Article:

Delaney, Laura and Thijssen, Jacco (2026) An alternative perspective on the classical solution to an optimal stopping problem with a spectrally negative Levy process. European Journal of Operational Research. ISSN: 0377-2217

https://doi.org/10.1016/j.ejor.2026.01.047

European Journal of Operational Research xxx (xxxx) xxx

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/eor

Innovative Applications of O.R.

An alternative perspective on the classical solution to an optimal stopping problem with a spectrally negative Lévy process

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A R T I C L E   I N F O

Keywords:
Decision analysis
Real put option
Spectrally negative Lévy process
Optimal stopping
Viscosity solutions.

A B S T R A C T

This study considers an optimal stopping problem for a real put option in which the underlying dynamics characterizing the payoff uncertainty are modeled as a spectrally negative Lévy process. It demonstrates that the solution, obtained using the conventional "value-matching" and "smooth-pasting" conditions, may not be optimal when the jumps are large in expectation and/or frequent. Specifically, at that threshold, the value of waiting can exceed the value of stopping.

The objective of this study is to emphasize this potential limitation in the classical solution for optimal stopping problems of the described type, and to propose an alternative solution approach. It proves that our proposed solution is a viscosity sub-solution to the Hamilton–Jacobi–Bellman equation.

1. Introduction

Over the past few decades, optimal stopping theory has enabled decision-makers in operations research to understand how uncertainty over future outcomes from irreversible decisions affects the optimal timing of actions (see (Trigeorgis & Tsekrekos, 2018) for a review). Generally, the standard approach usilized to solve such problems assumes that the underlying dynamics characterizing uncertainty are modeled as a geometric Brownian motion (GBM), which implicitly assumes that the sample paths are continuous. The primary reason for the GBM's popularity lies in its analytical tractability. Closed-form solutions are available for many problems, allowing detailed analyses of the comparative static properties of the model.

Often, however, it is more realistic to model the underlying stochastic dynamics of a state variable as a process that makes random discrete jumps (see, for example, Armerin, 2023). A sudden price drop due to the entry of a new competitor in the market is one such example, and a very turbulent economic environment is another. Naturally, this type of model is less tractable; however, in the case of only downward jumps, where the process is modeled as a spectrally negative Lévy process, a closed-form solution to the optimal stopping problem for a real call option can often be obtained almost as readily as in the GBM case (see, for example, Øksendal & Sulem, 2005, Theorem 2.2). Beyond its tractability, a justification for assuming a model that captures only downside risk is the "bad news principle" introduced by Bernanke (1983) which essentially implies that only bad news affects investment. Therefore, the spectrally negative Lévy model has attracted increasing attention in the context of optimal stopping (see, for example, Alvarez & Rakkolainen, 2010; Boyarchenko & Levendorskiĭ, 2002; Egami & Yamazaki, 2014; Thijssen, 2015; Yamazaki, 2015).

However, many real-world optimal stopping problems concern real put options, and the solution obtained through the standard optimal stopping approach may not be optimal or even appropriate to apply. This occurs when the expected jump size is large and/or the jumps are frequent (for example, sudden reductions in the economics of operating a coal-fired power plant due to policy change (Nagy et al., 2021), or when a patient's clinical decline is rapid and extreme (Everitt et al., 2011). In particular, the classical trigger solution can lie *above* the current period's breakeven level, implying that stopping will occur when the value of waiting is higher, which can never represent an optimal solution.

Kyprianou and Surya (2007) demonstrated that when the underlying dynamics characterizing uncertainty over asset values are modeled as a spectrally negative Lévy process, applying the standard approach to solve an optimal bankruptcy level is not always appropriate. Specifically, the smooth-pasting condition requiring the value function to be differentiable at the threshold fails to hold. However, based on the fluctuation theory of spectrally negative Lévy processes (see Bertoin, 1996 and Kyprianou, 2006), they addressed this issue by employing scale functions to examine the problem analytically. Egami and Yamazaki (2014) derived a first-order condition for maximization over threshold strategies by expressing the value function in terms of the scale function,

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https://doi.org/10.1016/j.ejor.2026.01.047
Received 21 October 2025; Accepted 27 January 2026
Available online 5 February 2026

while Surya and Yamazaki (2014) incorporated scale effects into their framework to obtain an optimal capital structure that balances minimizing bankruptcy costs and maximizing tax benefits. Although the result in the present study is similar, the modeling framework differs substantially from theirs.

