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THE LOCAL/LOGARITHMIC CORRESPONDENCE AND THE DEGENERATION FORMULA FOR QUASIMAPS

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Abstract We study the relationship between the enumerative geometry of rational curves in local geometries and various versions of maximal contact logarithmic curve counts. Our approach is via quasimap theory, and we show versions of the [vGGR19] local/logarithmic correspondence for quasimaps, and in particular for normal crossings settings, where the Gromov-Witten theoretic formulation of the correspondence fails. The results suggest a link between different formulations of relative Gromov-Witten theory for simple normal crossings divisors via the mirror map. The main results follow from a rank reduction strategy, together with a new degeneration formula for quasimaps.

1	Introduction	2
1.1.	Results	2
1.2.	Motivation	3
1.3.	The role of quasimaps	4
1.4.	Quasimaps and the degeneration formula	4
1.5.	Future directions	5
2	Absolute quasimaps	6
2.1.	Quasimaps to a double point degeneration	8
3	Logarithmic quasimaps	11
3.1.	The Ciocan-Fontanine–Kim obstruction theory	15
4	The degeneration formula	16
4.1.	Decomposition	18

Keywords: quasimaps; degeneration formula; logarithmic quasimap invariants

Mathematics Subject Classification: Primary 14H10; 14N10; 14N35

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4.2. Gluing	20
5 Local/logarithmic correspondence	26
5.1. Smooth divisor	27
5.2. Simple normal crossings divisors	35
References	42

1. Introduction

Let X be a smooth projective variety and $D = D_1 + \cdots + D_r \subset X$ be a simple normal crossings divisor whose irreducible components are nef. The *local/logarithmic correspondence* connects two enumerative theories associated to (X, D) : rational curves in X with maximal tangency to D , and rational curves in the total space of $\mathcal{O}_X(-D)$. This correspondence was conjectured by Takahashi [Tak01] and proved by Gathmann [Gat03], for the Gromov–Witten theory of $X = \mathbb{P}^2$ and D a smooth cubic, and then generalised in [vGGR19] to arbitrary X and D any *smooth*, nef divisor.

1.1. Results

We prove a quasimap local/logarithmic correspondence, for simple normal crossings divisors, under a mild positivity assumption. Let $X = W//G$ be a GIT quotient and let $D = D_1 + \cdots + D_r \subset X$ be a simple normal crossings divisor satisfying Assumptions 2.1 and 3.1.

This includes toric varieties and partial flag varieties, with any simple normal crossings divisor, as well as complete intersections in these spaces with divisors pulled back from the ambient space. There are two different directions in which to generalise the local/logarithmic correspondence for smooth divisors. When tangency is imposed with respect to all divisor components simultaneously we prove a local/logarithmic correspondence for arbitrary r .

Theorem A (Theorem 5.20). *Suppose $D_1 \cap \cdots \cap D_r \neq \emptyset$ and the components of D are very ample. Then we have the following equality of virtual classes*

$$[\mathcal{Q}_{0,(\underline{d},0)}^{\log}(X|D,\beta)]^{\text{vir}} = \prod_{i=1}^r (-1)^{d_i+1} \cdot \text{ev}_1^* D_i \cap [\mathcal{Q}_{0,2}(\oplus_{i=1}^r \mathcal{O}_X(-D_i), \beta)]^{\text{vir}},$$

where on the left-hand side maximal tangency is imposed to $D_1 \cap \cdots \cap D_r$ at a single point.

For $r = 1$, in Gromov–Witten theory, this is the strong form of the smooth local/logarithmic correspondence (see [TY23b, Conjecture 18]). The original form is obtained by using the divisor equation and is the main theorem of [vGGR19]. The authors of [vGGR19] conjectured a *different* generalisation for D an s.n.c. divisor, where maximal tangency is imposed to the components of the divisor at distinct points. Figure 4 exhibits the difference between the generalisations. Although there is some numerical evidence for either generalisation [BBvG24, BBvG22], the authors of [NR22] showed that even if $D = D_1 + D_2$, both generalisations can fail in Gromov–Witten theory. In addition to Theorem A, we show that the strong form version of the conjecture of [vGGR19] is true for quasimaps for $r = 2$.

Theorem B (Theorem 5.19). *Suppose that $D = D_1 + D_2$ has two components, which are very ample. Then we have the following equality of virtual classes*

$$[\mathcal{Q}_{0,(d_1,d_2)}^{\log}(X|D_1+D_2,\beta)]^{\text{vir}} = (-1)^{d_1+d_2} \cdot \prod_{i=1}^2 \text{ev}_i^* D_i \cap [\mathcal{Q}_{0,2}(\oplus_{i=1}^2 \mathcal{O}_X(-D_i),\beta)]^{\text{vir}}.$$

In fact, [TY23b, Conjecture 18] gives a much more general local/logarithmic conjecture for simple normal crossings divisors. The situations in Theorems A and B are the two extremes.

In order to prove Theorem A (as well as Theorem B), we reduce to the case where $r = 1$ in Theorems 5.21 and 5.26 by showing that, for products of projective spaces relative toric boundary divisors, the simple normal crossings logarithmic quasimap spaces can be obtained as fibre products of logarithmic quasimap spaces for smooth divisors. Then, following the strategy of [vGGR19], we prove Theorem A in the case where $r = 1$ using a degeneration formula for quasimaps which we develop.

Let $W \rightarrow \mathbb{A}^1$ be a toric double point degeneration. An example is the deformation to the normal cone of a smooth projective toric variety along a smooth toric divisor. Let X_0 denote the central fibre, which is a union of two toric varieties X_1 and X_2 glued along a common toric divisor D .

Theorem C (Degeneration formula for quasimaps, Theorem 4.20).

$$[\mathcal{Q}_{g,n}^{\log}(X_0,\beta)]^{\text{vir}} = \sum_{\tilde{\Gamma}} \frac{m_{\tilde{\Gamma}}}{|\text{Aut}(\tilde{\Gamma})|} j_{\tilde{\Gamma},*} c_{\tilde{\Gamma}}^* \Delta^! \left(\prod_{V \in V(\tilde{\Gamma})} [\mathcal{Q}_{g_V,\alpha_V}^{\log}(X_{r(V)}|D,\beta_V)]^{\text{vir}} \right)$$

Both the proof and application of the degeneration formula for quasimaps entail additional subtleties, when compared with stable maps. The main one is based on the fact that quasimap spaces do not embed along closed embeddings, see [BN21, Remark 2.3.3]. As a consequence, if V is a toric subvariety of a toric variety X , then quasimaps to X which factor through V contain more information than quasimaps to V with its toric presentation, as we illustrate in Example 2.9. We show in Example 2.8 that this phenomenon is relevant when studying toric double point degenerations, and it forces us to consider non-toric presentations for the irreducible components of the special fibre. As a result, the output of the degeneration formula, Theorem 4.20, may contain curve classes which live on the degeneration family, rather than on the irreducible components of the special fibre. We call such curve classes *ghost* classes (see Definition 2.10). For the application to the local/logarithmic correspondence, these contributions are ruled out in Proposition 5.17. To prove Proposition 5.17 we show that these contributions have a trivial excess line bundle.

1.2. Motivation

Relative or logarithmic Gromov–Witten theory [Che14b, GS13, AC14, Li01] has significantly shaped the landscape of enumerative geometry. The resulting invariants (and their generalisations) have proved important in mirror symmetry constructions [GS22], degeneration formulas [Ran22, Gro23, KLR21, Li02, ACGS21], tropical correspondence

theorems such as [MR20, Gro18, Gra22, NS06] as well as understanding the tautological ring of the moduli space of curves [GV05, RK23, MR23, HMP+22]. Consequently, computing logarithmic Gromov–Witten invariants is an important problem in enumerative algebraic geometry.

An interesting feature of the local/logarithmic correspondence is that the two sides can be understood by qualitatively different techniques, especially when the logarithmic divisor is genuinely simple normal crossings. Thus, conceptually moving to one side and then using its features can be profitable. Although there have been many numerical generalisations and variations of this correspondence [BFGW21, vGNS24, BBvG21, BBvG24, BBvG22], the original cycle level correspondence holds in the case where the divisor is smooth. On the other hand, the major difficulty in computing logarithmic Gromov–Witten invariants, at least in genus zero, lies in the case where the divisor is genuinely simple normal crossings.

1.3. The role of quasimaps

There have been conjectures to generalise the local/logarithmic correspondence to the case where the divisor is simple normal crossings, but as we have remarked these conjectures are false in general, in Gromov–Witten theory [NR22]. Moreover, the authors noticed [NR22, Remark 5.4] that one point of view for the underlying reason for the failure of these conjectures was the presence of excess components in the moduli space containing *rational tails*. Quasimap theory [MOP11, CFK10, CFKM14] provides an alternative framework for counting curves which excludes rational tails, and so it is conceivable that these are no longer counterexamples in the quasimap setting.

The origins of quasimap theory are intertwined with its connections to mirror symmetry [CFK14] and the main applications of the theory have been wall-crossing formulas which relate Gromov–Witten invariants to quasimap invariants [CFK14, CFK17, Zho22]. These wall-crossing formulas reduce computations in Gromov–Witten theory to quasimap theory, which are typically easier. Indeed, almost all results that allow us to compute genus-zero Gromov–Witten invariants [Giv98] rely at heart on the wall-crossing formula. Recently, the third author has developed a theory of logarithmic quasimaps [Sha24] with the expectation of proving *logarithmic* wall-crossing formulas, first proposed in [BN21]. A consequence of Theorem A and Theorem B, which confirm the predictions of [NR22], is that the techniques of ordinary Gromov–Witten/quasimap theory can be imported over to compute logarithmic quasimap invariants in the case where the divisor is genuinely simple normal crossings. Combining this with a conjectural wall-crossing should be a powerful method for computing logarithmic Gromov–Witten invariants.

1.4. Quasimaps and the degeneration formula

The degeneration formula for double point degenerations [Li02, Che14a, KLR21, AF16] has been a well-used tool in Gromov–Witten theory since its first introduction to the subject [MP06, OP09, PP17]. For all the same reasons the degeneration formula for quasimaps, Theorem C, will provide similar computational capabilities. In this paper

we use the degeneration formula to prove the local/logarithmic correspondence for smooth pairs. There is a related direction to this, which is the prospect of investigating holomorphic anomaly equations in the relative setting, for which quasimaps are better suited.

In the absolute setting, for Calabi–Yau threefolds, the wall-crossing formula coincides with the mirror map and hence quasimap invariants are exactly equal to the B-model invariants of the mirror. This link has been exploited to give a direct geometric proof, for local $\mathbb{P}^2$, of the holomorphic anomaly equation [LP18], which gives profound structure to the Gromov–Witten theory of a Calabi–Yau, coming from the B-model. Using the double point degeneration formula, the authors of [BFGW21] prove a holomorphic anomaly equation for the logarithmic Gromov–Witten theory of $\mathbb{P}^2$ relative an elliptic curve. This suggests that more general holomorphic anomaly equations in relative geometries could be pursued using the degeneration formula for quasimaps, after generalising this beyond the toric setting.

1.5. Future directions

1.5.1. Local/logarithmic correspondence in higher genus. There are two ways in which one can generalise the original smooth local/logarithmic correspondence. Either to increase the rank of the divisor, as we have addressed in the present paper, or to generalise from rational curves to higher genus curves, where on the logarithmic side one adds a λ_g class for the dimensions to match. In Gromov–Witten theory both generalisations fail on the nose, for the higher genus situation there are correction terms described in [BFGW21]. One can speculate about whether, analogous to the situation for higher rank divisors, there are instances where the correction terms vanish for the quasimap local/logarithmic correspondence in higher genus.

1.5.2. Holomorphic anomaly equations. In the Gromov–Witten situation, the relationship between local and logarithmic invariants in higher genus is used to prove a holomorphic anomaly equation for $\mathbb{P}^2$ relative to an elliptic curve. Since quasimaps are better suited to understanding holomorphic anomaly equations, a natural is to pursue this for other geometries.

In order to accomplish this, the degeneration formula would need to be generalised beyond the toric setting. This should be possible however. The main difficulty in quasimap theory is the restriction of the theory to spaces admitting a certain GIT presentation. Even the deformation to the normal cone of a toric variety along a smooth, non-toric divisor is no longer toric. On the other hand, one can rewrite the total space, as well as more general blow-ups, as a GIT quotient, see, for example, [CLS22, Proposition 6.2].

1.5.3. Orbifold/logarithmic correspondence. The local/logarithmic correspondence concerns invariants with maximal contact to the divisor which makes it fertile testing ground, but many of the ideas pursued here should extend to a more general context. An alternative approach to tangency involves replacing components of divisor with roots [Cad07], and using the framework of orbifold Gromov–Witten theory to

produce invariants [TY23a]. There has been effort to compare the logarithmic theories with the resulting orbifold theory [ACW17, BNR23, TY20], since the latter is significantly easier to compute in practice. However, it computes a different set of invariants in general, which do not immediately interact with degeneration formulas and lacks some of the conceptual advantages of the logarithmic theory. Furthermore, no known structure organises the difference between the theories.

Our work here sheds some light on this difference. One can view the local theory defined by an s.n.c. divisor as a sector of this orbifold by the results of [BNTY23], and so our results suggest that the difference between the computable orbifold theory and the conceptually better logarithmic theory, might be controlled in part by mirror maps. There is a related proposal in [You22].

Taking these issues together, it seems reasonable that orbifold and logarithmic Gromov–Witten theory for pairs should be compared via the quasimap theory, where at least in some sectors such as the one we study here, the invariants coincide. The general picture appears to be more complicated, but the significantly simplified combinatorics of quasimaps provides a clear path to computations.

2. Absolute quasimaps

Here we recall the definition of absolute GIT quasimaps from [CFKM14].

Assumption 2.1. Let $Z = \text{Spec} A$ be an affine algebraic variety with an action by a reductive algebraic group G . Let θ be a character inducing a linearisation for the action. We insist that

- $Z^s = Z^{ss}$
- Z^s is nonsingular
- G acts freely on Z^s
- Z has only l.c.i. singularities

Then quasimaps to $Z//G$ are defined as follows.

Definition 2.2. Fix non-negative integers g, n and $\beta \in \text{Hom}(\text{Pic}^G Z, \mathbb{Z})$. An n -marked *stable quasimap* of genus g and degree β to $Z//G$ is

$$((C, p_1, \dots, p_n), u)$$

where

- (1) $(C, p_1, \dots, p_n)$ is an n -marked prestable curve of genus g .
- (2) $u : C \rightarrow [Z/G]$ of class β (i.e. $\beta(L) = \deg_C u^*L$).

which satisfy

- (1) (Non-degeneracy) there is a finite (possibly empty) set $B \subset C$, distinct from the nodes and markings, such that $\forall c \in C \setminus B$ we have $u(c) \in Z^s$.
- (2) (Stability) $\omega_C(p_1 + \dots + p_n) \otimes L_\theta^\epsilon$ is ample $\forall \epsilon \in \mathbb{Q}_{>0}$, where L_θ comes from the linearisation.

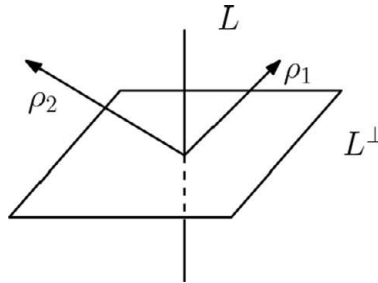


Figure 1. A double point degeneration has two special rays ρ_1, ρ_2 , which project to the generator of the fan of $\mathbb{A}^1$ inside a line L . All other rays are contained in the hyperplane $L^\perp$.

Theorem 2.3 [CFKM14]. *Let $X = Z//G$ with Z and G as in Assumption 2.1. The moduli stack of stable quasimaps $\mathcal{Q}_{g,n}(X, \beta)$ is a Deligne–Mumford stack. If X is proper then so is $\mathcal{Q}_{g,n}(X, \beta)$. Moreover, it admits a perfect obstruction over $\mathfrak{M}_{g,n}$ leading to a virtual fundamental class $[\mathcal{Q}_{g,n}(X, \beta)]^{\text{vir}}$.*

In an earlier iteration of the theory in [CFK10], *toric* quasimaps are defined for toric varieties using the description of the functor of points of a smooth toric variety in terms of line bundle-section pairs according to rays of the fan, proved in [Cox95]. In the general context of Definition 2.2, toric quasimaps are recovered by taking the standard GIT presentation coming from toric geometry.

We will prove a degeneration formula in a toric context, which means the double point degeneration will be a *toric morphism*. However, we will still require the more general definition of quasimaps, Definition 2.2, even in this context, because the process of gluing a quasimap is more subtle than in the case of maps (see the discussion and examples in Section 2.1). Specifically in order to glue together quasimaps to the components of the central fibre we require the GIT presentation induced by the embedding in the total space, rather than the ordinary toric presentation.

Definition 2.4. A *toric double point degeneration* $W \rightarrow \mathbb{A}^1$ is a double point degeneration with W a toric variety and $W \rightarrow \mathbb{A}^1$ a toric morphism. By double point degeneration we mean that $W \rightarrow \mathbb{A}^1$ is a flat, proper morphism, with W smooth. The general fibre is smooth and the fibre over $0 \in \mathbb{A}^1$ is a reduced union of two irreducible components glued along a common smooth divisor.

Notation 2.5. We denote the general fibre of a toric double point degeneration $W \rightarrow \mathbb{A}^1$ by X . The central fibre X_0 is a union of the two components X_1 and X_2 glued along a common smooth divisor D . Necessarily the fan of W has two distinguished rays which map surjectively to the ray in the fan of $\mathbb{A}^1$. These rays correspond to the divisors X_1 and X_2 of W , and will typically be denoted by ρ_1 and ρ_2 . All the other rays are contained in a codimension 1 sublattice, as in Figure 1. We will denote the toric GIT presentation of W by $\mathbb{A}^M // \mathbb{G}_m^s$ and the ambient stack $[\mathbb{A}^M / \mathbb{G}_m^s]$ by $\mathcal{W}$.

Remark 2.6. The morphism $W \rightarrow \mathbb{A}^1$ extends to $\mathcal{W}$ because restriction induces an isomorphism $\text{Pic}(\mathcal{W}) \rightarrow \text{Pic}(W)$. Indeed the unstable locus in the GIT presentation of W has codimension at least 2 [CLS11, Proposition 5.1.6], so W and $\mathcal{W}$ are isomorphic in codimension 1.

Remark 2.7. Just as in Gromov–Witten theory one can consider quasimaps over a base. Since this is not addressed in the literature we just make the observation that for a double point degeneration $W \rightarrow \mathbb{A}^1$ we have that $\mathcal{Q}_{g,n}(W/\mathbb{A}^1, \beta) = \mathcal{Q}_{g,n}(W, \beta)$, because any quasimap to W necessarily lives in a fibre over $\mathbb{A}^1$.

2.1. Quasimaps to a double point degeneration

The degeneration formula involves the study of quasimaps to a double point degeneration $W \rightarrow \mathbb{A}^1$, which we assume to be toric. Let $\mathbb{A}^M // \mathbb{G}_m^s$ be the toric GIT presentation of W . We define quasimaps to X_0 via the following Cartesian diagram.

$$\begin{array}{ccc} \mathcal{Q}_{g,n}(X_0, \beta) & \longrightarrow & \mathcal{Q}_{g,n}(W, \beta) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{A}^1 \end{array}$$

The degeneration formula will express the virtual class of $[\mathcal{Q}_{g,n}(X_0, \beta)]^{\text{vir}}$, defined via Gysin pullback in the above diagram, with the virtual classes of logarithmic quasimap spaces of X_1 and X_2 relative to D . For stable maps, one has that a map to X_0 is the same as a collection of maps to X_i gluing at the nodes. The analogue statement for quasimaps is true only if we take the GIT presentation $X_i \simeq \mathbb{A}^{M-1} // \mathbb{G}_m^s$ induced by W , which in general does not agree with the standard toric presentation. Example 2.8 provides a concrete example of this phenomenon.

