal{F}_n$, which is defined in the following way (see [3] for more details on stochastic approximations)

$$Z(n+1) - Z(n) = \frac{1}{n+1}\left(F(Z(n)) + E_{n+1} + R_{n+1}\right), \tag{2.1}$$

where E_n, R_n and the function F satisfy the following conditions. There exists $U \subset \mathbb{R}^r$ such that $Z_n \in U$ for all n almost surely (a.s. for brevity) and

A1 The function $F : \mathbb{R}^r \to \mathbb{R}^r$ is continuous and bounded in some neighbourhood of U, has a unique root θ in U, such that in some neighbourhood of the root (which may have some elements outside of U),

$$F(x) = H(x-\theta) + O\left(|x-\theta|^a\right)$$

[6] A connected graph with exactly one cycle.

for some $a > 1$, where the matrix H is stable, i.e. the real parts of the eigenvalues of H are strictly negative. The smallest of the absolute values of these real parts is bigger than $1/2$.

A2 For any $\epsilon > 0$

$$\sup_{|x-\theta|>\epsilon, x\in U} F^t(x)(x-\theta) < 0.$$

Note that for condition A2 to hold it is enough for the derivative matrix of $F(x)$ to exist and to be stable in U. We also note that the matrix H in the condition A1 is the derivative of F at $x = \theta$.

A3 E_n is a martingale difference with respect to $\mathcal{F}_n$ (recall that a process E_n is a *martingale difference* process with respect to a filtration $\mathcal{F}_n$ if $\mathbb{E}(E_{n+1} \mid \mathcal{F}_n) = 0$ for all n) and for some $\delta \in (0, 1/2)$, $R_n = O(n^\delta)$ a.s. (i.e. there exists a non-random constant C, such that $\limsup_{n\to\infty} \frac{|R_n|}{n^\delta} \leq C$ a.s.), and

$$\sum_{n=1}^{\infty} \frac{E_{n+1}}{n^{1-\delta}} < \infty \quad \text{a.s.}$$

Note that due to convergence theorems for martingale differences (see [3, Appendix B]) for the last condition to hold it is enough that $\sup_n \mathbb{E}(|E_{n+1}|^2 \mid \mathcal{F}_n) < \infty$ a.s.

We need the following result (see the proof in [3, Theorem 3.1.1]):

Theorem 2.1. *Under the above conditions, $Z(n) \to \theta$ a.s. with the convergence rate*

$$|Z(n) - \theta| = o(n^{-\delta}) \quad \textit{a.s.} \left(\textit{i.e.}\ \frac{|Z(n)-\theta|}{n^{-\delta}} \to 0 \ \textit{a.s.}\right).$$

3 Number of vertices of fixed degree

In our model, at step $n+1$, the probability to draw an edge to a given existing vertex equals to

$$1 - \frac{\binom{n-N_d(n)-1}{m}}{\binom{n-N_d(n)}{m}} = 1 - \frac{n - N_d(n) - m}{n - N_d(n)} = \frac{m}{n - N_d(n)}, \tag{3.1}$$

where $N_k(n)$ is the number of vertices with degree k at time n for $k \in \{m, \ldots, d\}$. In order to use this formula, we study an asymptotical behaviour of $N_k(n)$. Let $X_k(n) := N_k(n)/n$, $m \leq k \leq d$. Let us consider the equation

$$\left(\frac{m}{m+1-x}\right)^{d-m} = x. \tag{3.2}$$

Recall that $d > 2m$. Since

$$\frac{\partial}{\partial x}\left((d-m)\ln\frac{m}{m+1-x} - \ln x\right) = \frac{d-m}{m+1-x} - \frac{1}{x} = \frac{x(d-m+1)-m-1}{x(m+1-x)}$$

is negative on $\left(0, \frac{m+1}{d-m+1}\right)$ and is positive on $\left(\frac{m+1}{d-m+1}, 1\right)$ and since the equation (3.2) has root $x = 1$, it also has a unique root $x = \rho_d$ on $(0,1)$, which is strictly less than $\frac{m+1}{d-m+1}$, which is, in turn, less than $\frac{2m}{d}$.

Let us define

$$\rho_k := \frac{(1-\rho_d)m^{k-m}}{(m+1-\rho_d)^{k-m+1}}, \quad k = m, \ldots, d-1. \tag{3.3}$$

Lemma 3.1. *$X_k(n) \to \rho_k$ with rate $|X_k(n) - \rho_k| = o(n^{-1/2+\delta})$ for any $\delta > 0$ a.s.*

Proof. Let $\mathcal{F}_n$ be the filtration that corresponds to the graphs G_n. Let $v_n^{(1)},\dots,v_n^{(m)}$ be the vertices that the vertex added at step $n+1$ sends the edges to. In what follows, we denote by $\deg_n(v)$ the degree of a vertex v in the graph G_n. We get

$$\begin{aligned}\mathbb{E}\left(N_m(n+1)-N_m(n)\mid\mathcal{F}_n\right)&=1-\sum_{i=1}^m\mathbb{P}\left(\deg_n(v_n^{(i)})=m\mid\mathcal{F}_n\right)=1-\frac{m}{n-N_d(n)}N_m(n),\\ \mathbb{E}\left(N_k(n+1)-N_k(n)\mid\mathcal{F}_n\right)&=\sum_{i=1}^m\left[\mathbb{P}\left(\deg_n(v_n^{(i)})=k-1\mid\mathcal{F}_n\right)-\mathbb{P}\left(\deg_n(v_n^{(i)})=k\mid\mathcal{F}_n\right)\right]\\ &=\frac{m}{n-N_d(n)}\left(N_{k-1}(n)-N_k(n)\right),\qquad k=m+1,\dots,d-1,\\ \mathbb{E}\left(N_d(n+1)-N_d(n)\mid\mathcal{F}_n\right)&=\sum_{i=1}^m\mathbb{P}\left(\deg_n(v_n^{(i)})=d-1\mid\mathcal{F}_n\right)=\frac{m}{n-N_d(n)}N_{d-1}(n).\end{aligned}$$

Since the total number of edges in G_n is strictly less than mn and is twice less than the sum of all degrees in the graph, we get that the number of vertices of degree d does not exceed $\frac{2mn}{d}$. Therefore, if $d>2m$ we get that $X_d(n)\le\frac{2m}{d}<1$. Note that for $X_k(n)$ we get

$$\mathbb{E}\left(X_k(n+1)-X_k(n)\mid\mathcal{F}_n\right)=\frac{1}{n+1}\left(\mathbb{E}\left(N_k(n+1)-N_k(n)\mid\mathcal{F}_n\right)-X_k(n)\right).\tag{3.4}$$

Hence, if we define functions (on $[0,1)^{d-m}\times[0,\frac{2m}{d})$)

$$\begin{aligned}f_m(x_m,\dots,x_d)&=1-\left(\frac{m}{1-x_d}+1\right)x_m,\\ f_k(x_m,\dots,x_d)&=\frac{m}{1-x_d}x_{k-1}-\left(\frac{m}{1-x_d}+1\right)x_k,\quad k=m+1,\dots,d-1,\\ f_d(x_m,\dots,x_d)&=\frac{m}{1-x_d}x_{d-1}-x_d,\end{aligned}\tag{3.5}$$

we would get that for all $k\in\{m,\dots,d\}$,

$$\mathbb{E}\left(X_k(n+1)-X_k(n)\mid\mathcal{F}_n\right)=\frac{1}{n+1}f_k\left(X_m(n),\dots,X_d(n)\right).\tag{3.6}$$