The contribution of this study is threefold. (i) It derives a condition under which the classical approach to solving an optimal stopping problem for a real put option is not appropriate. This situation can arise when the process characterizing the uncertainty is modeled as a spectrally negative Lévy process with large and/or frequent jumps. Such a specific condition is not defined in, for example, Kyprianou and Surya (2007). (ii) The second contribution is the provision of a practical alternative approach, that does not rely on fluctuation theory or scale functions, for analytically addressing such optimal stopping problems. In particular, another solution is proposed for the optimal stopping problem when the classical solution is not suitable. This solution is also a trigger and corresponds to the level at which the per-period net payoff from continuing is equal to the per-period expected benefit from stopping. Unlike the trigger obtained through the classical approach, the value functions do not meet smoothly at that point. However, it is proven that this constitutes a viscosity sub-solution to the optimal stopping problem (see Øksendal & Sulem, 2005 for a discussion of viscosity solutions to optimal stopping problems). The advantage of this solution, compared to the scale function approach, is that it is readily obtained, is practically meaningful to decision-makers across disciplines, and does not require a technically complex specification of the value function. In addition, it is closely aligned with the classical solution which is shown to represent the appropriate trigger when jumps are modest in size and frequency. (iii) The third contribution is the provision of a multidisciplinary perspective on the applicability of the results in operations research beyond the fields of economics and management, where optimal stopping models are most commonly applied.

This study is organized such that the next section describes the model and provides some preliminary results. The main results are discussed in Section 3. Section 4 discusses the multidisciplinary applications of the model and results, and Section 5 concludes.

2. Problem set up and preliminary results

Consider a risk-neutral firm that has the option to replace an existing stream of cash flows with greater ones at a sunk cost. This can, for example, be the abandonment of production of some product if its profitability deteriorates significantly; i.e., it is a real put option. There is uncertainty about future cash flows which is modeled on a probability space $(\Omega, \mathcal{F}, \mathsf{P}_x)$, with expectation operator E_x. The dynamic revelation of information is modeled by a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$. The set of stopping times with respect to $\mathbf{F}$ is denoted by $\mathbf{T}$. The stochastic evolution of a state variable that affects the stream of cash flows is modeled by a geometric Lévy process $X = (X_t)_{t\geq 0}$, adapted to $\mathbf{F}$, which, on $(\Omega, \mathcal{F}, \mathsf{P}_x)$, is taken to be the unique strong solution to the stochastic differential equation (SDE)

$$\frac{\mathrm{d}X_t}{X_{t-}} = \alpha \mathrm{d}t + \sigma \mathrm{d}B_t - \int_0^1 zN(\mathrm{d}t, \mathrm{d}z), \quad X_0 = x \quad \mathsf{P}_x-\text{ a.s.} \tag{1}$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion with $B_0 = 0$, P_x-a.s., and $N(\mathrm{d}t, \mathrm{d}z)$ is an independent Poisson random measure with Lévy measure $\lambda\nu(\mathrm{d}z)$. The parameters α (expected growth rate), $\sigma > 0$ (volatility), and $\lambda \geq 0$ (jump frequency) are assumed to be constant. It is important to note that X is *spectrally negative* so that only downward jumps occur.

The firm's problem is to choose a stopping time $\tau \in \mathbf{T}$ at which to abandon current production so that its net present value (NPV) from doing so is maximized. If $\rho > \alpha$ is the discount rate, $c > 0$ is the per-period production cost, and $K \geq 0$ is the cost of abandonment, the optimal stopping problem is as follows:

$$F^*(x) : = \sup_{\tau\in\mathbf{T}} \mathsf{E}_x\left[\int_0^\tau e^{-\rho t}(X_t - c)\mathrm{d}t - e^{-\rho\tau}K\right]$$

$$= L_0(x) + \mathsf{E}_x\left[e^{-\rho\tau^*}G\left(X_{\tau^*}\right)\right], \tag{2}$$

where

$$L_0(x) : = \mathsf{E}_x\left(\int_0^\infty e^{-\rho s}(X_s - c)\mathrm{d}s\right) = \frac{x}{\rho - \alpha + \lambda\mathsf{E}_\nu[z]} - \frac{c}{\rho}, \tag{3}$$

$$G(X_t) : = -L_0(X_t) - K, \tag{4}$$

and, by assumption, $c > \rho K$ ensuring the solution is well-defined.

