



Cosmographic Footprints of Dynamical Dark Energy

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Abstract

We introduce a novel cosmographic framework to trace the late-time kinematics of the Universe without assuming any underlying dynamics. The method relies on generalized Padé (2, 1) expansions around arbitrary pivot redshifts, which, compared to state-of-the-art calculations, reduce truncation errors by up to two orders of magnitude at high redshift and yield more precise constraints by defining cosmographic parameters exactly where the data lie. This avoids extrapolations, mitigates degeneracies, and enables a clean disentangling of their effects. Using the latest low-redshift datasets, we center the generalized expansion in multiple bins across $z \in [0, 1]$ and obtain precise constraints on the redshift evolution of cosmographic parameters. We find that all key parameters deviate from their Λ CDM predictions in a redshift-dependent way that can be naturally explained within dynamical dark energy scenarios. The deceleration parameter $q(z)$ follows a redshift evolution consistent with the Chevallier–Polarski–Linder (CPL) parameterization, while the generalized $Om(z)$ diagnostic shows deviations of up to $\sim 4\sigma$ from the constant Λ CDM expectation, closely matching the CPL predictions. Taken together, these results point to footprints of dynamical dark energy in the kinematics of the Universe at $z \lesssim 1$.

Unified Astronomy Thesaurus concepts: Deceleration parameter (364); Dark energy (351); Cosmology (343); Cosmological constant (334)

1. Introduction

Baryon acoustic oscillation (BAO) measurements released by the Dark Energy Spectroscopic Instrument (DESI; R. Calderon et al. 2024; M. Abdul Karim et al. 2025a, 2025b; A. G. Adame et al. 2025a, 2025b, 2025c; U. Andrade et al. 2025; K. Lodha et al. 2025a) have revitalized interest in one of the most profound open problems in cosmology and fundamental physics: the nature of the current accelerated expansion of the Universe.

DESI percent-level constraints on the transverse comoving distance, the Hubble rate, and their combination (all relative to the sound horizon at the drag epoch, r_d) directly challenge the cosmological constant (Λ) interpretation of cosmic acceleration within the standard Λ cold dark matter (Λ CDM) model, pointing to a preference for evolving dark energy (DE; M. Abdul Karim et al. 2025a; G. Gu et al. 2025).

This has sparked intense debate and a wide range of new analyses and interpretations that, broadly speaking, can be grouped into two main categories (V. Sahni & A. Starobinsky 2006). The first comprises model-dependent analyses, where one assumes a theoretical framework to describe the late-time dynamics of the Universe, for instance, by specifying how the DE equation of state (EoS), $w(a)$, evolves with expansion. This is the strategy first adopted by the DESI collaboration, which, under the assumption of a Chevallier–Polarski–Linder (CPL) parameterization, $w(a) = w_0 + w_a(1 - a)$ (M. Chevallier & D. Polarski 2001; E. V. Linder 2003), reported a preference for evolving DE at

the level of $\sim 2.8\sigma$ – 4.2σ , depending on the specific combination of DESI BAO, Type Ia supernovae (SNIa), and cosmic microwave background (CMB) data (M. Abdul Karim et al. 2025a; G. Gu et al. 2025; see also W. Giarè et al. 2025; W. Giarè 2025). While this preference remains fairly robust across different parameterizations (W. Giarè et al. 2024; W. J. Wolf et al. 2025), more exotic scenarios (or the inclusion of additional free parameters) can shift the evidence from negligible (S. Nesseris et al. 2025) to $\gtrsim 5\sigma$ (M. Scherer et al. 2025), laying bare the inherent dependence of such conclusions on the assumed *Ansatz*.

A second assumption-free approach is to bypass any parameterization and reconstruct quantities such as $w(a)$ directly from data via nonparametric or machine learning techniques; see, e.g., L. W. K. Goh et al. (2024), A. N. Ormondroyd et al. (2025a, 2025b), M. Berti et al. (2025), K. Lodha (2025b), A. González-Fuentes & A. Gómez-Valent (2025), P. Mukherjee & A. A. Sen (2025). These methods, however, face other well-known challenges: reconstructions tend to overfit noise, depend on binning choices, and generally suffer from reduced precision and large uncertainties, preventing them from delivering conclusive statements about the nature of DE.

Yet a third possible approach—potentially able to capture the advantages of both model-dependent and model-independent analyses—is provided by cosmography (V. Sahni et al. 2003; M. Visser 2004, 2005; V. Vitagliano et al. 2010; A. Aviles et al. 2012; K. Bamba et al. 2012; P. K. S. Dunsby & O. Luongo 2016; S. Capozziello et al. 2020). Unlike traditional methods, cosmography makes no assumptions about the underlying dynamics of the Universe (M. Visser 2005; P. K. S. Dunsby & O. Luongo 2016) and instead expands observable quantities such as the expansion rate $H(z)$ or the



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luminosity distance $D_L(z)$, which encode its geometry and kinematics. These functions are typically expanded around $z_0 = 0$ (the present epoch), using a Taylor series or, preferably, a Padé approximation to improve convergence at higher redshift (A. Aviles et al. 2014; C. Gruber & O. Luongo 2014; H. Wei et al. 2014). The coefficients of the expansion map directly onto cosmographic parameters with clear physical interpretation, such as the present-day expansion rate H_0 , the deceleration parameter q_0 , and, at higher order, the jerk j_0 and the snap s_0 . Once specified, these parameters uniquely fix the truncated series for $H(z)$, $D_L(z)$, and related quantities at higher redshift, which can then be fitted to BAO, SNIa, and other low-redshift data, yielding constraints with a precision competitive with specific models.

While elegant, cosmography is not free of limitations (V. C. Busti et al. 2015). In its standard form, it relies on truncated series expansions of kinematic and geometric quantities around $z_0 = 0$. The fitted parameters (e.g., H_0 , q_0 , j_0 , s_0) describe the Universe at the present epoch, while the evolution at $z > 0$ is inferred indirectly from the truncated expansion. At higher redshift, the approximation deteriorates and truncation errors can become comparable to, or even larger than, observational uncertainties—precisely where most of the data lie (i.e., $0.1 \lesssim z \lesssim 2$). A common work-around is to extend the expansion to higher orders, but this introduces additional free parameters, leading to degeneracies and reduced predictive power. In practice, this entails a double risk: either we fail to exploit the full potential of the data, or, worse, we introduce biases.

To overcome these limitations, we introduce a generalized cosmographic framework that allows us to trace the kinematics and expansion history of the Universe up to $z \gtrsim 1$ with unprecedented precision, revealing clear departures from Λ CDM that can be interpreted as footprints of dynamical DE.

2. Methodology

We construct a generalized Padé (2, 1) expansion around an arbitrary pivot redshift $z_0 \neq 0$. This formulation expresses all relevant observables directly in terms of the cosmographic parameters $q(z_0)$, $j(z_0)$, and $s(z_0)$ at the chosen pivot. The explicit derivation is provided in Appendix A.

Generalizing the Padé (2, 1) expansion to an arbitrary pivot redshift z_0 offers several key advantages. First, because the accuracy of any truncated expansion inevitably deteriorates with distance from the expansion center, our framework allows the pivot redshift to be chosen strategically so as to minimize truncation errors across the redshift range populated by low- z data. To quantify this effect, we fix the cosmographic parameters to their Λ CDM values and compare Padé approximations of the key observables with the exact Λ CDM functions for different choices of z_0 . When expanded around $z_0 = 0$, the Padé (2, 1) already produces theoretical uncertainties comparable to current DESI BAO error bars. By contrast, for higher pivot redshifts, $0.4 \lesssim z_0 \lesssim 1$, the accuracy of the reconstructed observables improves by up to two orders of magnitude across the full data range. In this regime, the generalized Padé (2, 1) reproduces Λ CDM predictions with negligible relative error, while avoiding the need for higher-order expansions and the additional free parameters they entail.

Second, in our approach we expand the Hubble rate $H(z)$ and the luminosity distance $D_L(z)$ using the Padé framework,

and from these two quantities we obtain consistent expressions for all other observables in terms of the cosmographic parameters $H(z_0)$, $q(z_0)$, $j(z_0)$, $s(z_0)$ at the chosen pivot z_0 . By selecting $z_0 \in [0, 1]$ and fitting predictions to the data, these parameters are constrained exactly where most observations lie, without the need for extrapolation into regions with no data. This contrasts with the standard approach, which fixes parameters at $z = 0$ and extrapolates across the entire data range.

Finally, building on the previous point, our framework allows us to choose multiple pivot redshifts z_0 , yielding constraints on the cosmographic parameters $H(z_0)$, $q(z_0)$, $j(z_0)$, and $s(z_0)$ at different pivots. For quantities with direct physical interpretation such as $H(z_0)$ and $q(z_0)$, this provides a direct mapping of the expansion rate and the deceleration of the Universe as a function of redshift—not through extrapolation, but via direct measurements at multiple redshift bins. In practice, this delivers a precise trend of their redshift evolution, offering a powerful, model-independent probe of cosmic acceleration. For similar pioneering discussions see also Y. Liu et al. (2024).

To capitalize on the advantages of our generalized cosmographic framework, we develop a dedicated numerical package fully interfaced with the Cobaya sampler (J. Torrado & A. Lewis 2021), which we use to perform parameter inference via Markov Chain Monte Carlo (MCMC) sampling. In practice, we explore 11 pivot scenarios, fixing z_0 to values uniformly distributed between $z_0 = 0$ and $z_0 = 1$ in steps of $\Delta z_0 = 0.1$. For each choice of pivot, we run full MCMC chains to constrain the cosmographic parameters $\{H(z_0), q(z_0), j(z_0), s(z_0)\}$ using different combinations of the most up-to-date cosmological datasets.

Our baseline dataset includes the second DESI BAO data release (M. Abdul Karim et al. 2025a), calibrated with the Planck value of the sound horizon $r_d = (147.09 \pm 0.26)$ Mpc (N. Aghanim et al. 2020), distance moduli from SNIa (from both the PantheonPlus (PP) catalog (D. Brout et al. 2022) and the five-year Dark Energy Survey Supernova Program (DESy5; T. M. C. Abbott et al. 2024) sample), and 15 direct measurements of $H(z)$ from cosmic chronometers (CC) with a full covariance matrix (R. Jimenez & A. Loeb 2002; N. Borghi et al. 2022). A more detailed discussion of the methodology, datasets, and precision tests of the generalized Padé expansion is provided in Appendix B.

3. Results

The full numerical results for all dataset combinations are reported in Table 1 in Appendix C. Here we focus on the two most relevant cases: DESI+ r_d +CC+PP and DESI+ r_d +CC+DESy5. While these combinations are the most informative in terms of independent probes, we note that, as shown in Appendix C, CC contribute very little statistical weight in these cases, and removing them leaves the cosmographic constraints essentially unchanged.

We begin with the deceleration parameter $q(z)$, which we recall is defined as

$$q(z) \equiv -\frac{\ddot{a}a}{\dot{a}^2} = \frac{d \ln H}{d \ln(1+z)} - 1. \quad (1)$$

The sign of $q(z)$ indicates whether the Universe is accelerating ($q(z) < 0$) or decelerating ($q(z) > 0$) at a given redshift, while its value reflects how the expansion rate changes with time.

Table 1
Constraints at 68% CL on the Expansion rate $H(z_0)$ (in $\text{km s}^{-1} \text{Mpc}^{-1}$) and on the Cosmographic Parameters $q(z_0)$, $j(z_0)$, and $s(z_0)$ at Different Fixed Pivot Redshifts z_0 , for Various Combinations of Datasets