Example 2.8. Consider the deformation to the normal cone of $\mathbb{P}^1$ relative ∞ , $W = \text{Bl}_{\infty \times 0}(\mathbb{P}^1 \times \mathbb{A}^1)$. The fan of W , pictured in Figure 2, induces the following GIT presentation

$$W = \mathbb{A}^4 // \mathbb{G}_m^2, \text{ where } (\lambda, \mu) \cdot (x_0, x_1, v_1, v_2) = (\lambda \mu x_0, \lambda x_1, \mu v_1, \mu^{-1} v_2).$$

Setting $v_1 = 0$ in the presentation of W induces the following presentation of $X_1 = \mathbb{P}^1$:

$$X_1 = \mathbb{A}^3 // \mathbb{G}_m^2.$$

On the other hand, $\mathbb{P}^1$ has the following toric presentation

$$\mathbb{P}^1 = \mathbb{A}^2 // \mathbb{G}_m \text{ where } \alpha \cdot (z_0, z_1) = (\alpha z_0, \alpha z_1).$$

There is a morphism $f : [\mathbb{A}^3 / \mathbb{G}_m^2] \rightarrow [\mathbb{A}^2 / \mathbb{G}_m]$ given by

$$\begin{aligned} (x_0, x_1, v_2) &\mapsto (x_0 \cdot v_2, x_1) \\ (\lambda, \mu) &\mapsto \lambda \end{aligned}$$

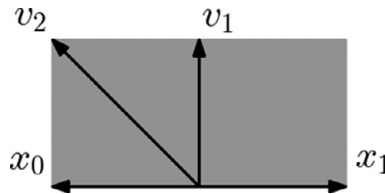


Figure 2. The fan of $W = \text{Bl}_{\infty \times 0}(\mathbb{P}^1 \times \mathbb{A}^1)$.

and a morphism $g : [\mathbb{A}^2/\mathbb{G}_m] \rightarrow [\mathbb{A}^3/\mathbb{G}_m^2]$ given by

$$\begin{aligned} (z_0, z_1) &\mapsto (1, z_1, 0, z_0) \\ \alpha &\mapsto (\alpha, \alpha^{-1}). \end{aligned}$$

Note that $f \circ g = \text{id}$ but $g \circ f \neq \text{id}$ (one must restrict to the stable locus for f and g to be inverse). As a consequence, every quasimap to $\mathbb{P}^1$ with the toric presentation appears as the restriction of a quasimap to $\mathbb{P}^1$ with the presentation induced by W , but the converse is not true.

Loosely speaking, Example 2.8 shows that a quasimap to X_1 with the presentation induced by W requires strictly more information than with the standard presentation. For instance, the degree of a quasimap to X_i in the first case lies in $H_2(W, \mathbb{Z})$, which in general may differ from $H_2(X_i, \mathbb{Z})$.

Example 2.9. For any $k \geq 0$, consider the quasimap

$$\begin{aligned} q_k : \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ [s : t] &\mapsto ([0 : s^k], [s : t]) \end{aligned}$$

Note that $x = [0 : 1]$ is a basepoint of q_k for $k \geq 1$. We can think of q_k as giving a quasimap to $\mathbb{P}^1$ with the presentation

$$\mathbb{P}^1 = \mathbb{A}^1 \times \mathbb{A}^2 // (\mathbb{G}_m \times \mathbb{G}_m) \text{ where } (\lambda, \mu) \cdot (x_1, y_0, y_1) = (\lambda x_1, \mu y_0, \mu y_1)$$

On the other hand, since q_k factors through $[0 : 1] \times \mathbb{P}^1$, there is an obvious induced quasimap to $\mathbb{P}^1$ with the standard presentation given by

$$\begin{aligned} \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \\ [s : t] &\mapsto [s : t]. \end{aligned}$$

But this is independent of k and so it is impossible to recover back the original quasimap q_k .

Definition 2.10. Let W be a toric variety and $\iota : V \hookrightarrow W$ the closure of a torus orbit. A *ghost class* of $V \hookrightarrow W$ is a curve class in $H_2(W, \mathbb{Z}) \setminus \iota_*(H_2(V, \mathbb{Z}))$.

For example, the quasimap q_k from Example 2.9 has degree $(k, 1)$, which is a ghost class unless $k = 0$. The following example describes ghost classes when the ambient W is

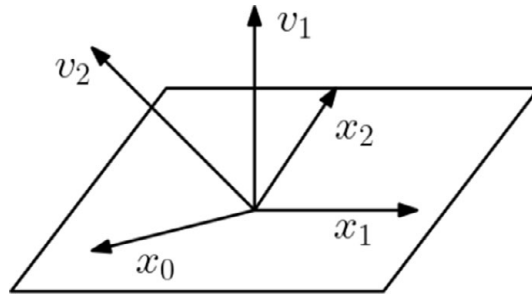


Figure 3. The fan of $W(\mathbb{P}^2, H) = \text{Bl}_{H \times 0} \mathbb{P}^2 \times \mathbb{A}^1$.

the deformation to the normal cone of a hyperplane in projective space, which we will encounter when dealing with the degeneration formula in Section 5.

Example 2.11. Let $W(\mathbb{P}^N, H) = \text{Bl}_{H \times 0}(\mathbb{P}^N \times \mathbb{A}^1)$ be the deformation to the normal cone of a hyperplane H in $\mathbb{P}^N$, which is a toric double point degeneration with toric GIT presentation

$$\mathbb{A}^{N+3} // \mathbb{G}_m^2, \text{ where } (\lambda, \mu) \cdot (x_0, x_1, \dots, x_N, v_1, v_2) = (\lambda \mu x_0, \lambda x_1, \dots, \lambda x_N, \mu v_1, \mu^{-1} v_2).$$

The fan of $W(\mathbb{P}^N, H)$ is pictured in Figure 2 for $N = 1$ and in Figure 3 for $N = 2$. We compute the ghost classes of the inclusion of $X_1 \simeq \mathbb{P}^N$ in $W(\mathbb{P}^N, H)$.

Let $H_0, \dots, H_N$ be the toric divisors on $\mathbb{P}^N$ with $H = H_0$, and let $\tilde{H}_i$ be the strict transform of $H \times \mathbb{A}^1$. The toric divisors of $W(\mathbb{P}^N, H)$ are:

- $\tilde{H}_1, \dots, \tilde{H}_N$;
- $\tilde{H}$, the infinity section;
- E , the exceptional divisor;
- $\mathbb{P}^N$ embedded in the special fiber.

They satisfy the following relations in $\text{Pic}(W(\mathbb{P}^N, H))$:

- $[\tilde{H}_1] = \dots = [\tilde{H}_N] = [\tilde{H}] + [E]$ and
- $[E] + [\mathbb{P}^N] = 0$.

In order to describe ghost classes of $\mathbb{P}^N$, we need to describe the morphism

$$H_2(\mathbb{P}^N, \mathbb{Z}) \rightarrow H_2(W(\mathbb{P}^N, H), \mathbb{Z}) \quad (1)$$

induced by the inclusion $\mathbb{P}^N \hookrightarrow W(\mathbb{P}^N, H)$ in the special fibre. On the one hand, $H_2(\mathbb{P}^N, \mathbb{Z}) \simeq \mathbb{Z}\ell$, with ℓ the class of a line. On the other hand, $H_2(W(\mathbb{P}^N, H), \mathbb{Z}) \simeq \mathbb{Z}l \oplus \mathbb{Z}f$ with f the class of a fiber $X_2 \rightarrow H$ and l the class of a line in the general fibre of $W(\mathbb{P}^N, H) \rightarrow \mathbb{A}^1$. We fix this basis of $H_2(W(\mathbb{P}^N, H), \mathbb{Z})$ from now on. Note that a class $(d, e) \in H_2(W(\mathbb{P}^N, H), \mathbb{Z})$ is effective if and only if $d \geq 0$ and $d + e \geq 0$.

The transpose $\varphi: \text{Pic}(W(\mathbb{P}^N, H)) \rightarrow \text{Pic}(\mathbb{P}^N)$ of the morphism in (1) is determined by $\varphi([\tilde{H}_1]) = [H_0]$ and $\varphi([\mathbb{P}^N]) = -[H_0]$. Combining this with the fact that the basis

$\{[\tilde{H}_1], [\mathbb{P}^N]\}$ of $\text{Pic}(W(\mathbb{P}^N, H))$ and $\{l, f\}$ of $H_2(W(\mathbb{P}^N, H))$ are dual, we conclude that

$$H_2(\mathbb{P}^N, \mathbb{Z}) \rightarrow H_2(W(\mathbb{P}^N, H), \mathbb{Z})$$

$$\ell \mapsto (1, -1).$$

This means that an effective curve class $(d, e) \in H_2(W(\mathbb{P}^N, H), \mathbb{Z})$ is ghost for $\mathbb{P} \hookrightarrow W(\mathbb{P}^N, H)$ if and only if $d \neq -e$.

3. Logarithmic quasimaps

In this section we will introduce logarithmic quasimaps to a toric double point degeneration analogous to the logarithmic quasimaps of [Sha24]. We show that this moduli space admits a virtual fundamental class using the construction of [BF97]. First, we recall the main theorem of [Sha24] in the specific case where the divisor is smooth.

Assumption 3.1. Let $X = Z//G$ be a GIT quotient and $D \subset X$ a simple normal crossings divisor. In addition to the requirements of the previous section we impose that $Z//G$ is a subvariety of a vector space quotient $V//G$ from which D is pulled back.

Suppose that D is a smooth divisor and let $\mathcal{A} = [\mathbb{A}^1/\mathbb{G}_m]$.

Theorem 3.2 [Sha24, Theorems 0.1 & 0.2]. Let $g, n \in \mathbb{N}$, let β be a quasimap degree and let $\alpha \in \mathbb{N}^n$ such that $\sum_{i=1}^n \alpha_i = D \cdot \beta$. The moduli space $\mathcal{Q}_{g, \alpha}^{\log}(X|D, \beta)$ parametrising logarithmic quasimaps to (X, D) is a proper Deligne–Mumford stack. If u is the universal map, and π is the projection from the universal curve, then the complex $(\mathbf{R}\pi_* u^* \mathbb{T}_{[Z/G]}^{\log})^\vee$ is a perfect obstruction theory for the morphism $\mathcal{Q}_{g, \alpha}^{\log}(X|D, \beta) \rightarrow \mathfrak{M}_{g, \alpha}^{\log}(\mathcal{A})$.

Remark 3.3. Assumption 3.1 differs slightly from [Sha24, Section 1] in that we do not make the simplifying assumption that the divisor D corresponds to a unique line-bundle section pair on $[V/G]$. The more general setup of Assumption 3.1 is necessary to work with the induced presentation of components of the central fibre of a toric double point degeneration (see Example 2.8). As explained in [Sha24, End of Section 1] this assumption is not necessary, but, in general, requires a choice of a line bundle section pair on $[V/G]$. In the case of the induced presentation of components of the central fibre of a toric double point degeneration, there is a natural choice given by the equation cutting out the other central fibre component.

Now we will move to the case of a double point degeneration, in the toric setting.

Let $W \rightarrow \mathbb{A}^1$ be a toric double point degeneration. Equipping W (resp. $\mathbb{A}^1$) with the divisorial logarithmic structure with respect to the central fibre (resp. the origin) makes $W \rightarrow \mathbb{A}^1$ into a logarithmically smooth morphism. The Artin fans [AW18] of the respective targets are $\mathcal{A}^2 = [\mathbb{A}^2/\mathbb{G}_m^2]$ and $\mathcal{A} = [\mathbb{A}^1/\mathbb{G}_m]$. Consequently, we have a diagram

$$\begin{array}{ccc} W & \longrightarrow & \mathcal{A}^2 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathcal{A} \end{array} \quad (2)$$

where the right-hand vertical arrow is given by the product of the two coordinates. This diagram is not Cartesian, but W maps to the fibre product $\mathcal{A}' = \mathcal{A}^2 \times_{\mathcal{A}} \mathbb{A}^1$, which lives over $\mathbb{A}^1$. This fibre product can be taken in the category of ordinary stacks or the fine and saturated category, as the bottom horizontal morphism is strict.

As in Notation 2.5, we denote by $\mathcal{W}$ the ambient quotient stack of W . The morphism $W \rightarrow \mathcal{A}^2$ factors through $\mathcal{W}$ by Remark 2.6.

Definition 3.4. Let $C \rightarrow \mathcal{W}$ be a quasimap to W (over $\mathbb{A}^1$). Define the *induced map* $C \rightarrow \mathcal{A}'$ via the composition

$$C \rightarrow \mathcal{W} \rightarrow \mathcal{A}'.$$

The morphism $\mathcal{A}' \rightarrow \mathbb{A}^1$ is a logarithmic morphism. Write $\mathfrak{M}_{g,n}^{\log}(\mathcal{A}'/\mathbb{A}^1)$ for the moduli space of minimal prestable logarithmic maps to $\mathcal{A}'/\mathbb{A}^1$.

Definition 3.5. Let g, n be non-negative integers, let β be an effective curve class on W . Define $\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta)$ as the fibre product of stacks

$$\begin{array}{ccc} \mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta) & \longrightarrow & \mathfrak{M}_{g,n}^{\log}(\mathcal{A}'/\mathbb{A}^1) \\ \downarrow & & \downarrow \epsilon_2 \\ \mathcal{Q}_{g,n}(W/\mathbb{A}^1, \beta) & \xrightarrow{\epsilon_1} & \mathfrak{M}_{g,n}(\mathcal{A}'/\mathbb{A}^1) \end{array}$$

Here $\mathfrak{M}_{g,n}(\mathcal{A}'/\mathbb{A}^1)$ is the stack of maps from n -marked, genus g prestable curves to $\mathcal{A}'$ over $\mathbb{A}^1$. The morphism ϵ_1 is given by associating to a stable quasimap the induced map to $\mathcal{A}'$, and ϵ_2 is given by forgetting the logarithmic structure.

Lemma 3.6. *The moduli space $\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta)$ is a Deligne–Mumford stack and $\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta) \rightarrow \mathcal{Q}_{g,n}(W/\mathbb{A}^1, \beta)$ is proper.*

Proof. Follows from the fact that ϵ_2 is representable and proper (see [Sha24, 2.6–2.8]). $\square$

Next we show that $\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1)$ admits a perfect obstruction theory over $\mathfrak{M}_{g,n}^{\log}(\mathcal{A}'/\mathbb{A}^1)$ following a similar argument to [Sha24, Section 4]. Note that we have a diagram

$$\begin{array}{ccccc} & & u & & \\ & \searrow & & \searrow & \\ \mathcal{C}_{\mathcal{Q}} & \xrightarrow{\quad} & \mathcal{C}_{\mathfrak{M}} & \xrightarrow{\quad} & W \longrightarrow \mathcal{A}' \\ \downarrow \pi & & \downarrow & \searrow & \downarrow \\ \mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta) & \longrightarrow & \mathfrak{M}_{g,n}^{\log}(\mathcal{A}'/\mathbb{A}^1) & \longrightarrow & \mathbb{A}^1 \end{array}$$

Lemma 3.7. *There is a morphism in the derived category of $\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta)$*

$$\phi : \mathbf{R}\pi_*(\mathbf{L}u^* \mathbb{L}_{\mathcal{W}/\mathbb{A}^1}^{\log} \otimes \omega_{\pi}) \rightarrow \mathbb{L}_{\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta)/\mathfrak{M}_{g,n}^{\log}(\mathcal{A}'/\mathbb{A}^1)}.$$

Proof. We have that $\mathbb{L}_{\mathcal{W}/\mathbb{A}^1}^{\log} \cong \mathbb{L}_{\mathcal{W}/\mathcal{A}'}$ by [Ols03, Corollary 5.25] and so there is a morphism

$$\mathbf{L}u^*\mathbb{L}_{\mathcal{W}/\mathbb{A}^1}^{\log} \rightarrow \mathbb{L}_{\mathcal{C}_{\mathcal{Q}/\mathcal{C}_{\mathfrak{M}}}} \cong \mathbf{L}\pi^*\mathbb{L}_{\mathcal{Q}/\mathfrak{M}}.$$

By tensoring with the dualising sheaf and applying $\mathbf{R}\pi_*$ we get

$$\phi : \mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{L}_{\mathcal{W}/\mathbb{A}^1}^{\log} \otimes \omega_{\pi}) \rightarrow \mathbb{L}_{\mathcal{Q}/\mathfrak{M}}. \quad \square$$

Proposition 3.8. *The morphism ϕ defines an obstruction theory and $\mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{L}_{\mathcal{W}/\mathbb{A}^1}^{\log} \otimes \omega_{\pi})$ is of perfect amplitude contained in $[-1, 0]$.*

Proof. The proof follows the argument of [Sha24, Proposition 4.5]. The only change to the argument is that we are working over the base $\mathbb{A}^1$ and so [Ols03, Theorem 5.9] instead gets applied to the following diagram

$$\begin{array}{ccc} \mathcal{C}_T & \longrightarrow & \mathcal{C}_{\bar{T}} \\ & \searrow & \swarrow \text{dashed} \\ & \mathcal{W} \times T & \longrightarrow \mathcal{W} \times \bar{T} \\ & \downarrow & \downarrow \\ & \mathbb{A}^1 \times T & \longrightarrow \mathbb{A}^1 \times \bar{T} \end{array}$$

Let $\mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log}$ denote the derived dual of $\mathbb{L}_{\mathcal{W}/\mathbb{A}^1}^{\log}$.

To prove that ϕ defines a perfect obstruction theory we use an argument similar to [Sha24, Proposition 4.7]. More precisely, by Corollary 3.10 and generic non-degeneracy on the space of quasimaps, we have that the complex $\mathbf{L}u^*\mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log}$ has cohomology supported in 0. This shows that the complex $\mathbf{R}\pi_*\mathbf{L}u^*\mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log} \cong (\mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{L}_{\mathcal{W}/\mathbb{A}^1}^{\log} \otimes \omega_{\pi}))^{\vee}$ is of perfect amplitude in $[0, 1]$. $\square$

Corollary 3.10 will follow from Proposition 3.9 below.

Proposition 3.9. *Let $W \rightarrow \mathbb{A}^1$ be a toric double point degeneration and let $\mathcal{W} = [\mathbb{A}^M/\mathbb{G}_m^s]$ be the ambient stack. Let $\{D_{\rho_i}\}_{i=1}^M$ denote the toric divisors on W with $X_1 = D_{\rho_1}$ and $X_2 = D_{\rho_2}$. Then the logarithmic tangent complex of $\mathcal{W}$ relative to $\mathbb{A}^1$ is quasi-isomorphic to the following complex supported in $[-1, 1]$*

$$\mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log} = [\mathcal{O}_{\mathcal{W}}^{\oplus s} \xrightarrow{f_1} \oplus_{\rho \neq \rho_1, \rho_2 \in \Sigma_W(1)} \mathcal{O}_{\mathcal{W}}(D_{\rho}) \oplus_{i=1}^2 \mathcal{O}_{\mathcal{W}} \xrightarrow{f_2} \mathcal{O}_{\mathcal{W}}], \quad (3)$$

where f_2 maps the generator of each $\mathcal{O}_{\mathcal{W}}(D_{\rho})$ to 0 and is the identity on each of the trivial factors.