The integro-differential operator $\mathscr{A}$ associated with X is defined by

$$\mathscr{A}\varphi(x) := \frac{1}{2}\sigma^2x^2\varphi''(x) + \alpha x\varphi'(x) + \lambda\int_0^1 \left\{\varphi(x - xz) - \varphi(x) + xz\varphi'(x)\right\}\nu(\mathrm{d}z) \tag{5}$$

for some function $\varphi \in C^2$.

Let $S \in \mathbb{R}_+$ be some fixed domain for which

$$F^* \text{ is continuous on its closure } \bar{S} \tag{6}$$

and whose boundary δS is *regular* for the process X; i.e.,

$$\mathbf{T} := \inf\{t > 0 | X \notin S\} = 0, \quad \mathsf{P}_x-\text{ a.s.}, \quad x \in \delta S. \tag{7}$$

Let the *continuation region* be defined by the set

$$C := \left\{ x \in S \mid F^*(x) > -K \right\} \subset S, \tag{8}$$

with

$$\tau_C := \inf\{ t \geq 0 \mid X \notin C \} \in \mathbf{T}$$

denoting the *first exit time* of X from C.

Provided F^* is sufficiently smooth and satisfies the following verification conditions,

$$\max\{\mathscr{A}F^*(x) - \rho F^*(x) + x - c, -K - F^*(x)\} = 0 \quad \forall x \in S \tag{9}$$

and

$$F^*(x) = -K \quad \forall x \in \delta S, \tag{10}$$

it is the unique solution to (2) and the optimal stopping time is $\tau^* = \tau_C$. This is a common result in the optimal stopping literature (see, for example, (Øksendal, 2005, Theorem 10.4.1) for a real call option).

Applying the standard optimal stopping approach of ensuring continuity and differentiability at $X_{\tau^*} := x^*$ (as outlined in, for example, Peskir & Shiryaev, 2006) to solve this specific problem yields the following solution to (2):[1]

$$F^*(x) = \begin{cases} L_0(x) - \left(\frac{x}{x^*}\right)^{\beta_2}\left(L_0(x^*) + K\right) & \text{for } x > x^* \\ -K & \text{for } x \leq x^* \end{cases}, \tag{11}$$

with

$$x^* = \frac{\beta_2}{\beta_2 - 1}(\rho - \gamma)\left(\frac{c - \rho K}{\rho}\right), \tag{12}$$

such that $\beta_2 < 0$ is the smaller root of the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + (\alpha + \lambda\bar{z})\beta - (\rho + \lambda) + \lambda\mathsf{E}_\nu[(1 - z)^\beta] = 0, \tag{13}$$

$$\gamma := \alpha - \lambda\bar{z}, \tag{14}$$

and $\bar{z} = E_\nu[z]$.

However, F^* will not always be the solution to the optimal stopping problem when X is spectrally negative.

Proposition 1. *The solution F^* to the optimal stopping problem (2) is given by (11) if, and only if,*

$$x^* < c - \rho K. \tag{15}$$

The condition may be violated if the jumps are large in expectation and/or frequent.

[1] Note that, in general, the solution to the optimal stopping problem with Lévy processes fails to be written as (11). However, since a specific problem *and* a specific type of process is being considered, our solution appears similar to the case where there are no Lévy jumps.

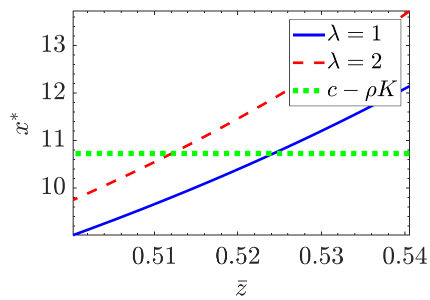

Fig. 1. The condition for the solution.