Redshift	Dataset	$H(z_0)$	$q(z_0)$	$j(z_0)$	$s(z_0)$
$z_0 = 0.0$	DESI+ r_d (+CC)	67.49 ± 0.56 (66.4 ± 1.0)	$-0.427^{+0.058}_{-0.051}$ ($-0.336^{+0.099}_{-0.078}$)	$0.78^{+0.23}_{-0.30}$ ($0.50^{+0.23}_{-0.36}$)	$-0.26^{+0.26}_{-0.50}$ ($-0.37^{+0.29}_{-0.57}$)
	PP (+CC)	Unconstrained (66.8 ± 4.3)	-0.504 ± 0.089 (-0.507 ± 0.071)	$1.33^{+0.69}_{-0.81}$ (1.35 ± 0.53)	Unconstrained ($1.8^{+2.5}_{-1.5}$)
	DESy5 (+CC)	Unconstrained (66.8 ± 4.1)	-0.405 ± 0.092 (-0.426 ± 0.086)	$0.80^{+0.63}_{-0.74}$ (0.96 ± 0.59)	Unconstrained ($1.4^{+2.8}_{-1.6}$)
	DESI+ r_d +PP (+CC)	67.51 ± 0.58 (67.49 ± 0.56)	$-0.426^{+0.063}_{-0.054}$ ($-0.427^{+0.058}_{-0.051}$)	$0.77^{+0.25}_{-0.34}$ ($0.78^{+0.23}_{-0.30}$)	$-0.23^{+0.28}_{-0.62}$ ($-0.26^{+0.26}_{-0.50}$)
	DESI+ r_d +DESy5 (+CC)	66.89 ± 0.53 (66.90 ± 0.53)	-0.362 ± 0.041 ($-0.366^{+0.048}_{-0.042}$)	$0.53^{+0.15}_{-0.19}$ ($0.55^{+0.16}_{-0.20}$)	$-0.491^{+0.068}_{-0.11}$ ($-0.475^{+0.099}_{-0.16}$)
$z_0 = 0.1$	DESI+ r_d (+CC)	71.61 ± 0.45 (71.55 ± 0.87)	-0.379 ± 0.039 ($-0.38^{+0.14}_{-0.12}$)	$0.87^{+0.23}_{-0.27}$ ($0.95^{+0.53}_{-0.75}$)	$-0.42^{+0.28}_{-0.48}$ (< -0.0559)
	PP (+CC)	Unconstrained (70.6 ± 4.3)	-0.392 ± 0.047 (-0.401 ± 0.044)	$1.08^{+0.53}_{-0.59}$ (1.23 ± 0.46)	Unconstrained (1.3 ± 1.9)
	DESy5 (+CC)	Unconstrained (70.8 ± 4.5)	-0.325 ± 0.048 (-0.338 ± 0.049)	$0.65^{+0.48}_{-0.54}$ (0.84 ± 0.49)	Unconstrained ($0.97^{+2.5}_{-2.0}$)
	DESI+ r_d +PP (+CC)	71.62 ± 0.46 (71.61 ± 0.45)	-0.374 ± 0.041 (-0.379 ± 0.039)	$0.84^{+0.23}_{-0.28}$ ($0.87^{+0.23}_{-0.27}$)	$-0.45^{+0.24}_{-0.47}$ ($-0.42^{+0.28}_{-0.48}$)
	DESI+ r_d +DESy5 (+CC)	71.34 ± 0.44 (71.31 ± 0.43)	-0.324 ± 0.037 (-0.325 ± 0.035)	$0.62^{+0.18}_{-0.22}$ ($0.63^{+0.17}_{-0.15}$)	$-0.65^{+0.13}_{-0.20}$ ($-0.671^{+0.094}_{-0.15}$)
$z_0 = 0.2$	DESI+ r_d (+CC)	75.83 ± 0.41 (75.91 ± 0.46)	-0.319 ± 0.032 (-0.333 ± 0.087)	0.96 ± 0.23 ($1.06^{+0.49}_{-0.63}$)	$-0.58^{+0.33}_{-0.54}$ ($-0.08^{+0.53}_{-1.7}$)
	PP (+CC)	Unconstrained (74.5 ± 4.6)	-0.325 ± 0.032 (-0.325 ± 0.031)	1.31 ± 0.45 (1.41 ± 0.40)	Unconstrained ($1.0^{+2.7}_{-2.1}$)
	DESy5 (+CC)	Unconstrained (74.9 ± 4.5)	-0.276 ± 0.032 (-0.280 ± 0.032)	$0.95^{+0.60}_{-0.67}$ (1.14 ± 0.55)	Unconstrained ($1.0^{+2.9}_{-2.0}$)
	DESI+ r_d +PP (+CC)	75.86 ± 0.42 (75.83 ± 0.41)	-0.322 ± 0.031 (-0.319 ± 0.032)	$0.98^{+0.23}_{-0.26}$ (0.96 ± 0.23)	$-0.55^{+0.35}_{-0.59}$ ($-0.58^{+0.33}_{-0.54}$)
	DESI+ r_d +DESy5 (+CC)	75.81 ± 0.39 (75.79 ± 0.41)	-0.274 ± 0.028 (-0.276 ± 0.029)	$0.69^{+0.17}_{-0.20}$ ($0.72^{+0.19}_{-0.22}$)	$-0.89^{+0.11}_{-0.17}$ ($-0.85^{+0.15}_{-0.24}$)
$z_0 = 0.3$	DESI+ r_d (+CC)	80.71 ± 0.35 (80.42 ± 0.46)	-0.235 ± 0.022 (-0.249 ± 0.057)	0.73 ± 0.11 ($0.97^{+0.40}_{-0.49}$)	$-1.186^{+0.077}_{-0.098}$ ($-0.62^{+0.51}_{-1.2}$)
	PP (+CC)	Unconstrained (80.7 ± 4.7)	-0.260 ± 0.032 (-0.255 ± 0.031)	0.66 ± 0.14 (0.69 ± 0.14)	Unconstrained (0.8 ± 2.0)
	DESy5 (+CC)	Unconstrained (79.9 ± 4.8)	-0.232 ± 0.031 (-0.233 ± 0.030)	1.02 ± 0.31 (1.10 ± 0.29)	Unconstrained (0.6 ± 2.3)
	DESI+ r_d +PP (+CC)	80.73 ± 0.35 (80.71 ± 0.35)	-0.236 ± 0.022 (-0.235 ± 0.022)	0.72 ± 0.11 (0.73 ± 0.11)	$-1.187^{+0.077}_{-0.096}$ ($-1.186^{+0.073}_{-0.098}$)
	DESI+ r_d +DESy5 (+CC)	80.41 ± 0.41 (80.38 ± 0.41)	-0.234 ± 0.027 (-0.238 ± 0.026)	$0.88^{+0.22}_{-0.25}$ (0.92 ± 0.22)	$-0.96^{+0.25}_{-0.45}$ ($-0.92^{+0.28}_{-0.47}$)
$z_0 = 0.4$	DESI+ r_d (+CC)	85.85 ± 0.31 (85.40 ± 0.57)	-0.165 ± 0.018 (-0.170 ± 0.038)	0.591 ± 0.042 ($0.87^{+0.34}_{-0.41}$)	-1.444 ± 0.064 ($-1.06^{+0.44}_{-0.79}$)
	PP (+CC)	Unconstrained (85.1 ± 5.2)	-0.209 ± 0.034 (-0.200 ± 0.033)	0.561 ± 0.050 (0.567 ± 0.049)	Unconstrained ($0.8^{+2.7}_{-2.3}$)
	DESy5 (+CC)	Unconstrained (85.4 ± 5.2)	-0.180 ± 0.034 (-0.177 ± 0.035)	0.690 ± 0.098 (0.69 ± 0.10)	Unconstrained (0.8 ± 2.2)
	DESI+ r_d +PP (+CC)	85.84 ± 0.31 (85.85 ± 0.31)	-0.165 ± 0.018 (-0.165 ± 0.018)	0.590 ± 0.043 (0.591 ± 0.042)	-1.444 ± 0.067 (-1.444 ± 0.064)
	DESI+ r_d +DESy5 (+CC)	85.65 ± 0.33 (85.64 ± 0.33)	-0.166 ± 0.020 (-0.167 ± 0.020)	0.694 ± 0.091 (0.698 ± 0.090)	$-1.458^{+0.090}_{-0.10}$ ($-1.463^{+0.087}_{-0.10}$)
$z_0 = 0.5$	DESI+ r_d (+CC)	91.10 ± 0.31 (90.70 ± 0.60)	-0.110 ± 0.018 (-0.108 ± 0.026)	0.611 ± 0.026 ($0.84^{+0.30}_{-0.35}$)	-1.703 ± 0.082 ($-1.41^{+0.37}_{-0.60}$)
	PP (+CC)	Unconstrained (90.2 ± 5.2)	-0.156 ± 0.038 (-0.147 ± 0.038)	0.571 ± 0.036 (0.577 ± 0.038)	Unconstrained ($0.5^{+2.7}_{-2.4}$)
	DESy5 (+CC)	Unconstrained (90.3 ± 5.2)	-0.132 ± 0.037 (-0.122 ± 0.036)	0.642 ± 0.041 (0.646 ± 0.042)	Unconstrained (0.7 ± 2.3)
	DESI+ r_d +PP (+CC)	91.08 ± 0.31 (91.10 ± 0.31)	-0.112 ± 0.017 (-0.110 ± 0.018)	0.608 ± 0.024 (0.611 ± 0.026)	-1.704 ± 0.078 (-1.703 ± 0.082)
	DESI+ r_d +DESy5 (+CC)	91.01 ± 0.31 (90.99 ± 0.31)	-0.107 ± 0.018 (-0.108 ± 0.019)	0.652 ± 0.038 (0.653 ± 0.038)	$-1.717^{+0.097}_{-0.11}$ ($-1.724^{+0.095}_{-0.11}$)
$z_0 = 0.6$	DESI+ r_d (+CC)	96.54 ± 0.33 (96.25 ± 0.64)	-0.060 ± 0.018 (-0.050 ± 0.022)	0.654 ± 0.024 ($0.85^{+0.29}_{-0.32}$)	$-1.99^{+0.11}_{-0.13}$ ($-1.62^{+0.46}_{-0.72}$)
	PP (+CC)	Unconstrained (95.1 ± 5.7)	-0.103 ± 0.040 (-0.089 ± 0.040)	$0.607^{+0.043}_{-0.049}$ ($0.623^{+0.046}_{-0.051}$)	Unconstrained (-0.2 ± 2.1)
	DESy5 (+CC)	Unconstrained (95.4 ± 5.5)	-0.075 ± 0.043 (-0.067 ± 0.041)	0.664 ± 0.044 (0.672 ± 0.044)	Unconstrained (0.0 ± 2.2)
	DESI+ r_d +PP (+CC)	96.54 ± 0.32 (96.54 ± 0.33)	-0.062 ± 0.018 (-0.060 ± 0.018)	0.652 ± 0.025 (0.654 ± 0.024)	$-1.99^{+0.11}_{-0.13}$ ($-1.99^{+0.11}_{-0.13}$)
	DESI+ r_d +DESy5 (+CC)	96.52 ± 0.34 (96.54 ± 0.34)	-0.057 ± 0.019 (-0.056 ± 0.019)	0.680 ± 0.026 (0.682 ± 0.027)	$-1.99^{+0.12}_{-0.15}$ ($-1.98^{+0.13}_{-0.16}$)
$z_0 = 0.7$	DESI+ r_d (+CC)	102.30 ± 0.39 (102.08 ± 0.64)	-0.008 ± 0.020 (0.001 ± 0.023)	0.712 ± 0.028 (0.88 ± 0.27)	$-2.25^{+0.18}_{-0.23}$ ($-1.85^{+0.50}_{-0.79}$)
	PP (+CC)	Unconstrained ($99.98^{+7.5}_{-6.7}$)	-0.039 ± 0.045 (-0.026 ± 0.045)	0.673 ± 0.059 (0.690 ± 0.061)	Unconstrained ($-0.7^{+2.0}_{-2.2}$)
	DESy5 (+CC)	Unconstrained (100.8 ± 6.0)	-0.019 ± 0.046 (-0.009 ± 0.046)	$0.717^{+0.057}_{-0.065}$ (0.729 ± 0.059)	Unconstrained ($-0.9^{+2.0}_{-2.4}$)
	DESI+ r_d +PP (+CC)	102.28 ± 0.38 (102.30 ± 0.39)	-0.0097 ± 0.019 (-0.008 ± 0.020)	0.710 ± 0.028 (0.712 ± 0.028)	$-2.26^{+0.17}_{-0.22}$ ($-2.25^{+0.18}_{-0.23}$)
	DESI+ r_d +DESy5 (+CC)	102.27 ± 0.39 (102.29 ± 0.39)	-0.007 ± 0.019 (-0.005 ± 0.019)	0.729 ± 0.027 (0.732 ± 0.028)	$-2.26^{+0.18}_{-0.23}$ ($-2.23^{+0.19}_{-0.25}$)

Table 1
(Continued)

Redshift	Dataset	$H(z_0)$	$q(z_0)$	$j(z_0)$	$s(z_0)$
$z_0 = 0.8$	DESI+ r_d (+CC)	108.33 ± 0.47 (108.23 ± 0.67)	0.043 ± 0.021 (0.049 ± 0.024)	0.777 ± 0.033 (0.87 ± 0.25)	$-2.46^{+0.30}_{-0.38}$ ($-2.12^{+0.54}_{-0.77}$)
	PP (+CC)	Unconstrained (106.9 ± 6.3)	0.029 ± 0.053 (0.042 ± 0.051)	0.758 ± 0.081 (0.779 ± 0.081)	Unconstrained ($-1.5^{+1.7}_{-2.3}$)
	DESy5 (+CC)	Unconstrained (107.4 ± 6.3)	0.048 ± 0.051 (0.057 ± 0.049)	$0.800^{+0.077}_{-0.087}$ (0.812 ± 0.076)	Unconstrained ($-1.6^{+1.6}_{-2.4}$)
	DESI+ r_d +PP (+CC)	108.33 ± 0.48 (108.33 ± 0.47)	0.041 ± 0.021 (0.043 ± 0.021)	0.774 ± 0.033 (0.777 ± 0.033)	$-2.49^{+0.29}_{-0.39}$ ($-2.46^{+0.30}_{-0.38}$)
	DESI+ r_d +DESy5 (+CC)	108.36 ± 0.48 (108.35 ± 0.48)	0.046 ± 0.020 (0.046 ± 0.021)	0.793 ± 0.032 (0.793 ± 0.032)	$-2.43^{+0.33}_{-0.45}$ ($-2.44^{+0.32}_{-0.44}$)
$z_0 = 0.9$	DESI+ r_d (+CC)	114.78 ± 0.58 (114.61 ± 0.75)	0.098 ± 0.023 (0.098 ± 0.025)	0.856 ± 0.039 ($0.91^{+0.25}_{-0.22}$)	$-2.49^{+0.56}_{-0.78}$ ($-2.29^{+0.67}_{-0.90}$)
	PP (+CC)	Unconstrained (113.8 ± 7.3)	0.118 ± 0.059 (0.123 ± 0.060)	$0.90^{+0.11}_{-0.12}$ ($0.91^{+0.11}_{-0.12}$)	Unconstrained ($-2.1^{+1.3}_{-2.2}$)
	DESy5 (+CC)	Unconstrained (113.9 ± 6.9)	0.123 ± 0.057 (0.133 ± 0.058)	0.91 ± 0.10 ($0.93^{+0.10}_{-0.12}$)	Unconstrained ($-2.18^{+0.92}_{-2.6}$)
	DESI+ r_d +PP (+CC)	114.70 ± 0.59 (114.78 ± 0.58)	0.095 ± 0.022 (0.098 ± 0.023)	0.851 ± 0.038 (0.856 ± 0.039)	$-2.58^{+0.53}_{-0.73}$ ($-2.49^{+0.56}_{-0.78}$)
	DESI+ r_d +DESy5 (+CC)	114.78 ± 0.58 (114.78 ± 0.59)	0.099 ± 0.022 (0.099 ± 0.022)	0.867 ± 0.038 (0.867 ± 0.038)	$-2.47^{+0.58}_{-0.81}$ ($-2.47^{+0.58}_{-0.79}$)
$z_0 = 1.0$	DESI+ r_d (+CC)	$121.32^{+0.84}_{-0.70}$ (121.12 ± 0.85)	$0.144^{+0.030}_{-0.025}$ (0.141 ± 0.024)	$0.926^{+0.054}_{-0.046}$ (0.96 ± 0.20)	$-2.64^{+0.87}_{-1.0}$ ($-2.67^{+0.71}_{-0.88}$)
	PP (+CC)	Unconstrained ($122.0^{+8.3}_{-7.5}$)	0.216 ± 0.071 (0.218 ± 0.070)	$1.07^{+0.15}_{-0.16}$ ($1.08^{+0.14}_{-0.16}$)	Unconstrained ($-2.6^{+1.0}_{-1.9}$)
	DESy5 (+CC)	Unconstrained ($121.8^{+8.3}_{-7.3}$)	0.220 ± 0.069 (0.220 ± 0.066)	$1.09^{+0.14}_{-0.16}$ ($1.09^{+0.13}_{-0.15}$)	Unconstrained ($-2.7^{+1.0}_{-1.8}$)
	DESI+ r_d +PP (+CC)	121.48 ± 0.66 ($121.32^{+0.84}_{-0.70}$)	0.151 ± 0.023 ($0.144^{+0.030}_{-0.025}$)	0.939 ± 0.042 ($0.926^{+0.054}_{-0.046}$)	$-2.37^{+0.87}_{-1.1}$ ($-2.64^{+0.87}_{-1.0}$)
	DESI+ r_d +DESy5 (+CC)	121.44 ± 0.66 (121.43 ± 0.63)	0.151 ± 0.022 (0.149 ± 0.021)	0.947 ± 0.041 (0.945 ± 0.039)	$-2.38^{+0.85}_{-1.1}$ ($-2.50^{+0.78}_{-0.90}$)

Note. For each dataset combination, results are reported both without including CC and including them, the latter shown in parentheses.