For the vector $Z(n):=(X_m(n),\dots,X_d(n))$ we have the following representation

$$Z(n+1)-Z(n)=\frac{1}{n+1}\Big(F(Z(n))+(n+1)(Z(n+1)-\mathbb{E}(Z(n+1)\mid\mathcal{F}_n))\Big),$$

where $F(x_m,\dots,x_d)=(f_m(x_m,\dots,x_d),\dots,f_d(x_m,\dots,x_d))^t$. Set

$$E_{n+1}=(n+1)(Z(n+1)-\mathbb{E}(Z(n+1)\mid\mathcal{F}_n)),\quad R_{n+1}=0.$$

Let us find solutions of $F(x_m,\dots,x_d)=0$, i.e. of the system of equations

$$\begin{cases}1-\left(\frac{m}{1-x_d}\right)x_m&=&x_m,\\ \frac{m}{1-x_d}(x_{k-1}-x_k)&=&x_k,\qquad k=m+1,\dots,d-1,\\ \frac{m}{1-x_d}x_{d-1}&=&x_d.\end{cases}\tag{3.7}$$

We get

$$\begin{aligned}x_m&=\frac{1-x_d}{m+1-x_d},\\ x_k&=\frac{m}{m+1-x_d}x_{k-1},\quad k=m+1,\dots,d-1.\end{aligned}$$

Hence for $k = m+1, \dots, d-1$

$$x_k = \frac{(1-x_d)m^{k-m}}{(m+1-x_d)^{k-m+1}}.$$

For x_d we get that

$$\frac{m}{1-x_d}\frac{(1-x_d)m^{d-1-m}}{(m+1-x_d)^{d-1-m+1}} - x_d = 0,$$

which is equivalent to

$$\left(\frac{m}{m+1-x_d}\right)^{d-m} = x_d.$$

This equation has a unique root $x_d = \rho_d$ in $(0, 2m/d)$, which results in the existence of a unique solution $x_k = \rho_k$, $k = m, \dots, d-1$. Note that the system (3.7) is equivalent (by summing all rows) to the system

$$\begin{cases} 1 - \left(\frac{m}{1-x_d}\right) x_m & = & x_m, & \\ \frac{m}{1-x_d}(x_{k-1} - x_k) & = & x_k, & k = m+1, \dots, d-1, \\ 1 & = & x_m + \dots + x_d. & \end{cases} \tag{3.8}$$

Let us check the conditions of Theorem 2.1. For non-zero partial derivatives of functions f_k, $k = m, \dots, d$, we would get:

$$\begin{cases} \frac{\partial f_m}{\partial x_m}(x_m, \dots, x_d) & = & -\frac{m}{1-x_d} - 1, & \\ \frac{\partial f_m}{\partial x_d}(x_m, \dots, x_d) & = & -\frac{m}{(1-x_d)^2} x_m, & \\ \frac{\partial f_k}{\partial x_{k-1}}(x_m, \dots, x_d) & = & \frac{m}{1-x_d}, & k = m+1, \dots, d-1, \\ \frac{\partial f_k}{\partial x_k}(x_m, \dots, x_d) & = & -\frac{m}{1-x_d} - 1, & k = m+1, \dots, d-1, \\ \frac{\partial f_k}{\partial x_d}(x_m, \dots, x_d) & = & \frac{m}{(1-x_d)^2}(x_{k-1} - x_k), & k = m+1, \dots, d-1, \\ \frac{\partial f_d}{\partial x_{d-1}}(x_m, \dots, x_d) & = & \frac{m}{1-x_d}, & \\ \frac{\partial f_d}{\partial x_d}(x_m, \dots, x_d) & = & -1 + \frac{m}{(1-x_d)^2} x_{d-1}. & \end{cases} \tag{3.9}$$

Hence, the characteristic polynomial of the derivative matrix is

$$\begin{aligned} P(\lambda) = (-1)^{d-m} &\left(\frac{m}{1-x_d}\right)^{d-m}\left(-\frac{m}{(1-x_d)^2}x_m\right) \\ &+ \sum_{k=m+1}^{d-1}(-1)^{d-k}\left(\frac{m}{1-x_d}\right)^{d-k}\left(-\frac{m}{1-x_d} - 1 - \lambda\right)^{k-m}\frac{m}{(1-x_d)^2}(x_{k-1} - x_k) \\ &+ \left(-\frac{m}{1-x_d} - 1 - \lambda\right)^{d-m}\left(-1 + \frac{m}{(1-x_d)^2}x_{d-1} - \lambda\right) \\ = (-1)^{d-m} &\sum_{k=m+1}^{d}\left(\frac{m}{1-x_d}\right)^{d-k}\left(\frac{m}{1-x_d} + 1 + \lambda\right)^{k-m}\frac{m}{(1-x_d)^2}x_{k-1} \\ &- (-1)^{d-m}\sum_{k=m}^{d-1}\left(\frac{m}{1-x_d}\right)^{d-k}\left(\frac{m}{1-x_d} + 1 + \lambda\right)^{k-m}\frac{m}{(1-x_d)^2}x_k \\ &- (-1)^{d-m}\left(\frac{m}{1-x_d} + 1 + \lambda\right)^{d-m}(1+\lambda) \\ = (-1)^{d-m} &\sum_{k=m}^{d-1}\left(\frac{m}{1-x_d}\right)^{d-k-1}\left(\frac{m}{1-x_d} + 1 + \lambda\right)^{k-m}\frac{m}{(1-x_d)^2}x_k(1+\lambda) \\ &- (-1)^{d-m}\left(\frac{m}{1-x_d} + 1 + \lambda\right)^{d-m}(1+\lambda). \end{aligned}$$

Let us denote $t = 1 + \lambda$, $c = \frac{m}{1-x_d}$. Then

$$P(\lambda) = (-1)^{d-m+1} t Q(t),$$

where

$$Q(t) := (c+t)^{d-m} - \sum_{k=m}^{d-1} c^{d-k}(c+t)^{k-m}\frac{x_k}{1-x_d}.$$

For $Q(t)$ we get

$$\begin{aligned} Q(t) &= \sum_{i=0}^{d-m}\binom{d-m}{i}c^{d-m-i}t^i - \sum_{k=m}^{d-1} c^{d-k}\sum_{i=0}^{k-m}\binom{k-m}{i}c^{k-m-i}t^i\frac{x_k}{1-x_d} \\ &= \sum_{i=0}^{d-m}\binom{d-m}{i}c^{d-m-i}t^i - \sum_{i=0}^{d-m-1} c^{d-m-i}t^i\sum_{k=m+i}^{d-1}\binom{k-m}{i}\frac{x_k}{1-x_d} \\ &= t^{d-m} + \sum_{i=0}^{d-m-1} c^{d-m-i}t^i\left(\binom{d-m}{i} - \sum_{k=m+i}^{d-1}\binom{k-m}{i}\frac{x_k}{1-x_d}\right). \end{aligned}$$

Note that

$$\binom{d-m}{i} - \sum_{k=m+i}^{d-1}\binom{k-m}{i}\frac{x_k}{1-x_d} \geq \binom{d-m-1}{i}\left(1 - \frac{\sum_{k=m+i}^{d-1} x_k}{1-x_d}\right)$$

since $\binom{d-m}{i} = \frac{d-m}{d-m-i}\binom{d-m-1}{i} \geq \binom{d-m-1}{i}$ and $\binom{k-m}{i} \leq \binom{d-m-1}{i}$ for all $k \leq d-1$. Therefore, if $\sum_{k=m}^{d} x_k \leq 1$ (in particular, when $x_i = \rho_i$), $Q(t)$ has non-negative coefficients and, therefore, does not have roots with positive real parts. Note that $P(-1) = 0$. As a result, we get that the largest real part of eigenvalues of the derivative matrix equals -1 if $\sum_{k=m}^{d} x_k \leq 1$. Note that $\sum_{k=m}^{d} X_k(n) = 1$. Therefore the process $Z(n)$ satisfies the conditions A1, A2 of Theorem 2.1 on the set

$$U = \left\{x_m + \ldots + x_d = 1, x_k \geq 0, k = m, \ldots, d, x_d \leq \frac{2m}{d}\right\}.$$

To check condition A3 we first recall that $R_{n+1} = 0$. At each step we draw m edges, so we change degrees of exactly m vertices, while adding one new vertex. Hence, $|N_k(n+1) - N_k(n)| \leq m+1$ and $|X_k(n+1) - X_k(n)| \leq \frac{m+1}{n}$. Therefore, for E_{n+1} we get

$$\begin{aligned} |E_{n+1}| &\leq (n+1)\left(|Z(n+1) - Z(n)| + |\mathbb{E}(Z(n+1) - Z(n) \mid \mathcal{F}_n)|\right) \\ &\leq 2\frac{(n+1)(m+1)(d-m+1)}{n}, \end{aligned}$$

which results in condition $A3$. By Theorem 2.1, we get the conclusion of Lemma 3.1. □

4 Probability to have degree less than d

We will need the following variant of the Chernoff bound.