Proof. See Appendix A. □

To examine Proposition 1 numerically, it is assumed that the jumps follow a Beta distribution with mean $\bar{z} = a/(a+b)$ and

$$v'(z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} z^{a-1}(1-z)^{b-1}.$$

The results are plotted in Fig. 1 for the following parameter values: $K = 95$, $c = 15$, $r = 4.5\%$, $\alpha = 0.01$, $\sigma = 0.2$, $a = 1$ and $b \in [0.85, 1]$. It is shown that it can be the case that $x^* > c - \rho K$ when expected jump sizes are relatively large. In addition, this effect is stronger when jumps arrive more frequently (i.e., for higher values of λ). Therefore, in this example, based on Proposition 1, F^* given by (11) will not be the solution to the optimal stopping problem (2) when jumps are large in expectation and/or frequent in arrival.

In the proof of Proposition 1 (see Appendix A), it is demonstrated that the violation of condition (15) occurs not via the option effect (i.e., via β_2), but rather via the effect of the jumps on the NPV of payoffs from continuing operations; specifically, through their effect on γ. Large and/or frequent jumps decrease the NPV (4), indicating a higher x^* and earlier stopping. This implies that the volatility associated with large and frequent jumps excessively discounts this NPV, causing the firm to abandon production too early, specifically when the value of waiting exceeds the value of stopping. In other words, if the firm abandons at some $x^* > c - \rho K$, it does so at a point at which its cash flow exceeds its operating cost net of its opportunity cost from continuing. Therefore the presence of jumps erodes, to some degree, the value of waiting. However, the classical NPV criterion, which recommends stopping as soon as the present value of expected payoffs equals the cost of stopping, is not optimal either. This is because $x^* < x_{NPV}$ always holds, where x_{NPV} is the unique root of $G(x) = 0$, i.e. $x_{NPV} = (\rho - \gamma)(c/\rho - K)$.

3. Proposed solution to the optimal stopping problem

This section focuses on the case in which stopping at the trigger x^*, given by (12), is *not* a solution to the optimal stopping problem (2), specifically when (15) is violated. Based on the discussion above, rather than stopping at x^*, waiting until $X_t = c - \rho K$ appears the sensible approach. Therefore, the proposed solution to (2) is as follows:

$$F^*(x) = \begin{cases} L_0(x) - \left(\frac{x}{\bar{x}}\right)^{\beta_2}\left(L_0(\bar{x}) + K\right) & \text{for } x > \bar{x} \\ -K & \text{for } x \le \bar{x} \end{cases}, \tag{16}$$

where $\bar{x} := x^* \wedge (c - \rho K)$.

To clarify the reasoning, the value and payoff functions are depicted in Fig. 2 for $\lambda = 1$ and $\bar{x} = c - \rho K$ (while the other parameter values remain the same as those in Fig. 1). It is observed that F^* is not C^1 at the proposed trigger $\bar{x}$. Moreover, for some values of x just to the right of $\bar{x}$, $F^*(x) < -K$. This suggests that stopping should be optimal *before* $\bar{x}$ is reached.

Let $\varphi \in C^2$ denote the value of waiting function, where $\varphi(x) = F^*(x)$ for $x > \bar{x}$. It is the solution to the equation

$$\mathscr{A}\varphi(x) - \rho\varphi(x) + x - c = 0.$$

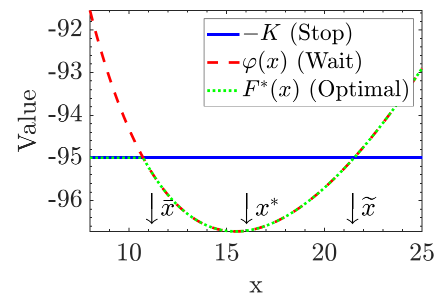

Fig. 2. Value Functions for $\bar{z} = 0.75$ and $\bar{x} = c - \rho K$.

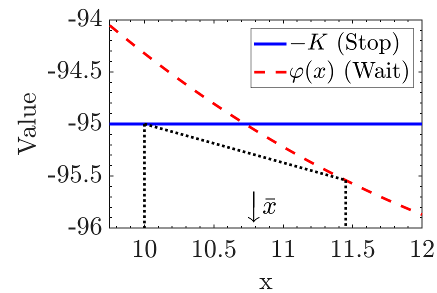

Fig. 3. Upward pointing (concave) kink.

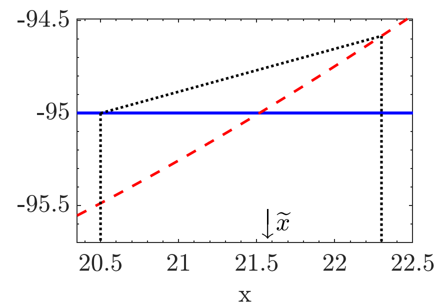

Fig. 4. Downward pointing (convex) kink.