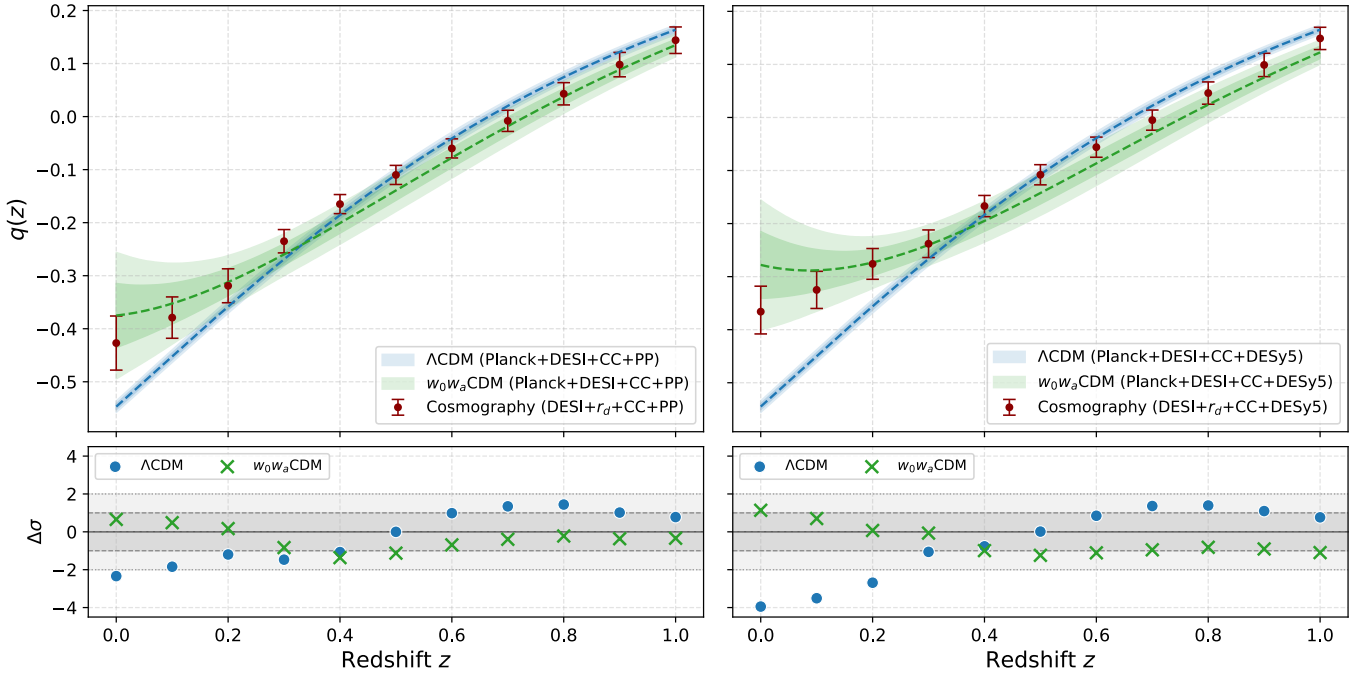


Figure 1. Cosmographic constraints on the deceleration parameter $q(z)$ at different redshift bins with 68% CL uncertainties (red points). The blue band shows the Λ CDM predictions, while the green band corresponds to those from a CPL model of dynamical DE, referred to as w_0w_a CDM. The left panel shows results obtained by combining Planck (or Planck r_d), DESI, and CC with PP SNIa. The right panel shows results from the same data combination but with DESy5 instead of PP. The bottom panels display the distance of the cosmographic points from the model predictions in units of the combined uncertainty $\Delta\sigma$. Blue circles correspond to Λ CDM and green crosses to w_0w_a CDM, providing a direct measure of the level of agreement between the cosmographic constraints and those derived within the respective cosmological models.

The constraints on $q(z_0)$ obtained with the Padé (2, 1) expansion at different pivot redshifts are shown in Figure 1 as red points with their 68% confidence level (CL) error bars. As seen in the figure, for both dataset combinations the error bars on $q(z_0)$ are largest when the expansion is centered at $z_0 = 0$ (standard approach) and shrink markedly when the pivot is shifted to intermediate redshifts, $z_0 \simeq 0.3\text{--}0.6$. This behavior has two main origins. First, expansions centered at higher z_0 provide a more accurate reconstruction across the redshift range covered by the data, improving the theoretical precision of the series by up to two orders of magnitude compared to $z_0 = 0$. Second, changing the pivot rotates the correlation between cosmographic parameters. As seen in Figure 2, at $z_0 \simeq 0$ a strong degeneracy exists between $q(z_0)$ and $H(z_0)$, with more negative values of the former driving larger values of the latter. As the pivot moves to $z_0 \simeq 0.3\text{--}0.6$ this correlation is progressively reduced and eventually lost, yielding much tighter constraints.

This happens because that redshift window is populated by both distance data (DESI, PP/DESy5) and direct $H(z)$ measurements (DESI, CC), making it far easier to disentangle their effects. At higher pivots ($z_0 \simeq 0.7\text{--}1$), however, the correlation between $q(z_0)$ and $H(z_0)$ reemerges with opposite sign, and uncertainties increase again. Overall, fixing the pivot around $z_0 \sim 0.3\text{--}0.6$ provides the optimal balance: it maximizes the theoretical accuracy of the truncated series while yielding significantly tighter constraints thanks to reduced degeneracies. Additional validation of these statements is provided in Appendix C. We stress that these results are crucial: given the increasing precision of present and future background probes, our generalized framework ensures that theoretical errors remain smaller than observational uncertainties without requiring higher-order expansions or introducing new parameters. This allows cosmographic tests to fully exploit the constraining power of current and forthcoming surveys.

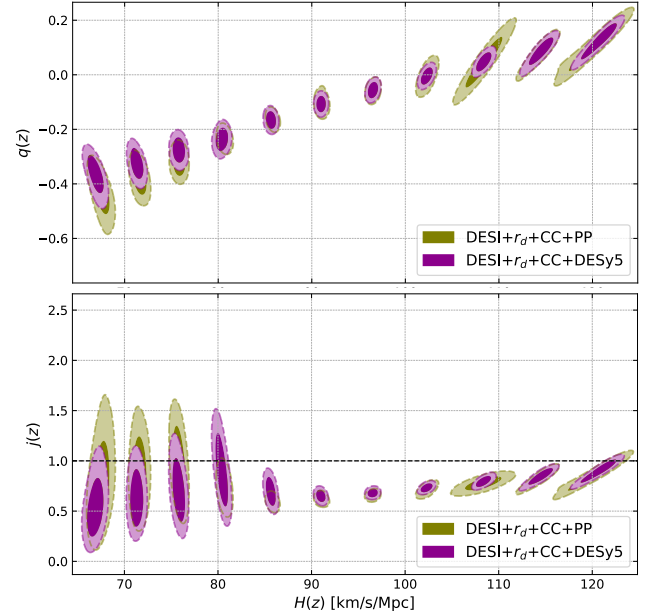


Figure 2. 2D correlations between the constraints on the expansion rate, the deceleration parameter, and the jerk parameter obtained at 11 different pivot redshifts, from $z_0 = 0$ (far left) to $z_0 = 1$ (far right).

That being said, the most critical information lies in the distribution of the red points in Figure 1. By constraining $q(z)$ at different pivots with high precision, we obtain snapshots of the deceleration parameter at multiple epochs. This provides a direct reconstruction of its redshift evolution, derived purely from cosmography without assuming any underlying model. By contrast, within a cosmological framework such as Λ CDM or CPL, once the free parameters

are constrained the behavior of $q(z)$ is fixed by the model itself.

In Figure 1, the red points from our generalized cosmographic expansion are compared with the blue band, which represents Λ CDM predictions (68% and 95% CL) obtained for the same datasets (Planck+DESI+CC+PP in the left panel and Planck+DESI+CC+DESy5 in the right panel). The bottom panels quantify these differences by showing the distance of the cosmographic points from the model predictions in units of the combined uncertainty $\Delta\sigma$. Across the full redshift range, the cosmographic reconstruction departs from the Λ CDM prediction in a coherent, redshift-dependent way. At very low redshift ($z \lesssim 0.2$), the mismatch is sharp: with DESy5, the red points lie $\sim 4\sigma$ above Λ CDM, while with PP the offset is smaller but still close to 2σ . At intermediate redshifts ($0.3 \lesssim z \lesssim 0.5$), the agreement improves, with the cosmographic points falling within 1σ – 2σ of the model. At higher redshifts ($z \gtrsim 0.5$), however, the trend reverses, with all cosmographic determinations lying systematically below the Λ CDM curve. Although each individual deviation remains at the 1.5σ – 2σ level, their persistence across consecutive bins makes the effect statistically meaningful: the cosmographic constraints on $q(z)$ do not simply scatter around the model expectation but reveal a structured, redshift-dependent trend.

This behavior is what one would expect if the late-time expansion of the Universe were driven by a dynamical DE component whose EoS evolves from a present-day quintessence phase ($w(z) > -1$) across the phantom divide ($w(z) < -1$). At low redshift, the cosmographic reconstruction shows $q(z)$ values higher than Λ CDM, indicating a weaker acceleration. This is consistent with a quintessence-like regime, where the negative pressure of DE is not strong enough to produce the same level of acceleration as a cosmological constant. At intermediate redshift, the quintessence-to-phantom transition occurs, so that around $w(z) \simeq -1$ the deceleration parameter approaches the Λ CDM prediction. Within benchmark dynamical DE scenarios, for the datasets analyzed in this work, the phantom crossing is estimated to occur around $z \sim 0.3$ – 0.4 (W. Giarè et al. 2024; E. Özlüker et al. 2025), precisely where we recover improved consistency with the Λ CDM predictions in Figure 1. Finally, at higher redshifts, after crossing into the phantom regime, the trend reverses: the negative pressure becomes more repulsive than that of a cosmological constant, the expansion is driven faster than in Λ CDM, and $q(z)$ falls systematically below the Λ CDM curve.

To corroborate this hypothesis, in Figure 1 we compare the cosmographic reconstruction with the green bands, which show the predictions (68% and 95% CL) obtained within the CPL parameterization of the DE EoS for the same datasets. The agreement with the red cosmographic points is remarkable. At low redshift, CPL successfully reproduces the $\gtrsim 2\sigma$ – 4σ departures from Λ CDM, remaining fully consistent with the cosmographic results within 1σ . This lends weight to the interpretation that the observed departures from Λ CDM may represent footprints of dynamical DE in the late-time expansion history of the Universe.

To further validate our interpretation, we generalize the $Om(z)$ diagnostic. Following V. Sahni et al. (2008), we define

$$Om(z) \equiv \frac{\tilde{h}^2(z) - 1}{(1+z)^3 - 1}, \quad (2)$$

where $\tilde{h}(z) = H(z)/H_0$ is the Hubble parameter normalized to its present value. This diagnostic has a distinctive property: if

DE is a cosmological constant, $Om(z)$ remains exactly constant at all redshifts, with $Om(z) = \Omega_m$. Any dynamical behavior in the DE sector instead produces a clear redshift dependence. For instance, as pointed out in Figure 3 of V. Sahni et al. (2008), a descending trend in $Om(z)$ should be observed for a quintessence-like DE EoS. Our method, which constrains $H(z)$ cosmographically at multiple pivots, delivers precise determinations of $H(z)$ across the redshift range. This enables us to constrain $Om(z)$ at several redshifts and directly test whether it is consistent with a constant value or reveals a redshift-dependent trend.

The results are shown in Figure 3, where the red points represent our cosmographic determinations of $Om(z)$ (the left panel corresponds to combinations including PP, and the right panel corresponds to those including DESy5). The red points are far from constant and deviate significantly from the Λ CDM prediction shown in blue. In both datasets, the discrepancy with Λ CDM emerges already at $z \gtrsim 0.1$ and persists across the entire redshift range. At low redshift, the uncertainties in $Om(z)$ are larger for the same reasons discussed for $q(z)$, but the trend is nonetheless visible. At intermediate redshifts, the benefit of our generalized cosmographic method becomes evident: the constraints are significantly tighter, and the red points remain systematically above the Λ CDM curve. In the bottom panels we quantify these differences by showing the distance of the cosmographic points from the model predictions in units of the combined uncertainty $\Delta\sigma$. Depending on the dataset, the cosmographic estimates of $Om(z)$ at $z \gtrsim 0.3$ exceed the Λ CDM prediction by 3σ – 3.5σ for combinations including PP and by as much as $\sim 4\sigma$ for DESy5. At higher redshift the trend continues, with all determinations consistently displaced from the Λ CDM expectation.

As in the case of $q(z)$, our determinations of $Om(z)$ do not scatter randomly around the Λ CDM prediction but reveal a coherent redshift-dependent trend. This becomes especially clear when comparing the red points with the green bands, which represent the CPL predictions for the same datasets. In this case, the agreement with dynamical DE models is remarkable: the red cosmographic points follow the CPL best-fit predictions (dotted green curve) almost perfectly across the full redshift range, for both Planck+DESI+CC+PP and Planck+DESI+CC+DESy5. This demonstrates that the cosmographic expansion, when translated into $Om(z)$ constraints, yields results in excellent agreement with dynamical DE (already within a simple CPL description) rather than with a cosmological constant.

Beyond $q(z)$ and $Om(z)$, further cosmographic parameters such as $H(z)$ and $j(z)$ also show systematic departures from Λ CDM, consistently pointing toward dynamical DE. In particular, our constraints on $j(z)$ reveal significant departures from the Λ CDM identity $j(z) \equiv 1$; see Figure 2. Similarly, by constraining the expansion rate $H(z)$ at different pivot redshifts, our methodology indicates a preference for a faster expansion at $z \gtrsim 0.3$, about $2 \text{ km s}^{-1} \text{ Mpc}^{-1}$ above the Λ CDM prediction, while coming into closer agreement at lower redshift ($z \lesssim 0.3$). At $z = 0$, the Padé series yields $H_0 = 67.49 \pm 0.56$ (66.90 ± 0.53) $\text{km s}^{-1} \text{ Mpc}^{-1}$ for DESI+ r_d +PP+CC (DESI+ r_d +DESy5+CC). These values are lower than the H_0 inferred from the same datasets under the assumption of Λ CDM by roughly $1 \text{ km s}^{-1} \text{ Mpc}^{-1}$, while remaining fully consistent with the values obtained assuming a CPL dynamical DE model. A more detailed discussion of these

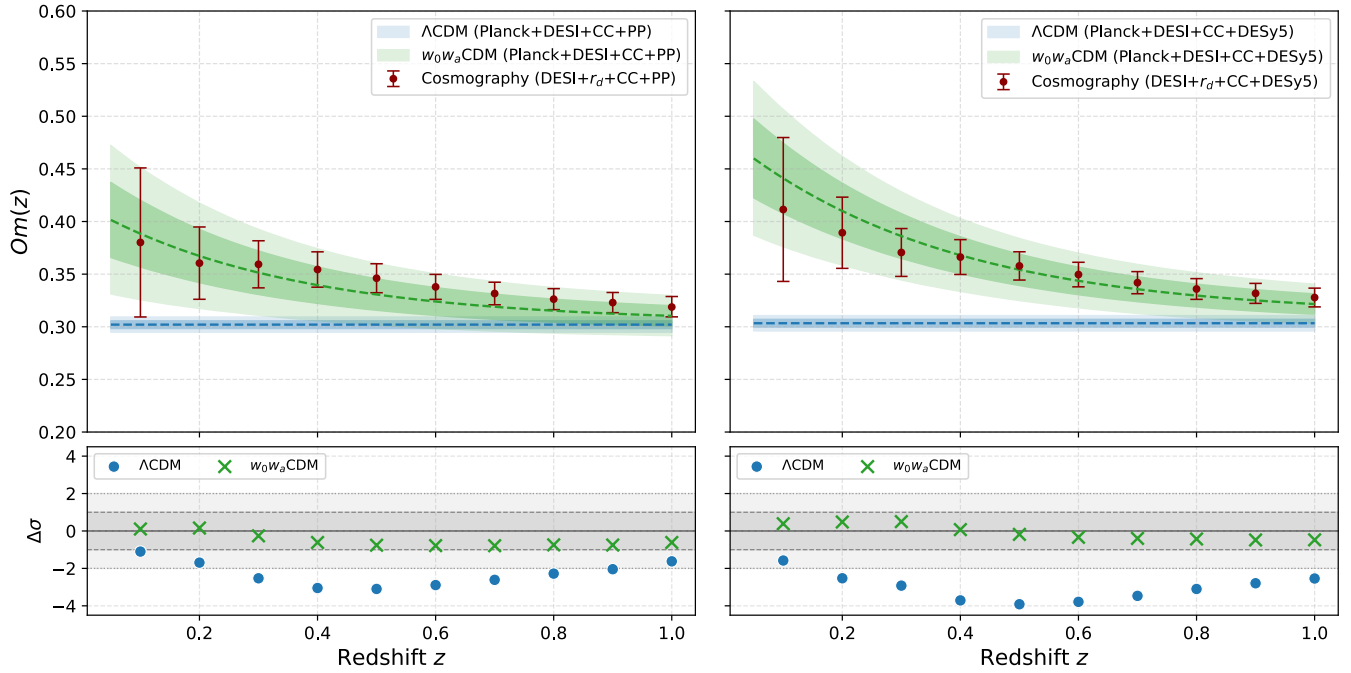


Figure 3. Cosmographic constraints on the $Om(z)$ diagnostic at different redshift bins with 68% CL uncertainties (red points). The blue band shows the Λ CDM predictions ($Om(z) = \Omega_m$), while the green band corresponds to those from a CPL model of dynamical DE, referred to as w_0w_a CDM. The left panel shows results obtained by combining Planck (or Planck r_d), DESI, and CC with PP SNIa. The right panel shows results from the same data combination but with DESy5 instead of PP. The bottom panels display the distance of the cosmographic points from the model predictions in units of the combined uncertainty $\Delta\sigma$. Blue circles correspond to Λ CDM and green crosses to w_0w_a CDM, providing a direct measure of the level of agreement between the cosmographic constraints and those derived within the respective cosmological models.

results is presented in Appendix C. We also note that the absolute normalization of $H(z)$ depends on the calibration adopted for BAO, which in our baseline analysis is anchored to the Planck determination of the sound horizon within Λ CDM. In Appendix D we present an extended analysis exploring alternative calibrations of the datasets (e.g., SH0ES-based calibration of SNIa or reduced- r_d priors motivated by models of interest for the Hubble tension), showing that such choices simply rescale $H(z)$ while leaving the cosmographic constraints on $q(z)$ and $Om(z)$ —and thus our main conclusions—completely unchanged.