Lemma 4.1 ([22, Theorem 4.4]). *Let X_i, $i \geq k$, $k > 1$, be independent Bernoulli random variables with $\mathbb{E}X_i = \frac{p}{i}$ for some $p > 0$. Let $S_n = \sum_{i=k}^{n} X_i$, $n > k$. Then, for any $\delta > 0$,*

$$\mathbb{P}\Big(S_n \leq (1-\delta)p(\ln(n+1) - \ln k)\Big) \leq \left(\frac{n+1}{k}\right)^{\frac{-\delta^2 p}{2}},$$

$$\mathbb{P}\Big(S_n \geq (1+\delta)p(\ln n - \ln(k-1))\Big) \leq \left(\frac{n}{k-1}\right)^{\frac{-\delta^2 p}{2+\delta}}.$$

Note that Lemma 4.1 indeed follows immediately from the standard Chernoff bound [22, Theorem 4.4] since $\mathbb{E}S_n = p\sum_{i=k}^n \frac{1}{i}$ is greater than $\ln(n+1) - \ln k$ and less than $\ln n - \ln(k-1)$ for all $n > k > 1$.

Let us consider the evolution of the degree of a given vertex. Fix a time s and consider the vertex s that appears at this time. It appears with the degree m. If its degree at time $t \geq s$ is less than d the probability to draw an edge to it (from the vertex $t+1$) equals to $\frac{m}{t-N_d(t)}$. For $x \in (0,1)$, let $X_i(x)$, $i \geq s$, be independent Bernoulli random variables with $\mathbb{E}X_i(x) = x$ and let each process $(X_i(x), x \in (0,1))$ be independent of G_i. Set $X_i := X_i\left(\frac{m}{i-N_d(i)}\right)$ for $i \geq s$. Clearly, there are independent Bernoulli X_i', $i \geq s$, such that $\mathbb{E}X_i' = \frac{m}{i}$ and $X_i' \leq X_i$ for all $i \geq s$. Then, due to Lemma 4.1, the probability that the vertex s has degree less than d at time $n > s$ does not exceed

$$\mathbb{P}\left(\sum_{i=s}^{n-1} X_i \leq d - m - 1\right) \leq \mathbb{P}\left(\sum_{i=s}^{n-1} X_i' \leq d - m - 1\right) \leq c\left(\frac{n}{s}\right)^{-m/2}$$

for some positive constant c. By repeating this estimate to vertices that appear at the beginning of our graph process, we would get the following result.

Lemma 4.2. *For any fixed s, with high probability (hereinafter we write 'w.h.p.' for brevity, i.e. with probability tending to 1 as $n \to \infty$) the degree of s in G_n equals d. In particular, for any fixed n_0 and a, w.h.p., the degree of each vertex in the a-neighbourhood of the first n_0 vertices has degree equal to d.*

5 Number of cycles: Upper bound

Let us estimate the probability that a new cycle of length r is formed at time $n+1$. To form a cycle of length r we have to connect a new vertex with two vertices joined by a path of length $r-2$ that are open to attachment (there are $n - N_d(n)$ such vertices). There are at most $(d^{r-2}-1)(n-N_d(n))$ ordered pairs (v,u) of vertices that are open to attachment and joined by an $(r-2)$-path. Indeed, there at most $n - N_d(n)$ ways to choose the first vertex v in the pair. Since v has degree at most $d-1$ and every other vertex has degree at most d, the ball around v of radius $r-2$ has at most $(d-1)d^{r-3} \leq d^{r-2}-1$ vertices (excluding v). Thus, there are at most $d^{r-2}-1$ ways to choose the vertex u.

Recall that the number of vertices of degree d, denoted by $N_d(n)$, does not exceed $\frac{2m}{d}n$. Hence, the probability to form a new cycle of length r does not exceed

$$m(m-1)\frac{(d^{r-2}-1)(n-N_d(n))}{(n-N_d(n))(n-N_d(n)-1)} \leq \frac{m(m-1)d^{r-2}}{n-N_d(n)} \leq \frac{m(m-1)d^{r-2}}{\left(1-\frac{2m}{d}\right)n}. \tag{5.1}$$

Let $n > m(m-1)d^{r-1}$. Let $C_r^{\max}$ be the maximum possible number of r-cycles on first $m(m-1)d^{r-1}+1$ vertices. For $i \in \{m(m-1)d^{r-1}+1, \ldots, n\}$, let X_i be the indicator random variable of the event that a new r-cycle appears with the introduction of the i-th vertex in G_i. As we have just proved, there exist independent Bernoulli random variable X_i' such that, for all $i \in \{m(m-1)d^{r-1}+1, \ldots, n\}$, $\mathbb{E}X_i' = p/i$, where $p = \frac{m(m-1)d^{r-2}}{1-\frac{2m}{d}}$, and $X_i' \geq X_i$. Clearly, the total number of r-cycles in G_n is at most $C_r^{\max} + \sum_{i=m(m-1)d^{r-1}+1}^n Y_i$, where Y_i is the number of r-cycles created at step i. We have that $Y_i = 0$ whenever $X_i = 0$. Moreover, $Y_i \leq d^{r-3}m(m-1)/2$ for all i almost surely, since any cycle that appears at step i involves two edges $\{i,v\}$ and $\{i,u\}$ (out of m) that contain the vertex i and a path between v and u in G_{i-1} of length $r-2$. Any such path consists of $r-3$ vertices other than u and v. So there are at most d^{r-3} ways to choose the path, since all vertices in G_{i-1} have degrees at most d. We conclude that the number of r-cycles in G_n is at most $C_r + \sum_{i=m(m-1)d^{r-1}+1}^n X_i' d^{r-3}m(m-1)/2$. Due to Lemma 4.1 there are constants $C, c > 0$, such that

$$\mathbb{P}\left(\frac{d^{r-3}m(m-1)}{2}\sum_{i=m(m-1)d^{r-1}+1}^{n} X_i' > C\ln n\right) \le cn^{-2}. \tag{5.2}$$

The total number of r-cycles that contain at least one vertex $i \in \{m(m-1)d^{r-1}+1, \dots, n\}$ equals

$$Y_n^{\Sigma} := \sum_{i=m(m-1)d^{r-1}+1}^{n} Y_i \le \frac{d^{r-3}m(m-1)}{2}\sum X_i \le \frac{d^{r-3}m(m-1)}{2}\sum X_i'.$$

Due to (5.2), we get $\sum_{n=m(m-1)d^{r-1}+1}^{\infty} \mathbb{P}(Y_n^{\Sigma} > C\ln n) < \infty$. Since the total number of r-cycles in G_n is at most $O(1)$-far from Y_n^{Σ}, due to the Borel–Cantelli lemma[7], the probability that there are more than $C\ln n$ cycles in G_n for some $n > N$ tends to 0 as $N \to \infty$, i.e. we proved the following result.

Lemma 5.1. *For any $r > 2$, the number of cycles of length r in G_n is $O(\ln n)$ a.s.*

Note that w.h.p. there are at most $C\ln n$ vertices in the a-neighbourhood of the union of all r-cycles. Therefore the probability to draw an edge to this neighbourhood at time n does not exceed $\frac{C\ln n}{n}$ (for some constant C), and to draw two edges does not exceed $\frac{C\ln^2 n}{n^2}$. Therefore, by the Borel–Cantelli lemma, we get the following result.