Proof. Recall that we have a diagram

$$\begin{array}{ccccc} \mathcal{W} & \xrightarrow{\epsilon} & \mathcal{A}' & \longrightarrow & \mathcal{A}^2 \\ & & \downarrow \alpha & & \downarrow \\ & & \mathbb{A}^1 & \longrightarrow & \mathcal{A} \end{array} \quad (4)$$

The logarithmic tangent complex of $\mathcal{W}$ is, equivalently, the relative tangent complex $\mathbb{T}_{\mathcal{W}/\mathcal{A}'}$. The top row of diagram (4) induces an exact triangle

$$\mathbb{T}_{\mathcal{W}/\mathcal{A}'} \rightarrow \mathbb{T}_{\mathcal{W}/\mathcal{A}^2} \rightarrow \mathbf{L}\epsilon^* \mathbb{T}_{\mathcal{A}'/\mathcal{A}^2} \rightarrow .$$

Since $\mathcal{A}^2 \rightarrow \mathcal{A}$ is flat, $\mathbb{T}_{\mathcal{A}'/\mathcal{A}^2} = \alpha^* \mathbb{T}_{\mathbb{A}^1/\mathcal{A}}$ is quasi-isomorphic to $[\mathcal{O}]$.

On the other hand, we can compute $\mathbb{T}_{\mathcal{W}/\mathcal{A}^2}$ from the exact triangle

$$\mathbb{T}_{\mathcal{W}/\mathcal{A}^2} \rightarrow \mathbb{T}_{\mathcal{W}} \rightarrow \mathbb{T}_{\mathcal{A}^2}.$$

The complex $\mathbb{T}_{\mathcal{W}}$ is quasi-isomorphic to

$$[\mathcal{O}^{\oplus s} \rightarrow \oplus_{\rho \in \Sigma_W(1)} \mathcal{O}(D_\rho)] \quad (5)$$

induced by the weight matrix of the action of $\mathbb{G}_m^s$ on $\mathbb{A}^M$, which can be made explicit as follows. Using that W is a toric double point degeneration with special fibre $D_{\rho_1} + D_{\rho_2}$, we can find a basis of $\text{Pic}(W)$ of the form $\{D_{\rho_1}, D_{\rho_{i_2}}, \dots, D_{\rho_{i_s}}\}$ with $i_j \notin \{1, 2\}$. This choice of basis for $\text{Pic}(W)$ induces a choice of generators $\{\lambda_1, \dots, \lambda_s\}$ of $\text{Hom}(\text{Pic}(W), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq \mathbb{G}_m^s$.

With this choice of generators, the action has expression

$$\lambda_i \cdot (x_1, \dots, x_M) = (\lambda_i^{a_{i,1}} x_1, \dots, \lambda_i^{a_{i,M}} x_M)$$

such that the weight matrix $(a_{i,j})_{1 \leq i \leq s, 1 \leq j \leq M}$ for the action of $\mathbb{G}_m^s$ in this basis satisfies that $a_{1,1} = 1$, $a_{1,2} = -1$ and $a_{i,1} = a_{i,2} = 0$ for each $2 \leq i \leq s$. Then the morphism in eq.(5) is given by

$$1_i \mapsto (y_{i,j})_{j=1}^M \text{ with } y_{i,j} = \partial_{\lambda_i} (\lambda_i^{a_{i,j}} x_j) |_{\lambda_i=1} = \begin{cases} 0 & \text{if } a_{i,j} = 0 \\ a_{i,j} x_j & \text{if } a_{i,j} \neq 0 \end{cases}$$

Using this presentation, we get that $\mathbb{T}_{\mathcal{W}/\mathcal{A}^2}$ is quasi-isomorphic to the complex

$$[\mathcal{O}^{\oplus s} \rightarrow \oplus_{\rho \neq \rho_1, \rho_2} \mathcal{O}(D_\rho) \oplus \mathcal{O}^{\oplus 2}]$$

with morphism

$$1_i \mapsto \begin{cases} (-y_{1,j})_{j=3}^M \oplus (-1, 1) & \text{if } i = 1 \\ (-y_{i,j})_{j=3}^M \oplus (0, 0) & \text{if } i \neq 1 \end{cases} \quad (6)$$

The result is obtained by computing the cone of $\mathbb{T}_{\mathcal{W}/\mathcal{A}^2} \rightarrow \mathbb{T}_{\mathcal{A}'/\mathcal{A}^2}$. $\square$

Corollary 3.10. *The complex $\mathbf{L}u^* \mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log}$ is a perfect complex in $[-1, 0]$ and it has cohomology supported in degree 0.*

Proof. The first statement follows from the fact that f_2 in (3) is surjective. Since f_1 is injective, the cohomology of the complex (3) is supported in degree zero. $\square$

Remark 3.11. The moduli space $\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta)$ lives over $\mathbb{A}^1$. We denote the central fibre by $\mathcal{Q}_{g,n}^{\log}(X_0, \beta)$, where X_0 is the central fibre of W . Let $i_0 : 0 \hookrightarrow \mathbb{A}^1$ be the natural embedding. There is an induced pullback perfect obstruction theory on $\mathcal{Q}_{g,n}^{\log}(X_0, \beta)$ where the associated virtual class coincides with $i_0^![\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta)]^{\text{vir}}$, by functoriality of virtual pullbacks [Man12, Theorem 4.8].

3.1. The Ciocan-Fontanine–Kim obstruction theory

In the following we define an analogue of the obstruction theory given in [CFK10, Section 5.3] for the situation of Theorem 3.2 as well as of Proposition 3.8.

Definition 3.12. Let g, n, d be non-negative integers and let $\alpha \in \mathbb{N}^n$. Let $\mathfrak{Pic}_{g,\alpha,d}^{\log}(\mathcal{A})$ be defined to be the fibre product

$$\begin{array}{ccc} \mathfrak{Pic}_{g,\alpha,d}^{\log}(\mathcal{A}) & \longrightarrow & \mathfrak{Pic}_{g,n,d} \\ \downarrow & & \downarrow \\ \mathfrak{M}_{g,\alpha}^{\log}(\mathcal{A}) & \longrightarrow & \mathfrak{M}_{g,n} \end{array}$$

where $\mathfrak{Pic}_{g,n,d}$ is the Picard stack over $\mathfrak{M}_{g,n}$ of degree d line bundles.

Definition 3.13. Let X be a smooth projective toric variety with GIT presentation $\mathbb{A}^M // \mathbb{G}_m^s$ induced by the fan Σ . Let D_i denote the divisor induced by the character of $\mathbb{G}_m^s$ given by the i^{th} projection and let $d_i = \beta \cdot D_i$.

We define

$$\mathfrak{Pic}_{g,\alpha,\underline{d}}^{\log,s}(\mathcal{A}) := \mathfrak{Pic}_{g,n,d_1}^{\log}(\mathcal{A}) \times_{\mathfrak{M}_{g,n}^{\log}(\mathcal{A})} \cdots \times_{\mathfrak{M}_{g,n}^{\log}(\mathcal{A})} \mathfrak{Pic}_{g,n,d_s}^{\log}(\mathcal{A}). \quad (7)$$

Let $q : \mathfrak{Pic}_{g,\alpha,\underline{d}}^{\log,s}(\mathcal{A}) \rightarrow \mathfrak{M}_{g,\alpha}^{\log}(\mathcal{A})$ denote the natural projection.

Proposition 3.14. Let $\rho_0 \in \Sigma(1)$, and let D_{ρ_0} denote the associated toric divisor. For $\rho \in \Sigma(1)$ let $\mathcal{O}_{\mathcal{X}}(D_{\rho})$ denote the associated line bundle on $\mathcal{X} = [\mathbb{A}^M / \mathbb{G}_m^s]$. Then

$$\mathbb{E}_{X|D_{\rho_0}} := \oplus_{\rho \neq \rho_0} \mathbf{R}\pi_* u^* \mathcal{O}_{\mathcal{X}}(D_{\rho}) \oplus \mathbf{R}\pi_* \mathcal{O}_{\mathcal{C}}$$

is a dual obstruction theory for $\nu : \mathcal{Q}_{g,\alpha}^{\log}(X|D_{\rho_0}, \beta) \rightarrow \mathfrak{Pic}_{g,\alpha,\underline{d}}^{\log,s}(\mathcal{A})$. Moreover, this obstruction theory induces the same class as the one in Theorem 3.2.

Proof. The logarithmic Euler sequence [Sha24, Equation 13] induces a distinguished triangle

$$\mathcal{O}_{\mathcal{X}}^{\oplus s} \rightarrow \oplus_{\rho \neq \rho_0} \mathcal{O}_{\mathcal{X}}(D_{\rho}) \oplus \mathcal{O}_{\mathcal{X}} \rightarrow \mathbb{T}_{\mathcal{X}}^{\log} \rightarrow \mathcal{O}_{\mathcal{X}}^{\oplus s}[1].$$

Taking the pullback via the universal map and then pushing down to the moduli space gives

$$\mathbf{R}\pi_* u^* \mathcal{O}_{\mathcal{X}}^{\oplus s} \rightarrow \oplus_{\rho \neq \rho_0} \mathbf{R}\pi_* u^* \mathcal{O}_{\mathcal{X}}(D_{\rho}) \oplus \mathbf{R}\pi_* \mathcal{O}_{\mathcal{C}} \rightarrow \mathbf{R}\pi_* \mathbf{L}u^* \mathbb{T}_{\mathcal{X}}^{\log}.$$

We see that we have a commutative diagram

$$\begin{array}{ccc} \mathbb{T}_{\mathcal{Q}_{g,\alpha}^{\log}(X|D_{\rho_0},\beta)/\mathfrak{M}_{g,\alpha}^{\log}(\mathcal{A})} & \longrightarrow & \nu^*\mathbb{T}_{\mathfrak{Pic}_{g,\alpha,\underline{d}}^{\log,s}(\mathcal{A})/\mathfrak{M}_{g,n}^{\log}(\mathcal{A})} \\ \downarrow & & \downarrow \\ \mathbf{R}\pi_*\mathbf{L}u^*\mathbb{T}_{\mathcal{X}}^{\log} & \longrightarrow & \mathbf{R}\pi_*\mathcal{O}_{\mathcal{C}}^{\oplus s}[1] \end{array}$$

This induces a morphism of distinguished triangles

$$\begin{array}{ccccc} \mathbb{T}_{\mathcal{Q}_{g,\alpha}^{\log}(X|D_{\rho_0},\beta)/\mathfrak{M}_{g,\alpha}^{\log}(\mathcal{A})} & \longrightarrow & \nu^*\mathbb{T}_{\mathfrak{Pic}_{g,\alpha,\underline{d}}^{\log,s}(\mathcal{A})/\mathfrak{M}_{g,\alpha}^{\log,s}(\mathcal{A})} & \longrightarrow & \mathbb{T}_{\mathcal{Q}_{g,\alpha}^{\log}(X|D_{\rho_0},\beta)/\mathfrak{Pic}_{g,\alpha,\underline{d}}^{\log,s}(\mathcal{A})}[1] \\ \downarrow & & \downarrow & & \downarrow \psi \\ \mathbf{R}\pi_*\mathbb{T}_{\mathcal{X}}^{\log} & \longrightarrow & \mathbf{R}\pi_*\mathcal{O}_{\mathcal{C}}^{\oplus s}[1] & \longrightarrow & \mathbf{R}\pi_*\oplus_{\rho\neq\rho_0}u^*\mathcal{O}_{\mathcal{X}}(D_{\rho})\oplus\mathbf{R}\pi_*\mathcal{O}_{\mathcal{C}}[1] \end{array}$$

Applying the four lemma to the long exact sequence in cohomology proves that ψ is a (dual) obstruction theory.

By functoriality of virtual pull-backs applied to the composition

$$\mathcal{Q}_{g,\alpha}^{\log}(X|D_{\rho_0},\beta) \xrightarrow{\nu} \mathfrak{Pic}_{g,\alpha,\underline{d}}^{\log,s}(\mathcal{A}) \longrightarrow \mathfrak{M}_{g,\alpha}^{\log}(\mathcal{A})$$

the result follows. $\square$

Consider the stack $\mathfrak{Pic}_{g,n,\underline{d}}^{\log,s}(\mathcal{A}')$ obtained by replacing $\mathcal{A}$ by $\mathcal{A}'$ in Definition 3.12 and taking X to be a double point degeneration in Definition 3.13. The following result has a proof analogous to that of Proposition 3.14.

Proposition 3.15. *Let $W \rightarrow \mathbb{A}^1$ be a toric double point degeneration. Let $\rho_1, \rho_2 \in \Sigma_W(1)$ correspond to the pieces of the central fibre as in Notation 2.5. The complex*

$$\mathbf{R}\pi_*\mathbf{L}u^*[\oplus_{\rho\neq\rho_1,\rho_2\in\Sigma_W(1)}\mathcal{O}_W(D_{\rho})\oplus_{i=1}^2\mathcal{O}_W \xrightarrow{f_2} \mathcal{O}_W]$$

is a dual obstruction theory for $\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1,\beta) \rightarrow \mathfrak{Pic}_{g,n,\underline{d}}^{\log,s}(\mathcal{A}')$. Moreover, this obstruction theory induces the same class on $\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1,\beta)$ as the one constructed in Proposition 3.8.

4. The degeneration formula

In this section we prove the degeneration formula for quasimaps to a toric double point degeneration. We do this in two steps, decomposition and gluing. The steps follow [ACGS20] and then the arguments of [KLR21].

First we need to verify that the general fibre of $\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1,\beta)$ recovers quasimaps to the general fibre of $W \rightarrow \mathbb{A}^1$.

Lemma 4.1. *Let $W \rightarrow \mathbb{A}^1$ be a toric double point degeneration, let $t \in \mathbb{A}^1$ with $t \neq 0$ and let $\iota_t: t \rightarrow \mathbb{A}^1$ be the corresponding embedding. Then the following diagram is Cartesian*

$$\begin{array}{ccc} \mathcal{Q}_{g,n}(X, \beta) & \longrightarrow & \mathcal{Q}_{g,n}(W/\mathbb{A}^1, \beta) \\ \downarrow & & \downarrow \\ t & \xrightarrow{\iota_t} & \mathbb{A}^1 \end{array}$$

Proof. A quasimap to W consists of line-bundle section pairs (L_ρ, s_ρ) for $\rho \in \Sigma_W(1)$. Let ρ_1, ρ_2 correspond to the pieces of the central fibre. An element of $\mathcal{Q}_{g,n}(W/\mathbb{A}^1, \beta) \times_{\mathbb{A}^1} t$ is an element of $\mathcal{Q}_{g,n}(W/\mathbb{A}^1, \beta)$ such that $s_{\rho_1} \otimes s_{\rho_2}$ is constant and equal to t , under the trivialisation of $L_{\rho_1} \otimes L_{\rho_2}$. This is enough to determine the pairs (L_{ρ_1}, s_{ρ_1}) and (L_{ρ_2}, s_{ρ_2}) up to a $\mathbb{G}_m$ -action. The remaining data is precisely the data of a quasimap to X . $\square$

Lemma 4.2. *Let $W \rightarrow \mathbb{A}^1$ be a toric double point degeneration, let $t \in \mathbb{A}^1$ with $t \neq 0$ and let $\iota_t: t \rightarrow \mathbb{A}^1$ be the corresponding embedding. Then the following diagram is Cartesian*

$$\begin{array}{ccc} \mathcal{Q}_{g,n}(X, \beta) & \longrightarrow & \mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta) \\ \downarrow & & \downarrow \\ t & \xrightarrow{i_t} & \mathbb{A}^1 \end{array}$$

Moreover, we have that $i_t^![\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta)]^{\text{vir}} = [\mathcal{Q}_{g,n}(X, \beta)]^{\text{vir}}$

Proof. There is a Cartesian diagram

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathcal{A}' \\ \downarrow & & \downarrow \\ \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \end{array}$$

Since the logarithmic structure on $\mathbb{G}_m$ is trivial this induces a Cartesian diagram

$$\begin{array}{ccc} \mathfrak{M}_{g,n} \times \mathbb{G}_m & \longrightarrow & \mathfrak{M}_{g,n}^{\log}(\mathcal{A}'/\mathbb{A}^1) \\ \downarrow & & \downarrow \\ \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \end{array}$$

Combining Lemma 4.1 with the definition of $\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta)$ implies the first claim. The second claim follows from the fact that the restriction of $\mathbb{T}_{W/\mathbb{A}^1}^{\log}$ to any non-zero fibre is $\mathbb{T}_X$ which follows from [Ols03, 1.1 (iv)] applied to the diagram (in the logarithmic category)

$$\begin{array}{ccc} \mathcal{X} \times \mathbb{G}_m & \longrightarrow & \mathcal{W} \\ \downarrow & & \downarrow \\ \mathbb{G}_m & \longrightarrow & \mathbb{A}^1. \end{array}$$

$\square$

4.1. Decomposition

Lemma 4.2 tells us that the virtual class of $\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta)$ restricts to $[\mathcal{Q}_{g,n}(X, \beta)]^{\text{vir}}$ on any non-zero fibre. By deformation invariance and Remark 3.11, this can be related to $[\mathcal{Q}_{g,n}^{\log}(X_0, \beta)]^{\text{vir}} = i_0^! [\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta)]^{\text{vir}}$. Now we use the decomposition formula of [ACGS20] to decompose this class into a sum of classes coming from combinatorics of how the components of the curve behave.

Definition 4.3. Fix non-negative integers g, n . A *decorated bipartite graph* is the following data:

- a connected graph Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$;
- a function $r: V(\Gamma) \rightarrow \{1, 2\}$;
- for each vertex V , a choice (n_V, g_V) where $g_V \geq 0$ and $n_V \subset \{1, \dots, n\}$;
- for each edge $E \in E(\Gamma)$, a positive integer $w_E \geq 0$, called the *weight*;

subject to the following conditions:

- (1) for an edge connecting two vertices V_1 and V_2 , $r(V_1) \neq r(V_2)$;
- (2) $\cup_V n_V = \{1, \dots, n\}$ and $n_{V_1} \cap n_{V_2} = \emptyset$ for any distinct $V_1, V_2 \in V(\Gamma)$;
- (3) $\sum_V g_V + g(\Gamma) = g$.

Abusing notation, we will refer to such a decorated bipartite graph by Γ .

Recall that the perfect obstruction theory on $\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta)$ from Proposition 3.8 is relative to $\mathfrak{M}_{g,n}^{\log}(\mathcal{A}'/\mathbb{A}^1)$. If $\mathcal{A}_0 = \mathcal{A}' \times_{\mathbb{A}^1} 0$, then there is a Cartesian tower

$$\begin{array}{ccc}
 \mathcal{Q}_{g,n}^{\log}(X_0, \beta) & \longrightarrow & \mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta) \\
 \downarrow & & \downarrow \\
 \mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0) & \longrightarrow & \mathfrak{M}_{g,n}^{\log}(\mathcal{A}'/\mathbb{A}^1) \\
 \downarrow & & \downarrow \\
 0 & \xrightarrow{i_0} & \mathbb{A}^1.
 \end{array} \tag{8}$$

In Diagram (8), $\mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0)$ refers to minimal prestable logarithmic maps to $\mathcal{A}_0$ over the standard logarithmic point. A consequence of Diagram (8) is that the perfect obstruction theory defining the virtual class $[\mathcal{Q}_{g,n}^{\log}(X_0, \beta)]^{\text{vir}} = i_0^! [\mathcal{Q}_{g,n}^{\log}(W/\mathbb{A}^1, \beta)]^{\text{vir}}$ is relative to $\mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0)$. In [ACGS20], the authors show that the fundamental class of $\mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0)$ decomposes into a sum of contributions coming from decorated bipartite graphs (introduced in Definition 4.3). We denote by $\text{Aut}(\Gamma)$ the subset of automorphisms of the underlying graph Γ that respect the extra data $r, (n_V, g_V)_V$ and $(w_E)_E$. We let $m_\Gamma \in \mathbb{N} \setminus \{0\}$ be the least common multiple¹ of the weights $\{w_E\}$.