Overall, the cosmographic reconstructions at multiple pivots are consistently better matched by simple dynamical DE models such as CPL, providing an additional and independent indication of DE dynamics and offering a clear, model-independent guideline for building more successful theoretical scenarios to interpret current data.

4. Conclusions

In this work we have developed a generalized cosmographic framework based on Padé (2, 1) expansions around arbitrary pivot redshifts z_0 , showing that it overcomes several intrinsic limitations of the standard approach, which relies on truncated Taylor or Padé series around $z_0 = 0$. Our method provides three key methodological advantages:

1. *Accuracy.* Expanding the Padé (2, 1) series around higher pivots reduces truncation errors across the full redshift range probed by current data ($0 \lesssim z \lesssim 2.5$), improving the accuracy of reconstructed observables by up to two orders of magnitude.

2. *Precision.* Expressing all observables directly in terms of cosmographic parameters defined at the chosen pivot ensures that parameters are constrained exactly where data are available. This removes the need for extrapolation, reduces degeneracies, and allows for a much cleaner disentangling of their effects, resulting in significantly more precise constraints.
3. *Flexibility.* By repeating the analysis at multiple pivots, we obtain high-precision constraints on cosmographic parameters at different epochs, directly mapping the redshift evolution of key quantities such as $H(z)$ and $q(z)$. In this way, cosmography becomes a precise, model-independent tracer of the late-time kinematics of the Universe.

To exploit these advantages, we derive cosmographic constraints at 11 pivot redshifts uniformly distributed between $z_0 = 0$ and $z_0 = 1$. We analyze the latest DESI BAO measurements calibrated with the Λ CDM Planck value of the sound horizon, SNIa from PP and DESy5, and CC. The main results can be summarized as follows:

1. *Pivot Redshift and Correlations.* As seen in Figure 2, at $z_0 \simeq 0$, $H(z_0)$ and $q(z_0)$ are strongly anticorrelated, and the uncertainties on their inferred values are large. Moving z_0 to higher redshift, the degeneracy line rotates so that around $z_0 \simeq 0.3$ – 0.6 the two parameters become nearly decorrelated and their effects can be cleanly disentangled, yielding significantly tighter constraints. At still higher pivots ($z_0 \simeq 0.7$ – 1), the correlation continues to rotate and reappears with opposite sign, again enlarging the inferred uncertainties. With current data, the optimal pivot range is therefore $z_0 \simeq 0.3$ – 0.6 .

2. *Footprints of Dynamical DE in $q(z)$.* The cosmographic constraints on the deceleration parameter shown in Figure 1 deviate from the Λ CDM predictions in a coherent, redshift-dependent way. At $z \lesssim 0.2$, $q(z)$ lies significantly above the Λ CDM curve (up to $\sim 4\sigma$ for DESy5 and $\sim 2\sigma$ for PP), while at $z \gtrsim 0.6$ it falls systematically below it. Overall, the reconstructed trend shows much closer agreement with the behavior predicted within a CPL parameterization of DE.
3. *Footprints of Dynamical DE in $Om(z)$.* We generalize the $Om(z)$ diagnostic to constrain this quantity at different pivot redshifts. The cosmographic constraints shown in Figure 3 are inconsistent with the constant behavior expected in Λ CDM, as they lie systematically above the Λ CDM curve (with deviations of 2.5σ – 3.5σ for combinations including PP and up to $\sim 4\sigma$ for DESy5), showing a redshift trend that aligns almost perfectly with the CPL best-fit predictions across the entire redshift range.

Taken together, these results provide a coherent and assumption-free indication of deviations from Λ CDM and suggest that we are observing the footprints of dynamical DE in the late-time kinematics of the Universe. Additional cosmographic constraints reinforce this interpretation: the jerk parameter $j(z)$ departs from the Λ CDM identity $j \equiv 1$, while the expansion rate $H(z)$ is faster than Λ CDM at $z \gtrsim 0.3$ by about $2 \text{ km s}^{-1} \text{ Mpc}^{-1}$, in close alignment with CPL predictions. More broadly, since our constraints on the cosmographic quantities are by definition independent of the underlying dynamics, they also represent universal consistency conditions that any model of cosmic acceleration should satisfy.

Looking ahead, our generalized cosmographic framework enables high-precision, model-independent tests of cosmic acceleration. With forthcoming data from DESI and Euclid, it will be crucial to keep theoretical truncation errors below observational uncertainties. Our approach ensures higher accuracy without higher-order expansions, reduces parameter degeneracies, and provides a robust and flexible tool to probe cosmic acceleration.

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Author contributions

E.F. derived the generalized cosmographic framework used in this work, implemented it numerically, performed the MCMC analyses, and contributed to editing the manuscript. W.G. conceived the initial research idea, supervised the

theoretical derivation, supervised and contributed to the numerical implementation of the cosmographic expansion within the cosmological pipeline, and wrote the first draft of the manuscript. E.D.V. validated both the theoretical and numerical analyses and contributed to editing the manuscript. All authors contributed to refining and finalizing the manuscript.

Appendix A

Generalized Cosmographic Expansion

Cosmography describes the late-time expansion history of the Universe by expressing cosmological observables as series expansions.⁵ The standard approach employs a Taylor expansion in redshift z , centered on the present epoch $z_0 = 0$. In this framework, observables such as the luminosity distance $D_L(z)$ and the expansion rate $H(z)$ can be written in terms of cosmographic parameters defined at $z = 0$ (M. Visser 2004, 2005). Constraining these parameters from data then determines the behavior of the series at any redshift, providing a model-independent description of the Universe’s kinematics.⁶

However, while the Taylor expansion performs well at low redshifts, it rapidly loses accuracy at higher redshifts, where it may even diverge, introducing biases and large truncation errors. One may attempt to improve the approximation by including higher-order terms, but any expansion in z with $a = 1/(1+z)$ has a singularity at $z = -1$ (C. Cattoen &

⁵ With no claim of completeness regarding the plethora of works employing cosmographic analyses over the past couple of decades to probe cosmological properties, we refer the reader to T. D. Saini et al. (2000), G. M. Bernstein & B. Jain (2004), M. V. John (2005), C. Cattoen & M. Visser (2007a), S. Capozziello & L. Izzo (2008), L. Xu & Y. Wang (2011), O. Luongo (2011), J. C. Carvalho & J. S. Alcaniz (2011), M. Demianski et al. (2012), K. Bamba et al. (2012), R. Lazkoz et al. (2013), C. M. Springob et al. (2014), H. Gao et al. (2014), I. Jee et al. (2016), Y.-N. Zhou et al. (2016), M. Demianski et al. (2017), M.-J. Zhang et al. (2017), C. J. A. P. Martins et al. (2016), A. J. Shajib et al. (2018), T. Yang et al. (2020), E.-K. Li et al. (2020), M. Benetti & S. Capozziello (2019), C. Escamilla-Rivera & S. Capozziello (2019), Z.-Y. Yin & H. Wei (2019), M. Rezaei et al. (2020b), G. C. F. Chen et al. (2021), A. Banerjee et al. (2021), C. Escamilla-Rivera et al. (2022), T. Treu et al. (2022), J. P. Hu & F. Y. Wang (2022), S. Pourojaghi et al. (2022), B. A. R. Rocha & C. J. A. P. Martins (2022), S. Pourojaghi et al. (2025), J. F. Jesus et al. (2025), G. Rodrigues et al. (2025), Z. Yang et al. (2025), and J.-L. Ling et al. (2025).

⁶ The nature of cosmic acceleration, and in particular the question of whether it is driven by a cosmological constant or emerges from dynamical models, has long attracted significant attention in the high-energy community, even before the release of DESI collaboration data. For pioneering works in this direction, we refer to A. R. Cooray & D. Huterer (1999), G. Efstathiou (1999), M. Chevallier & D. Polarski (2001), E. V. Linder (2003), C. Wetterich (2004), B. Feng et al. (2006), S. Hannestad & E. Mortsell (2004), J.-Q. Xia et al. (2005), Y.-g. Gong & Y.-Z. Zhang (2005), H. K. Jassal et al. (2005), S. Nesseris & L. Perivolaropoulos (2005), D.-J. Liu et al. (2008), E. M. Barboza & J. S. Alcaniz (2008), E. M. Barboza et al. (2009), J.-Z. Ma & X. Zhang (2011), I. Sendra & R. Lazkoz (2012), L. Feng & T. Lu (2011), E. M. Barboza & J. S. Alcaniz (2012), A. De Felice et al. (2012), C.-J. Feng et al. (2012), H. Wei et al. (2014), J. Magaña et al. (2014), Ö. Akarsu et al. (2015), S. Pan et al. (2016), E. Di Valentino et al. (2016); R. C. Nunes et al. (2017a, 2017b), J. Magana et al. (2017), W. Yang et al. (2018), S. Pan et al. (2018), G. Panotopoulos & Á. Rincón (2018); W. Yang et al. (2019b), L. G. Jaime et al. (2018), A. Das et al. (2018), W. Yang et al. (2019a), X. Li & A. Shafieloo (2019), W. Yang et al. (2019c), S. Pan et al. (2020a), D. Tamayo & J. A. Vazquez (2019), S. Pan et al. (2020b), E. Di Valentino et al. (2021), M. Rezaei et al. (2020a); D. Perkovic & H. Stefancic (2020), A. Banihashemi et al. (2021), M. Jaber-Bravo et al. (2020), H. B. Benaoum et al. (2022), W. Yang et al. (2021), M. Jaber et al. (2022), G. Alestas et al. (2022), J. Yang et al. (2023), H. G. Escudero et al. (2022), M. N. Castillo-Santos et al. (2023), W. Yang et al. (2023), S. Dahmani et al. (2023), L. A. Escamilla et al. (2024), M. Rezaei et al. (2024), S. A. Adil et al. (2024), J. A. Lozano Torres (2024), J. K. Singh et al. (2024), M. Rezaei (2024), and M. Reyhani et al. (2024).

M. Visser 2007b; E. Ó Colgáin & M. M. Sheikh-Jabbari 2021), which fixes the radius of convergence to $|z| \leq 1$ according to the Cauchy–Hadamard theorem. As a result, adding more terms does not necessarily extend the validity of the expansion and may even worsen it beyond $z \simeq 1$. This limits the applicability of standard cosmographic expansions in the regime $z \gtrsim 1$, precisely where recent datasets such as DESI BAO and SNIa provide the most constraining power. A common work-around is to reformulate the expansion in terms of the auxiliary variable $y = z/(1+z)$, which removes the singularity and ensures convergence over the full redshift range. However, the y variable compresses the redshift interval and acts as a smaller expansion parameter, leading to larger propagated uncertainties and requiring more coefficients to achieve the same accuracy (V. C. Busti et al. 2015).

To address the shortcomings of Taylor series, Padé approximants have been proposed, replacing the polynomial expansion with a ratio of polynomials (A. Aviles et al. 2014; C. Gruber & O. Luongo 2014; H. Wei et al. 2014). Among them, the Padé (2, 1) expansion has been shown to offer the best compromise between accuracy, stability, and the number of cosmographic parameters involved (S. Capozziello et al. 2019; S. Capozziello et al. 2020; J. P. Hu & F. Y. Wang 2022; B. Yu et al. 2025).

Despite these improvements, significant limitations remain. In its standard formulation, the Padé expansion is still built from truncated series around $z_0 = 0$, so the fitted parameters characterize only the present epoch, while the behavior at $z > 0$ is inferred indirectly from the series itself. As redshift increases, the approximation still deteriorates and truncation errors can reach or even exceed observational uncertainties precisely in the range where most data are available ($0.1 \lesssim z \lesssim 2$). Extending the series to higher orders can mitigate this, but at the cost of introducing extra parameters, which increases degeneracies and undermines predictivity.

To overcome these limitations, in this work we refine the cosmographic approach by formulating the Padé (2, 1) expansion around a generic pivot redshift z_0 . This enables us to fit cosmological data at higher redshifts with improved precision and to constrain cosmographic parameters directly at different redshift values. In turn, this allows us to reconstruct the full cosmographic functions, which can then be employed to test the nature of cosmic acceleration.⁷

⁷ Following the DESI results, this interest has been widely revitalized, with a multitude of analyses and reinterpretations of the observations. For a noncomplete yet broad survey of the existing literature in this direction, see, e.g., M. Cortés & A. R. Liddle (2024), D. Shlivko & P. J. Steinhart (2024), O. Luongo & M. Muccino (2024), W. Yin (2024), I. D. Gialamas et al. (2025), B. R. Dinda (2024), M. Najafi et al. (2024), H. Wang & Y.-S. Piao (2024), G. Ye et al. (2025), Y. Tada & T. Terada (2024), Y. Carloni et al. (2025), C.-G. Park et al. (2024), K. Lodha et al. (2025a), S. Bhattacharya et al. (2024), O. F. Ramadan et al. (2024), A. Notari et al. (2024), L. Orchard & V. H. Cárdenas (2024), A. Hernández-Almada et al. (2024), M. Malekjani et al. (2025), W. Giarè et al. (2024), J. Rebouças et al. (2025), W. Giarè (2025), C.-G. Park et al. (2025), N. Menci et al. (2024), T.-N. Li et al. (2025b), J.-X. Li & S. Wang (2025), T.-N. Li et al. (2024), A. Notari et al. (2025), Q. Gao et al. (2025), R. Fikri et al. (2025), J.-Q. Jiang et al. (2024), J. Zheng et al. (2025), A. Gómez-Valent & J. Solà Peracaula (2025), S. Roy Choudhury & T. Okumura (2024), A. Lewis & E. Chamberlain (2025), W. J. Wolf et al. (2025), A. J. Shajib & J. A. Frieman (2025), W. Giarè et al. (2025), E. Chaussidon et al. (2025), D. A. Kessler et al. (2025), Y.-H. Pang et al. (2025), S. Roy Choudhury (2025), M. Scherer et al. (2025), E. M. Teixeira et al. (2025), E. Specogna et al. (2025), H. Cheng et al. (2025a, 2025b), E. Özlülker et al. (2025), D. H. Lee et al. (2025), E. Silva & R. C. Nunes (2025), M. Ishak & L. Medina-Varela (2025), T.-N. Li et al. (2025a), Z. Lu et al. (2025), and J.-L. Ling et al. (2025).