Lemma 5.2. *For any $\epsilon > 0$ and ℓ there is s such that with probability at least $1-\epsilon$ in $[n]\setminus[s]$ there are no connected subgraphs with at most ℓ vertices and at least 2 cycles.*

6 Number of rooted trees

For a rooted tree T, let $N_T(n)$ be the number of vertices that are roots of maximal subtrees of G_n (a subtree is *maximal* in G_n if all its non-leaf vertices are adjacent only to vertices of that tree) isomorphic to T. Note that the set of all isomorphism classes of rooted trees with degrees at most d of a given depth is finite. We would refer to a maximal subtree of G_n isomorphic to a tree T from that set as *having the type* T (i.e. when we talk about the type of a tree in G_n we assume it is rooted and maximal). Also, we call a tree T *max-admissible*, if with positive probability its isomorphic copy is a maximal subtree of G_n for large enough n. In the current section, we prove the following statement:

Lemma 6.1. *For any max-admissible tree T there is a constant $\rho_T \in (0,1)$, such that for any $\delta > 0$*

$$N_T(n) = \rho_T n + o(n^{1/2+\delta}) \quad \text{a.s.}$$

In particular, for all $s \in \mathbb{N}$ and any max-admissible tree T w.h.p. there are at least s vertices in G_n that are roots of maximal subtrees of G_n that are isomorphic to T.

Proof. Let us fix $b \in \mathbb{N}$ and consider variables $X_T(n) := N_T(n)/n$ and vector $Z_b(n) := (X_{T_i}(n))$ over all max-admissible rooted trees T_i of depth b (there are only finitely many such trees). Note that the case $b = 1$ refer to the number of stars and was already considered in Section 3. Let $b > 1$. The order of the elements of $Z_b(n)$ (or, in other words, the order on the set of all max-admissible trees of depth b) is defined in a way such that an addition of new branches (that preserves the depth of the tree) increases the order. It could be done by induction on b in the following way. If T_1, T_2 are stars (i.e. $b = 1$),

[7] Let us recall that Borel–Cantelli lemma states the following. Let $\mathcal{B}_n$, $n \in \mathbb{N}$, be a sequence of events such that $\sum_{n\in\mathbb{N}} \mathbb{P}(\mathcal{B}_n) < \infty$. Then $\mathbb{P}(\cap_{N=1}^{\infty} \cup_{N=n}^{\infty} \mathcal{B}_n) = \lim_{N\to\infty} \mathbb{P}(\cup_{N=n}^{\infty}\mathcal{B}_n) = 0$. In our case, $\mathcal{B}_n = \{Y_n^{\Sigma} > C\ln n\}$.

then $T_1 \prec T_2$ if and only if T_1 has less leaves than T_2. Assume that $\prec$ on the set of all max-admissible trees of depth $b-1$ is defined. Let s_1, s_2 be the number of children of roots of trees T_1, T_2 of depth b respectively. If $s_1 < s_2$, then $T_1 \prec T_2$. If $s_1 = s_2 =: s$, then let $T_j^1, \ldots, T_j^s$ be the subtrees of T_j rooted at the children $v_j^1, \ldots, v_j^s$ of the root of T_j comprising all descendants of these children and ordered in the decreasing order. Then $T_1 \prec T_2$ if and only if $(T_1^1, \ldots, T_1^s) \prec_s (T_2^1, \ldots, T_2^s)$, where $\prec_s$ is the lexicographical order on the set of s-vectors of trees of depth $b-1$ induced by the order $\prec$.

Note that

$$\mathbb{E}(X_T(n+1) - X_T(n) \mid \mathcal{F}_n) = \frac{1}{n+1}\Big(\mathbb{E}(N_T(n+1) - N_T(n) \mid \mathcal{F}_n) - X_T(n)\Big).$$

There are two ways to change $N_T(n)$ at time $n+1$. We could draw an edge to a maximum tree isomorphic to T or we could create a new copy of T rooted at $n+1$. Recall that due to equation (3.1) for each given vertex of degree less than d the probability to draw an edge to it is

$$\frac{m}{n - N_d(n)} = \frac{1}{n}\frac{m}{1 - \frac{N_d(n)}{n}}.$$

In a rooted tree T, fix a non-leaf vertex u. Then the expected number (conditioned on G_n) of trees T' in G_n of type T such that an edge is drawn from $n+1$ to a vertex u' of T' and there exists an isomorphism of rooted trees $T \to T'$ sending u to u' equals

$$C\frac{m\frac{N_T(n)}{n}}{1 - \frac{N_d(n)}{n}} = C\frac{mX_T(n)}{1 - X_d(n)},$$

where the constant $C = C(T, u)$ corresponds to the number of vertices that belong to the orbit of u under the action of the automorphism group of the rooted tree T. We stress that all automorphisms of T preserve the root R, so $C(T, R) = 1$. For instance, let us consider a "regular" tree T of depth 3 such that all leaves are at distance 3 from the root and all non-leaf vertices have degree k. Then any vertex u at distance 1 from the root has $C(T, u) = k$ since the group of automorphisms of T induces the symmetric group S_k on the k vertices adjacent to the root (any two branches adjacent to the root can be permuted). Since, on the next layer, there are $k(k-1)$ vertices, any vertex u at distance 2 from the root has $C(T, u) = k(k-1)$ (clearly, for every two vertices at distance 2 from the root there exists an automorphism that maps one to another).

Recall that $X_d(n) \leq \frac{2m}{d}$, and hence, due to the condition $2m < d$, $X_d(n)$ is bounded away from 1.

The type of the maximal tree T_{n+1}^* of depth b with root $n+1$ would correspond to the probability distribution induced by the numbers of maximal trees of depth $b-1$ at time n. It is defined by the types of trees of depth $b-1$, to whose roots we draw m edges from vertex $n+1$. For a max-admissible tree T of depth b such that its root has exactly m children, let $\mathcal{M}(T)$ be the set of children of the root of T and let $\mathcal{T}(T)$ be the multiset of types of the m subtrees of T rooted in $u \in \mathcal{M}(T)$ and containing all descendants of u. For $t \in \mathcal{T}(T)$, let $\nu(t)$ be the multiplicity of the type t in the multiset $\mathcal{T}(T)$. Let $t_1, \ldots, t_s$ be all the different types from $\mathcal{T}(T)$. Note that, for $t \in \mathcal{T}(T)$ and $i \in [m]$, the probability (subject to G_n) that the i-th edge emanating from the vertex $n+1$ meets the root of a maximal subtree of G_n of type t, equals $\frac{N_t(n)}{n - N_d(n)} = \frac{X_t(n)}{1 - X_d(n)}$. Then

$$\left|\mathbb{P}(T_{n+1}^* \cong T \mid G_n) - \binom{m}{\nu(t_1), \ldots, \nu(t_s)}\frac{\prod_{i=1}^s N_{t_i}(n) \cdot \ldots \cdot (N_{t_i}(n) - \nu(t_i) + 1)}{(n - N_d(n)) \cdot \ldots \cdot (n - N_d(n) - m + 1)}\right| \leq P_{\text{cycle}},$$

where P_{cycle} is the probability to create a cycle of length at most $2b+1$ at step $n+1$ which is $O(1/n)$ due to the bound (5.1). Therefore, the conditional probability (subject to G_n) to create a tree of type T rooted in the vertex $n+1$ at step $n+1$ is a polynomial function of $\frac{X_{t_i}(n)}{1-X_d(n)}$, for every $i \in [s]$, up to $O(\frac{1}{n})$ error term.

In order to change the type of a given maximal tree in G_n (to another given type) of depth b we need to draw an edge from the vertex $n+1$ to one of its vertices and draw the rest of the edges to the roots of trees of depth at most $b-2$ of given types (that depends on the type of a tree we want to obtain). We also need to make sure that all trees are disjoint — the probability of drawing edges to "intersecting" trees is of order $O\left(\frac{1}{n}\right)$. So, the probability of changing the type of the given tree in the described way (subject to G_n) is a polynomial function of $\frac{1}{1-X_d(n)}$ and $X_{T_i}(n)$, up to a term $O(\frac{1}{n})$, where T_i are max-admissible trees of depth $b-2$.