The figure indicates that when the smooth-pasting condition is not applied, there is some other value of the state process $\tilde{x} > \bar{x}$ at which the values of waiting and stopping intersect; i.e., $\tilde{x}$ is such that $\varphi(\tilde{x}) = -K$. For values of x slightly to the left of $\tilde{x}$, $-K > F^*(x)$. This suggests $\tilde{x}$ ought to be the appropriate stopping trigger. However, by analysing Figs. 3 and 4, it is argued that waiting until $\bar{x}$ to stop is the better strategy.

The first step is to examine the point of intersection depicted in Fig. 4, which is a snapshot of Fig. 2 focused on $\tilde{x}$. If the firm stops at $\tilde{x}$, its payoff is $-K$. However, it could adopt an alternative strategy of continuing as follows: i) wait at $\tilde{x}$, (ii) continue to wait if the process moves upward in the next instant, and (iii) stop if it moves downward, either by a jump or a continuous decrease. This is represented on the plot by the black vertical dotted lines, and the expected value of this strategy is represented by the diagonal dotted black line connecting the two. The expected value exceeds $-K$, implying that the firm will be better off waiting at $\tilde{x}$.

Next, the point of intersection depicted in Fig. 3, which is focussed on $\bar{x} = c - \rho K$, is examined. If $\bar{x}$ is the trigger, then the strategy is to stop at $\bar{x}$, but not before. However, the expected value of the alternative strategy of continuing is below the value of stopping, which implies that the firm should stop producing at $\bar{x}$.

Since F^* is not C^1 at the proposed trigger $\bar{x}$, it does not satisfy the variational inequality (9) for all $x \in S$. The theory of viscosity solutions provides a means of addressing this issue (cf. Øksendal & Reikvam, 1998). The general idea of the theory is that the value function is set equal to some alternative function $\psi \in C^2$ at the point of non-

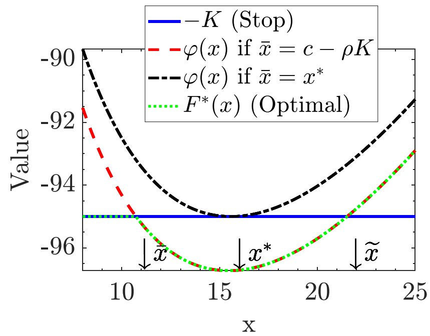

Fig. 5. Stopping at $\bar{x} = c - \rho K$.

differentiability. It is a viscosity solution if it satisfies the features of ψ' at that point (Achdou et al., 2022, Online Appendix D) because it will therefore also satisfy the standard verification conditions for optimal stopping problems (similar to Eqs. (9) and (10) in the problem).

By (Øksendal & Reikvam, 1998, Theorem 2.1), (6) and (7) ensure F^* is a possible candidate viscosity solution to (2). In Appendix B it is proven that (16) is a solution *in the viscosity sense* to (2). This means that it is a viscosity solution for $\bar{x} = x^*$, and a viscosity *sub*-solution for $\bar{x} = c - \rho K$.

First, the definition of these types of solutions is stated in the context of the problem (cf. Øksendal & Sulem, 2005, Chapter 9).

Definition 1 (Viscosity solution)**.** Let F^* be given by (16).

1. F^* is a *viscosity sub-solution* to (2) if, for all $\psi \in C^2(S)$ and all $x_0 \in S$ such that $\psi \geq F^*$ and $\psi(x_0) = F^*(x_0)$, it holds that

$$\max\{\mathscr{A}\psi(x_0) - \rho\psi(x_0) + x_0 - c, -K - F^*(x_0)\} \geq 0. \tag{17}$$

2. F^* is a *viscosity super-solution* to (2) if, for all $\psi \in C^2(S)$ and all $x_0 \in S$ such that $\psi \leq F^*$ and $\psi(x_0) = F^*(x_0)$, it holds that

$$\max\{\mathscr{A}\psi(x_0) - \rho\psi(x_0) + x_0 - c, -K - F^*(x_0)\} \leq 0. \tag{18}$$

3. F^* is a *viscosity solution* to (2) if it is both a viscosity sub-solution and a viscosity super-solution.

The following proposition can now be proven.