In this appendix, we derive, step by step, the generalized cosmographic expansion (Section A.1) and show that our formulation reproduces the known results when $z_0 = 0$ (Section A.2).

A.1. Derivation of Padé (2, 1) Expansion around a Pivot Redshift z_0

In general mathematical terms, the Padé (2, 1) expansion of a function is defined as (A. Aviles et al. 2014; C. Gruber & O. Luongo 2014)

$$f^{(2,1)}(z) = \frac{a_0 + a_1x + a_2x^2}{1 + b_1x}, \quad (\text{A1})$$

where a_i and b_i are the Padé coefficients. The Padé approximants are degenerate under a rescaling factor; for this reason b_0 is usually fixed to $b_0 = 1$.

To express these coefficients explicitly in terms of cosmographic parameters, we follow the standard procedure of comparing the Taylor expansion with the Padé form. The first step is to expand the denominator of the Padé expression in Equation (A1) as a Taylor series, obtaining

$$\begin{aligned} f^{(2,1)}(z) &= a_0 + (a_1 - a_0b_1)x + (a_2 - a_1b_1 + a_0b_1^2)x^2 \\ &\quad + (-a_2b_1 + a_1b_1^2 - a_0b_1^3)x^3 + \mathcal{O}(x^4). \end{aligned} \quad (\text{A2})$$

In this way the Padé (2, 1) expression can be compared with the third-order Taylor expansion of a function, written in general as

$$f^{\text{Tay}}(z) = c_0 + c_1x + c_2x^2 + c_3x^3 + \mathcal{O}(x^4), \quad (\text{A3})$$

Equating term by term Equations (A3) and (A2), we can fix the Padé parameters:

$$\begin{aligned} c_0 &= a_0, \\ c_1 &= a_1 - a_0b_1, \\ c_2 &= a_2 - a_1b_1 + a_0b_1^2, \\ c_3 &= -a_2b_1 + a_1b_1^2 - a_0b_1^3. \end{aligned} \quad (\text{A4})$$

Solving this triangular system, we finally obtain

$$a_0 = c_0, \quad a_1 = c_1 + c_0b_1, \quad a_2 = c_2 + c_1b_1, \quad b_1 = -\frac{c_3}{c_2}. \quad (\text{A5})$$

In the following subsection, we apply this methodology to derive a generalized Padé (2, 1) expansion centered around a generic redshift. We proceed as follows. First, in Section A.1.1 we present the derivation of a third-order Taylor expansion for the Hubble parameter $H(z)$ around an arbitrary pivot redshift $z = z_0$. In Section A.1.2, we present the derivation of a third-order Taylor expansion for the luminosity distance $D_L(z)$ around an arbitrary pivot redshift $z = z_0$. Building on this, we then construct the Padé (2, 1) approximation for both the luminosity distance (Section A.1.3) and the expansion rate (Section A.1.4) by adopting the definition of the Padé (2, 1) series and matching terms order by order to determine the Padé coefficients.

A.1.1. Taylor Expansion of the Hubble Parameter around a Pivot Redshift z_0

Here we present the derivation of a third-order Taylor expansion for the Hubble parameter $H(z)$ and the luminosity distance $D_L(z)$ around an arbitrary pivot redshift $z = z_0$.

We begin with the Taylor expansion of the Hubble parameter, $H(z)^{\text{Tay}}$, around $z = z_0$, truncated at third order, which can be written as

$$H(z)^{\text{Tay}} = H(z_0) + H'(z_0)(z - z_0) + \frac{1}{2} H''(z_0)(z - z_0)^2 + \frac{1}{6} H'''(z_0)(z - z_0)^3 + \mathcal{O}((z - z_0)^4). \quad (\text{A6})$$

It is convenient to factor out $H(z_0)$ and write

$$H^{\text{Tay}}(z) = H_{z_0}[1 + Ax + Bx^2 + Cx^3 + \mathcal{O}(x^4)], \quad (\text{A7})$$

where $x \equiv z - z_0$, we adopt the notation $Y_{z_0} \equiv Y(z_0)$ for any generic function Y evaluated at z_0 (so $H_{z_0} \equiv H(z_0)$), and we define the dimensionless coefficients

$$A = \frac{H'_{z_0}}{H_{z_0}}, \quad B = \frac{H''_{z_0}}{2H_{z_0}}, \quad C = \frac{H'''_{z_0}}{6H_{z_0}}. \quad (\text{A8})$$

Our goal is to express these coefficients in terms of cosmographic parameters evaluated at z_0 . For a third-order expansion, the relevant quantities are the deceleration parameter $q(z)$, the jerk parameter $j(z)$, and the snap parameter $s(z)$, respectively defined as (M. Visser 2004)

$$q = -\frac{d^2a/dt^2}{aH^2}, \quad j = \frac{d^3a/dt^3}{aH^3}, \quad s = \frac{d^4a/dt^4}{aH^4}. \quad (\text{A9})$$

To express A , B , and C in terms of q_{z_0} , j_{z_0} , and s_{z_0} , we first rewrite the derivatives of the scale factor $a(t)$ in terms of derivatives of the Hubble parameter $H(t)$. Starting from the definition of the Hubble parameter $H \equiv \frac{\dot{a}}{a}$ and differentiating it with respect to time, we find

$$\frac{\dot{a}}{a} = \dot{H} + H^2. \quad (\text{A10})$$

Using the relation between time and redshift derivatives, $\frac{d}{dt} = -(1+z)H\frac{d}{dz}$, we get $\dot{H}(t) = -(1+z)H(z)H'(z)$. Substituting these relations into Equation (A9), one obtains

$$q(z) = -1 + \frac{(1+z)H'(z)}{H(z)}. \quad (\text{A11})$$

Evaluating this at $z = z_0$, we finally find

$$A = \frac{H'_{z_0}}{H_{z_0}} = \frac{1 + q_{z_0}}{1 + z_0}. \quad (\text{A12})$$

To obtain the jerk parameter, we have to reiterate the above process starting from the relation

$$\frac{d}{dt}\left(\frac{\ddot{a}}{a}\right) = \frac{\ddot{a}}{a} - \frac{\ddot{a}H}{a}, \quad (\text{A13})$$

and using Equation (A10) and the definition of the deceleration parameter in Equation (A9), we find

$$j(t) = \frac{1}{H^3}(\ddot{H} + 2H\dot{H}) - q(t). \quad (\text{A14})$$

Now, switching to the redshift variable (through the derivative rule) and rearranging the expression, we obtain

$$j(z) = q(z)^2 + (1+z)^2 \frac{H''(z)}{H(z)}, \quad (\text{A15})$$

from which it is straightforward to find

$$B = \frac{H''_{z_0}}{2H_{z_0}} = \frac{j_{z_0} - q_{z_0}^2}{2(1+z_0)^2}. \quad (\text{A16})$$

Lastly, from the definition of the snap parameter in Equation (A9) and from

$$\frac{d}{dt}\left(\frac{\ddot{a}}{a}\right) = \frac{a^{(4)}}{a} - \frac{\ddot{a}H}{a}, \quad (\text{A17})$$

and using also the definition of the jerk parameter, we find

$$s(t) = \frac{1}{H^4} \frac{d}{dt}(j(t)H^3) + j(t). \quad (\text{A18})$$

Using the definition in Equation (A15) and switching to the redshift variable, we obtain

$$s(z) = -(1+z) \left[2q(z)q'(z) + 3q(z)^2 \frac{H'}{H} + \frac{H''}{H} + (1+z) \left(\frac{H'''}{H} + 2 \frac{H'H''}{H^2} \right) \right] + j(z). \quad (\text{A19})$$

Applying the first derivative to Equation (A11), and after some algebraic manipulations, we finally get

$$s(z) = 3q(z)^3 + 3q(z)^2 - 4q(z)j(z) - 3j(z) - (1+z)^3 \frac{H'''}{H}, \quad (\text{A20})$$

from which it is immediate to obtain

$$C = \frac{H'''_{z_0}}{6H_{z_0}} = \frac{3q_{z_0}^3 + 3q_{z_0}^2 - (4q_{z_0} + 3)j_{z_0} - s_{z_0}}{6(1+z_0)^3}. \quad (\text{A21})$$

A.1.2. Taylor Expansion of the Luminosity Distance around a Pivot Redshift z_0

The luminosity distance in a flat, homogeneous, and isotropic Universe described by the Friedmann–Lemaître–Robertson–Walker metric is defined as

$$D_L(z) = (1+z) \int_0^z \frac{dz'}{H(z')}. \quad (\text{A22})$$

Because of the inverse Hubble parameter in its formulation, it is inversely proportional to the coefficients A , B , and C . In order to obtain an analytically solvable integral, it is possible to expand $1/H$ as a Taylor series around z_0 , keeping terms up to second order in x (since we are interested in the luminosity

distance up to third order). This yields

$$\frac{1}{H(z)} = \frac{1}{H_{z_0}}(1 - Ax + (A^2 - B)x^2). \quad (\text{A23})$$

Integrating this expression leads to the final expression for the Taylor-expanded luminosity distance:

$$D_L^{\text{Taylor}}(z) = \frac{(1+z)}{H_{z_0}} \left\{ z - \frac{A}{2} [(z - z_0)^2 - z_0^2] + \frac{A^2 - B}{3} [(z - z_0)^3 + z_0^3] \right\}. \quad (\text{A24})$$

A.1.3. Padé (2, 1) Expansion of the Luminosity Distance around a Pivot Redshift z_0

Following the methodology discussed in Section A.1, after deriving the expression for the Taylor series, we can obtain the Padé (2, 1) expansion of the luminosity distance by rewriting the Taylor expansion of $D_L(z)$ in Equation (A24) as

$$D_L^{\text{Taylor}}(z) = c_0 + c_1x + c_2x^2 + c_3x^3 + \mathcal{O}(x^4), \quad (\text{A25})$$

where

$$\begin{aligned} c_0 &= \frac{(1+z_0)}{H_{z_0}} \left[z_0 + \frac{A}{2}z_0^2 + \frac{(A^2 - B)}{3}z_0^3 \right], \\ c_1 &= \frac{1}{H_{z_0}} \left[1 + 2z_0 + \frac{A}{2}z_0^2 + \frac{(A^2 - B)}{3}z_0^3 \right], \\ c_2 &= \frac{1}{2H_{z_0}} [2 - A(1 + z_0)], \\ c_3 &= \frac{1}{6H_{z_0}} [2(1 + z_0)(A^2 - B) - 3A]. \end{aligned} \quad (\text{A26})$$

This directly fixes the Padé coefficients a_i and b_i in terms of the parameters c_i above (and thus in terms of H_{z_0} , q_{z_0} , j_{z_0} , and s_{z_0}) by means of Equation (A5).

A.1.4. Padé (2, 1) Expansion of the Expansion Rate around a Pivot Redshift z_0

To derive the expression of the Padé (2, 1) expansion of the Hubble parameter, we analogously begin from the Taylor expansion of the Hubble parameter in Equation (A7), rewritten as

$$H^{\text{Taylor}}(z) = c_0 + c_1x + c_2x^2 + c_3x^3 + \mathcal{O}(x^4), \quad (\text{A27})$$

where

$$\begin{aligned} c_0 &= H_{z_0}, \\ c_1 &= H_{z_0}A, \\ c_2 &= H_{z_0}B, \\ c_3 &= H_{z_0}C. \end{aligned} \quad (\text{A28})$$

Again, this directly fixes the Padé coefficients a_i and b_i in terms of the parameters c_i above (and thus in terms of H_{z_0} , q_{z_0} , j_{z_0} , and s_{z_0}) by means of Equation (A5).

A.2. Consistency Checks

As a consistency test, we check that for $z_0 = 0$ we recover the expressions for the Hubble parameter and luminosity distance corresponding to the standard Taylor and Padé (2, 1) expansions commonly found in the literature (see, e.g., M. Visser 2005).

We start with the Taylor expansions, H^{Taylor} and D_L^{Taylor} . Evaluating Equations (A12), (A16), and (A21) at $z_0 = 0$, we obtain

$$H^{\text{Taylor}}(z; z_0 = 0) = H_0 \left[1 + (1 + q_0)z + \frac{j_0 - q_0^2}{2}z^2 + \frac{3q_0^3 + 3q_0^2 - (4q_0 + 3)j_0 - s_0}{6}z^3 \right], \quad (\text{A29})$$

$$D_L^{\text{Taylor}}(z; z_0 = 0) = \frac{1}{H_0} \left(z + \frac{1 - q_0}{2}z^2 - \frac{1 - q_0 - 3q_0^2 + j_0}{6}z^3 \right), \quad (\text{A30})$$

which correctly reproduce the results usually found in the literature.

Similarly, it is straightforward to verify that the generalized Padé (2, 1) expansion of the luminosity distance $D_L^{\text{P}(2,1)}$ and the expansion rate $H^{\text{P}(2,1)}$ reduces to the known expressions when fixing $z_0 = 0$. Indeed, computing Equations (A12), (A16), and (A21) at $z_0 = 0$, we find

$$\begin{aligned} c_0 &= 0, \\ c_1 &= \frac{1}{H_0}, \\ c_2 &= \frac{1}{2H_0}(1 - q_0), \\ c_3 &= -\frac{1}{6H_0}(1 - q_0 - 3q_0^2 + j_0), \end{aligned} \quad (\text{A31})$$

which are exactly the Taylor coefficients of the luminosity distance in Equation (A30). Substituting these coefficients into the Padé expressions of Equation (A5), we obtain

$$\begin{aligned} D_L^{\text{P}(2,1)}(z; z_0 = 0) &= \frac{1}{2H_0} \left(\frac{6(1 - q_0)z + (5 - 8q_0 - 3q_0^2 + 2j_0)z^2}{3(1 - q_0) + (1 - q_0 - 3q_0^2 + j_0)z} \right), \end{aligned} \quad (\text{A32})$$

which corresponds to the known expressions in A. Aviles et al. (2014) and S. Capozziello et al. (2020). On the other hand, for the expansion rate, we obtain

$$\begin{aligned} H^{\text{P}(2,1)}(z; z_0 = 0) &= \frac{H_0(3j_0^2z^2 + 2j_0[z((q_0(q_0 + 7) + 3)z + 7q_0 + 6) + 3] + 9q_0^4z^2 - 6q_0^2(z + 1)^2 + 2q_0s_0z^2 + 2s_0z(z + 1))}{j_0((8q_0 + 6)z + 6) + 6q_0^3z - 6q_0^2(z + 1) + 2s_0z}. \end{aligned} \quad (\text{A33})$$

Appendix B

Numerical Implementation and Precision Tests

Building on the cosmographic framework outlined in the previous appendix, we have developed a dedicated Python module that fully implements the Padé (2, 1) expansion around an arbitrary pivot redshift. Once a pivot redshift z_0 and the corresponding cosmographic parameters $\{H_{z_0}, q_{z_0}, j_{z_0}, s_{z_0}\}$ are specified, the code makes direct use of the expressions derived in this work to compute the luminosity distance $D_L(z)$ via Equation (A26) and the Hubble parameter $H(z)$ via Equation (A28). From these quantities, all related distance measures are automatically obtained (such as the angular diameter distance $D_A(z)$ and the comoving distance $D_M(z)$) by enforcing the distance duality relation (I. M. H. Etherington 1933; G. F. R. Ellis 2007).⁸ This framework offers several advantages. On the one hand, it allows us to validate the Padé expansion against exact predictions from numerical Boltzmann codes such as CAMB (A. Lewis et al. 2000; C. Howlett et al. 2012) and CLASS (D. Blas et al. 2011; J. Lesgourgues & T. Tram 2011; J. Lesgourgues 2011a, 2011b) within specific cosmological models. For instance, by fixing the cosmographic parameters to their Λ CDM values inferred from the Planck best-fit cosmology, we can directly test how well the Padé-based reconstruction of $D_L(z)$ matches the exact model predictions. On the other hand, the module is fully interfaced with state-of-the-art cosmological samplers and will be released as part of a broader package for background analyses. Therefore, it provides a practical tool for confronting theoretical predictions with observational background data, enabling MCMC analyses directly in terms of cosmographic parameters.