Therefore[8]

$$\mathbb{E}(Z_b(n+1) - Z_b(n) \mid \mathcal{F}_n) = \frac{1}{n+1}\left(A_b Z_b(n) - Z_b(n) + Y_b + O\left(\frac{1}{n}\right)\right)$$

where $A_b = A_b(Z_1(n), \dots, Z_{b-2}(n))$ is a lower-triangular matrix with negative elements on the diagonal and non-negative under the diagonal and $Y_b = Y_b(Z_{b-1}(n), X_d(n))$ is a vector, such that the elements of both A_b and Y_b are polynomials of $\frac{1}{1-X_d(n)}$ and $X_{T_i}(n)$, where T_i are trees of depth at most $b-2$ (for A_b) or exactly $b-1$ (for Y_b). Let us consider $F_b(Z_1, \dots, Z_b) := A_b Z_b(n) - Z_b(n) + Y_b$ (note that A_b and Y_b are functions of $Z_1, \dots, Z_{b-1}$ itself). Recall that Z_1 contains X_d, so F_b is deterministic. We would use induction over b to prove that there is a unique solution of the system $F_i(z_1, \dots, z_i) = 0, i = 1, \dots, b$ (in an appropriate area). We already established the existence of the unique (non-zero) root for the case $b = 1$. Assume there are unique non-zero solutions $z_1^*, \dots, z_{b-1}^*$ of the systems $F_i(z_1, \dots, z_i) = 0, i = 1, \dots, b-1$. If we define $H_b(z_b) = F_b(z_1^*, \dots, z_{b-1}^*, z_b)$, then $H_b(z_b) = 0$ is a system of linear equations with the unique root z_b^* since A_b is lower-triangular with negative elements on the diagonal. Now let us show that all components of z_b^* are positive. Recall that all elements under the diagonal of A_b are non-negative and each (except first) row has at least one positive element outside the diagonal (if a tree is not the smallest possible, we could remove one vertex with its children from it to make it smaller). All components of $Y_b(z_{b-1}^*, \rho_d)$ are non-negative as well. Hence it is enough to show that the first element of Y_b is positive. It follows from the fact that the smallest max-admissible tree of depth b (which corresponds to the first coordinate of z_b) could be obtained by drawing edges from a new vertex to the smallest max-admissible trees of depth $b-1$ and the first coordinate of z_{b-1}^* is positive by the induction hypothesis.

Let us consider the vector $W_b(n) = (Z_1(n), \dots, Z_b(n))$. We get that

$$\mathbb{E}(W_b(n+1) - W_b(n) \mid \mathcal{F}_n) = \frac{1}{n+1}\left(F_1 + O\left(\frac{1}{n}\right), \dots, F_b + O\left(\frac{1}{n}\right)\right).$$

The derivative matrix of function $(F_1, \dots, F_b)(z_1, \dots, z_b)$ is of the following form. Around the diagonal, it has b blocks: the i-th block is the derivative matrix of F_i with respect to z_i. For $i > 1$, the i-th block is a lower-triangular matrix (since $F_i = A_i z_i - z_i + Y_i$) with diagonal elements at most -1. The block that corresponds to $i = 1$ was studied in Section 3 and has characteristic polynomial $P(\lambda)$ with the biggest root -1. Since each F_i depends only on $z_1, \dots, z_i$, all elements above the blocks are 0. Therefore the highest eigenvalue of the derivative matrix of $(F_1, \dots, F_b)$ is -1 (for all possible values of the process). Hence $W_b(n)$ satisfies condition A2 of Theorem 2.1. Since functions $(F_1, \dots, F_b)$ have second-order derivatives, condition A1 is satisfied as well. To check condition A3

[8] For a sequence of r-dimensional vectors $a_n \in \mathbb{R}^r$, $n \in \mathbb{N}$, where $r \in \mathbb{N}$ does not depend on n, we write $a_n = O(1/n)$, if there exists a constant $C > 0$ such that, for all n, $\|a_n\| < C/n$. Here, $\|a_n\|$ is the ℓ_2-norm of a_n.

note that if we take

$$E_{n+1} = (n+1)\Big(W_b(n+1) - \mathbb{E}(W_b(n+1) \mid \mathcal{F}_n)\Big),$$

then

$$\begin{aligned} R_{n+1} :&= (n+1)(W_b(n+1) - W_b(n)) - (F_1, \dots, F_b) - E_{n+1} \\ &= (n+1)\mathbb{E}(W_b(n+1) - W_b(n) \mid \mathcal{F}_n) - (F_1, \dots, F_b) = O\left(\frac{1}{n}\right) \quad \text{a.s.} \end{aligned}$$

and

$$|E_{n+1}| \le (n+1)|W_b(n+1) - W_b(n)| + (n+1)|\mathbb{E}(W_b(n+1) - W_b(n) \mid \mathcal{F}_n)| \le C$$

for some constant C since the number of trees of depth b that could be impacted by the vertex $n+1$ is bounded from above by a constant, which results in condition A3. Therefore, due to Theorem 2.1 $W_b(n)$ converges a.s. to $(z_1^*, \dots, z_b^*)$ with the rate $o(n^{-1/2+\delta})$ for any $\delta > 0$ a.s. □

7 Number of cycles: Lower bound

By Lemma 6.1, recall that for any max-admissible rooted tree T of depth $r-1$ there exists $\rho_T > 0$, such that

$$N_T(n) = \rho_T n + o(n^{2/3}) \quad \text{a.s.} \tag{7.1}$$

Let $\mathcal{T}_{r-1}$ be the set of all max-admissible trees T of depth $r-1$, such that the root of T and at least one vertex at distance $r-2$ from the root have degrees less than d each. Note that this set is not empty. For every tree $T \in \mathcal{T}_{r-1}$, its root R and its vertex u of degree less than d and at distance exactly $r-2$ from R, the addition of edges $\{n+1, R\}$ and $\{n+1, u\}$ creates an r-cycle. The number of such pairs $\{R, u\}$ over all $T \in \mathcal{T}_{r-1}$ is at least $\frac{1}{2}\sum_{T \in \mathcal{T}_{r-1}} N_T(n)$ since each pair is counted at most twice. Then, since $m \ge 2$ and due to (7.1), the probability to draw a cycle of length r at step $n+1$ (subject to G_n) is at least

$$\sum_{T \in \mathcal{T}_{r-1}} N_T(n) \frac{m(m-1)}{2(n - X_d(n))^2} \ge \frac{1}{n} \sum_{T \in \mathcal{T}} \rho_T - o(n^{-4/3}) \quad \text{a.s.}$$

Therefore, there exist $p > 0$, $n_0 \in \mathbb{N}$, and independent Bernoulli random variables $\zeta_n \sim \mathrm{Bern}(p/n)$, $n \ge n_0$, such that a.s., for all $n \ge n_0$, the increase in the number of cycles at step $n+1$ is not less than ζ_n. For any n_0 due to Lemma 4.2 w.h.p. all vertices in the r-neighbourhood of $[n_0]$ have degrees equal to d, and, hence, w.h.p. if a cycle arises at step $n+1$, then it entirely belongs to $[n+1] \setminus [n_0]$. Due to Lemma 4.1 and the approximation by Bernoulli random variables ζ_n, there are constants $c, C, \delta > 0$, such that the number of r-cycles in $[n] \setminus [n_0]$ exceeds $c \ln n$ with probability at least $1 - Cn^{-\delta}$. Therefore, we get the following result.