Proposition 2. *Let $\beta_2 < 0$ be the smaller of the roots of the characteristic function given by (13). Furthermore, let L_0 be given by (3), x^* by (12), and $\bar{x} := x^* \wedge (c - \rho K)$. Then*

$$F^*(x) = \begin{cases} -K & \text{if} x \leq \bar{x} \\ \left(\frac{x}{\bar{x}}\right)^{\beta_2}\left[-L_0(\bar{x}) - K\right] + L_0(x) & \text{if} x > \bar{x} \end{cases}, \tag{19}$$

is a solution in the viscosity sense to the optimal stopping problem (2).

Proof. See Appendix B. □

Typically, the trigger in the solution to an optimal stopping problem using classical techniques is the greatest lower bound on the state process at which the value of waiting exceeds that of stopping. However, it may not be the greatest lower bound when the expected jumps are large and/or frequent. Therefore, the problem can be rephrased as determining the optimal time to exercise a real put option such that the opportunity cost from doing so, i.e., $\varphi(x)$, is minimized whenever (15) is not satisfied. In Fig. 5, the value and payoff functions are depicted for stopping at x^* or at $c - \rho K$. Since $x^* > c - \rho K$ in this example, $\bar{x} = c - \rho K$ and it is optimal to stop there - i.e., $F^*(x)$ corresponds with $\varphi(x)|_{\bar{x}}$. Clearly this is below $\varphi(x)|_{x^*}$. If stopping were to occur at some $x \in (\bar{x}, x^*)$, then the associated opportunity cost would lie between $\varphi(x)|_{\bar{x}}$ and $\varphi(x)|_{x^*}$. This is represented by the magenta dashed line in Fig. 6. Hence, stopping should occur no earlier than at $\bar{x}$ for $\varphi(x)$ to be minimized.

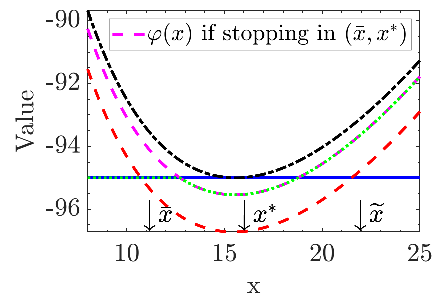

Fig. 6. Stopping at some $x \in (\bar{x}, x^*)$.

4. Applications to other disciplines

The model is set up and presented in Section 2 within the context of an economics and management application, and the associated results in Section 3 are interpreted from that perspective. In this section, the relevance and applicability of the modeling framework and results are discussed in two further unrelated operations research disciplines: medical decision-making and environmental policy.

4.1. Medical decision-making

One area in which optimal stopping theory has significant potential but is not widely utilized is medical decision-making (Liberatore & Nydick, 2008). It is the responsibility of a clinician to ensure that the decisions made regarding patients' care and treatment are optimal. This includes determining the most appropriate course of treatment. However, equally important is determining the right time to initiate a specific treatment when patients' responses are uncertain. Some recent medical studies on complex conditions have highlighted the importance of timing, while also acknowledging that determining it can be challenging. For example, Crossland et al. (2019) conducted a study of cardiac transplantation in a group of patients born with a congenital heart condition. They pointed out that "the optimal timing for listing and transplanting these patients is key to improving outcomes". This view is echoed by others (for example, Kenny et al., 2018; Polyviou et al., 2018) who argue that determining the optimal treatment time is challenging and that a greater understanding of this issue will help guide decision-making.

In the medical context, the decision-maker is a clinician rather than a firm. The issue concerns determining when to initiate treatment for a patient whose clinical decline can be sudden and severe. The cash flows $(X_t)_{t \geq 0}$ correspond to the patient's clinical state, which can be measured, for instance, by a specific blood level. Once the clinical state deteriorates sufficiently, treatment is initiated (this corresponds to abandonment in the economic context), and the patient's clinical state improves by an amount $|\widetilde{K}|$ where $\widetilde{K} < 0$ (the negative value maintains correspondence with the abandonment cost $K > 0$). Waiting for treatment is costly, as the clinical state continues to decline (c represents the per-period decline in clinical state from waiting). Given these interpretations, the technical analysis of determining when to treat follows exactly as in the economic context described in Section 2 and Proposition 1 remains applicable.