B.1. Precision Tests

To assess the robustness of our method and to quantify the accuracy gained or lost when adopting different choices of the pivot redshift z_0 , we perform a dedicated consistency test. The idea is to isolate the truncation error of the Padé (2, 1) expansion and verify how it propagates across redshift when the series is expanded around different pivots. As a theoretical reference, we adopt a flat Λ CDM cosmology with all parameters fixed to their Planck 2018 best-fit values (N. Aghanim et al. 2020). Using CLASS (D. Blas et al. 2011; J. Lesgourgues & T. Tram 2011; J. Lesgourgues 2011a, 2011b), we

⁸ For works spanning the past two decades testing the distance duality relation against a multitude of astrophysical and cosmological datasets in both model-dependent and model-independent manners, we refer to J.-P. Uzan et al. (2004), F. De Bernardis et al. (2006), S. More et al. (2009), R. F. L. Holanda et al. (2011, 2010), Z. Li et al. (2011), R. Nair et al. (2011), N. Liang et al. (2013), X.-L. Meng et al. (2012), R. F. L. Holanda et al. (2012b), S. Khedekar & S. Chakraborti (2011), R. S. Gonçalves et al. (2012), J. A. S. Lima et al. (2011), R. F. L. Holanda et al. (2012a), V. F. Cardone et al. (2012), X. Yang et al. (2013), Y. Zhang (2014), S. Santos-da Costa et al. (2015), P. Wu et al. (2015), K. Liao et al. (2016), A. Rana et al. (2016), C. Ma & P.-S. Corasaniti (2018), R. F. L. Holanda et al. (2017), S. More et al. (2016), A. Rana et al. (2017), X. Li & H. N. Lin (2018), H.-N. Lin et al. (2018), J.-Z. Qi et al. (2019), R. F. L. Holanda et al. (2019), C. Zhou et al. (2021), J. Qin et al. (2021), K. Bora & S. Desai (2021), P. Mukherjee & A. Mukherjee (2021), T. Liu et al. (2021), F. Renzi et al. (2022), B. Xu et al. (2022), L. Tonghua et al. (2023), J.-Z. Qi et al. (2025), F. Yang et al. (2024), A. Favale et al. (2024), L. Tang et al. (2025), J. F. Jesus et al. (2025), J.-Z. Qi et al. (2025), A. C. Alfano & O. Luongo (2025), F. Yang et al. (2025), F. Keil et al. (2025), E. M. Teixeira et al. (2025), M. Ruchika et al. (2025), C. De Leo et al. (2025). For projected limits from future surveys, see, e.g., V. F. Cardone et al. (2012), T. Yang et al. (2019), X. Fu et al. (2019), N. B. Hogg et al. (2020), F. Renzi et al. (2021), M. Martinelli et al. (2020), and C. De Leo et al. (2025).

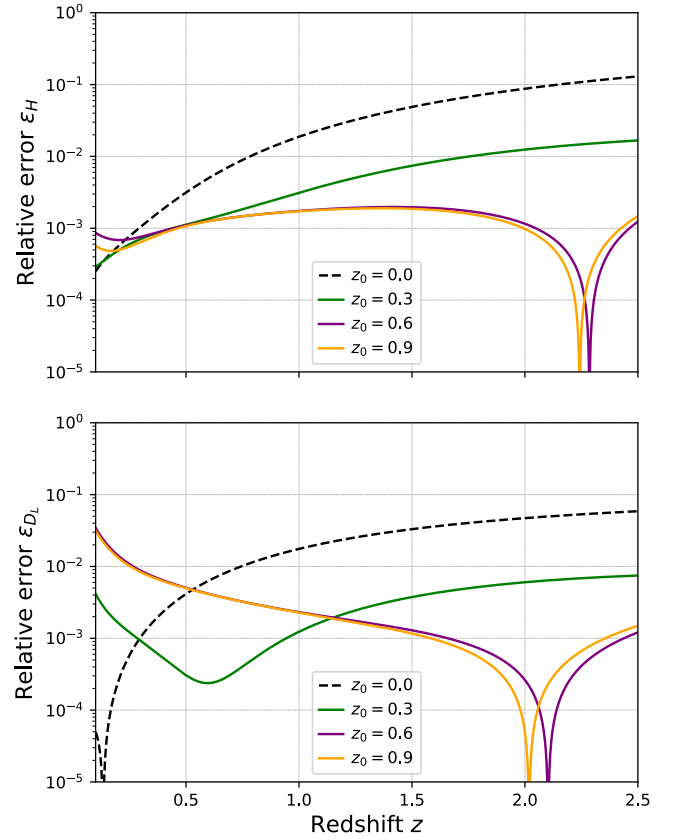


Figure 4. Comparison of the relative precision of the Padé expansion for $H(z)$ and $D_L(z)$ at different redshifts z (using exact Λ CDM predictions computed with CLASS as reference) for four representative choices of the pivot redshift z_0 . The black dashed line shows the Padé expansion around $z_0 = 0$, corresponding to the standard case commonly discussed in the literature.

compute the exact background observables on a dense redshift grid up to $z = 2.5$, which serves as the benchmark for our comparison.

From this reference cosmology we derive the exact values of the cosmographic parameters $H(z)$, $q(z)$, $j(z)$, and $s(z)$ at any redshift z . These are obtained by combining the definitions in Equations (A11), (A15), and (A20) with the Hubble parameter in a flat Λ CDM model,

$$H_{z_0} = H_0 \sqrt{\Omega_m(1+z_0)^3 + (1-\Omega_m)}, \quad (\text{B1})$$

which leads to the following explicit expressions for the cosmographic parameters in Λ CDM, keeping z_0 as a free pivot:

$$q_{z_0} = \frac{\Omega_m(1+z_0)^3 - 2(1-\Omega_m)}{2[\Omega_m(1+z_0)^3 + (1-\Omega_m)]}, \quad (\text{B2})$$

$$j_{z_0} = 1, \quad (\text{B3})$$

$$s_{z_0} = 1 - \frac{9\Omega_m(1+z_0)^3}{2[\Omega_m(1+z_0)^3 + (1-\Omega_m)]}, \quad (\text{B4})$$

where $H_0 \equiv H(z=0)$ is the present-day Hubble constant and Ω_m denotes the present-day matter density parameter. For every chosen pivot z_0 , we fix the cosmographic parameters to their exact values in Λ CDM at that pivot, i.e., $H_{z_0}, q_{z_0}, j_{z_0} \equiv 1$, and s_{z_0} . This ensures that the only difference between the Padé reconstruction and the exact Λ CDM observables is due to the truncation of the series, not to parameter estimation uncertainties. With these inputs, we build the Padé (2, 1) expansion

around z_0 and reconstruct the observables $H(z)$ and $D_L(z)$ across the full redshift range $[0, 2.5]$. We then compare the reconstructed quantities with the exact Λ CDM results from CLASS, defining the relative error as the absolute difference normalized to the exact value:

$$\epsilon_X = \frac{X_{\Lambda\text{CDM}} - X_{\Lambda\text{CDM}}^{\text{P}(2,1)}}{X_{\Lambda\text{CDM}}}, \quad (\text{B5})$$

where X indicates a cosmological function.

The results of this exercise are shown in Figure 4. The black dashed line corresponds to the standard case with $z_0 = 0$, which is commonly adopted in the literature. As expected, the

any dynamical assumptions. This demonstrates that the generalized Padé (2, 1) expansion, when implemented around arbitrary pivots, provides a highly flexible and accurate tool for cosmographic reconstructions across the whole redshift range of interest.

We conclude this section by noting that the same precision tests remain valid when the cosmographic parameters are fixed to the values predicted by the best-fit Planck+DESI w_0w_a CDM model, where the DE EoS follows the CPL parameterization $w(a) = w_0 + w_a(1 - a)$. In particular, performing the same derivation as in Λ CDM but using the Hubble parameter for a flat w_0w_a CDM model, we obtain the following relations for the cosmographic parameters:

$$q(z) = \frac{(1 - \Omega_m)(1 + z)^{3(w_0+w_a)}(1 + z + 3(w_0 + (w_0 + w_a)z)) + \Omega_m(1 + z)e^{\frac{3w_az}{1+z}}}{2(1 - \Omega_m)(1 + z)^{1+3w_0+3w_a} + 2\Omega_m(1 + z)e^{\frac{3w_az}{1+z}}}, \quad (\text{B6})$$

$$j(z) = \frac{(1 - \Omega_m)(1 + z)^{3(w_0+w_a)}[2 + 3w_a + 4(1 + 3w_a)z + (2 + 9w_a(1 + w_a))z^2 + 9w_0^2(1 + z)^2]}{2(1 + z)^2[(1 - \Omega_m)(1 + z)^{3(w_0+w_a)} + \Omega_me^{\frac{3w_az}{1+z}}] + 9w_0(1 + z)(1 + z + 2w_az)] + 2\Omega_m(1 + z)^2e^{\frac{3w_az}{1+z}}}, \quad (\text{B7})$$

relative error is very small close to the pivot but grows as one moves to higher redshift. In contrast, when adopting a pivot at $z_0 = 0.3$ (green line), the truncation error is slightly larger in the immediate vicinity of $z = 0$ but is up to an order of magnitude smaller in the range $0.3 \lesssim z \lesssim 2.5$, precisely where most of the data are concentrated. The same pattern holds for higher pivots: when setting $z_0 = 0.6$ (purple line) or $z_0 = 0.9$ (orange line), the error at low redshift $z \rightarrow 0$ somewhat increases, but the expansion becomes up to two orders of magnitude more accurate at intermediate and high redshifts. The gain in accuracy is consistent for both $H(z)$ and $D_L(z)$, and we have verified that it also applies to $D_V(z)$ and to the ratio $D_M(z)/D_H(z)$.

Overall, this test illustrates a clear trade-off. Choosing a pivot at $z_0 = 0$ maximizes the accuracy at the present epoch but introduces increasingly large truncation errors at higher redshifts. Shifting the pivot to $z_0 > 0$ slightly reduces the accuracy at very low z but substantially improves the reconstruction at the redshifts where most of the data lie.

We stress that the loss of precision at $z \simeq 0$ is not problematic for our analysis, since when constraining cosmographic parameters at low z_0 we consistently use the Padé (2, 1) expansion centered at low z_0 . On the other hand, when extracting constraints at higher redshift, adopting an expansion centered closer to the redshift of interest yields clear gains in both accuracy and precision. This validates our multipivot strategy: by selecting different z_0 values, one can minimize the truncation error in the region most relevant for the data, achieving accurate reconstructions without relying on

where w_0 and w_a are the two parameters governing the evolution of DE. We have explicitly verified that the same precision tests performed for Λ CDM remain valid in the CPL framework when the cosmographic parameters are fixed to their best-fit values obtained under CPL priors, exactly as we did for Λ CDM. In addition, these are the expressions used in the main text to compare the CPL predictions with the cosmographic reconstructions.

B.2. Markov Chain Monte Carlo Analyses

The results presented in the main text are obtained from MCMC analyses carried out for different fixed values of the pivot redshift z_0 . In particular, we explore a uniform grid in the range $z_0 \in [0, 1]$ with step size $\Delta z_0 = 0.1$. To compare the theoretical predictions with data, we employ the following background-level datasets:

1. SNIa. Distance modulus measurements relating the theoretical luminosity distance to the magnitude–redshift relation $\mu(z) = 5 \log(D_L(z)) + 25$. We use the PP (D. Brout et al. 2022) catalog and the 5 yr dataset of the Dark Energy Survey (DESy5; T. M. C. Abbott et al. 2024). We consider these two catalogs because the DESI Collaboration (M. Abdul Karim et al. 2025a) showed that they represent, respectively, the combinations with the minimum and maximum preference for evolving DE scenarios.

2. *CC*. Measurements of the Hubble parameter $H(z)$ (R. Jimenez & A. Loeb 2002; N. Borghi et al. 2022), which can be directly compared against the theoretical Padé expansion of the same function. We selected 15 data points from M. Moresco et al. (2012), M. Moresco (2015), and M. Moresco et al. (2016), since for this subset full estimates of the covariance matrix’s non-diagonal terms and systematic contributions are available, as outlined in M. Moresco et al. (2018, 2020). We exclude some earlier measurements owing to concerns expressed in A. A. Kjerrgren & E. Mortsell (2022), which do not apply to our selected data. This dataset is referred to as “CC.”
3. *BAO*. Data from the DESI second data release (M. Abdul Karim et al. 2025a). These measurements include the transverse comoving distance $D_M(z)$, the Hubble horizon $D_H(z)$, and the angle-averaged distance $D_V(z)$, all normalized to the sound horizon at the drag epoch r_d . Observational data are compared with the respective theoretical functions obtained from the Padé expansions of the expansion rate $H(z)$ and luminosity distance $D_L(z)$, using the relations $D_L(z) = (1 + z)D_M(z)$, $D_H(z) = c/H(z)$, and $D_V(z) = (zD_M(z)^2 D_H(z))^{1/3}$. Since we are performing a purely late-time background analysis, the sound horizon r_d is calibrated using Planck Λ CDM cosmology (N. Aghanim et al. 2020), with a Gaussian prior $r_d = 147.09 \pm 0.26$ Mpc. This choice is both natural and robust: models of dynamical DE cannot alter the physics before recombination that determines r_d . Moreover, Padé and Taylor expansions are only accurate at low redshift, so we cannot extrapolate them up to $z \sim 1100$ and include the CMB as an effective BAO point, as is sometimes done in the literature. Instead, calibrating BAO with the Planck-inferred r_d provides a consistent way to anchor our low-redshift analysis. We denote this combined dataset as “DESI+ r_d .” For completeness, in Appendix D we explore alternative BAO (and SNIa) calibrations motivated by the Hubble tension, showing that they only rescale $H(z)$ and do not affect the cosmographic conclusions.

The MCMC analysis is carried out with our dedicated numerical package, fully interfaced with the *Cobaya* software (J. Torrado & A. Lewis 2021), where an MCMC sampler is implemented to compare the cosmological functions derived from the Padé expansion with the observational datasets. Uniform priors are adopted for the free cosmographic parameters: $H_{z_0} \in \mathcal{U}[40, 140]$ km s^{−1} Mpc^{−1}, $q_{z_0} \in \mathcal{U}[-2, 2]$, $j_{z_0} \in \mathcal{U}[-5, 5]$, and $s_{z_0} \in \mathcal{U}[-5, 5]$. Convergence of the chains is assessed through the Gelman–Rubin criterion, requiring $R - 1 < 0.01$ (A. Gelman & D. B. Rubin 1992).