¹This definition of the multiplicity m_Γ agrees with the one in [ACGS20] as the projection of the generator of the dual basic monoid $Q_\Gamma^\vee \simeq \mathbb{N}$ associated to Γ (see [ACGS20, Equation (2.9)]) to the tropicalisation of the standard log point.

Theorem 4.4 [ACGS20, Corollary 3.8 & Lemma 3.9].

$$[\mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0)] = \sum_{\Gamma} \frac{m_{\Gamma}}{|\mathrm{Aut}(\Gamma)|} i_{\Gamma,*} [\mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0, \Gamma)]$$

As in the case of stable maps [ACGS20, Theorem 3.11] we will use Theorem 4.4 to decompose the virtual class $[\mathcal{Q}_{g,n}^{\log}(X_0, \beta)]^{\mathrm{vir}}$ into contributions coming from quasimap moduli spaces associated to bipartite graphs with extra information about the quasimap degree on each curve component. We first make the formal definition.

Definition 4.5. For each decorated bipartite graph Γ we define $\mathcal{Q}_{g,n}^{\log}(X_0, \Gamma)$ via the Cartesian diagram

$$\begin{array}{ccc} \mathcal{Q}_{g,n}^{\log}(X_0, \Gamma) & \xrightarrow{j_{\Gamma}} & \mathcal{Q}_{g,n}^{\log}(X_0, \beta) \\ \downarrow q & & \downarrow p \\ \mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0, \Gamma) & \xrightarrow{i_{\Gamma}} & \mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0). \end{array} \quad (9)$$

Lemma 4.6. The space $\mathcal{Q}_{g,n}^{\log}(X_0, \Gamma)$ decomposes as a disjoint union $\bigcup_{\tilde{\Gamma}=(\Gamma, (\beta_V)_V)} \mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma})$ where each β_V is a quasimap degree of W and a point of $\mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma})$ is

- an n -pointed genus g logarithmic quasimap from C/pt_Q to $W/\mathbb{A}^1$ which factors through the zero fibre, i.e. an element of $\mathcal{Q}_{g,n}^{\log}(X_0, \beta)$;
- for each vertex V , a stable quasimap to $X_{r(V)}$, where the GIT presentation is induced from the embedding in W , from a curve C_V with marked points x_e for each edge containing V ;

such that the gluing of the curves C_V along the points corresponding to the edges in $E(\tilde{\Gamma})$ gives the underlying curve of C . Furthermore, the underlying quasimap of the element of $\mathcal{Q}_{g,n}^{\log}(X_0, \beta)$ is glued from the quasimaps from each vertex.

Proof. An element of $\mathcal{Q}_{g,n}^{\log}(X_0, \Gamma)$ comes with the data of a decomposition of the underlying curve C defining the quasimap into C_V for each $V \in V(\Gamma)$, with marked points x_e corresponding to the edge containing V and morphisms $C_V \rightarrow \mathcal{A}$ such that the gluing the C_V according to the graph and gluing the morphisms $C_V \rightarrow \mathcal{A}$ produces the underlying morphism of $C \rightarrow \mathcal{A}_0$ induced by the logarithmic quasimap. On the other hand there is a Cartesian diagram

$$\begin{array}{ccccc} \mathcal{X}_{r(V)} & \longrightarrow & \mathcal{W}_0 & \longrightarrow & \mathcal{W} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}_0 & \longrightarrow & \mathcal{A}' \end{array}$$

where $\mathcal{X}_{r(V)}$ is the quotient stack inside of which $X_{r(V)}$ lives when the GIT presentation is induced by the embedding in W . Therefore the morphism $C_V \rightarrow \mathcal{A}$ induces a morphism $C_V \rightarrow \mathcal{X}_{r(V)}$, i.e. a quasimap to $X_{r(V)}$. $\square$

Diagram (9) implies that the spaces $\mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma})$ admit a perfect obstruction theory over $\mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0, \Gamma)$.

Proposition 4.7.

$$[\mathcal{Q}_{g,n}^{\log}(X_0, \beta)]^{\text{vir}} = \sum_{\tilde{\Gamma}=(\Gamma, (\beta_V)_V)} \frac{m_{\Gamma}}{|\text{Aut}(\Gamma)|} j_{\Gamma,*} [\mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma})]^{\text{vir}}.$$

Proof. We have that

$$\begin{aligned} [\mathcal{Q}_{g,n}^{\log}(X_0, \beta)]^{\text{vir}} &= p^! [\mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0/0)] \\ &= \sum_{\Gamma} \frac{m_{\Gamma}}{|\text{Aut}(\Gamma)|} p^! i_{\Gamma,*} [\mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0/0, \Gamma)] \\ &= \sum_{\tilde{\Gamma}=(\Gamma, (\beta_V)_V)} \frac{m_{\Gamma}}{|\text{Aut}(\Gamma)|} j_{\Gamma,*} [\mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma})]^{\text{vir}} \end{aligned}$$

where sum is over Γ up to isomorphism, and then over choices of $(\beta_V)_V$ on Γ . $\square$

Remark 4.8. One may wonder why in Lemma 4.6, the data of an element of $\mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma})$ is a quasimap to $X_{r(V)}$ with presentation induced by the embedding in W . In fact $X_{r(V)}$ is a toric variety and its fan induces a GIT presentation. On the other hand a quasimap to $X_{r(V)}$ with this toric presentation contains strictly less information and hence Lemma 4.6 would not be true with this presentation. We explain this phenomenon in Section 2.1, see Example 2.8.

Remark 4.9. Analogously to Section 3.1, we have that the obstruction theory of $\mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma})$ can be defined relative to the fibre product

$$\begin{array}{ccc} \mathfrak{Pic}_{g,n,d}^{\log}(\mathcal{A}_0, \Gamma) & \longrightarrow & \mathfrak{Pic}_{g,n,d}(\mathcal{A}_0) \\ \downarrow & & \downarrow \\ \mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0, \Gamma) & \longrightarrow & \mathfrak{M}_{g,n}^{\log}(\mathcal{A}_0) \end{array}$$

4.2. Gluing

Fix a stable decorated bipartite graph $\tilde{\Gamma}$. Now we relate $[\mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma})]^{\text{vir}}$ to the logarithmic moduli spaces corresponding to the vertices of $\tilde{\Gamma}$. More precisely, let Γ_V denote the graph with a single vertex, with half edges for any edge with vertex V recording the degree β_V , genus g_V as well as the contact order at each half edge $\alpha_V = (w_E : V \in E)$. For each vertex we can form the moduli space $\mathcal{Q}_{g_V, \alpha_V}^{\log}(X_V | D, \beta_V)$ as in [Sha24].

Remark 4.10. Strictly speaking, we need an extension of the moduli spaces defined in [Sha24] to allow for the presentation induced here by the degenerations (as well as non-proper targets), but this can be done and is addressed in [Sha24, End of Section 1].

Observe that for each edge E of $\tilde{\Gamma}$ there are two natural maps

$$\prod_V \mathcal{Q}_{g_V, \alpha_V}^{\log}(X_V | D, \beta_V) \rightarrow D$$

because each edge is adjacent to two vertices. We build a map

$$\prod_V \mathcal{Q}_{g_V, \alpha_V}^{\log}(X_V | D, \beta_V) \rightarrow \prod_E D^2.$$

Define $\times_V \mathcal{Q}_V$ via the Cartesian diagram

$$\begin{array}{ccc} \times_V \mathcal{Q}_V & \longrightarrow & \prod_{V \in \Gamma} \mathcal{Q}_{g_V, \alpha_V}^{\log}(X_V | D, \beta_V) \\ \downarrow & & \downarrow \\ \prod_E D & \xrightarrow{\Delta} & \prod_E D^2 \end{array} \quad (10)$$

Proposition 4.11. *There is a morphism*

$$c_{\tilde{\Gamma}} : \mathcal{Q}_{g, n}^{\log}(X_0, \tilde{\Gamma}) \rightarrow \times_V \mathcal{Q}_V$$

which is étale of degree $\frac{\prod_E w_E}{\text{lcm}(w_E)_E}$.

Proof. The existence of the morphism follows from the description of $\mathcal{Q}^{\log}(X_0, \tilde{\Gamma})$ of Lemma 4.6. Since nodes are not basepoints, the rest of the result follows from the analogous statement for stable maps which is proved in [KLR21, Lemma 9.2]. $\square$

Remark 4.12. In the setting of stable maps, the reference [KLR21, (8.1)] defines the moduli space of maps to X_0 marked by Γ using $\mathfrak{M}'_{\Gamma} \rightarrow \mathfrak{M}'_0$, where $\mathfrak{M}'_{\Gamma}$ is defined using a certain splitting stack, and $\mathfrak{M}'_0$ is a sufficiently small open substack of the space of all logarithmically smooth curves. On the other hand [ACGS20] uses $\mathfrak{M}_{g, n}^{\log}(\mathcal{A}_0, \Gamma)$. In fact these give the same resulting space for the moduli space of maps to X_0 marked by Γ . Indeed, there is a Cartesian diagram

$$\begin{array}{ccc} \mathfrak{M}_{g, n}^{\log}(\mathcal{A}_0, \Gamma) & \longrightarrow & \mathfrak{M}'_{\Gamma} \\ \downarrow & & \downarrow \\ \mathfrak{M}_{g, n}^{\log}(\mathcal{A}_0) & \longrightarrow & \mathfrak{M}'_0 \end{array}$$

To sketch this, we note that the fibre of the right-hand vertical morphism [KLR21, Proposition 7.4 (4)] is given by a certain choice of element ρ in the dual of the basic monoid on the base of the logarithmic curve. On the other hand the left-hand vertical morphism is a choice of marking of the underlying prestable map by $\mathcal{A}_0$. The fact that these are equivalent can be seen from the construction of [KLR21, Eq (4.8)], where it is shown how to associate a (bipartite) graph to a choice of ρ . Since ρ is required to be in the finite set of primitive generators for the rays, it necessarily contracts edges adjacent to a vertex corresponding to a curve component mapping to D in such a way that the graph becomes bipartite, and hence defines a marking; see [KLR21, Figure 4.1], for an example.

Theorem 4.13 (Gluing).

$$[\mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma})]^{\text{vir}} = c_{\tilde{\Gamma}}^* \Delta^! \prod_V [\mathcal{Q}_{g_V, \alpha_V}^{\log}(X_{r(V)} | D, \beta_V)]^{\text{vir}}$$

As in [KLR21, Bou19], Theorem 4.13 will be proved by showing that both classes come from compatible perfect obstruction theories with respect to different bases.

In the following we fix a decorated bipartite graph $\tilde{\Gamma}$ and use the following simplified notation. Let $\mathcal{Q}_V$ denote $\mathcal{Q}_{g_V, \alpha_V}^{\log}(X_{r(V)} | D, \beta_V)$. Similarly, we denote

$$\mathcal{Q}_{\tilde{\Gamma}} = \mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma}), \quad \prod_V \mathcal{Q}_V = \prod_V \mathcal{Q}_{g_V, \alpha_V}^{\log}(X_{r(V)} | D, \beta_V), \quad (11)$$

$$\mathfrak{M}_{\tilde{\Gamma}} = \mathfrak{M}_{g,n}^{\log}(\mathcal{A}, \Gamma), \quad \prod_V \mathfrak{M}_V = \prod_V \mathfrak{M}_{g_V, \alpha_V}^{\log}(\mathcal{A}). \quad (12)$$

Let $\pi_V : \mathcal{C}_V \rightarrow \mathcal{Q}_V$ be the universal curve over $\mathcal{Q}_V$, and let $\mathcal{C}_{V, \tilde{\Gamma}}$ denote the piece of the partial normalisation of $\mathcal{C}$ along the nodes associated to the edges of $\tilde{\Gamma}$, corresponding to the vertex V . With this, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{W} & \xleftarrow{i_{r(V)}} & \mathcal{X}_{r(V)} & & \\ \uparrow u & & \uparrow u_V & & \\ \mathcal{C} & \xleftarrow{j_V} & \mathcal{C}_{V, \tilde{\Gamma}} & \xrightarrow{\ell_V} & \mathcal{C}_V \\ \downarrow \pi & & & & \downarrow \pi_V \\ \mathcal{Q}_{\tilde{\Gamma}} & \xrightarrow{g_V} & \mathcal{Q}_V & & \end{array} \quad (13)$$

as well as

$$\begin{array}{ccc} \mathcal{Q}_{\tilde{\Gamma}} & \xrightarrow{\text{ev}_E} & D \subset \mathcal{D} \\ \downarrow i_E & & \downarrow i \\ \mathcal{C} & \xrightarrow{u} & \mathcal{W} \end{array}$$

where ev_E and i_E utilise the fact that the universal curve has a distinguished node for the edge E .

Recall that by Theorem 3.2, for each vertex $V \in V(\tilde{\Gamma})$, the complex $(\mathbf{R}\pi_{V,*} u_V^* \mathbb{T}_{\mathcal{X}_{r(V)}}^{\log})^{\vee}$ is a perfect obstruction theory for the morphism

$$\mathcal{Q}_{g_V, \alpha_V}^{\log}(X_{r(V)} | D, \beta_V) \rightarrow \mathfrak{M}_{g_V, \alpha_V}^{\log}(\mathcal{A}).$$

Let $\mathcal{E}$ denote the complex $u^* \mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log}$ on the universal curve $\mathcal{C}_{\tilde{\Gamma}}$ over $\mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma})$. Let

$$\mathbb{E} = (\mathbf{R}\pi_* \mathcal{E})^{\vee}.$$

By Proposition 3.8, $\mathbb{E}$ defines a perfect obstruction theory of $\mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma})$ relative to $\mathfrak{M}_{g,n}^{\log}(\mathcal{A}, \Gamma)$.

Lemma 4.14. *There is a distinguished triangle*

$$\mathcal{E} \rightarrow \bigoplus_{V \in V(\tilde{\Gamma})} j_{V,*} \ell_V^* u_V^* \mathbb{T}_{\mathcal{X}_{r(V)}}^{\log} \rightarrow \bigoplus_{E \in E(\tilde{\Gamma})} i_{E,*} \text{ev}_E^* \mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log} |_D \rightarrow \mathcal{E}[1] \quad (14)$$

where $i_E : \mathcal{Q}_{\tilde{\Gamma}} \rightarrow \mathcal{C}_{\tilde{\Gamma}}$ is the inclusion of the node corresponding to E and $\text{ev}_E : \mathcal{Q}_{\tilde{\Gamma}} \rightarrow D$ is the evaluation at the node.

Proof. There is a partial normalisation exact sequence on the universal curve $\mathcal{C}_{\tilde{\Gamma}}$ given by

$$0 \rightarrow \mathcal{O}_{\mathcal{C}_{\tilde{\Gamma}}} \rightarrow \oplus_{V \in V(\tilde{\Gamma})} j_{V,*} \mathcal{O}_{\mathcal{C}_{V,\tilde{\Gamma}}} \rightarrow \oplus_{E \in E(\tilde{\Gamma})} i_{E,*} \mathcal{O}_{\mathcal{Q}_{\tilde{\Gamma}}} \rightarrow 0$$

We claim the result follows from tensoring the associated distinguished triangle with $\mathcal{E}$. We have that $i_{E,*} \mathcal{O}_{\mathcal{Q}_{\tilde{\Gamma}}} \otimes u^* \mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log} \cong i_{E,*} (i_E^* u^* \mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log}) \cong i_{E,*} \text{ev}_E^* \mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log}|_D$. For the middle term we can notice from our explicit description above that $\mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log}|_{\mathcal{X}_r(V)} \cong \mathbb{T}_{\mathcal{X}_r(V)}^{\log}$ using the explicit description for the latter given in [Sha22, 3.5.20]. We have

$$j_{V,*} \mathcal{O}_{\mathcal{C}_{V,\tilde{\Gamma}}} \otimes u^* \mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log} \cong j_{V,*} j_V^* u^* \mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log} \cong j_V^* \ell_V^* u_V^* \mathbb{T}_{\mathcal{X}_r(V)}^{\log}$$

by the projection formula and commutativity of Diagram (13). This proves the claim. $\square$

Lemma 4.15. *We have a morphism*

$$\oplus_{V \in V(\tilde{\Gamma})} j_{V,*} \ell_V^* u_V^* \mathbb{T}_{\mathcal{X}_r(V)}^{\log} \rightarrow \oplus_{E \in E(\tilde{\Gamma})} i_{E,*} \text{ev}_E^* T_D. \quad (15)$$

Proof. Let $\mathcal{D}_{\rho}$ be the toric divisors of $\mathcal{X}_r(V)$ and without loss of generality we assume that $\mathcal{D} = \mathcal{D}_{\rho_1}$. The logarithmic tangent complex of $\mathcal{X}_r(V)$ is

$$\mathcal{O}_{\mathcal{X}_r(V)}^{\oplus s} \rightarrow \oplus_{\rho \neq \rho_1} \mathcal{O}_{\mathcal{X}_r(V)}(\mathcal{D}_{\rho}) \oplus \mathcal{O}_{\mathcal{X}_r(V)}$$

and the tangent complex of $\mathcal{D}$ is

$$\mathcal{O}_{\mathcal{D}}^{\oplus s} \rightarrow \oplus_{\rho \neq \rho_1} \mathcal{O}_{\mathcal{D}}(\mathcal{D}_{\rho})$$

To construct the morphism (15) it suffices to construct the morphism for $\mathbb{T}_{\mathcal{X}_r(V)}^{\log}$ and T_D replaced with either the first or second terms in the above complexes. Consider the diagram

$$\begin{array}{ccccc} \mathcal{C} & \xleftarrow{j_V} & \mathcal{C}_{V,\tilde{\Gamma}} & \xrightarrow{\ell_V} & \mathcal{C}_V \\ \uparrow i_E & \nearrow j_V^E & & \searrow u'_V & \downarrow u_V \\ \mathcal{Q}_{\tilde{\Gamma}} & \xrightarrow{\text{ev}_E} & D & \hookrightarrow & \mathcal{X}_r(V) \end{array}$$

For the second term in the complex (the first is similar) we see that

$$j_{V,*} u'_V{}^* (\oplus_{\rho \neq \rho_1} \mathcal{O}_{\mathcal{X}_r(V)}(\mathcal{D}_{\rho}) \oplus \mathcal{O}_{\mathcal{X}_r(V)}) \rightarrow j_{V,*} u'_V{}^* (\oplus_{\rho \neq \rho_1} \mathcal{O}_{\mathcal{D}}(\mathcal{D}_{\rho}))$$

by the restriction and the projection. And then we have a morphism

$$j_{V,*} u'_V{}^* (\oplus_{\rho \neq \rho_1} \mathcal{O}_{\mathcal{D}}(\mathcal{D}_{\rho})) \rightarrow j_{V,*} j_V^E{}^* j_V^E{}^* u'_V{}^* (\oplus_{\rho \neq \rho_1} \mathcal{O}_{\mathcal{D}}(\mathcal{D}_{\rho})) = i_{E,*} \text{ev}_E^* (\oplus_{\rho \neq \rho_1} \mathcal{O}_{\mathcal{D}}(\mathcal{D}_{\rho}))$$

By adjunction and commutativity of the diagram. This procedure builds a morphism like in (15), but only for one summand of the source and target for a vertex V contained in an edge. We form the morphism (15) by taking the difference of sections for every pair of vertices contained in an edge. $\square$

Remark 4.16. The geometric origin of this morphism comes from taking logarithmic tangent vectors in X_1 and X_2 , restricting them to D and taking their difference, see [Bou19, Proof of Proposition 19]. The construction is slightly more complicated in our case as we are working with the ambient quotient stacks $\mathcal{X}_1, \mathcal{X}_2$.