The interpretation of Proposition 1 is that the threshold on clinical status derived using classical techniques should be followed only if the patient's condition and clinical state are relatively stable and their clinical decline is gradual (similar to the absence of jumps). However, if the clinical decline can be rapid and sudden, as occurs in some complex conditions, this threshold should not be strictly followed by clinicians optimizing their patients' treatment strategies. For example, in cardiac transplantation, listing a patient too early is not ideal (see Delaney, 2021). The "possibility of survival must be suitably diminished" (Sadat-Hossieny et al., 2022) because surgery carries a high risk. In addition, after transplantation, the patient remains on immunosuppression therapy for life, placing them at high risk of serious infection. This is one of several considerations outlined by Kenny et al. (2018) as arguments against early listing. Instead, initiating treatment at $\bar{x} = c - \rho K < x^*$ aligns with these arguments, as it implies avoiding treatment too early and proceeding only when patients are in sufficiently poor clinical states to justify the surgical risks.

4.2. Environmental policy

Another field where optimal stopping theory proves useful is environmental policy (see, for example, Bigerna et al., 2019 and references therein). The problem of determining when to abandon fossil fuels as the primary energy source in many industrial contexts is a real put option.

In this setting, the decision-maker can be an electricity-generating firm that currently employs a carbon-intensive technology. The revenues generated by the plant are represented by $(X_t)_{t\geq 0}$, which incurs a constant running cost $c > 0$. Once the revenues deteriorate sufficiently, the plant is abandoned at cost $K > 0$. The interpretation of Proposition 1 in this context is that the threshold derived using classical techniques should be followed only if negative shocks in revenues, due to, for instance, policy changes or sudden price hikes in the firm's carbon fuel for powering the plant, are not excessively abrupt or large. Otherwise, the firm should wait until $\bar{x} = c - \rho K$ is reached. Within the context of this model, the *possibility* of an adverse change in the policy landscape (from the firm's perspective) can lead to a delay in abandonment.

5. Conclusion

This study examines an optimal stopping problem for a real put option where the underlying dynamics characterizing payoff uncertainty are modeled as a spectrally negative Lévy process. It derives a condition under which the classical free-boundary solution to such an optimal stopping problem is not appropriate. It shows that when the jumps are large in expectation and/or frequent in arrival, the smooth-pasting condition can break down. Therefore, an alternative solution to the optimal stopping problem is proposed, which is proven to be a viscosity sub-solution. From this, an important and novel implication regarding the appropriate stopping time is obtained.

If the modeled problem resembles a real put option and the underlying dynamics characterizing the uncertainty are represented by a spectrally negative Lévy process, then applying the classical approach to solve the optimal stopping problem is not always appropriate. When the state process is volatile and erratic, the expected payoffs from continuing can be excessively discounted, leading to premature action, i.e., when the benefits from waiting exceed those from stopping. Therefore, for such problems, decision-makers are advised to refrain from acting at the threshold derived through standard optimal stopping techniques and instead act once the net value from continuing becomes at least as low as the benefit from stopping. Essentially, they should reframe the problem so that their objective is to act once the opportunity cost from stopping is minimized, rather than when the expected net present value from stopping is maximized. When the process exhibits limited volatility, both solutions are equivalent; however, in highly volatile conditions, the solution to the reframed problem indicates stopping later than the classical approach implies.

It is worth noting that in this problem, the decision-maker is assumed to be risk-neutral. However, assuming risk aversion toward the large and frequent jumps, an assumption that is both plausible and realistic, especially given the challenges of hedging such risks for non-traded assets, the problem of excessive discounting becomes even more pronounced. In essence, the stopping value is paid as a one-off lump sum. Previous studies (see, for example, Henderson, 2007 and Miao & Wang, 2007) demonstrated that when a manager is risk-averse and the payoff is received as a lump-sum, there is a tendency to exercise early relative to the risk-neutral case to eliminate the uncertainty by locking in the payoff. In other words, when uncertainty is high, the Sharpe ratio associated with the investment asset is insufficient to compensate the manager for waiting. The corresponding implication in this model is that the classical threshold becomes even higher than in the risk-neutral scenario. Therefore, the proposed solution, to wait until the opportunity cost from stopping is minimized, is particularly appropriate for a risk-averse manager.