Appendix C Supplementary Results and Discussions

In this appendix we present the results of the comprehensive analysis carried out for the different data combinations listed previously. Table 1 reports the numerical constraints on the expansion rate $H(z_0)$ and on the cosmographic parameters $q(z_0)$, $j(z_0)$, and $s(z_0)$, derived at various pivot redshifts and for all dataset combinations considered. For every dataset combination in Table 1, we repeat the full cosmographic analysis both including and excluding CC.

These measurements are known to rely more strongly on astrophysical modeling than BAO or SNIa, since they are derived from differential age estimates of passively evolving galaxies. In combinations that include DESI BAO calibrated with the Planck prior on r_d together with SNIa (i.e., the “most informative” combinations discussed in the main text), the contribution of CC is naturally limited. DESI+ r_d constrain the expansion rate very tightly, and both PP and DESy5 SNIa provide high-precision distance modulus measurements, leaving CC with comparatively little statistical weight. Their leverage on $H(z_0)$, $q(z_0)$, $j(z_0)$, and $s(z_0)$ is therefore small, as reflected in Table 1: for these combinations, the constraints obtained with and without CC are nearly identical at all pivot redshifts. This confirms that the BAO+ r_d +SNIa results shown in the main text are not significantly affected by the inclusion of CC. A similar argument applies when comparing BAO+ r_d alone with BAO+ r_d +CC: CC do not modify the qualitative conclusions because the expansion history is already tightly constrained by calibrated BAO. By contrast, the role of CC becomes important in SNIa-only combinations. SNIa alone cannot constrain $H(z_0)$ because of the well-known degeneracy with the absolute magnitude M . This degeneracy directly propagates into the snap parameter $s(z_0)$, which depends sensitively on the absolute expansion scale. This is explicitly reflected in the fact that the coefficients c_i in Equation (A26) for the expansion of $D_L(z)$ in Equation (A25) depend only on the quantities A and B defined in Equations (A12) and (A16), which in turn involve only $q(z)$ and $j(z)$. Conversely, the coefficient c_3 in Equation (A28) for the expansion of $H(z)$ in Equation (A27) also depends on the quantity C defined in Equation (A21), which depends on $s(z)$. As a result, in SNIa-only analyses both $H(z_0)$ and $s(z_0)$ remain unconstrained when CC are excluded, as shown in Table 1. Adding CC provides an absolute measurement of the Hubble rate at multiple redshifts, breaking the H – M degeneracy and enabling meaningful (though still relatively broad) constraints on $H(z_0)$ and $s(z_0)$.

Having clarified the role of CC (i.e., the fact that they do not significantly alter the constraints in combinations where parameters are already tightly constrained by DESI+ r_d or DESI+ r_d +SNIa, while being essential for constraining $H(z_0)$ and $s(z_0)$ in SNIa-only analyses), we now focus on the results obtained from datasets that include the additional information provided by CC. A compact visual representation of these results is provided in the whisker plot in Figure 5. While the main highlights and implications of these findings have been discussed in the main text, here we present complementary consistency checks and extended discussions. These results, on the one hand, reinforce the robustness of our analysis and, on the other, provide additional perspective on the behavior of the parameters across datasets.

The global picture that emerges is one of remarkable consistency across the results obtained with different data combinations. The whisker plot shows that, at each pivot, the constraints from independent probes are always compatible within 1σ , with only very rare cases approaching 2σ . This cross-consistency is an important outcome of our analysis, since it demonstrates that the cosmographic reconstruction delivers stable results across datasets of very different nature, from Planck-calibrated BAO to SNIa combined with CC.

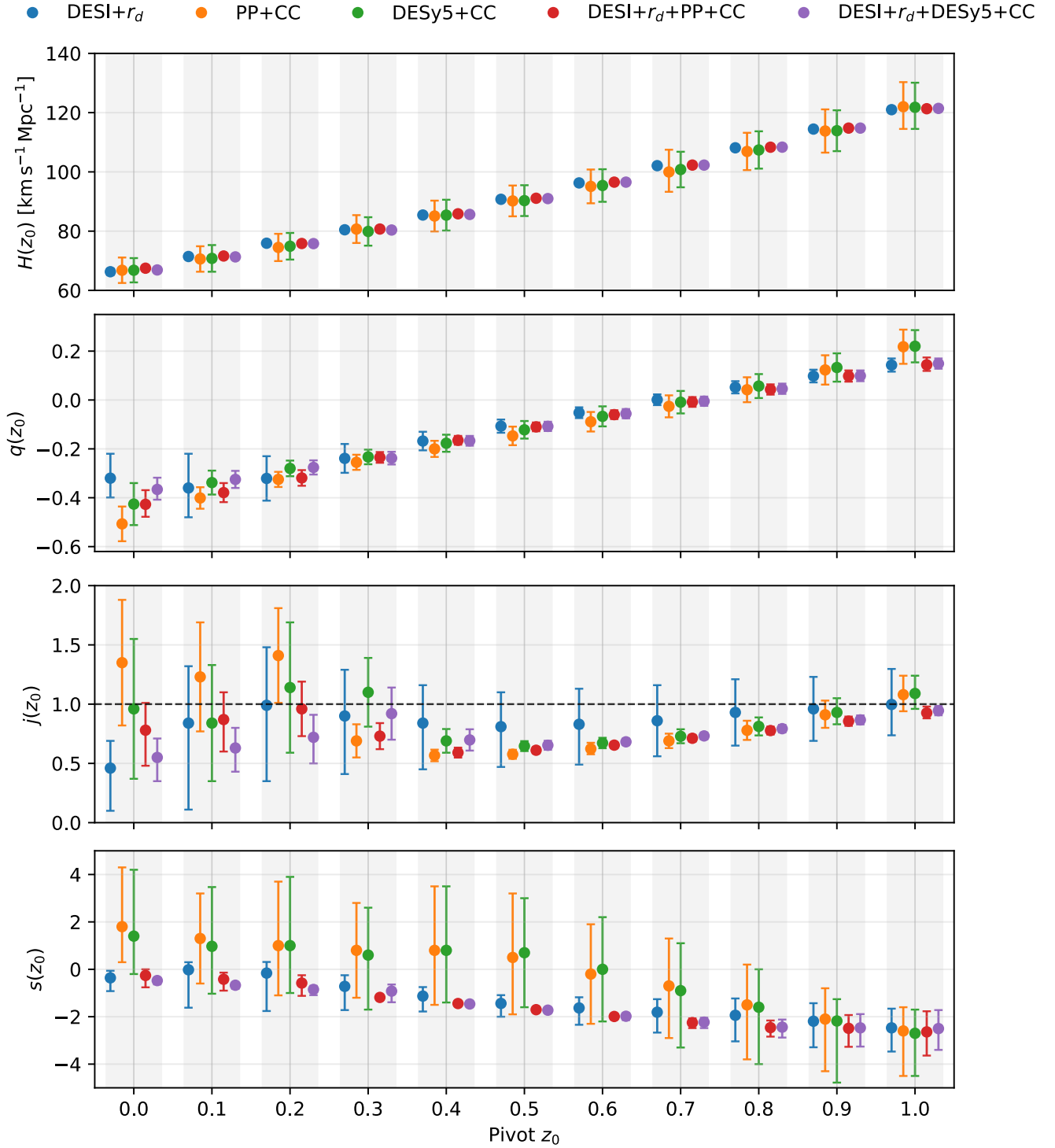


Figure 5. Whisker plot summarizing the 68% CL constraints on the cosmographic parameters $H(z_0)$, $q(z_0)$, $j(z_0)$, and $s(z_0)$ at different pivot redshifts $z_0 \in [0, 1]$, for all dataset combinations considered in this work. For clarity, the constraints are slightly displaced horizontally around each pivot.

Focusing first on the deceleration parameter, the results follow the expected trend: $q(z_0)$ is negative at low redshift, approaches zero around $z_0 \simeq 0.7$, and becomes positive afterward. Interestingly, at very low redshift DESI+ r_d prefers less negative values of q_0 than the SNIa-based combinations, which point to a more strongly accelerated expansion. For instance, at $z_0 = 0$ we find $q = -0.320^{+0.10}_{-0.079}$ from DESI+ r_d , while PP+CC yields $q = -0.507 \pm 0.071$. Although these differences remain compatible within $\sim 2\sigma$, it is worth noting

that larger values of q_0 are naturally expected in dynamical DE scenarios. Therefore, this suggests that DESI BAO data may strengthen the shift toward the predictions of dynamical DE compared to SNIa, while still remaining consistent overall.

The behavior of the jerk parameter is more subtle. At very low pivot redshifts, both BAO and SNIa provide only weak constraints on $j(z_0)$. This is expected: at low redshift the contribution of the jerk parameter is subdominant compared to the deceleration parameter, and its effect can only be

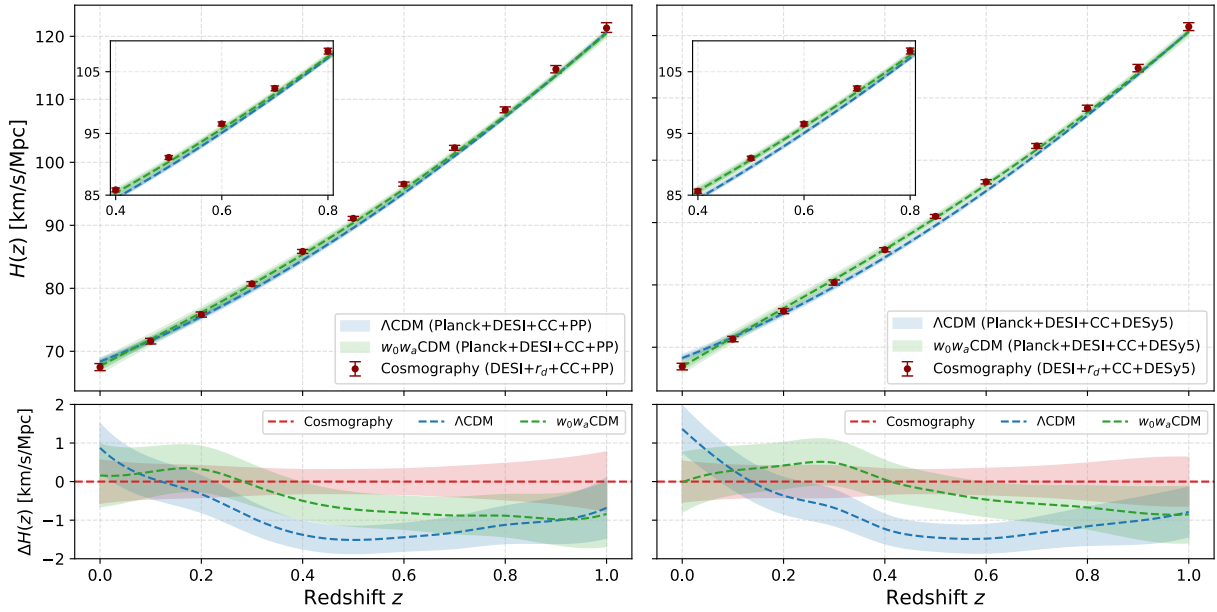


Figure 6. Cosmographic constraints on the Hubble parameter $H(z)$ at different redshift bins with 68% CL uncertainties (red points). The blue band shows the Λ CDM predictions, while the green band corresponds to those from a CPL model of dynamical DE. The left panel shows results obtained by combining Planck-calibrated DESI BAO, CC, and PP SNIa, while the right panel shows results from the same data combination but with DESy5 SNIa instead of PP. The bottom panels display the residuals, showing that the cosmographic reconstruction consistently prefers values of $H(z)$ about $2 \text{ km s}^{-1} \text{ Mpc}^{-1}$ higher than Λ CDM, in better agreement with CPL expectations.

constrained through extrapolation to higher redshift. In addition, fixing the pivot redshift to $z_0 \sim 0$, the inferred constraints are broadened by a strong correlation between $j(z_0)$ and $q(z_0)$. From Figure 5, we see that at $z_0 \simeq 0$ DESI+ r_d alone even favors values below unity, though with large uncertainties. Conversely, when the expansion is pivoted at higher redshifts, the sensitivity to j increases significantly. This is due to three effects: first, the impact of j on the cosmographic expansion naturally grows with redshift; second, expanding around higher pivots reduces correlations among parameters, allowing a cleaner determination of their contributions, as described in the main text; and third, the large number of SNIa in the range $0.3 \lesssim z \lesssim 0.8$ makes their contribution especially powerful at these redshifts. In this regime SNIa deliver precise constraints on j and reveal the most pronounced deviations from the Λ CDM prediction $j = 1$. For example, DESI+ r_d +PP+CC gives $j(0.4) = 0.591 \pm 0.042$ and $j(0.8) = 0.777 \pm 0.033$. Notably, SNIa in some cases even outperform BAO in constraining j , since BAO are intrinsically more sensitive to the snap parameter, as discussed below. Overall, the complementarity between probes is very clear in the combined analyses, where the inclusion of DESI+ r_d systematically reduces uncertainties while preserving the SNIa-driven trends.

When it comes to the snap parameter, by contrast, it is most strongly constrained by BAO. This reflects the fact that $s(z)$ affects most directly the expansion rate $H(z)$ in our parameterization and thus impacts BAO measurements of D_H more strongly than it affects SNIa luminosity distances. SNIa constraints on $s(z)$ are therefore broader, while BAO consistently provide sharper measurements. However, despite the larger uncertainties of SNIa, we again find overall consistency between probes, with the combined datasets delivering robust constraints on $s(z)$ at the few-percent level.

The cosmographic reconstruction of $H(z)$ itself is shown both in Figure 5 and in more detail in Figure 6. In the latter

figure, the cosmographic points are compared with Λ CDM (blue bands) and CPL (green bands), which, as in the main text, are derived from Equations (B2)–(B4) and Equations (B6)–(B7), respectively. The trend observed in the figure directly supports the conclusion drawn in the main text: at $z \gtrsim 0.2$, the reconstructed expansion rate lies systematically above the Λ CDM prediction by about $2 \text{ km s}^{-1} \text{ Mpc}^{-1}$. The residuals in the bottom panel make this shift evident: all dataset combinations consistently prefer $H(z)$ values larger than Λ CDM, and the alignment with CPL predictions is visibly closer. This behavior is stable across pivots and is seen in both DESI+ r_d +PP+CC and DESI+ r_d +DESy5+CC, confirming that it is not driven by a single dataset but represents a coherent feature across independent probes.

Taken together, these supplementary results strengthen the conclusions of the main text. Not only do the different probes provide mutually consistent constraints, but their complementarity allows us to sharpen the reconstruction of late-time cosmic kinematics: BAO and SNIa both contribute significantly to $q(z_0)$ and $H(z_0)$, SNIa dominate the sensitivity to the jerk parameter at intermediate redshifts, BAO provide the sharpest constraints on the snap parameter, and the combined analyses consistently improve precision while preserving coherence. The overall consistency across probes, combined with the coherent redshift-dependent trends of the cosmographic parameters, provides strong evidence for the robustness of the framework and indicates that the observed deviations can hardly be attributed to noise or to any single dataset.