Notation 4.17. Let $\mathcal{F}$ be defined by the following distinguished triangle

$$\mathcal{F} \rightarrow \oplus_{V \in V(\tilde{\Gamma})} j_{V,*} \ell_V^* u_V^* \mathbb{T}_{\mathcal{X}_{r(V)}}^{\log} \rightarrow \oplus_{E \in E(\tilde{\Gamma})} i_{E,*} \text{ev}_E^* T_D \rightarrow \mathcal{F}[1]$$

on the universal curve over $\mathcal{Q}_{\tilde{\Gamma}}$. Let $\mathbb{F} := (\mathbf{R}\pi_* \mathcal{F})^\vee$.

Lemma 4.18. *In notation as above, we have that $\mathbb{F}$ is a perfect obstruction theory for the morphism*

$$\mathcal{Q}_{g,n}^{\log}(X_0, \tilde{\Gamma}) \rightarrow \prod_V \mathfrak{M}_{g_V, \alpha_V}^{\log}(\mathcal{A}).$$

The induced class is $c_{\tilde{\Gamma}}^* \Delta^! \prod_V [\mathcal{Q}_{g_V, \alpha_V}^{\log}(X_V | D, \beta_V)]^{\text{vir}}$.

Proof. Consider the diagram

$$\begin{array}{ccccc} & & g & & \\ & \nearrow c_{\tilde{\Gamma}} & & \searrow \Delta_{\mathcal{Q}} & \\ \mathcal{Q}_{\tilde{\Gamma}} & \xrightarrow{\quad} & \times_V \mathcal{Q}_V & \xrightarrow{\quad} & \prod_V \mathcal{Q}_V \longrightarrow \prod_V \mathfrak{M}_V \\ & \searrow \text{ev}_E & & \downarrow & \\ & \prod_{E \in E(\tilde{\Gamma})} D & \xrightarrow{\Delta} & \prod_{E \in E(\tilde{\Gamma})} D^2 & \end{array}$$

The map $g: \mathcal{Q}_{\tilde{\Gamma}} \rightarrow \prod_V \mathcal{Q}_V$ is the composition of the étale map $c_{\tilde{\Gamma}}$ and $\Delta_{\mathcal{Q}}$, so it admits a perfect obstruction theory induced by the regular embedding Δ . Let

$$\mathbb{B}^\vee := \oplus_{E \in E(\tilde{\Gamma})} \text{ev}_E^*[0 \rightarrow T_D]$$

in degrees $[0, 1]$. Then $\mathbb{B}$ is a perfect obstruction theory for g . We know that $\prod_V \mathcal{Q}_V \rightarrow \prod_V \mathfrak{M}_V$ admits a dual perfect obstruction theory

$$\mathbb{A}^\vee := \oplus_V \mathbf{R}\pi_{V,*} u_V^* \mathbb{T}_{\mathcal{X}_{r(V)}}^{\log}.$$

In the following we use $\mathbb{A}$ and $\mathbb{B}$ to construct a perfect obstruction theory for $\mathcal{Q}_{\tilde{\Gamma}} \rightarrow \prod_V \mathfrak{M}_V$. We first prove that $g^* \mathbb{A}^\vee \cong \oplus_{V \in V(\tilde{\Gamma})} \mathbf{R}\pi_* j_{V,*} \ell_V^* u_V^* \mathbb{T}_{\mathcal{X}_{r(V)}}^{\log}$. Consider the following commutative diagram with the square being Cartesian

$$\begin{array}{ccccc} \mathcal{C} & \xleftarrow{j_V} & \mathcal{C}_{V, \tilde{\Gamma}} & \xrightarrow{\ell_V} & \mathcal{C}_V \\ & \searrow \pi & \downarrow \pi_{V, \tilde{\Gamma}} & & \downarrow \pi_V \\ & & \mathcal{Q}_{\tilde{\Gamma}} & \xrightarrow{g_V} & \mathcal{Q}_V. \end{array}$$

The statement follows from $\pi_{V, \tilde{\Gamma},*} = \pi_* j_{V,*}$ and cohomology and base change for $u_V^* \mathbb{T}_{\mathcal{X}_{r(V)}}^{\log}$ in the above square.

Note that $\pi_* i_{E,*} \text{ev}_E^* T_D = \text{ev}_E^* T_D$. With these we can apply Equation (15) to get a morphism $g^* \mathbb{A}^\vee[-1] \rightarrow \mathbb{B}^\vee$. By the definition of $\mathbb{F}$ in Notation 4.17, we have a distinguished triangle

$$\mathbb{F} \rightarrow \mathbb{B} \rightarrow g^* \mathbb{A}[1] \rightarrow \mathbb{F}[1]$$

Moreover, we obtain a morphism of distinguished triangles

$$\begin{array}{ccccccc} \mathbb{B} & \longrightarrow & g^* \mathbb{A}[1] & \longrightarrow & \mathbb{F}[1] & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{L}_{\mathcal{Q}_{\tilde{\Gamma}}/\Pi \mathcal{Q}_V} & \longrightarrow & g^* \mathbb{L}_{\Pi \mathcal{Q}_V/\Pi \mathfrak{M}_\Gamma}[1] & \longrightarrow & \mathbb{L}_{\mathcal{Q}_{\tilde{\Gamma}}/\Pi \mathfrak{M}_V}[1] & \longrightarrow & \end{array}$$

where the commutativity of left square above is commutativity of Diagram 9.10 in [KLR21]. A repeated application of the Four Lemma implies that $\mathbb{F}$ is an obstruction theory for $\mathcal{Q}_{\tilde{\Gamma}} \rightarrow \prod_V \mathfrak{M}_V$. Moreover, the diagram above shows that we have a compatible triple of obstruction theories. By functoriality of pull-backs we have that the class induced by $\mathbb{F}$ is equal to the class

$$c_{\tilde{\Gamma}}^* \Delta^! \prod_V [\mathcal{Q}_{g_V, \alpha_V}^{\log}(X_V | D, \beta_V)]^{\text{vir}}.$$

□

We have morphisms

$$\mathcal{Q}_{\tilde{\Gamma}} \longrightarrow \mathfrak{M}_\Gamma \xrightarrow{\delta} \prod_V \mathfrak{M}_V.$$

Proposition 4.19. *Let $\mathbb{E}^\vee$ be the complex $\mathbf{R}\pi_* u^* \mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log}$. The dotted arrow in the following diagram can be filled in to form a commutative diagram in the derived category of $\mathcal{Q}_{\tilde{\Gamma}}$.*

$$\begin{array}{ccccccc} \mathbb{F} & \dashrightarrow & \mathbb{E} & \longrightarrow & \mathbb{L}_\delta[1] & \longrightarrow & \\ \downarrow & & \downarrow & & \Downarrow & & \\ \mathbb{L}_{\mathcal{Q}_{\tilde{\Gamma}}/\Pi \mathfrak{M}_V} & \longrightarrow & \mathbb{L}_{\mathcal{Q}_{\tilde{\Gamma}}/\mathfrak{M}_\Gamma} & \longrightarrow & \mathbb{L}_\delta[1] & \longrightarrow & \end{array}$$

Proof. If we take the pushforward of Equation (14) under π we obtain the distinguished triangle

$$\mathbf{R}\pi_* u^* \mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log} \rightarrow \oplus_{V \in V(\tilde{\Gamma})} \mathbf{R}\pi_* j_{V,*} \ell_V^* u_V^* \mathbb{T}_{\mathcal{X}_V}^{\log} \rightarrow \oplus_{E \in E(\tilde{\Gamma})} \text{ev}_E^* \mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log} |_D \rightarrow \quad (16)$$

To construct the dotted arrow we use the definition of $\mathbb{F}$, the distinguished triangle (16) and Equation (15). More precisely, we have solid arrows which induce the dashed arrow making the following diagram commute

$$\begin{array}{ccccccc} \mathbb{E}^\vee & \longrightarrow & \oplus_{V \in V(\tilde{\Gamma})} \mathbf{R}\pi_* j_{V,*} \ell_V^* u_V^* \mathbb{T}_{\mathcal{X}_V}^{\log} & \longrightarrow & \oplus_{E \in E(\tilde{\Gamma})} \text{ev}_E^* \mathbb{T}_{\mathcal{W}/\mathbb{A}^1}^{\log} |_D & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{F}^\vee & \longrightarrow & \oplus_{V \in V(\tilde{\Gamma})} \mathbf{R}\pi_* j_{V,*} \ell_V^* u_V^* \mathbb{T}_{\mathcal{X}_V}^{\log} & \longrightarrow & \oplus_{E \in E(\tilde{\Gamma})} \text{ev}_E^* T_D & \longrightarrow & \end{array}$$

We can complete the diagram

$$\begin{array}{ccccccc}
 & & & & \oplus_{E \in E(\tilde{\Gamma})} \mathrm{ev}_E^* \mathcal{O}_D & & \\
 & & & & \downarrow & & \\
 \mathbb{E}^\vee & \longrightarrow & \oplus_{V \in V(\tilde{\Gamma})} \mathbf{R}\pi_{*,*} j_{V,*} \ell_V^* u_V^* \mathbb{T}_{\mathcal{X}_V}^{\log} & \longrightarrow & \oplus_{E \in E(\tilde{\Gamma})} \mathrm{ev}_E^* T_{\mathcal{W}/\mathbb{A}^1}^{\log}|_D & \longrightarrow & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \mathbb{F}^\vee & \longrightarrow & \oplus_{V \in V(\tilde{\Gamma})} \mathbf{R}\pi_{*,*} j_{V,*} \ell_V^* u_V^* \mathbb{T}_{\mathcal{X}_V}^{\log} & \longrightarrow & \oplus_{E \in E(\tilde{\Gamma})} \mathrm{ev}_E^* T_D & \longrightarrow &
 \end{array}$$

The octahedron axiom gives a distinguished triangle

$$\mathbb{F} \rightarrow \mathbb{E} \rightarrow \oplus_{E \in E(\tilde{\Gamma})} \mathrm{ev}_E^* \mathcal{O}_D[1] \rightarrow \mathbb{F}[1].$$

By [KLR21], Lemma 9.2, we have $\oplus_{E \in E(\tilde{\Gamma})} \mathrm{ev}_E^* \mathcal{O}_D \simeq \mathbb{L}_\delta$. The only thing left to prove is that the map $\mathbb{F} \rightarrow \mathbb{E}$ obtained above is the same obtained by completing the morphism $\mathbb{E} \rightarrow \mathbb{L}_\delta$ to a distinguished triangle. This is the commutativity of diagram 9.12 in [KLR21]. $\square$

Proof of Theorem 4.13. We have a compatible triple and by functoriality of pull-backs we get the statement. $\square$

The following is the main result of this section. Let $W \rightarrow \mathbb{A}^1$ be a toric double point degeneration such that the associated quasimap moduli space is proper. Let $X_0 = X_1 \cup_D X_2$ denote the central fibre.

Theorem 4.20 (Degeneration formula).

$$[\mathcal{Q}_{g,n}^{\log}(X_0, \beta)]^{\mathrm{vir}} = \sum_{\tilde{\Gamma}} \frac{m_{\tilde{\Gamma}}}{|\mathrm{Aut}(\tilde{\Gamma})|} j_{\tilde{\Gamma},*} c_{\tilde{\Gamma}}^* \Delta^! \left(\prod_{V \in V(\tilde{\Gamma})} [\mathcal{Q}_{g_V, \alpha_V}^{\log}(X_{r(V)}|_D, \beta_V)]^{\mathrm{vir}} \right).$$

Proof. The statement follows from Proposition 4.7 together with Theorem 4.13. $\square$

Remark 4.21. In Section 5 we will use the degeneration formula with a certain point condition imposed on the moduli space directly. The proof of this result is analogous to this section. For details, see for example [Bou19, Section 7.3].

5. Local/logarithmic correspondence

In this section we prove the local/logarithmic correspondence for quasimaps. Mirroring the Gromov–Witten version [vGGR19], we will use the degeneration formula proved in the previous section to prove this correspondence for projective space and a coordinate hyperplane. We will then use this to pull back the local/logarithmic correspondence to any very ample set up. We will then show that stronger results hold than in the stable maps setting, see Theorem 5.19 and Theorem 5.20. The proof of these theorems will utilise the same strategy to prove these correspondences for any very ample set up.

5.1. Smooth divisor

Theorem 5.1. *Let X be a GIT quotient $W//G$ and let $D \subset X$ be a smooth, very ample divisor satisfying Assumptions 2.1 and 3.1. Let β be an effective curve class on X and let $d = D \cdot \beta \geq 0$, then we have the following equality of virtual classes in $A_*(\mathcal{Q}_{0,2}(X, \beta))$*

$$[\mathcal{Q}_{0,(d,0)}^{\log}(X|D, \beta)]^{\text{vir}} = (-1)^{d+1} \cdot \text{ev}_2^* D \cap [\mathcal{Q}_{0,2}(\mathcal{O}_X(-D), \beta)]^{\text{vir}}.$$

As the degeneration formula for quasimaps in Theorem 4.20 only applies in a toric setting we will first use the degeneration formula to prove

Theorem 5.2. *Let $X = \mathbb{P}^N$ for $N > 1$ and $D = H$ a coordinate hyperplane, and let $d \in \mathbb{N}$. Then we have the following equality in $A_*(\mathcal{Q}_{0,2}(\mathbb{P}^N, d))$*

$$[\mathcal{Q}_{0,(d,0)}^{\log}(\mathbb{P}^N|H, d)]^{\text{vir}} = (-1)^{d+1} \cdot \text{ev}_2^* H \cap [\mathcal{Q}_{0,2}(\mathcal{O}_{\mathbb{P}^N}(-H), d)]^{\text{vir}}.$$

Assuming Theorem 5.2, the proof of Theorem 5.1 is straightforward.

Proof of Theorem 5.1. We have a Cartesian diagram (see [Sha24, Proof of Theorem 5.5]),

$$\begin{array}{ccc} \mathcal{Q}_{0,(d,0)}^{\log}(X|D, \beta) & \xrightarrow{i} & \mathcal{Q}_{0,(d,0)}^{\log}(\mathbb{P}^N|H, d) \\ \downarrow p & & \downarrow p' \\ \mathcal{Q}_{0,2}(X, \beta) & \xrightarrow{i'} & \mathcal{Q}_{0,2}(\mathbb{P}^N, d) \end{array} \quad (17)$$

where the vertical arrows are the forgetful morphisms which only remember the underlying quasimap ([Sha24, Definition 2.4]). The existence of the horizontal arrows follows similarly from [CFK14, Section 3.1]. We can use the fact [Sha22, Proposition 3.5.10] that the virtual class on $\mathcal{Q}_{0,(d,0)}^{\log}(X|D, \beta)$ can be defined relative to $\mathcal{Q}_{0,(d,0)}^{\log}(\mathbb{P}^N|H, d)$ and is in fact pulled back from the obstruction theory for i' , which defines the virtual class on $\mathcal{Q}_{0,2}(X, \beta)$. Let $\mathcal{L}$ (resp. $\mathcal{L}_D$) denote the universal line bundle on the universal curve $\mathcal{Q}_{0,2}(\mathbb{P}^N, d)$ (resp. $\mathcal{Q}_{0,2}(X, \beta)$) corresponding to $\mathcal{O}_{\mathbb{P}^N}(H)$ (resp. $\mathcal{O}_X(D)$). We then have that

$$\begin{aligned} p_*[\mathcal{Q}_{0,(d,0)}^{\log}(X|D, \beta)]^{\text{vir}} &= p_*(i')^! [\mathcal{Q}_{0,(d,0)}^{\log}(\mathbb{P}^N|H, d)] \\ &= i'^! p'_* [\mathcal{Q}_{0,(d,0)}^{\log}(\mathbb{P}^N|H, d)] \\ &= i'^! (-1)^{d+1} \cdot \text{ev}^* H \cap [\mathcal{Q}_{0,2}(\mathcal{O}_{\mathbb{P}^N}(-H), d)]^{\text{vir}} \\ &= i'^! (-1)^{d+1} \cdot \text{ev}^* H \cap e(\mathbf{R}^1 \pi_* \mathcal{L}^\vee) \cap [\mathcal{Q}_{0,2}(\mathbb{P}^N, d)] \\ &= (-1)^{d+1} \cdot \text{ev}^* D \cap e(\mathbf{R}^1 \pi_* \mathcal{L}_D^\vee) \cap [\mathcal{Q}_{0,2}(X, \beta)]^{\text{vir}} \\ &= (-1)^{d+1} \cdot \text{ev}^* D \cap [\mathcal{Q}_{0,2}(\mathcal{O}_X(-D), \beta)]^{\text{vir}}. \end{aligned}$$

□

Remark 5.3. Although we only prove Theorem 5.2 for projective spaces of dimension greater than one, Theorem 5.1 implies the corresponding result for $\mathbb{P}^1$. The reason for

the restriction is that for $\mathbb{P}^1$ the induced presentation of the central fibre components are slightly more complicated so we exclude this analysis. On the other hand, the fact that the local/logarithmic correspondence holds for quasimaps for $\mathbb{P}^1$ may be surprising. For example, a consequence is that the Gromov–Witten/quasimap wall-crossing for $\mathcal{O}_{\mathbb{P}^1}(-1)$ is trivial, after capping with a hyperplane class. This follows from the fact that the logarithmic spaces are birational. This is not at all an obvious conclusion from [CFK17, Theorem 4.2.1]. On the other hand there is precedent for this, in [MOP11, Section 9] the authors show that quasimap and Gromov–Witten invariants of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ coincide in all genera.

In the remainder of Section 5.1, we prove Theorem 5.2. The proof, which is deferred until the end of Section 5.1, is a combination of the equalities in Lemma 5.7 and Propositions 5.8 and 5.9. Furthermore, Proposition 5.8 is the result of applying the degeneration formula (Theorem 4.20) and ruling out all the contributions except for the graph

$$\bullet \xrightarrow{E} \bullet$$

$V_1 \quad E \quad V_2$

with V_1 an L_1 -vertex, V_2 an L_2 -vertex, $w_E = d$, $\beta_{V_1} = (d, -d)$, $\beta_{V_2} = (0, d)$ and exactly one marking on each vertex. To do so, we follow the next steps:

- (1) Lemma 5.11 shows there must be two vertices with a single bounded edge connecting them.
- (2) Lemma 5.12 shows there are no contributions from graphs with *at least one* L_2 -vertex with degree different from a multiple class of a fibre.
- (3) Proposition 5.17 shows there are no contributions from graphs with one edge, but ghost classes. This uses Corollary 5.16, which follows from Lemma 5.15.
- (4) Lemma 5.18 shows there are no contributions from graphs contained in L_1 .

This argument follows [vGGR19], with two alterations.

The first alteration is minor. The main theorem of [vGGR19] compares the one marked logarithmic space with the unmarked local space and involves a forgetful morphism. Since neither of these spaces exist in quasimap theory we use the alteration to the degeneration argument discussed in [FW21, TY23b], in addition to adding a redundant marking.

The second alteration requires an extra argument. On the one hand the stability condition of quasimap theory allows us to exclude certain graphs without effort. On the other hand we require an additional vanishing of contributions where the degrees contains ‘ghost’ curve classes in the sense of Definition 2.10. These occur because, as in Examples 2.8 and 2.9, the involved spaces of quasimaps depend on the GIT presentation of the target space.