Lemma 7.1. *For any s, r and n_0 w.h.p. there are at least s cycles of length r that are entirely in $[n] \setminus [n_0]$.*

8 Number of unicyclic graphs

Let us recall that a graph U is *unicyclic* if it is connected and contains exactly one cycle. In other words, a unicyclic graph comprises a cycle (of length ℓ) with disjoint trees growing from this cycle (we assume that all trees have the same depth k; ℓ and k are fixed for the rest of the section). Let U be a max-admissible unicyclic (maximality and max-admissibility in the case of unicyclic graphs are defined exactly in the same way as for trees) graph. We say that a maximal unicyclic subgraph of G_n *has type* U if it is isomorphic to U. We have one specific type U_0 of unicyclic graphs with all non-leaf vertices having degree d. Let us call such unicyclic graphs *complete*. As above, $N_U(n)$ is the number of maximal subgraphs in G_n isomorphic to U. Let us consider the vector $Z(n) = (N_{U_i}(n))_{i=1,\ldots,K}$, where U_i are all non-complete unicycle graphs of *depth* k (i.e. the depth of trees growing from the cycle) comprising an ℓ-cycle, ordered from the smallest to the largest (the linear order on unicyclic graphs could be defined in the same way as on rooted trees), and $K = K(k,\ell)$ is the number of unicyclic graphs of such kind. Process $Z(n)$ takes values in $\mathbb{Z}_+^K$. Note that the complete unicyclic graph U_0 could only be obtained by adding a leaf (since the degree of a new vertex equals m) to a unique non-leaf vertex of U_K with degree less than d. In this section, we prove that $Z(n)$ has a limiting probability distribution.

Lemma 8.1. *For any* $i_1,\ldots,i_K$ *there exists a constant* $c = c(i_1,\ldots,i_K)$ *such that*

$$\mathbb{P}(N_{U_1}(n) = i_1,\ldots,N_{U_K}(n) = i_K) \to c$$

as $n\to\infty$*, and* $\sum_{i_1,\ldots,i_K\in\mathbb{Z}_+} c(i_1,\ldots,i_k) = 1$*. Moreover for any* n_0

$$\mathbb{P}(N_{U_0} > n_0) \to 1$$

as $n\to\infty$*.*

Proof. For a fixed max-admissible unicyclic graph U, at time $n+1$ the value of N_U may change due to the following reasons (similar to the changing of the number of rooted trees from the previous section).

- A new graph may be created by drawing 2 edges from the vertex $n+1$ to a single tree of a certain type (recall that by Lemma 3.1, the probability to draw an edge to a given vertex (subject to G_n) equals $\frac{1}{(1-\rho_d)n} + o(n^{-4/3})$ a.s.), and the rest of the edges to roots of “disjoint” (without common non-leaf vertices) trees of certain types. By Lemma 6.1, for a max-admissible tree T, we have $N_T = \rho_T n + \theta_T(n)n^{2/3}$, where, for every C, $\max_{T:\,|V(T)|\le C}\theta_T(n)\to 0$ a.s. Hence, in the same way as in the previous section, the conditional probability of creating a unicyclic graph of type U this way (given G_n) equals $\frac{c_U}{n} + o(n^{-4/3})$ a.s. for some constant $c_U \ge 0$.
- A U-isomorphic graph may be created from a fixed smaller unicyclic subgraph H of the type U', if the vertex $n+1$ sends an edge to a non-leaf vertex of H and the rest of the edges to roots of “disjoint” trees of certain fixed types in a way that H becomes of type U (i.e. the maximal subgraph comprising the same cycle and having the same depth as H becomes of type U). The conditional probability of creating a maximal subgraph of type U in this way (given H) equals $\frac{c_{U',U}}{n} + o(n^{-4/3})$ for some constant $c_{U',U}\ge 0$.

If U has at least one non-leaf vertex of degree less than d, the previous procedure could reduce $N_U(n)$ by drawing an edge to a unicyclic graph of the type U. Once a maximal unicyclic subgraph becomes complete, it never changes its type.

Note that the conditional probability (given G_n) to perform more than one of such operations (maybe for different types of U) at the same time equals $O\left(\frac{1}{n^2}\right)$ a.s. We prove the existence of a limiting probability distribution for $Z(n)$ by considering an auxiliary process which is defined below.

Let us consider a Markov chain $S(n) = (S_1(n),\ldots,S_K(n))$ on $\mathbb{Z}_+^K$ (see, e.g., [8, Chapter 6] for more details on Markov chains and corresponding terminology) with transition probabilities (we denote $c_i := c_{U_i}$, $c_{j,i} := c_{U_j,U_i}$ for brevity)

- for $i \in [K]$,

$$\mathbb{P}(S_i(n+1) = S_i(n) + 1,\, S_j(n+1) = S_j(n), j \neq i) = \frac{c_i}{n};$$

- for $1 \le j < i \le K$,

$$\mathbb{P}(S_i(n+1) = S_i(n) + 1,\, S_j(n+1) = S_j(n) - 1,\, S_k(n+1) = S_k(n), k \neq i, j) = \frac{c_{j,i} S_j}{n};$$

- $c_{K,0} = 1/(1-\rho_d)$ and

$$\mathbb{P}(S_K(n+1) - S_K(n) = -1,\, S_j(n+1) = S_j(n), j \neq K) = \frac{c_{K,0} S_K}{n};$$

- $$\mathbb{P}(\forall i\, S_i(n+1) = S_i(n)) = 1 - \sum_{i=1}^{K} \frac{c_i}{n} - \sum_{1 \le j < i \le K} \frac{c_{j,i} S_j}{n} - \frac{c_{K,0} S_K}{n}.$$

Since sums of error terms $o\left(n^{-4/3}\right)$ and $O\left(\frac{1}{n^2}\right)$ converge, such terms would not impact process $Z(n)$ after some random moment N, and hence the existence of the limiting probability distribution for $Z(n)$ follows from its existence for $S(n)$ for any initial distribution.

Note that $c_1 \neq 0$, $C_{K,0} \neq 0$ and from the definition of $c_{U,U'}$ and the ordering, it follows that for any i, j strictly between 1 and K, we get that

- there are $1 = i_1 < \ldots < i_t = i$, such that $c_{i_s, i_{s+1}} \neq 0$ for all $s \in [t-1]$,
- there are $j = j_1 < \ldots < j_p = K$, such that $c_{i_s, i_{s+1}} \neq 0$ for all $s \in [p-1]$.

This implies that $S(n)$ is aperiodic and irreducible. Note that $S(n)$ is not time-homogeneous. Let us consider a random walk $S'(t)$ on $\mathbb{Z}_+^K$ that reflects only those moves of $S(n)$ when it changes its state (i.e. for every t, $S'(t) := S(n_t)$, where n_t is the t-th moment n such that $S(n) \neq S(n-1)$). Since $\frac{c}{n}$, $c \neq 0$, forms a divergent series, by Borel–Cantelli lemma, all coordinates of S change infinitely many times a.s., so S' is well defined. Also, since the conditional probability (given $S(n-1) = x$) to change the state at time n is $\frac{c}{n-1}$, where c depends only on x, we get that the conditional probability that the state at time n becomes y (for a fixed $y \neq x$), subject to $S_n = x$ and the event that the state is changed, does not depend on n, and only depends on x and y. Thus, $S'(t)$ is time-homogeneous and its transition probabilities are given by

- $\mathbb{P}(S'_i(t+1) = S'_i(t) + 1,\, S'_j(t+1) = S'_j(t), j \neq i) = \frac{c_i}{D(S'(t))}$,
- $\mathbb{P}(S'_i(t+1) = S'_i(t) + 1,\, S'_j(t+1) = S'_j(t) - 1,\, S'_k(t+1) = S_k(t), k \neq i, j) = \frac{c_{j,i} S'_j(t)}{D(S'(t))}$,
- $\mathbb{P}(S'_K(t+1) - S'_K(t) = -1,\, S'_j(t+1) = S'_j(t), j \neq K) = \frac{c_{K,0} S'_K(t)}{D(S'(t))}$,

where

$$D(S'(t)) = \sum_{i=1}^{K} c_i + \sum_{1 \le j < i \le K} c_{j,i} S'_j(t) + c_{K,0} S'_K(t).$$