The problem is discussed from a multidisciplinary perspective by considering three operations research disciplines where it is relevant and applicable: economics and management, medical decision-making, and environmental policy. The implications of the general theoretical result are explained within the context of each discipline-relevant problem.

CRediT authorship contribution statement

Laura Delaney: Conceptualization, Formal analysis, Methodology, Writing – original draft, Writing – review & editing; **Jacco J.J. Thijssen:** Conceptualization, Formal analysis, Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We thank consultant cardiologists Prof. Gruschen Veldtman, Dr. Sihong Huang and Dr. Katrijn Jansen for their specialist advice on the medical application of our model. We also thank seminar participants at the 2022 Annual Meeting of the International Society of Heart and Lung Transplantation for helpful insights.

Appendix A. Proof of Proposition 1

Suppose that F^* given by (11) solves the optimal stopping problem (2). Then F^* solves the variational inequality,

$$0 = \max\{-K - F^*(x), \mathscr{A}F^*(x) - \rho F^*(x) + (x - c)\}, \quad \text{for all} x \in S. \tag{A.1}$$

Let $x \leq x^*$. Then, $F^*(x) = -K$ so that based on (A.1), it holds that $\mathscr{A}F^*(x) - \rho F^*(x) + (x - c) < 0$. Now since $\mathscr{A}F^* = 0$ the latter implies $x < c - \rho K$. Therefore, in the stopping region $x \in [0, x^*]$, $x < c - \rho K$ must hold. However,

$$x^* < c - \rho K \iff \frac{\beta_2}{\beta_2 - 1}\left(\frac{\rho - \gamma}{\rho}\right) < 1. \tag{A.2}$$

Since $\beta_2 < 0$, the condition will definitely be satisfied if $\frac{\rho-\gamma}{\rho} < 1$; i.e., if $\gamma > 0$, where $\gamma = \alpha - \lambda\bar{z}$. However, if $\alpha > 0$ and λ and/or $\bar{z}$ are high, then γ can be negative and high in absolute terms implying it is possible for $x^* > c - \rho K$ to hold. In that case, (9) will be violated, and (11) will not be the solution to (2).

Appendix B. Proof of Proposition 2

Choose a function $\psi \in C^2(S)$ with $\psi(x_0) = F^*(x_0)$ based on Conditions 1 and 2 of Definition 1. Applying the operator (5) to ψ for $x_0 > \bar{x}$ gives

$$\mathscr{A}\psi(x_0) - \rho\psi(x_0) + x_0 - c = \frac{1}{\rho - \gamma}\left[(\alpha - \gamma)x_0 - \gamma\bar{x}^{-\beta_2}(c - \rho K)\right] + \rho\left[\left(\frac{x_0}{\bar{x}}\right)^{\beta_2}\left[L_0(\bar{x}) + K\right]\right]. \tag{B.1}$$

1. If $\bar{x} = c - \rho K$, the expected jumps are very large and/or frequent and (15) does not hold. Hence, $\gamma < 0$ and $|\gamma|$ is necessarily high (cf. Appendix A).

(a) If $x_0 \leq \bar{x}$. Then $F^*(x_0) = -K$ and

$$\max\{\mathscr{A}\psi(x_0) - \rho\psi(x_0) + x_0 - c, -K - F^*(x_0)\} = \rho K + x_0 - c > 0.$$

(b) If $x_0 > \bar{x}$, then substituting $\bar{x}$ with $c - \rho K$ in (B.1) gives

$$\mathscr{A}\psi(x_0) - \rho\psi(x_0) + x_0 - c = \frac{1}{\rho - \gamma}\left[(\alpha - \gamma)x_0 - \gamma\bar{x}^{-\beta_2}(c - \rho(K + x_0^{\beta_2}))\right] > 0 \tag{B.2}$$

implying the variational inequality (17) is satisfied in this region also.

Given points (a) and (b), (17) holds for all $x_0 \in S$. Therefore, F^* is a viscosity *sub*-solution for $\bar{x} = c - \rho K$.

2. Say $\bar{x} = x^*$ so that (15) holds. F^* is then the classical solution to the optimal stopping problem and will be differentiable at x^*. It therefore satisfies the *sub*- and *super*-solution conditions implying it is a viscosity solution to (2) (cf. Achdou et al., 2022, Online Appendix D).

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