Notation 5.4. With the notations of Example 2.11, let $W(\mathbb{P}^N, H) = \text{Bl}_{H \times 0}(\mathbb{P}^N \times \mathbb{A}^1)$ be the deformation to the normal cone of H in $\mathbb{P}^N$ for $N > 1$. The central fibre is a union of $X_1 \simeq X = \mathbb{P}^N$ and $X_2 \simeq \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(H)|_H)$ glued along H .

Let $L = \mathcal{O}(-\tilde{H})$ denote the total space of the line bundle over $W(\mathbb{P}^N, H)$, where $\tilde{H}$ is the strict transform of $H \times \mathbb{A}^1$. Over a non-zero point of $\mathbb{A}^1$ the fibre is just $\mathcal{O}_{\mathbb{P}^N}(-H)$. The central fibre L_0 is a union of two components $L_1 = \mathbb{P}^N \times \mathbb{A}^1$ and $L_2 = \mathcal{O}(-H_\infty)$

over $X_2 \simeq \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(H)|_H)$, glued along $L_H = H \times \mathbb{A}^1$. We follow the convention from Notation 2.5 to denote ambient stacks with calligraphic letters, such as $\mathcal{L}$ for the stack corresponding to L and $\mathcal{L}_0$ for L_0 .

The degree of a quasimap to $W(\mathbb{P}^N, H)$, with its toric presentation, is given by $\text{Pic}^{\mathbb{G}_m^2} \mathbb{A}^5 \simeq \mathbb{Z}^2$. We choose the same basis $\{l, f\}$ as in Example 2.11.

Remark 5.5. Although the degeneration formula is written in terms of logarithmic quasimap spaces with presentation induced by the total space we have that

$$\mathcal{Q}_{g,\alpha}^{\log}(\mathbb{P}^N|H, (d, -d)) \simeq \mathcal{Q}_{g,\alpha}^{\log}(\mathbb{P}^N|H, d)$$

where the former space uses the presentation $\mathbb{P}^N = \mathbb{A}^{N+2} // \mathbb{G}_m^2$ induced by $W(\mathbb{P}^N, H)$ and the latter uses the standard presentation of $\mathbb{P}^N$. The same is true for $L_1|L_H$.

Definition 5.6. Let ev_2 denote the evaluation at the second marked point. We define $\mathcal{Q}_{0,2}^{\log}(L, (d, 0))^{\tilde{H}}$ as the fibre product

$$\begin{array}{ccc} \mathcal{Q}_{0,2}^{\log}(L, (d, 0))^{\tilde{H}} & \longrightarrow & \mathcal{Q}_{0,2}^{\log}(L, (d, 0)) \\ \downarrow & & \downarrow \text{ev}_2 \\ \tilde{H} & \xrightarrow{i_{\tilde{H}}} & W(\mathbb{P}^N, H) \end{array}$$

Note that although L is not proper, $\mathcal{Q}_{0,2}^{\log}(L, (d, 0))$ is proper over $\mathbb{A}^1$ (as in [vGGR19, Section 2.2]). The moduli space $\mathcal{Q}_{0,2}^{\log}(L, (d, 0))^{\tilde{H}}$ comes with a virtual class given by Gysin pullback by $i_{\tilde{H}}$. The following follows from the argument of [vGGR19, Lemma 2.2].

Lemma 5.7. *There is a morphism $P : \mathcal{Q}_{0,2}^{\log}(L_0, (d, 0))^{\tilde{H}} \rightarrow \mathcal{Q}_{0,2}(\mathbb{P}^N, d)$. Moreover,*

$$P_*[\mathcal{Q}_{0,2}^{\log}(L_0, (d, 0))^{\tilde{H}}]^{\text{vir}} = \text{ev}_2^* H \cap [\mathcal{Q}_{0,2}(\mathcal{O}_{\mathbb{P}^N}(-H), d)]^{\text{vir}}.$$

Next we apply the degeneration formula. We will prove that the analogue of [vGGR19, Theorem 2.3] also applies in this situation.

Let us first introduce the notation. Consider the following diagram

$$\begin{array}{ccc} \mathcal{Q}_{0,(d,0)}^{\log}(L_1|L_H, d) \times_{L_H} \mathcal{Q}_{(d,0)}^{\log}(L_2|L_H, (0, d))^{H_{\infty}} & \xleftarrow{c_{\tilde{\Gamma}}} \mathcal{Q}_{\tilde{\Gamma}} & \xrightarrow{j_{\tilde{\Gamma}}} \mathcal{Q}_{0,2}^{\log}(L, (d, 0))^{\tilde{H}} \\ \downarrow \text{pr}_1 & & \downarrow P \\ \mathcal{Q}_{0,(d,0)}^{\log}(\mathbb{P}^N|H, d) & \xrightarrow{F} & \mathcal{Q}_{0,2}(\mathbb{P}^N, d). \end{array} \quad (18)$$

The notation $\mathcal{Q}_{\tilde{\Gamma}}$ was introduced in Equation (11). Here $\tilde{\Gamma}$ is the graph with one L_1 -vertex of degree $(d, -d)$, one L_2 -vertex of degree $(0, d)$ (that is, d times a fibre class), an edge of weight d connecting the two vertices and an extra marking on each vertex. In addition there is a constraint to pass through $\tilde{H}$ as in Definition 5.6, corresponding to the leg on the L_2 -vertex. This graph determines the spaces $\mathcal{Q}_{0,(d,0)}^{\log}(L_1|L_H, (d, -d))$ and $\mathcal{Q}_{0,(d,0)}^{\log}(L_2|L_H, (0, d))^{H_{\infty}}$ by the degeneration formula. In general the former is a quasimap

space with $L_1 = \mathbb{P}^N \times \mathbb{A}^1$ given by a non-toric presentation. However, by Remark 5.5, for the degree $(d, -d)$ this space can be identified with the logarithmic quasimap space to L_1 with its toric presentation and with degree d .

Proposition 5.8. *Let $\Delta : L_H \rightarrow L_H \times L_H$ be the diagonal embedding. We have that*

$$[\mathcal{Q}_{0,2}^{\log}(L_0, (d, 0))^{\tilde{H}}]^{\text{vir}} = d \cdot j_{\tilde{\Gamma},*} \left(c_{\tilde{\Gamma}}^* \Delta^! \left([\mathcal{Q}_{0,(d,0)}^{\log}(L_1 | L_H, d)]^{\text{vir}} \right. \right. \\ \left. \left. \times [\mathcal{Q}_{0,(d,0)}^{\log}(L_2 | L_H, (0, d))^{H_\infty}]^{\text{vir}} \right) \right),$$

Proof. Applying the degeneration formula in Theorem 4.20 to L , we get

$$[\mathcal{Q}_{0,2}^{\log}(L_0, (d, 0))^{\tilde{H}}]^{\text{vir}} = \sum_{\tilde{\Gamma}} \frac{m_{\tilde{\Gamma}}}{|\text{Aut}(\tilde{\Gamma})|} j_{\tilde{\Gamma},*} c_{\tilde{\Gamma}}^* \Delta^! \left(\prod_{V \in V(\tilde{\Gamma})} [\mathcal{Q}_{0,\alpha_V}^{\log}(L_{r(V)} | L_H, \beta_V)]^{\text{vir}} \right)$$

We now need to show that all terms except one vanish.

This follows from combining Lemma 5.11, Lemma 5.12, Proposition 5.17, Lemma 5.18. Finally, we see that for $\tilde{\Gamma}$ the graph described above, we have $\frac{m_{\tilde{\Gamma}}}{|\text{Aut}(\tilde{\Gamma})|} = d$ since there are no automorphisms and $m_{\tilde{\Gamma}}$ can be given by the lowest common multiple of the weights. $\square$

Proposition 5.9. *With the previous notations,*

$$\text{ev}_{1,*} \left([\mathcal{Q}_{(d,0)}^{\log}(L_2 | L_H, (0, d))^{H_\infty}]^{\text{vir}} \right) = \frac{(-1)^{d+1}}{d} [H].$$

Proof. As in [vGGR19, Proposition 2.4], it is enough to show via localisation that

$$\deg \left(\text{ev}_2^*(H_\infty) \cap e(\mathbf{R}^1 \pi_* u^* \mathcal{O}_{\mathbb{P}^1}(-1)) \cap [\mathcal{Q}_{(d,0)}^{\log}(\mathbb{P}^1 | 0, d)]^{\text{vir}} \right) = \frac{(-1)^{d+1}}{d}. \quad (19)$$

By Lemma 5.10, there is a unique graph γ contributing to the localisation computation. By the proof of [BP05, Theorem 5.1], the same holds for stable maps. Since the fixed locus corresponding to γ lies in the interior of both moduli spaces, we have that

$$\deg \left(\text{ev}_2^*(H_\infty) \cap e(\mathbf{R}^1 \pi_* u^* \mathcal{O}_{\mathbb{P}^1}(-1)) \cap [\mathcal{Q}_{(d,0)}^{\log}(\mathbb{P}^1 | 0, d)]^{\text{vir}} \right) = \quad (20) \\ = \deg \left(\text{ev}_2^*(H_\infty) \cap e(\mathbf{R}^1 \pi_* f^* \mathcal{O}_{\mathbb{P}^1}(-1)) \cap [\overline{\mathcal{M}}_{(d,0)}^{\log}(\mathbb{P}^1 | 0, d)]^{\text{vir}} \right)$$

Therefore it is enough to show via localisation that

$$\deg \left(\text{ev}_2^*(H_\infty) \cap e(\mathbf{R}^1 \pi_* f^* \mathcal{O}_{\mathbb{P}^1}(-1)) \cap [\overline{\mathcal{M}}_{(d,0)}^{\log}(\mathbb{P}^1 | 0, d)]^{\text{vir}} \right) = \frac{(-1)^{d+1}}{d}. \quad (21)$$

The fixed locus associated to γ is isomorphic to $[\text{pt}/\mathbb{Z}_d]$. The equivariant Euler class of the virtual normal bundle can be written as a product of three terms corresponding to the vertices, edges and flags of γ , see [GP99, GV05]. We choose a 1-dimensional subtorus with equivariant parameter $\hbar$ and a lift of the action to $\mathcal{O}_{\mathbb{P}^1}(-1)$ analogous to those in the proof of [BP05, Theorem 5.1] (with the roles of 0 and ∞ reversed). We compute that

$$e_\gamma^F = -\frac{1}{\hbar^2}, \quad e_\gamma^v = \hbar^2, \quad e_\gamma^e = (-1)^d \frac{d! \hbar^d}{d^d}.$$

Therefore the equivariant Euler class of the virtual normal bundle at the locus corresponding to γ is

$$e(N_{\gamma}^{\text{vir}}) = e_{\gamma}^F e_{\gamma}^v e_{\gamma}^e = (-1)^{d+1} \frac{d! \hbar^d}{d^d}. \quad (22)$$

The contribution of the obstruction bundle is

$$e(\mathbf{R}^1 \pi_* f^* \mathcal{O}_{\mathbb{P}^1}(-1)) = \frac{(d-1)! \hbar^{d-1}}{d^{d-1}}. \quad (23)$$

Multiplying together (23), the inverse of (22), a factor $1/d$ coming from the automorphism group of the fixed point corresponding to γ and a factor $\hbar$ coming from the condition that the second mark is mapped to a hyperplane class, we get (21). We conclude by combining (19), (20) and (21). $\square$

Lemma 5.10. *Consider the setting of Proposition 5.9. In the localisation computation of the invariant*

$$\deg \left(\text{ev}_2^*(H_{\infty}) \cap e(\mathbf{R}^1 \pi_* f^* \mathcal{O}_{\mathbb{P}^1}(-1)) \cap [\mathcal{Q}_{(d,0)}^{\log}(\mathbb{P}^1|0,d)]^{\text{vir}} \right)$$

the unique graph whose contribution is non-zero is

$$\gamma = \begin{array}{ccc} \{1\} & d & \{2\} \\ \bullet & \text{---} & \bullet \\ V_1 & & V_2 \end{array}$$

with V_1 mapped to 0 and V_2 mapped to ∞ .

Proof. Recall that, in the context of localisation, each vertex in a graph γ corresponds to a connected component of the preimage of the fixed points on the target and each edge corresponds to a multiple cover of the target $\mathbb{P}^1$. We decorate edges with the labels of the marked points and edges with the degree of the cover.

We would like to do the localisation calculation directly on $\mathcal{Q}_{0,(d,0)}^{\log}(\mathbb{P}^1|0,d)$. For a technical reason this is not possible, since the base of the perfect obstruction theory is not smooth. On the other hand one can use the procedure of [Ran22, Section 3.4] to produce a logarithmic modification

$$\widetilde{\mathcal{Q}}_{0,(d,0)}^{\log}(\mathbb{P}^1|0,d) \rightarrow \mathcal{Q}_{0,(d,0)}^{\log}(\mathbb{P}^1|0,d) \quad (24)$$

whose obstruction theory is defined relative to the space of logarithmic maps to expansions of $\mathcal{A}$.

Consider a fixed locus $\mathcal{Q}_{\gamma}$ in $\widetilde{\mathcal{Q}}_{0,(d,0)}^{\log}(\mathbb{P}^1|0,d)$.

Elements in $\mathcal{Q}_{(d,0)}^{\log}(\mathbb{P}^1|0,d)$ have irreducible source by stability. Indeed, if we have two rational components, as rational tails are disallowed, we must have a marking on each component, and neither component is allowed to be contracted. Consequently, the component carrying the tangency must come from the line bundle $\mathcal{O}(d)$ but then the other component would be contracted. So the morphism (24) forces quasimaps in $\mathcal{Q}_{\gamma}$ to have irreducible source.

The tangency condition $(d, 0)$ ensures that the first marked point is mapped to 0. By the factor $\text{ev}_2^*(H_\infty)$, the second marked point must be mapped to ∞ . This is enough to conclude that the graph γ in the statement is indeed the only contributing graph. $\square$

Lemma 5.11. *Let $\tilde{\Gamma}$ be a decorated bipartite graph appearing in the degeneration formula. Then if $\tilde{\Gamma}$ is not of the form, two vertices with a single edge connecting them, then*

$$[\mathcal{Q}_{0,2}^{\log}(L_0, \tilde{\Gamma})]^{\text{vir}} = 0.$$

Proof. The exact same argument of [vGGR19, Lemma 3.1] tells us that if $\tilde{\Gamma}$ contains an L_1 -vertex with at least two adjacent bounded edges then $[\mathcal{Q}_{0,2}^{\log}(L_0, \tilde{\Gamma})]^{\text{vir}} = 0$. On the other hand, quasimap spaces in genus zero need to have at least 2 markings by stability. Since we have two markings in total, one of which is forced to correspond to an L_2 vertex, we have that any graph which contributes must have two vertices and a single bounded edge between them with the markings distributed to either side. $\square$

Lemma 5.12. *If V is an L_2 -vertex with $p_*\beta_V \neq 0$ then the class $[\mathcal{Q}_{0,\alpha_V}^{\log}(L_2|L_H, \beta_V)]^{\text{vir}}$ pushes forward to 0 in $\mathcal{Q}_{0,n_V+r_V}(H, p_*\beta_V)$.*

Proof. By the computation of virtual dimensions in [vGGR19, Lemma 5.1] we have that the virtual dimension of $\mathcal{Q}_{0,\alpha_V}^{\log}(L_2|L_H, \beta_V)$ is strictly greater than the virtual dimension of $\mathcal{Q}_{0,n_V+r_V}(H, p_*\beta_V)$ if $p_*\beta_V \neq 0$. On the other hand $\mathcal{Q}_{0,n_V+r_V}(H, p_*\beta_V)$ is smooth of the expected dimension and so the result follows. $\square$

Notation 5.13. Let $\tilde{\Gamma}$ be a decorated bipartite graph with two vertices and a single edge E . We use the following (slightly abusive) notation for the universal curve and the irreducible components of the universal curve.

$$\begin{array}{ccc} \mathcal{C}_{\tilde{\Gamma}} & \xrightarrow{f} & \mathcal{L}_0 \\ \downarrow \pi & & \downarrow \pi^i \\ \mathcal{Q}_{\tilde{\Gamma}} & & \mathcal{Q}_{\tilde{\Gamma}}^i \end{array}$$

We denote by P the section corresponding to the node.

In the following, we want to exclude the contribution from ghost classes (see Definition 2.10). Thanks to Lemma 5.11 and Lemma 5.12, we only need to rule out the following type of graphs. Below we use the basis $\{l, f\}$ of $H_2(W(\mathbb{P}, H))$ introduced in Example 2.11.

Definition 5.14. Consider the double point degeneration $W(\mathbb{P}^N, H) \rightarrow \mathbb{A}^1$. Let $\tilde{\Gamma}$ be a graph

$$\bullet \xrightarrow{E} \bullet$$

with exactly one marking on each vertex, V_1 a X_1 -vertex, V_2 an X_2 -vertex, $w_E = e$ and with curve classes $\beta_{V_1} = (d, -e)$ and $\beta_{V_2} = (0, e)$ with $d \geq e \geq 0$. We say that $\tilde{\Gamma}$ is a *ghost graph* if $d \neq e$. Equivalently, $\tilde{\Gamma}$ is a ghost graph if β_{V_1} is a ghost class for the embedding $\mathbb{P}^N \hookrightarrow W(\mathbb{P}^N, H)$ through the special fibre, in the sense of Definition 2.10.

Lemma 5.15. *Let $\tilde{\Gamma}$ be a ghost graph. The moduli space $\mathcal{Q}_{0,2}^{\log}(W_0(\mathbb{P}^N, H), \tilde{\Gamma})$ is smooth over $\mathfrak{Pic}_{0,2,\underline{d}}^{\log,s}(\mathcal{A}_0, \Gamma)$.*

Proof. We show that $\mathcal{Q}_{0,2}^{\log}(W_0(\mathbb{P}^N, H), \tilde{\Gamma})$ is unobstructed over $\mathfrak{Pic}_{0,2,\underline{d}}^{\log,s}(\mathcal{A}_0, \Gamma)$. Using the description of the toric divisors of $W(\mathbb{P}^N, H)$ in Example 2.11, we see that

$$\begin{aligned} \tilde{H}_1 \cdot (d, -e) &= d & \tilde{H}_1 \cdot (0, e) &= 0 \\ \tilde{H} \cdot (d, -e) &= d - e & \tilde{H} \cdot (0, e) &= e. \end{aligned} \quad (25)$$

We consider the normalisation sequence of $\mathcal{C}$, the universal curve over $\mathcal{Q}_{0,2}^{\log}(W_0(\mathbb{P}^N, H), \tilde{\Gamma})$,

$$0 \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow \mathcal{O}_{\mathcal{C}_1} \oplus \mathcal{O}_{\mathcal{C}_2} \rightarrow \mathcal{O}_P \rightarrow 0.$$

Given the obstruction theory in Proposition 3.15, we take the sheaf

$$\mathcal{E}(W) := \bigoplus_{i=1}^N \mathcal{O}_{\mathcal{C}}(\tilde{H}_i) \oplus \mathcal{O}_{\mathcal{C}}(\tilde{H}) \oplus \mathcal{O}_{\mathcal{C}}. \quad (26)$$

on $\mathcal{C}$ and the sheaves

$$\mathcal{E}_{X_i|H} := \bigoplus_{\rho \neq \rho_0 \in \Sigma_{X_i}(1)} \mathcal{O}_{\mathcal{C}_i}(D_{\rho}) \oplus \mathcal{O}_{\mathcal{C}_i}$$

on $\mathcal{C}_i$ for $i = 1, 2$, with $D_{\rho_0} = H$ in both cases.