Let us consider $S'_1(t)$. There are constants c_- and c_+, such that

$$\mathbb{P}(S'_1(t+1) - S'_1(t) = -1 \mid S'(t)) \ge c_- \frac{S'_1(t)}{|S'(t)| + 1},$$
$$\mathbb{P}(S'_1(t+1) - S'_1(t) = 1 \mid S'(t)) \le c_+ \frac{1}{|S'(t)| + 1}.$$

Hence, for large enough $S_1'(t)$ (i.e. with $S_1'(t) \geq N$ for some $N \in \mathbb{N}$),

$$\mathbb{E}\Big(S_1'(t+1) - S_1'(t) \mid S'(t),\, S_1'(t) > N,\, S_1'(t+1) \neq S_1'(t)\Big) < C < 0$$

for some constant C. Therefore $S_1'(t)$ is positively persistent. Consider $W_i(t) = (S_1'(t), \dots, S_i'(t))$, $i = 1, \dots, K$. Let us assume that $W_i(t)$ is positively persistent, and prove that the same is true for $W_{i+1}(t)$. Note that there are constants $C_1, C_2 > 0$, such that

$$\mathbb{P}(S_{i+1}'(t+1) - S_{i+1}'(t) = 1 \mid S'(t)) < C_1 \frac{|W_{i+1}(t)| + 1}{|S'(t)| + 1},$$
$$\mathbb{P}(S_{i+1}'(t+1) - S_{i+1}'(t) = -1 \mid S'(t)) > C_2 \frac{S_{i+1}'(t)}{|S'(t)| + 1}.$$

Let $N > \frac{C_1}{C_2}$, $N \in \mathbb{N}$. We get

$$\mathbb{E}\Big(S_{i+1}'(t+1) - S_i'(t) \mid S'(t),\, S_{i+1}'(t) > N|W_{i+1}(t)|,\, S_{i+1}'(t+1) \neq S_{i+1}'(t)\Big) < C < 0$$

for some constant C. Hence, the probability $\mathbb{P}(S_{i+1}'(t+t') \leq N|W_{i+1}(t+t')| \mid S_{i+1}'(t) \leq N|W_{i+1}(t)|)$ is bounded away from 0 as $t' \to \infty$. Since $W_i(t)$ is positively persistent, it implies that $W_{i+1}(t) = (W_i(t), S_{i+1}'(t))$ is positively persistent as well.

As result, for each state $s = (s_1, \dots, s_K)$ probabilities $\mathbb{P}(S'(t+t') = s \mid S'(t) = s)$ (as $t' \to \infty$) are bounded away from 0. Hence, the same is true for probabilities $\mathbb{P}(S(n_{t+t'}) = s \mid S(n_t) = s)$ as $t' \to \infty$, and for $\mathbb{P}(S(t+t') = s \mid S(t) = s)$ as well. Therefore, there exists a limiting distribution for $S(n)$ (and for $Z(n)$).

The second part of Lemma 8.1 follows from Lemma 7.1 and the existence of the limiting distribution for $Z(n)$. □

9 Convergence laws

Fix $R \in \mathbb{N}$ and set $a = 3^R$. For $r \in \mathbb{N}$, let us call a unicyclic graph comprising a cycle of length at most r and trees of depth exactly r an *r-graph*. The cycle of a unicyclic graph is called its *kernel*. An r-graph is *complete* if all its vertices have degrees either 1 or d, and all its trees are perfect and of the same depth.

Below we define graph properties Q1 and Q2 that imply the existence of a winning strategy of Duplicator. Consider some integer numbers $n > N_0 > n_0$. We say that a graph G with maximum degree d on $[n]$ has *property* Q1, if

1. any two cycles of length at most a with vertices outside of $[n_0]$ are at distance at least $3a$ from each other;
2. any vertex outside of $[N_0]$ is at distance at least $3a$ from $[n_0]$;
3. any vertex from $[N_0]$ has degree d;
4. for any max-admissible tree T of depth at most a, there are at least R maximal subgraphs in G isomorphic to T at distance at least a from $[N_0]$ and each other, and the same is true for any complete a-graph U.

Now, assume that $n_1 > n_2 > N_0$, and G^1, G^2 are graphs on $[n_1]$ and $[n_2]$ respectively such that $G^1|_{[N_0]} = G^2|_{[N_0]}$. We say that the pair of graphs (G^1, G^2) has *property* Q2, if for any non-complete max-admissible a-graph U,

- either the numbers of maximal subgraphs in G^i isomorphic to U are equal for $i \in \{1, 2\}$,

- or, in both graphs, there are at least R maximal copies of U that are distance at least a from $[N_0]$ and each other.

Note that, if G^1, G^2 have maximum degree d, property Q1, and the pair (G^1, G^2) has property Q2, then, for any positive integer $\delta \le 2^R$, the numbers of maximal subgraphs in G^i isomorphic to a given non-complete max-admissible a-graph U with all vertices at distance at least δ from $[n_0]$ are equal for $i \in \{1,2\}$.

Lemma 9.1. *If both graphs G^1, G^2 have maximum degree d, have property* Q1*, and the pair (G^1, G^2) has property* Q2*, then Duplicator has a winning strategy in the Ehrenfeucht-Fraïssé game on graphs G^1, G^2 in R rounds.*

Proof. Let us define the winning strategy of Duplicator. For a vertex v and $r \in \mathbb{N}$, let $B_r(v)$ be the r-neighbourhood of v (i.e., the closed ball in the graph metric of radius r and the center at v). In the same way, for a set of vertices U, $B_r(U) = \cup_{v \in U} B_r(v)$ is the r-neighbourhood of U. Note that we omit a reference to a graph in the notation for these balls — each time we use the notation, the host graph would be clear from the context. For every round $i \in [R]$, we denote by $x_1, \dots, x_i$ and $y_1, \dots, y_i$ the vertices chosen in graphs where Spoiler and Duplicator made the i-th move respectively (say, G^1 and G^2 respectively). For x_j and y_j, $j \in [i]$, let us denote by X_j and Y_j the unions of sets of vertices of all kernels of non-complete 2^R-graphs in G^1 and G^2 respectively such that these kernels are completely outside $[n_0]$, and are at distance at most 2^{R-j+1} from x_j and y_j respectively. Since G^1, G^2 have property Q1, each of the sets X_j, Y_j comprises at most 1 cycle. For a set $A \subset V(G^1)$ and a set $B \subset V(G^2)$, we say that they are *i-equivalent*, and write $A \equiv_i B$, if the following conditions are fulfilled:

- the sets of $j \in [i]$ such that the respective vertex x_j (y_j) belongs to A (B) are equal,
- there exists an isomorphism $\varphi : G^1|_A \to G^2|_B$ of the induced subgraphs on A and B that maps x_j to y_j for all j such that $x_j \in A$ and preserves (in both directions) all kernels that are outside $[n_0]$ of non-complete 2^R-graphs,
- if x_i is at distance at most 2^{R-i+1} from $[n_0]$, then φ can be extended to an isomorphism of $G^1|_{A \cup [N_0]}$ and $G^2|_{B \cup [N_0]}$ that maps every vertex of $[N_0]$ to itself.

We define the strategy by induction on the number of rounds that have been just played. Fix $i \in [R]$ and assume that, in round i, Spoiler makes a move in G_1 (without loss of generality — if the move was done in G_2, then the strategy is exactly the same), and that for all $j \le i-1$, $B_{2^{R-j+1}}(X_j \cup \{x_j\}) \equiv_j B_{2^{R-j+1}}(Y_j \cup \{y_j\})$. Note that, if $i = 1$, then there are no additional requirements on the graphs. We also note that, due to the assumption, the map sending x_j to y_j, $j \in [i-1]$, is an isomorphism of $G_1|_{\{x_1,\dots,x_{i-1}\}}$ and $G_2|_{\{y_1,\dots,y_{i-1}\}}$. So if we succeed with the induction step, then we get the winning strategy for Duplicator.