Tensoring the normalisation sequence of $\mathcal{O}_{\mathcal{C}}$ with $\mathcal{E}(W)$ and pushing forward to $\mathcal{Q}_{0,2}^{\log}(W_0(\mathbb{P}^N, H), \tilde{\Gamma})$, we get a long exact sequence

$$\begin{aligned} 0 \rightarrow \mathbf{R}^0 \pi_* \mathcal{E}(W) &\rightarrow \mathbf{R}^0 \pi_*^1 \mathcal{E}_{X_1|H} \oplus \mathbf{R}^0 \pi_*^2 \mathcal{E}_{X_2|H} \xrightarrow{\phi} \mathbf{R}^0 \pi_* \mathcal{E}|_P \rightarrow \\ &\rightarrow \mathbf{R}^1 \pi_* \mathcal{E}(W) \rightarrow \mathbf{R}^1 \pi_*^1 \mathcal{E}_{X_1|H} \oplus \mathbf{R}^1 \pi_*^2 \mathcal{E}_{X_2|H} \rightarrow 0. \end{aligned}$$

Since ϕ is surjective and since $\mathbf{R}^1 \pi_*^1 \mathcal{E}_{X_1|H} \oplus \mathbf{R}^1 \pi_*^2 \mathcal{E}_{X_2|H} = 0$ by (25), we conclude that $\mathbf{R}^1 \pi_* \mathcal{E}(W) = 0$. This proves the claim. $\square$

Corollary 5.16. *Let $\tilde{\Gamma}$ be a ghost graph. The moduli space $\mathcal{Q}_{0,2}^{\log}(L_0, \tilde{\Gamma})$ is smooth over $\mathfrak{Pic}_{0,2,\underline{d}}^{\log,s}(\mathcal{A}'_0, \Gamma)$.*

Proof. Since $\tilde{\mathcal{D}}$ is effective, we have that maps from (nodal) curves to $\mathcal{L}$ factor through the zero section. This shows that $\mathcal{Q}_{0,2}^{\log}(W_0(\mathbb{P}^N, H), \tilde{\Gamma}) \simeq \mathcal{Q}_{0,2}^{\log}(L/\mathbb{A}^1, \tilde{\Gamma})$. Lemma 5.15 implies the claim. $\square$

Proposition 5.17. *Let $\tilde{\Gamma}$ be a ghost graph. Then*

$$[\mathcal{Q}_{0,2}^{\log}(L_0, \tilde{\Gamma})]^{\text{vir}} = 0$$

Proof. In this proof we write W for $W(\mathbb{P}^N, H)$, as there is no risk of confusion.

Let $\mathcal{C}$ denote the universal curve over $\mathcal{Q}_{0,2}^{\log}(L_0, \tilde{\Gamma}) \simeq \mathcal{Q}_{0,2}^{\log}(W_0, \tilde{\Gamma})$. By Proposition 3.15, the obstruction theory on $\mathcal{Q}_{0,2}^{\log}(W_0, \tilde{\Gamma})$ is determined by the sheaf $\mathcal{E}(W)$ on $\mathcal{C}$, defined in (26), while the one on $\mathcal{Q}_{0,2}^{\log}(L_0, \tilde{\Gamma})$ is controlled by

$$\mathcal{E}(L) = \mathcal{E}(W) \oplus \mathcal{O}_{\mathcal{C}}(Z). \quad (27)$$

Tensoring the normalisation sequence of $\mathcal{O}_C$ with $\mathcal{E}(L)$, we get the following long exact sequence on $\mathcal{Q}_{0,2}^{\log}(L_0, \tilde{\Gamma})$

$$\begin{aligned} 0 \rightarrow \mathbf{R}^0 \pi_* \mathcal{E}(L) \rightarrow \mathbf{R}^0 \pi_*^1 \mathcal{E}_{L_1|H} \oplus \mathbf{R}^0 \pi_*^2 \mathcal{E}_{L_2|H} \xrightarrow{\phi} \mathbf{R}^0 \pi_* \mathcal{E}(L)|_P \rightarrow \\ \rightarrow \mathbf{R}^1 \pi_* \mathcal{E}(L) \rightarrow \mathbf{R}^1 \pi_*^1 \mathcal{E}_{L_1|H} \oplus \mathbf{R}^1 \pi_*^2 \mathcal{E}_{L_2|H} \rightarrow 0 \end{aligned} \quad (28)$$

with

$$\mathcal{E}_{L_i|H} = \mathcal{E}_{X_i|H} \oplus \mathcal{O}_{C_i}(Z). \quad (29)$$

By Corollary 5.16, $\mathcal{Q}_{0,2}^{\log}(W_0, \tilde{\Gamma}) \simeq \mathcal{Q}_{0,2}^{\log}(L_0, \tilde{\Gamma})$ is smooth over $\mathfrak{Pic}_{0,2,\underline{d}}^{\log,s}(\mathcal{A}'_0)$ and it has $\mathbf{R}^1 \pi_* \mathcal{E}(L)$ as an obstruction bundle. This shows that

$$[\mathcal{Q}_{0,2}^{\log}(L_0, \tilde{\Gamma})]^{\text{vir}} = e(\mathbf{R}^1 \pi_* \mathcal{E}(L)) \cap [\mathcal{Q}_{0,2}^{\log}(W_0, \tilde{\Gamma})]^{\text{vir}}. \quad (30)$$

In the following we show that the Euler class of $\mathbf{R}^1 \pi_* \mathcal{E}$ is zero.

We first show that we have an exact sequence

$$\mathbf{R}^0 \pi_*^1 \mathcal{E}_{L_1|H} \oplus \mathbf{R}^0 \pi_*^2 \mathcal{E}_{L_2|H} \xrightarrow{\phi} \mathbf{R}^0 \pi_* \mathcal{E}(L)|_P \rightarrow \mathbf{R}^0 \pi_* \mathcal{O}_C(Z)|_P \rightarrow 0. \quad (31)$$

For that, we use (27) and (29) to re-write the morphism ϕ as follows

$$\begin{aligned} (\mathbf{R}^0 \pi_*^1 \mathcal{E}_{X_1|H} \oplus \mathbf{R}^0 \pi_*^2 \mathcal{E}_{X_2|H}) \oplus (\mathbf{R}^0 \pi_*^1 \mathcal{O}_{C_1}(Z) \oplus \mathbf{R}^0 \pi_*^2 \mathcal{O}_{C_2}(Z)) \\ \xrightarrow{\phi} (\mathbf{R}^0 \pi_* \mathcal{E}(W)|_P) \oplus \mathbf{R}^0 \pi_* \mathcal{O}_C(Z)|_P. \end{aligned}$$

We have that

$$\mathbf{R}^0 \pi_*^1 \mathcal{E}_{X_1|H} \oplus \mathbf{R}^0 \pi_*^2 \mathcal{E}_{X_2|H} \rightarrow \mathbf{R}^0 \pi_* \mathcal{E}(W)|_P$$

is surjective and by Example 2.11 we have that

$$(d, -e) \cdot [Z] = e - d < 0 \quad \text{and} \quad (0, e) \cdot [Z] = -e < 0, \quad (32)$$

which shows that $\mathbf{R}^0 \pi_*^1 \mathcal{O}_{C_1}(Z) \oplus \mathbf{R}^0 \pi_*^2 \mathcal{O}_{C_2}(Z) = 0$. This shows that the cokernel of ϕ is $\mathbf{R}^0 \pi_* \mathcal{O}_C(Z)|_P$, which proves the exactness of (31).

Combining (31) and (28), we get a short exact sequence

$$0 \rightarrow \mathbf{R}^0 \pi_* \mathcal{O}_C(Z)|_P \rightarrow \mathbf{R}^1 \pi_* \mathcal{E}(L) \rightarrow \mathbf{R}^1 \pi_*^1 \mathcal{E}_{L_1|H} \oplus \mathbf{R}^1 \pi_*^2 \mathcal{E}_{L_2|H} \rightarrow 0, \quad (33)$$

which implies that

$$e(\mathbf{R}^1 \pi_* \mathcal{E}(L)) = e(\mathbf{R}^0 \pi_* \mathcal{O}_C(Z)|_P) \cdot e(\mathbf{R}^1 \pi_*^1 \mathcal{E}_{L_1|H} \oplus \mathbf{R}^1 \pi_*^2 \mathcal{E}_{L_2|H}). \quad (34)$$

Therefore, it is enough to show that the Euler class of $\mathbf{R}^0 \pi_* \mathcal{O}_C(Z)|_P$ is zero.

For this, we notice that $[Z] = -p^*[\tilde{H}]$ with $p: L \rightarrow W$ the natural projection. It follows that $e(\mathbf{R}^0 \pi_* \mathcal{O}_C(Z)|_P) = -\text{ev}_P^*(p^* \tilde{H})$. As $p^* \tilde{H}$ has a universal section which does not vanish at P , we have that $(p^* \tilde{H})|_P$ has a nowhere vanishing section and thus $\text{ev}_P^*(p^* \tilde{H})$ is a trivial line bundle. This shows that

$$e(\mathbf{R}^0 \pi_* \mathcal{O}_C(Z)|_P) = 0. \quad (35)$$

The equalities (30), (34) and (35) conclude the proof. $\square$

Lemma 5.18. *Graphs contained in L_1 do not contribute to the degeneration formula.*

Proof. Intersection of these with $\text{ev}^* D_\infty$ is zero. $\square$

Proof of Theorem 5.2. By Proposition 5.8, together with commutativity of diagram (18), we have that

$$\begin{aligned} P_*[\mathcal{Q}_{0,2}^{\log}(L_0, (d, 0))^{\tilde{H}}]_{\text{vir}} \\ &= d \cdot F_* \text{pr}_{1,*} c_{\tilde{\Gamma},*}^* \Delta^! \left([\mathcal{Q}_{0,(d,0)}^{\log}(L_1|L_H, d)]^{\text{vir}} \times [\mathcal{Q}_{0,(d,0)}^{\log}(L_2|L_H, (0, d))^{H_\infty}]^{\text{vir}} \right) \\ &= d \cdot F_* \text{pr}_{1,*} \Delta^! \left([\mathcal{Q}_{0,(d,0)}^{\log}(L_1|L_H, d)]^{\text{vir}} \times [\mathcal{Q}_{0,(d,0)}^{\log}(L_2|L_H, (0, d))^{H_\infty}]^{\text{vir}} \right). \end{aligned}$$

On the other hand, $[\mathcal{Q}_{0,(d,0)}^{\log}(L_1|L_H, d)]^{\text{vir}}$ Gysin-restricts to $[\mathcal{Q}_{0,(d,0)}^{\log}(\mathbb{P}^N|H, d)]^{\text{vir}}$ when confining the evaluation map to be in $H \times \{0\}$. Combining this with Proposition 5.9 tells us that

$$\begin{aligned} \text{pr}_{1,*} \Delta^! \left([\mathcal{Q}_{0,(d,0)}^{\log}(L_1|L_H, d)]^{\text{vir}} \times [\mathcal{Q}_{0,(d,0)}^{\log}(L_2|L_H, (0, d))^{H_\infty}]^{\text{vir}} \right) \\ = \frac{(-1)^{d+1}}{d} [\mathcal{Q}_{0,(d,0)}^{\log}(\mathbb{P}^N|H, d)]^{\text{vir}}. \end{aligned}$$

Finally, combining this with Lemma 5.7 tells us that

$$P_*[\mathcal{Q}_{0,2}^{\log}(L_0, (d, 0))^{\tilde{H}}]_{\text{vir}} = \text{ev}_2^* H \cap [\mathcal{Q}_{0,2}(\mathcal{O}_{\mathbb{P}^N}(-H), d)]^{\text{vir}} = (-1)^{d+1} F_*[\mathcal{Q}_{0,(d,0)}^{\log}(\mathbb{P}^N|H, d)]^{\text{vir}},$$

which completes the proof. $\square$

5.2. Simple normal crossings divisors

In [vGGR19] the authors conjecture a generalisation of the local/logarithmic correspondence to the simple normal crossings situation. In the strong form of the correspondence, this would be

$$[\overline{\mathcal{M}}_{0,((d_1, \dots, d_r))}^{\log}(X|D, \beta)]^{\text{vir}} = \prod_{i=1}^r (-1)^{d_i+1} \cdot \text{ev}_i^* D_i \cap [\overline{\mathcal{M}}_{0,r}(\oplus_{i=1}^r \mathcal{O}_X(-D_i), \beta)]^{\text{vir}}$$

where the left-hand side indicates r -marked curves in X with maximal tangency to D_i at the i^{th} marking. The classes are viewed as living in $\overline{\mathcal{M}}_{0,r}(X, \beta)$ but in this section we will omit the pushforward on the left-hand side. Although this generalisation often holds numerically, [BBvG22, BBvG24], at the level of virtual classes it is not true in complete generality, due to counterexamples in [NR22]. However, in [NR22, Remark 5.4] the authors note that the difference appears when there are components in the moduli space containing rational tails and so may not be present in the quasimap setting.

It will turn out that an easy corollary of the local/logarithmic correspondence for smooth divisors and properties of logarithmic quasimaps is the following.

Theorem 5.19. *Let X be a GIT quotient $W//G$ and let $D \subset X$ be a very ample divisor satisfying Assumptions 2.1 and 3.1. Suppose that $D = D_1 + D_2$ has two components, which*

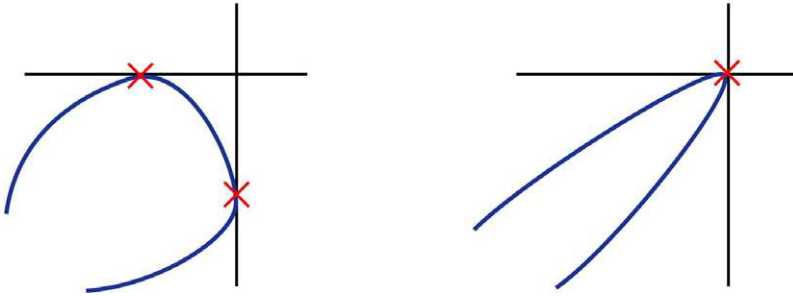


Figure 4. The two s.n.c. maximal tangency geometries. Left-hand side depicts [vGGR19] version and right-hand side depicts the *corner theory*.

are very ample. Let β be a curve class on X , let $d_1 = D_1 \cdot \beta$ and let $d_2 = D_2 \cdot \beta$. Then we have the following equality of virtual classes in $A_*(\mathcal{Q}_{0,2}(X, \beta))$

$$[\mathcal{Q}_{0,((d_1, d_2))}^{\log}(X|D_1 + D_2, \beta)]^{\text{vir}} = (-1)^{d_1 + d_2} \prod_{i=1}^2 \text{ev}_i^* D_i \cap [\mathcal{Q}_{0,2}(\oplus_{i=1}^2 \mathcal{O}_X(-D_i), \beta)]^{\text{vir}}.$$

In fact there is a much more general conjecture one could make for the local/logarithmic correspondence for simple normal crossings divisors [TY23b]. The conjecture of [vGGR19] is one extreme. The other extreme, we call the *corner theory*. On the logarithmic side the geometry is of curves with a single relative marking with maximal tangency to *all* divisor components simultaneously. Using exactly the same techniques we prove that this generalisation holds for quasimap theory in *any* rank.

Theorem 5.20. *Let X be a GIT quotient $W//G$ and let $D = D_1 + \cdots + D_r \subset X$ be a very ample divisor satisfying Assumptions 2.1 and 3.1 with $D_1 \cap \cdots \cap D_r \neq \emptyset$. Then we have the equality in $A_*(\mathcal{Q}_{0,2}(X, \beta))$*

$$[\mathcal{Q}_{0,(\underline{d}, 0)}^{\log}(X|D, \beta)]^{\text{vir}} = \prod_{i=1}^r (-1)^{d_i + 1} \cdot \text{ev}_1^* D_i \cap [\mathcal{Q}_{0,2}(\oplus_{i=1}^r \mathcal{O}_X(-D_i), \beta)]^{\text{vir}}.$$

We start with Theorem 5.19. We first prove the equality in the following special case.

Theorem 5.21. *Let $X = \mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$. Let $H'_1 = H_1 \times \mathbb{P}^{N_2}$ for H_1 a coordinate hyperplane in $\mathbb{P}^{N_1}$ and $H'_2 = \mathbb{P}^{N_1} \times H_2$ for H_2 a coordinate hyperplane in $\mathbb{P}^{N_2}$. Then the following diagram is Cartesian in the category of ordinary stacks.*

$$\begin{array}{ccc} \mathcal{Q}_{0,((d_1, d_2))}^{\log}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} | H'_1 + H'_2, (d_1, d_2)) & \longrightarrow & \mathcal{Q}_{0,(d_1, 0)}^{\log}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} | H'_1, (d_1, d_2)) \\ \downarrow & & \downarrow \\ \mathcal{Q}_{0,(0, d_2)}^{\log}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} | H'_2, (d_1, d_2)) & \longrightarrow & \mathcal{Q}_{0,2}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}, (d_1, d_2)) \end{array}$$

Proof. By stability, the boundary divisors of $\mathcal{Q}_{0,(d_1, 0)}^{\log}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} | H'_1, (d_1, d_2))$ consist of quasimaps from curves with at most two components [Sha24, Example 3.1]. As

a consequence the right-hand vertical morphism (or indeed the bottom horizontal morphism) is such that the morphism on tropicalisations is combinatorially flat. Therefore the fine and saturated fibre product coincides with ordinary fibre product [MR24, Proposition 1.7.3], but the fine and saturated fibre product is just $\mathcal{Q}_{0,((d_1,d_2))}^{\log}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} | H'_1 + H'_2, (d_1, d_2))$, [Sha24, Lemma 2.9]. $\square$

Proposition 5.22. *With the notation of Theorem 5.21, we have the following equality of (virtual) classes*

$$\begin{aligned} & [\mathcal{Q}_{0,((d_1,d_2))}^{\log}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} | H'_1 + H'_2, (d_1, d_2))] \\ &= (-1)^{d_1+d_2} \prod_{i=1}^2 \text{ev}_i^* H'_i \cap [\mathcal{Q}_{0,2}(\oplus_{i=1}^2 \mathcal{O}_{\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}}(-H'_i), (d_1, d_2))]^{\text{vir}} \end{aligned}$$

Proof. The local virtual class is

$$[\mathcal{Q}_{0,2}(\oplus_{i=1}^2 \mathcal{O}_{\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}}(-H'_i), (d_1, d_2))]^{\text{vir}} = e(\oplus_{i=1}^2 \mathbf{R}^1 \pi_*(\mathcal{L}_i^\vee)) \cap [\mathcal{Q}_{0,2}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}, (d_1, d_2))]$$

where $\mathcal{L}_i$ denotes the line bundle over the universal curve of $\mathcal{Q}_{0,2}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}, (d_1, d_2))$ corresponding to $\mathcal{O}_{\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}}(H'_i)$. But because of the Whitney sum formula, and the fact that pushforward commutes with direct sum, we have that

$$\begin{aligned} & e(\oplus_{i=1}^2 \mathbf{R}^1 \pi_*(\mathcal{L}_i^\vee)) \cap [\mathcal{Q}_{0,2}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}, (d_1, d_2))] \\ &= \prod_{i=1}^2 e(\mathbf{R}^1 \pi_* \mathcal{L}_i^\vee) \cap [\mathcal{Q}_{0,2}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}, (d_1, d_2))] \\ &= \prod_{i=1}^2 [\mathcal{Q}_{0,2}(\mathcal{O}_{\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}}(-H'_i), (d_1, d_2))]^{\text{vir}}. \end{aligned}$$

By Theorem 5.1 and the fact that the logarithmic quasimap moduli spaces for $\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$ relative to H'_i for $i = 1, 2$ are unobstructed we have

$$\begin{aligned} & [\mathcal{Q}_{0,(d_1,0)}^{\log}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} | H'_1, (d_1, d_2))] \cdot [\mathcal{Q}_{0,(0,d_2)}^{\log}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} | H'_2, (d_1, d_2))] \\ &= (-1)^{(d_1+d_2)} \prod_{i=1}^2 \text{ev}_i^* H'_i \cdot [\mathcal{Q}_{0,2}(\oplus_{i=1}^2 \mathcal{O}_{\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}}(-H'_i), (d_1, d_2))]^{\text{vir}} \end{aligned}$$

Therefore it suffices to show that

$$\begin{aligned} & [\mathcal{Q}_{0,((d_1,d_2))}^{\log}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} | H'_1 + H'_2, (d_1, d_2))] \\ &= [\mathcal{Q}_{0,(d_1,0)}^{\log}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} | H'_1, (d_1, d_2))] \cdot [\mathcal{Q}_{0,(0,d_2)}^{\log}(\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} | H'_2, (d_1, d_2))]. \end{aligned}$$

This follows from Theorem 5.21 as all spaces are unobstructed. $\square$

Remark 5.23. The natural generalisation of this result for rank greater than 2 will fail. The introduction of additional markings will allow for boundary divisors in the logarithmic quasimap space where the source curve contains more than two components and so the fine and saturated product and the ordinary fibre product may differ [Sha22, Remark 3.7.6].

Lemma 5.24. *Let $X = Z//G$ be a GIT quotient and D be a simple normal crossings divisor on X satisfying Assumptions 2.1 and 3.1. In addition suppose that $D = D_1 + D_2$ has two components, which are very ample. Let $j : X \rightarrow \mathbb{P} = \mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$ be the induced morphism with $j^{-1}(H_1 \times \mathbb{P}^{N_2}) = D_1$ and $j^{-1}(\mathbb{P}^{N_1} \times H_2) = D_2$ and assume H_1, H_2 are coordinate hyperplanes. Then there is an induced morphism*

$$i : \mathcal{Q}_{0,\alpha}^{\log}(X|D_1 + D_2, \beta) \rightarrow \mathcal{Q}_{0,\alpha}^{\log}(\mathbb{P}|H'_1 + H'_2, (d_1, d_2))$$

where $H'_1 = H_1 \times \mathbb{P}^{N_2}$, $H'_2 = \mathbb{P}^{N_1} \times H_2$ and $i_*\beta = (d_1, d_2)$. Moreover, this morphism admits a perfect obstruction theory and the virtual pullback of the class $[\mathcal{Q}_{0,\alpha}^{\log}(\mathbb{P}|H'_1 + H'_2, (d_1, d_2))]$ coincides with the virtual class $[\mathcal{Q}_{0,\alpha}^{\log}(X|D_1 + D_2, \beta)]^{\text{vir}}$ as in [Sha24, Theorem 14].

Proof. This proof is similar to [Sha24, 5.10]. The morphism j induces a morphism of quotient stacks, which we also denote j ,

$$j : [Z/G] \rightarrow [\mathbb{A}^{N_1+N_2+2}/\mathbb{G}_m^2]$$

such that $j^{-1}(\bar{H}'_1) = \bar{D}_1$ and $j^{-1}(\bar{H}'_2) = \bar{D}_2$. The divisors $\bar{D}_1 + \bar{D}_2, \bar{H}'_1 + \bar{H}'_2$ define logarithmic structures via morphisms to $[\mathbb{A}^2/\mathbb{G}_m^2]$ such that there is a commutative diagram

$$\begin{array}{ccc} & \bar{D}_1 + \bar{D}_2 & \\ & \curvearrowright & \\ [Z/G] & \xrightarrow{j} & [\mathbb{A}^{N_1+N_2+2}/\mathbb{G}_m^2] \xrightarrow{\bar{H}'_1 + \bar{H}'_2} [\mathbb{A}^1/\mathbb{G}_m]. \end{array}$$

Taking the induced distinguished triangle (on tangent complexes) gives

$$\mathbb{T}_j \rightarrow \mathbb{T}_{[Z/G]}^{\log} \rightarrow \mathbf{L}j^*\mathbb{T}_{[\mathbb{A}^{N_1+N_2+2}/\mathbb{G}_m^2]}^{\log} \rightarrow \mathbb{T}_j[1].$$

Next, consider the diagram

$$\begin{array}{ccccc} & & u & & \\ & \curvearrowright & & \curvearrowright & \\ \mathcal{C}_X & \xrightarrow{\quad} & \mathcal{C}_{\mathbb{P}^N} & \xrightarrow{\quad} & [Z/G] \xrightarrow{j} [\mathbb{A}^{N_1+N_2+2}/\mathbb{G}_m^2] \\ \downarrow \pi & & \downarrow \pi_{\mathbb{P}} & & \uparrow u_{\mathbb{P}} \\ \mathcal{Q}_{0,\alpha}^{\log}(X|D, \beta) & \xrightarrow{i} & \mathcal{Q}_{0,\alpha}^{\log}(\mathbb{P}|H'_1 + H'_2, (d_1, d_2)). & & \end{array} \quad (36)$$

If we apply $\mathbf{R}\pi_* \circ \mathbf{L}u^*$ and dualise, we get a distinguished triangle in $\mathcal{Q}_{0,\alpha}^{\log}(X|D, \beta)$

$$\begin{aligned} \mathbf{L}i^*(\mathbf{R}(\pi_{\mathbb{P}})_*\mathbf{L}u_{\mathbb{P}}^*\mathbb{T}_{[\mathbb{A}^{N_1+N_2+2}/\mathbb{G}_m^2]}^{\log})^{\vee} &\rightarrow (\mathbf{R}\pi_*\mathbf{L}u^*\mathbb{T}_{[Z/G]}^{\log})^{\vee} \\ &\rightarrow (\mathbf{R}\pi_*\mathbf{L}u^*\mathbb{T}_j)^{\vee} \rightarrow \mathbf{L}i^*(\mathbf{R}(\pi_{\mathbb{P}})_*\mathbf{L}u_{\mathbb{P}}^*\mathbb{T}_{[\mathbb{A}^{N_1+N_2+2}/\mathbb{G}_m^2]}^{\log})^{\vee}[1]. \end{aligned}$$

Note that the first two complexes in the triangle define the obstruction theories for $\mathcal{Q}_{0,\alpha}^{\log}(X|D, \beta)$ and $\mathcal{Q}_{0,\alpha}^{\log}(\mathbb{P}|H'_1 + H'_2, (d_1, d_2))$.

We also have the commutative diagram

$$\begin{array}{ccccc} & & \curvearrowright & & \\ \mathcal{Q}_{0,\alpha}^{\log}(X|D,\beta) & \xrightarrow{i} & \mathcal{Q}_{0,\alpha}^{\log}(\mathbb{P}|H'_1 + H'_2, (d_1, d_2)) & \longrightarrow & \mathfrak{M}_{0,\alpha}^{\log}([\mathbb{A}^2/\mathbb{G}_m^2]) \end{array}$$

inducing a distinguished triangle

$$\mathbf{L}i^* \mathbb{L}_{\mathcal{Q}^{\log}(\mathbb{P})/\mathfrak{M}^{\log}} \rightarrow \mathbb{L}_{\mathcal{Q}^{\log}(X)/\mathfrak{M}^{\log}} \rightarrow \mathbb{L}_i \rightarrow \mathbf{L}i^* \mathbb{L}_{\mathcal{Q}^{\log}(\mathbb{P})/\mathfrak{M}^{\log}}[1]$$

where $\mathcal{Q}^{\log}(X) = \mathcal{Q}_{0,\alpha}^{\log}(X|D,\beta)$, $\mathcal{Q}^{\log}(\mathbb{P}) = \mathcal{Q}_{0,\alpha}^{\log}(\mathbb{P}|H'_1 + H'_2, (d_1, d_2))$ and $\mathfrak{M}^{\log} = \mathfrak{M}_{0,\alpha}^{\log}([\mathbb{A}^2/\mathbb{G}_m^2])$. Putting these together we get

$$\begin{array}{ccccc} \mathbf{L}i^*(\mathbf{R}(\pi_{\mathbb{P}})_* \mathbf{L}u_{\mathbb{P}}^* \mathbb{T}_{[\mathbb{A}^{N_1+N_2+2}/\mathbb{G}_m^2]}^{\log})^{\vee} & \longrightarrow & (\mathbf{R}\pi_* \mathbf{L}u^* \mathbb{T}_{[Z/G]}^{\log})^{\vee} & \longrightarrow & (\mathbf{R}\pi_* \mathbf{L}u^* \mathbb{T}_j)^{\vee} \xrightarrow{[1]} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{L}i^* \mathbb{L}_{\mathcal{Q}^{\log}(\mathbb{P})/\mathfrak{M}^{\log}} & \longrightarrow & \mathbb{L}_{\mathcal{Q}^{\log}(X)/\mathfrak{M}^{\log}} & \longrightarrow & \mathbb{L}_i \xrightarrow{[1]} . \end{array}$$

The first two (and last) vertical arrows come from the perfect obstruction theory from [Sha24]. By [Man12, Construction 3.13] and [Man12, Remark 3.15] it follows that $(\mathbf{R}\pi_* \mathbf{L}u^* \mathbb{T}_j)^{\vee}$ defines a perfect obstruction theory. By [Man12, Theorem 4.8] we have that $i^![\mathcal{Q}_{0,\alpha}^{\log}(\mathbb{P}|H'_1 + H'_2, (d_1, d_2))] = [\mathcal{Q}_{0,\alpha}^{\log}(X|D,\beta)]^{\text{vir}}$. $\square$

Remark 5.25. One can show that this virtual pullback coincides with the pullback coming from a perfect obstruction theory of the morphism $\mathcal{Q}_{0,2}(X,\beta) \rightarrow \mathcal{Q}_{0,2}(\mathbb{P},(d_1,d_2))$.

Proof of Theorem 5.19. We have a Cartesian diagram (similar to in the proof of Theorem 5.1)

$$\begin{array}{ccc} \mathcal{Q}_{0,((d_1,d_2))}^{\log}(X|D_1 + D_2,\beta) & \xrightarrow{i} & \mathcal{Q}_{0,((d_1,d_2))}^{\log}(\mathbb{P}|H'_1 + H'_2, (d_1, d_2)) \\ \downarrow p & & \downarrow p' \\ \mathcal{Q}_{0,2}(X,\beta) & \xrightarrow{i'} & \mathcal{Q}_{0,2}(\mathbb{P},(d_1,d_2)) \end{array} \quad (37)$$

Let $\mathcal{L}_i$ (resp. $\mathcal{L}_{D_i}$) denote the universal line bundle on the universal curve $\mathcal{Q}_{0,2}(\mathbb{P},d)$ (resp. $\mathcal{Q}_{0,2}(X,\beta)$) corresponding to $\mathcal{O}_{\mathbb{P}}(H'_i)$ (resp. $\mathcal{O}_X(D_i)$).

$$\begin{aligned} & p_*[\mathcal{Q}_{0,((d_1,d_2))}^{\log}(X|D_1 + D_2,\beta)]^{\text{vir}} \\ &= p_*(i')^![\mathcal{Q}_{0,((d_1,d_2))}^{\log}(\mathbb{P}|H'_1 + H'_2, (d_1, d_2))] \\ &= i'^! p'_*[\mathcal{Q}_{0,((d_1,d_2))}^{\log}(\mathbb{P}|H'_1 + H'_2, (d_1, d_2))] \\ &= i'^!(-1)^{d_1+d_2} \prod_{i=1}^2 \text{ev}_i^* H'_i \cap [\mathcal{Q}_{0,2}(\oplus_{i=1}^2 \mathcal{O}_{\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}}(-H'_i), (d_1, d_2))]^{\text{vir}} \end{aligned}$$

$$\begin{aligned}
&= (-1)^{d_1+d_2} \prod_{i=1}^2 \operatorname{ev}_i^* D_i \cap i'^! [\mathcal{Q}_{0,2}(\oplus_{i=1}^2 \mathcal{O}_{\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}}(-H'_i), (d_1, d_2))]^{\operatorname{vir}} \\
&= (-1)^{d_1+d_2} \prod_{i=1}^2 \operatorname{ev}_i^* D_i \cap i'^! e(\oplus_{i=1}^2 \mathbf{R}^1 \pi_* \mathcal{L}_i^\vee) \cap [\mathcal{Q}_{0,2}(\mathbb{P}, (d_1, d_2))] \\
&= (-1)^{d_1+d_2} \prod_{i=1}^2 \operatorname{ev}_i^* D_i \cap e(\oplus_{i=1}^2 \mathbf{R}^1 \pi_* \mathcal{L}_{D_i}^\vee) \cap [\mathcal{Q}_{0,2}(X, \beta)] \\
&= (-1)^{d_1+d_2} \prod_{i=1}^2 \operatorname{ev}_i^* D_i \cap [\mathcal{Q}_{0,2}(\oplus_{i=1}^2 \mathcal{O}_X(-D_i), \beta)]^{\operatorname{vir}}.
\end{aligned}$$

□

Finally, we move on to proving Theorem 5.20. With the same argument of Theorem 5.21, we get the following

Theorem 5.26. *Let $\mathbb{P} = \prod_{i=1}^r \mathbb{P}^{N_i}$ and let $H'_i = H_i \times \prod_{j \neq i} \mathbb{P}^{N_j}$ for H_i a coordinate hyperplane in $\mathbb{P}^{N_i}$ and $H' = \sum_{i=1}^r H'_i$. There is a Cartesian diagram in the category of ordinary stacks*

$$\begin{array}{ccc}
\mathcal{Q}_{0,(\underline{d},0)}^{\log}(\mathbb{P}|H', \underline{d}) & \longrightarrow & \prod_{i=1}^r \mathcal{Q}_{0,(d_i,0)}^{\log}(\mathbb{P}|H'_i, \underline{d}) \\
\downarrow & & \downarrow \\
\mathcal{Q}_{0,2}(\mathbb{P}, \underline{d}) & \xrightarrow{\Delta} & \prod_{i=1}^r \mathcal{Q}_{0,2}(\mathbb{P}, \underline{d})
\end{array}$$

Proposition 5.27. *With the notations of Theorem 5.26, we have the following equality of (virtual) classes*

$$[\mathcal{Q}_{0,(\underline{d},0)}^{\log}(\mathbb{P}|H', \underline{d})] = \prod_{i=1}^r (-1)^{d_i+1} \cdot \operatorname{ev}_1^* H'_i \cap [\mathcal{Q}_{0,2}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}}(-H'_i), \underline{d})]^{\operatorname{vir}}.$$

Proof. As in Proposition 5.22, we have

$$[\mathcal{Q}_{0,2}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}}(-H'_i), \underline{d})]^{\operatorname{vir}} = \prod_{i=1}^r [\mathcal{Q}_{0,2}(\mathcal{O}_{\mathbb{P}}(-H'_i), \underline{d})]^{\operatorname{vir}}$$

so that, similar to before, it suffices to show that

$$[\mathcal{Q}_{0,(\underline{d},0)}^{\log}(\mathbb{P}|H', \underline{d})] = \prod_{i=1}^r [\mathcal{Q}_{0,(d_i,0)}^{\log}(\mathbb{P}|H'_i, \underline{d})].$$

As before this follows from Theorem 5.26. □

A similar argument to Lemma 5.24 shows

Lemma 5.28. *Let $X = Z//G$ be a GIT quotient and $D = D_1 + \cdots + D_r$ a simple normal crossings divisor, satisfying Assumptions 2.1 and 3.1, with $D_1 \cap \cdots \cap D_r \neq \emptyset$ and each*

component very ample. Let $j : X \rightarrow \prod_{i=1}^r \mathbb{P}^{N_i}$ be the induced morphism with $j^{-1}(H_i \times \prod_{k \neq i} \mathbb{P}^{N_k}) = D_i$, and assume H_i are coordinate hyperplanes. Then there is an induced morphism

$$i : \mathcal{Q}_{0,\alpha}^{\log}(X|D, \beta) \rightarrow \mathcal{Q}_{0,\alpha}^{\log}(\mathbb{P}|H', \underline{d})$$

with $i_*\beta = \underline{d}$. Moreover, this morphism admits a perfect obstruction theory and the virtual pullback of the class $[\mathcal{Q}_{0,\alpha}^{\log}(\mathbb{P}|H', \underline{d})]$ coincides with the virtual class $[\mathcal{Q}_{0,\alpha}^{\log}(X|D, \beta)]^{\text{vir}}$ as in [Sha24, Theorem 14].

Proof of Theorem 5.20. We have a Cartesian diagram (similar to in the proof of Theorem 5.1)

$$\begin{array}{ccc} \mathcal{Q}_{0,(\underline{d},0)}^{\log}(X|D, \beta) & \xrightarrow{i} & \mathcal{Q}_{0,(\underline{d},0)}^{\log}(\mathbb{P}|H', \underline{d}) \\ \downarrow p & & \downarrow p' \\ \mathcal{Q}_{0,2}(X, \beta) & \xrightarrow{i'} & \mathcal{Q}_{0,2}(\mathbb{P}, \underline{d}). \end{array} \quad (38)$$

Let $\mathcal{L}_i$ (resp. $\mathcal{L}_{D_i}$) denote the universal line bundle on the universal curve $\mathcal{Q}_{0,2}(\mathbb{P}, \underline{d})$ (resp. $\mathcal{Q}_{0,2}(X, \beta)$) corresponding to $\mathcal{O}_{\mathbb{P}}(H'_i)$ (resp. $\mathcal{O}_X(D_i)$).

$$\begin{aligned} p_*[\mathcal{Q}_{0,(\underline{d},0)}^{\log}(X|D, \beta)]^{\text{vir}} &= p_*(i')^*[\mathcal{Q}_{0,(\underline{d},0)}^{\log}(\mathbb{P}|H', (\underline{d}))] \\ &= i'^! p'_*[\mathcal{Q}_{0,(\underline{d},0)}^{\log}(\mathbb{P}|H', (\underline{d}))] \\ &= i'^! \prod_{i=1}^r (-1)^{d_i+1} \cdot \text{ev}_1^* H'_i \cap [\mathcal{Q}_{0,2}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}}(-H'_i), (\underline{d}))]^{\text{vir}} \\ &= \prod_{i=1}^r (-1)^{d_i+1} \cdot \text{ev}_1^* D_i \cap i'^! [\mathcal{Q}_{0,2}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}}(-H'_i), (\underline{d}))]^{\text{vir}} \\ &= \prod_{i=1}^r (-1)^{d_i+1} \cdot \text{ev}_1^* D_i \cap i'^! e(\oplus_{i=1}^r \mathbf{R}^1 \pi_* \mathcal{L}_i^{\vee}) \cap [\mathcal{Q}_{0,2}(\mathbb{P}, \underline{d})] \\ &= \prod_{i=1}^r (-1)^{d_i+1} \cdot \text{ev}_1^* D_i \cap e(\oplus_{i=1}^r \mathbf{R}^1 \pi_* \mathcal{L}_{D_i}^{\vee}) \cap [\mathcal{Q}_{0,2}(X, \beta)] \\ &= \prod_{i=1}^r (-1)^{d_i+1} \cdot \text{ev}_1^* D_i \cap [\mathcal{Q}_{0,2}(\oplus_{i=1}^r \mathcal{O}_X(-D_i), \beta)]^{\text{vir}}. \end{aligned}$$

□

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