1. If $d(x_i, [n_0]) \le 2^{R-i+1}$ then Duplicator chooses $y_i = x_i$. We need to check that $B_{2^{R-i+1}}(X_i \cup \{x_i\}) \equiv_j B_{2^{R-i+1}}(Y_i \cup \{y_i\})$. Due to property Q1, every cycle of length at most a which is completely outside $[N_0]$ is far from $x_i = y_i$, and so $X_i = Y_i = \varnothing$. Also, the balls $B_{2^{R-i+1}}(x_i)$ and $B_{2^{R-i+1}}(y_i)$ are equal and lie entirely in $[N_0]$. If there is $j < i$ such that $x_j \in B_{2^{R-i+1}}(x_i)$, then, by the induction hypothesis, $B_{2^{R-j+1}}(x_j) \equiv_j B_{2^{R-j+1}}(y_j)$, implying that $B_{2^{R-j+1}}(x_j) = B_{2^{R-j+1}}(y_j)$ due to the third condition in the definition of the relation $\equiv_j$. Then $x_j = y_j$ belongs to $B_{2^{R-i+1}}(x_i) = B_{2^{R-i+1}}(y_i)$. The relation $B_{2^{R-i+1}}(X_i \cup \{x_i\}) \equiv_j B_{2^{R-i+1}}(Y_i \cup \{y_i\})$ follows. If there are no such j, then there is also no j such that $y_j \in B_{2^{R-i+1}}(y_i)$, and the relation is immediate.

2. Assume that $d(x_i, [n_0]) > 2^{R-i+1}$ and that there exists $j < i$ such that $x_j \in B_{2^{R-i+1}}(X_i \cup \{x_i\})$. Let $j < i$ be the biggest such round. Then $B_{2^{R-i+1}}(X_i \cup \{x_i\}) \subset B_{2^{R-j+1}}(X_j \cup \{x_j\})$, and either $X_i = X_j$ or $X_i = \varnothing$. Let $\mathcal{J}$ be the set of all $j' < j$ such that $x_{j'} \in B_{2^{R-j+1}}(X_j \cup \{x_j\})$. Since $B_{2^{R-j+1}}(X_j \cup \{x_j\}) \equiv_j B_{2^{R-j+1}}(Y_j \cup \{y_j\})$ by the induction hypothesis, we may find an isomorphism $\varphi : G^1|_{B_{2^{R-j+1}}(X_j \cup \{x_j\})} \to G^2|_{B_{2^{R-j+1}}(Y_j \cup \{y_j\})}$ such that $\varphi(x_{j'}) = y'_j$ for all $j' \in \mathcal{J}$, there are no $y_{j'} \in B_{2^{R-j+1}}(Y_j \cup \{y_j\})$ for $j' \in [j-1] \setminus \mathcal{J}$, and $\varphi(X_j) = Y_j$. Duplicator chooses $y_i = \varphi(x_i)$. It is obvious that $\varphi' := \varphi|_{B_{2^{R-i+1}}(X_i \cup \{x_i\})}$ is the desired isomorphism that insures that $B_{2^{R-i+1}}(X_i \cup \{x_i\}) \equiv_i B_{2^{R-i+1}}(Y_i \cup \{y_i\})$.
3. Finally, we assume that $d(x_i, [n_0]) > 2^{R-i+1}$ and there are no $j < i$ such that $x_j \in B_{2^{R-i+1}}(X_i \cup \{x_i\})$. If $X_i \neq \varnothing$, then let U_1 be the unique maximal 2^R-graph with the kernel X_i. Due to the observation after the definition of property Q2 and due to property Q1, in the other graph there exists a maximal 2^R-graph U_2 isomorphic to U_1 such that

   - the kernel of U_2 is either at least at the same distance from $[n_0]$ as the kernel of U_1 from $[n_0]$, or is at distance at least 2^R from $[n_0]$;
   - U_2 is at distance at least $2a$ from the cycle of length at most 2^R that meets the 2^{R-j+1}-neighbourhood of y_j for every $j < i$;
   - if, for some $j < i$, the 2^{R-j+1}-neighbourhood of y_j does not meet a cycle of length at most 2^R, then the cycle of U_2 is at distance greater than 2^{R-j+1} from y_j.

   Consider an isomorphism $\varphi : U_1 \to U_2$, that sends the vertex x_i to a vertex which is at distance more than 2^{R-i+1} from $[n_0]$ and set $y_i = \varphi(x_i)$. The relation $B_{2^{R-i+1}}(X_i \cup \{x_i\}) \equiv_i B_{2^{R-i+1}}(Y_i \cup \{y_i\})$ is straightforward.

   Finally, if $X_i = \varnothing$, then the existence of a good choice of y_i follows from property Q1.(4). Indeed, there are only two options: 1) $B_{2^{R-i+1}}$ is a maximal subtree, and then there is an isomorphic maximal subtree in the other graph which is at distance at least a from the neighbourhoods of all y_j, $j < i$; 2) there is a complete maximal unicyclic graph comprising a cycle of length at most R which is at distance at most 2^{R-i+1} from x_i, and then there is an isomorphic maximal unicyclic subgraph in the other graph which is at distance at least a from the neighbourhoods of all y_j, $j < i$. The choice of y_i is straightforward. □

Theorem 1.3 follows from Corollary 1.2 and the following lemma.

Lemma 9.2. *For any $R \in \mathbb{N}$ and any $\varepsilon > 0$ there are $N_0 > n_0$, graph families $\mathcal{A}_i$, $i \in [M]$, and numbers $p_i > 0$, $i \in [M]$, $\sum_{i=1}^{M} p_i > 1 - \varepsilon$, such that*

- *all graphs in $\sqcup_{i \in [M]} \mathcal{A}_i$ have property* Q1*;*
- *if $n_1 > n_2 > N_0$ and graphs $G^1 \supset G^2$ on $[n_1]$ and $[n_2]$ respectively belong to the same family $\mathcal{A}_i$, then the pair (G^1, G^2) has property* Q2*;*
- *for every $i \in [M]$, $\lim_{n\to\infty} \mathbb{P}(G_n \in \mathcal{A}_i) = p_i$.*

Proof. Fix $R \in \mathbb{N}$ and $\varepsilon > 0$. By Lemmas 4.2, 5.2, 6.1, 8.1, there exist $N > N_0 > n_0$ such that with probability at least $1 - \varepsilon$, for all $n \geq N$, G_n has maximum degree d and property Q1, and all its vertices that are at distance at most $4a$ from $[N_0]$ have degree d. We let $\mathcal{A}$ to be the union over all $n \geq N$ of the families of graphs G on $[n]$ such that the maximum degree of G equals d, G has property Q1, and all the vertices of G at distance at most $4a$ from $[N_0]$ have degree d. It remains to partition $\mathcal{A} = \sqcup_{i=1}^{M} \mathcal{A}_i$ in an appropriate way.

Let $\mathcal{M}$ be all K-tuples (we refer to Section 8 to recall the definition of K) of non-negative integers that are at most R, and set $M := |\mathcal{M}| M_0$, where M_0 is the number of all admissible maximal subgraphs of G_N on $[N_0]$. For each $i \in [M]$, the respective tuple

$\mathbf{m}_i \in \mathcal{M}$, and the respective admissible H on $[N_0]$, let $\mathcal{A}_i \subset \mathcal{A}$ be the set of all graphs G from $\mathcal{A}$ such that $G|_{[N_0]} = H$ and $\tilde{Z}(G) = \mathbf{m}_i$, where $\tilde{Z}(G)$ consists of $\tilde{Z}_j = \min\{R, N_{U_j}\}$. By Lemma 8.1, for every $i \in [M]$, there exists $p_i := \lim_{n\to\infty} \mathbb{P}(G_n \in \mathcal{A}_i)$. Note that, for every graph from $\mathcal{A}$, every its maximal unicyclic subgraph with a cycle of length at most a and depth a that is at distance at most a from $[N_0]$, is complete (since all the vertices at distance at most $4a$ from $[N_0]$ have degree d). Property Q2 follows. □

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