Appendix D Implications for the Hubble Tension

A key aspect of our cosmographic reconstruction is that it directly traces the redshift evolution of the Hubble parameter without assuming any specific cosmological model. This

Table 2

Constraints at 68% CL on the Expansion Rate $H(z_0)$ (in $\text{km s}^{-1} \text{Mpc}^{-1}$) and on the Cosmographic Parameters $q(z_0)$, $j(z_0)$, and $s(z_0)$ at Different Fixed Pivot Redshifts z_0 , for Various Combinations of Datasets

Redshift	Dataset	$H(z_0)$	$q(z_0)$	$j(z_0)$	$s(z_0)$
$z_0 = 0.0$	DESI+PP+ $r_{d\text{SHOES}}$	71.39 ± 0.61	$-0.423^{+0.058}_{-0.051}$	$0.75^{+0.23}_{-0.30}$	$-0.28^{+0.22}_{-0.48}$
	DESI+DESY5+ $r_{d\text{SHOES}}$	70.81 ± 0.55	$-0.367^{+0.046}_{-0.041}$	$0.55^{+0.16}_{-0.21}$	$-0.47^{+0.10}_{-0.16}$
	DESI+PP+SHOES	73.5 ± 1.0	$-0.441^{+0.068}_{-0.056}$	$0.85^{+0.27}_{-0.38}$	$-0.11^{+0.35}_{-0.81}$
	DESI+DESY5+SHOES	73.0 ± 1.1	$-0.365^{+0.045}_{-0.040}$	$0.54^{+0.15}_{-0.19}$	$-0.488^{+0.076}_{-0.12}$
$z_0 = 0.1$	DESI+PP+ $r_{d\text{SHOES}}$	75.78 ± 0.49	-0.377 ± 0.038	$0.86^{+0.22}_{-0.26}$	$-0.45^{+0.25}_{-0.45}$
	DESI+DESY5+ $r_{d\text{SHOES}}$	75.47 ± 0.46	-0.323 ± 0.034	$0.62^{+0.17}_{-0.20}$	$-0.673^{+0.093}_{-0.15}$
	DESI+PP+SHOES	77.7 ± 1.1	-0.385 ± 0.040	$0.91^{+0.24}_{-0.27}$	$-0.37^{+0.31}_{-0.52}$
	DESI+DESY5+SHOES	77.6 ± 1.1	-0.324 ± 0.035	$0.62^{+0.17}_{-0.21}$	$-0.66^{+0.12}_{-0.18}$
$z_0 = 0.2$	DESI+PP+ $r_{d\text{SHOES}}$	80.27 ± 0.45	-0.322 ± 0.032	$0.98^{+0.24}_{-0.27}$	$-0.53^{+0.38}_{-0.62}$
	DESI+DESY5+ $r_{d\text{SHOES}}$	80.24 ± 0.43	-0.274 ± 0.029	$0.69^{+0.18}_{-0.21}$	$-0.87^{+0.12}_{-0.20}$
	DESI+PP+SHOES	82.3 ± 1.2	-0.313 ± 0.031	0.89 ± 0.22	$-0.70^{+0.24}_{-0.41}$
	DESI+DESY5+SHOES	82.3 ± 1.4	-0.274 ± 0.027	$0.69^{+0.17}_{-0.20}$	$-0.88^{+0.11}_{-0.17}$
$z_0 = 0.3$	DESI+PP+ $r_{d\text{SHOES}}$	85.41 ± 0.40	-0.236 ± 0.023	0.72 ± 0.12	$-1.178^{+0.081}_{-0.11}$
	DESI+DESY5+ $r_{d\text{SHOES}}$	85.06 ± 0.44	-0.236 ± 0.027	0.90 ± 0.23	$-0.93^{+0.28}_{-0.48}$
	DESI+PP+SHOES	87.8 ± 1.3	-0.229 ± 0.026	$0.67^{+0.14}_{-0.16}$	$-1.176^{+0.080}_{-0.11}$
	DESI+DESY5+SHOES	86.2 ± 1.7	-0.236 ± 0.027	0.90 ± 0.23	$-0.94^{+0.27}_{-0.45}$
$z_0 = 0.4$	DESI+PP+ $r_{d\text{SHOES}}$	90.83 ± 0.34	-0.164 ± 0.018	0.591 ± 0.043	-1.446 ± 0.067
	DESI+DESY5+ $r_{d\text{SHOES}}$	90.65 ± 0.36	-0.167 ± 0.021	0.696 ± 0.091	$-1.459^{+0.091}_{-0.11}$
	DESI+PP+SHOES	93.6 ± 1.3	-0.151 ± 0.019	0.497 ± 0.069	$-1.349^{+0.094}_{-0.11}$
	DESI+DESY5+SHOES	92.5 ± 1.6	-0.166 ± 0.020	0.694 ± 0.090	$-1.467^{+0.089}_{-0.099}$
$z_0 = 0.5$	DESI+PP+ $r_{d\text{SHOES}}$	96.38 ± 0.33	-0.112 ± 0.018	0.607 ± 0.026	-1.703 ± 0.079
	DESI+DESY5+ $r_{d\text{SHOES}}$	96.31 ± 0.34	-0.108 ± 0.019	0.653 ± 0.037	$-1.724^{+0.094}_{-0.11}$
	DESI+PP+SHOES	99.1 ± 1.4	-0.103 ± 0.018	0.570 ± 0.034	-1.665 ± 0.078
	DESI+DESY5+SHOES	99.0 ± 1.6	-0.108 ± 0.019	0.653 ± 0.039	$-1.722^{+0.095}_{-0.11}$
$z_0 = 0.6$	DESI+PP+ $r_{d\text{SHOES}}$	102.15 ± 0.35	-0.062 ± 0.019	0.652 ± 0.025	$-1.99^{+0.11}_{-0.13}$
	DESI+DESY5+ $r_{d\text{SHOES}}$	102.11 ± 0.36	-0.058 ± 0.019	0.680 ± 0.026	$-1.99^{+0.13}_{-0.15}$
	DESI+PP+SHOES	104.8 ± 1.6	-0.055 ± 0.019	0.636 ± 0.027	$-1.95^{+0.11}_{-0.14}$
	DESI+DESY5+SHOES	105.4 ± 1.7	-0.056 ± 0.019	0.681 ± 0.027	$-1.99^{+0.13}_{-0.16}$
$z_0 = 0.7$	DESI+PP+ $r_{d\text{SHOES}}$	108.24 ± 0.42	-0.010 ± 0.019	0.709 ± 0.027	$-2.26^{+0.17}_{-0.22}$
	DESI+DESY5+ $r_{d\text{SHOES}}$	108.24 ± 0.40	-0.007 ± 0.019	0.729 ± 0.027	$-2.26^{+0.18}_{-0.23}$
	DESI+PP+SHOES	110.9 ± 1.8	-0.005 ± 0.019	0.705 ± 0.029	$-2.23^{+0.17}_{-0.22}$
	DESI+DESY5+SHOES	112.0 ± 1.9	-0.006 ± 0.021	$0.731^{+0.028}_{-0.032}$	$-2.22^{+0.23}_{-0.31}$
$z_0 = 0.8$	DESI+PP+ $r_{d\text{SHOES}}$	114.59 ± 0.50	0.040 ± 0.021	0.773 ± 0.033	$-2.51^{+0.28}_{-0.37}$
	DESI+DESY5+ $r_{d\text{SHOES}}$	114.64 ± 0.47	0.045 ± 0.019	0.791 ± 0.030	$-2.46^{+0.30}_{-0.41}$
	DESI+PP+SHOES	117.3 ± 1.9	0.046 ± 0.021	0.776 ± 0.034	$-2.43^{+0.31}_{-0.44}$
	DESI+DESY5+SHOES	118.6 ± 2.3	0.046 ± 0.021	0.793 ± 0.032	$-2.43^{+0.34}_{-0.46}$
$z_0 = 0.9$	DESI+PP+ $r_{d\text{SHOES}}$	121.38 ± 0.63	0.095 ± 0.022	0.850 ± 0.039	$-2.59^{+0.52}_{-0.73}$
	DESI+DESY5+ $r_{d\text{SHOES}}$	121.33 ± 0.64	0.094 ± 0.024	0.858 ± 0.040	$-2.58^{+0.55}_{-0.79}$
	DESI+PP+SHOES	124.0 ± 2.1	0.099 ± 0.023	0.855 ± 0.040	$-2.44^{+0.59}_{-0.84}$
	DESI+DESY5+SHOES	$124.7^{+4.0}_{-3.0}$	0.098 ± 0.021	0.865 ± 0.036	$-2.51^{+0.54}_{-0.74}$
$z_0 = 1.0$	DESI+PP+ $r_{d\text{SHOES}}$	$128.1^{+1.4}_{-0.96}$	$0.134^{+0.047}_{-0.033}$	$0.910^{+0.082}_{-0.059}$	$-2.73^{+0.96}_{-1.3}$
	DESI+DESY5+ $r_{d\text{SHOES}}$	128.51 ± 0.69	0.151 ± 0.023	0.947 ± 0.043	$-2.38^{+0.86}_{-1.0}$
	DESI+PP+SHOES	130.8 ± 2.1	0.151 ± 0.023	0.938 ± 0.042	$-2.39^{+0.84}_{-1.0}$
	DESI+DESY5+SHOES	$128.4^{+9.4}_{-4.0}$	0.150 ± 0.022	0.946 ± 0.040	$-2.43^{+0.85}_{-0.97}$

Note. The table compares the impact of two distinct calibrations: (i) a reduced sound horizon $r_d^{\text{SHOES}} = 139 \pm 0.26 \text{ Mpc}$, representative of early-time solutions to the Hubble tension, and (ii) the SHOES calibration of SNIa. For PP+SHOES we directly adopt the SHOES SNIa calibration, while for DESy5+SHOES we impose an external Gaussian prior on H_0 centered on the SHOES value of $73.04 \pm 1.04 \text{ km s}^{-1} \text{Mpc}^{-1}$.

allows us to examine, in a fully model-independent manner, how the late-time expansion history inferred from DESI BAO, SNIa, and CC relates to the long-standing Hubble tension.

The extrapolated value of the expansion rate at $z = 0$ obtained from the Padé series is $H_0 = 67.49 \pm 0.56 \text{ km s}^{-1} \text{ Mpc}^{-1}$ for DESI+ r_d +PP+CC and $H_0 = 66.90 \pm 0.53 \text{ km s}^{-1} \text{ Mpc}^{-1}$ for DESI+ r_d +DESy5+CC. An immediate observation is that these values are systematically lower than the H_0 inferred from the same datasets under the assumption of Λ CDM by roughly $1 \text{ km s}^{-1} \text{ Mpc}^{-1}$, while being fully consistent with the H_0 inferred assuming a dynamical DE model such as CPL. This is clearly visible in the bottom panels of Figure 6: the cosmographic residuals (red) closely match the CPL predictions (green), whereas the Λ CDM curve (blue) lies noticeably higher at $z = 0$.

This behavior reinforces a trend already highlighted in the recent literature: although dynamical DE models are statistically favored over a cosmological constant, they do not alleviate the Hubble tension (M. Abdul Karim et al. 2025a; W. Giarè et al. 2025; E. Özlüker et al. 2025). On the contrary, they typically exacerbate it by shifting the inferred late-time H_0 to even lower values. This naturally raises the question of how models proposed to resolve the tension might affect our cosmographic analysis and the conclusions drawn from it.

To answer this question, we note that although our approach is model independent, it is still sensitive to how the observational data are calibrated, most notably through the sound horizon used to anchor BAO and the absolute magnitude calibration used for SNIa. Any mechanism able to address the Hubble tension can be broadly classified into two categories, which affect these calibrations differently:

1. *Early-time new physics.* These scenarios modify the expansion history before recombination, reducing the sound horizon r_d and increasing the inferred value of H_0 . Early DE (EDE) is a well-known example, typically requiring a 5%–7% reduction in r_d to shift H_0 toward the SH0ES value; see, e.g., V. Poulin et al. (2023).
2. *Late-time new physics.* These scenarios modify the expansion history after recombination, leaving the sound horizon essentially unchanged but increasing H_0 at low redshift. In these models, r_d usually remains consistent with Λ CDM, but the absolute expansion rate today shifts the inferred H_0 toward the SH0ES value.

These two classes of solutions have distinct implications for our cosmographic framework. Early-time solutions require modifying the BAO calibration, whereas late-time solutions can be represented by recalibrating SNIa distances using the SH0ES determination of H_0 , thus remaining agnostic on r_d . To explore these effects in a model-independent way, without committing to any specific new-physics scenario, we adopt two proxy implementations:

1. *Proxy for early-time solutions.* We repeat the full cosmographic analysis using an EDE-like prior on the sound horizon $r_d^{\text{SH0ES}} = 139.00 \pm 0.26 \text{ Mpc}$, which is representative of the reduction required by early-time solutions to significantly alleviate the Hubble tension. We therefore consider the combinations DESI+PP+ r_d^{SH0ES} and DESI+DESy5+ r_d^{SH0ES} , which test the

sensitivity of our conclusions to changes in the BAO anchor.

2. *Proxy for late-time solutions.* We impose the SH0ES H_0 (or equivalently M_b) prior to calibrating the SNIa absolute magnitude (A. G. Riess et al. 2022) while leaving BAO uncalibrated (i.e., without any prior on r_d). We therefore consider the combinations DESI+PP+SH0ES and DESI+DESy5+SH0ES, which probe how the SH0ES calibration of SNIa affects the results of our cosmographic analysis.

These two procedures allow us to map, in a controlled and model-agnostic way, how our cosmographic reconstruction and the inferred preference for dynamical DE respond to the types of modifications typically invoked to address the Hubble tension. The numerical results obtained for all cosmographic parameters in the SH0ES-calibrated cases are reported in Table 2 and visualized in Figure 7. In the figure, the red points in the left (right) panels correspond to DESI+ r_d +PP (DESI+ r_d +DESy5), i.e., BAO calibrated with the Planck r_d value assuming Λ CDM. The blue points correspond to DESI+PP+SH0ES (DESI+DESy5+SH0ES), where SNIa are calibrated with the SH0ES determination of H_0 (or M_b) and no prior is imposed on r_d . Finally, the green points correspond to DESI+ r_d^{SH0ES} +PP (DESI+ r_d^{SH0ES} +DESy5), where BAO are calibrated with the reduced sound horizon $r_d^{\text{SH0ES}} = 139 \pm 0.26 \text{ Mpc}$ discussed above.

As shown in the bottom panel of Figure 7 (and in the corresponding $H(z_0)$ values in Table 2), adopting either a SH0ES-based calibration for SNIa or an EDE-like prior on r_d^{SH0ES} shifts all reconstructed values of $H(z)$ upward. As expected, the SH0ES-calibrated SNIa produce the highest values of H_0 , fully consistent with SH0ES, while the r_d^{SH0ES} prior yields intermediate values that move significantly closer to SH0ES. We also note that the prior on r_d^{SH0ES} has a much smaller uncertainty than the SH0ES prior on H_0 , which explains why the green points display smaller error bars than the blue ones. Overall, the net effect on the expansion rate can be summarized as an overall rescaling,

$$H(z)^{\text{SH0ES}} = \alpha H(z)^{\text{Planck}}, \quad (\text{D1})$$

between the values inferred from cosmography using either of the SH0ES-based calibrations (green and blue points) and those inferred when BAO are calibrated with the Planck sound horizon (red points). This behavior is clearly visible in the bottom panel of Figure 7.

We now show, both explicitly and analytically, that such a rescaling has no impact on the two key quantities used in the main text to assess the preference for dynamical DE: the deceleration parameter $q(z)$ and the $Om(z)$ diagnostic. This is immediately evident from the middle and top panels of Figure 7. The constraints on $q(z)$ and $Om(z)$ obtained from SH0ES-calibrated data (blue and green) and Planck-calibrated data (red) remain in excellent mutual agreement across all redshift bins. This confirms that the upward shift in $H(z)$ does not affect the quantities most relevant to assessing deviations from Λ CDM.

The reason is that both $q(z)$ and $Om(z)$ depend only on the *shape* of $H(z)$, not on its overall amplitude. In more quantitative terms, H_0 sets the normalization of $H(z)$, but by

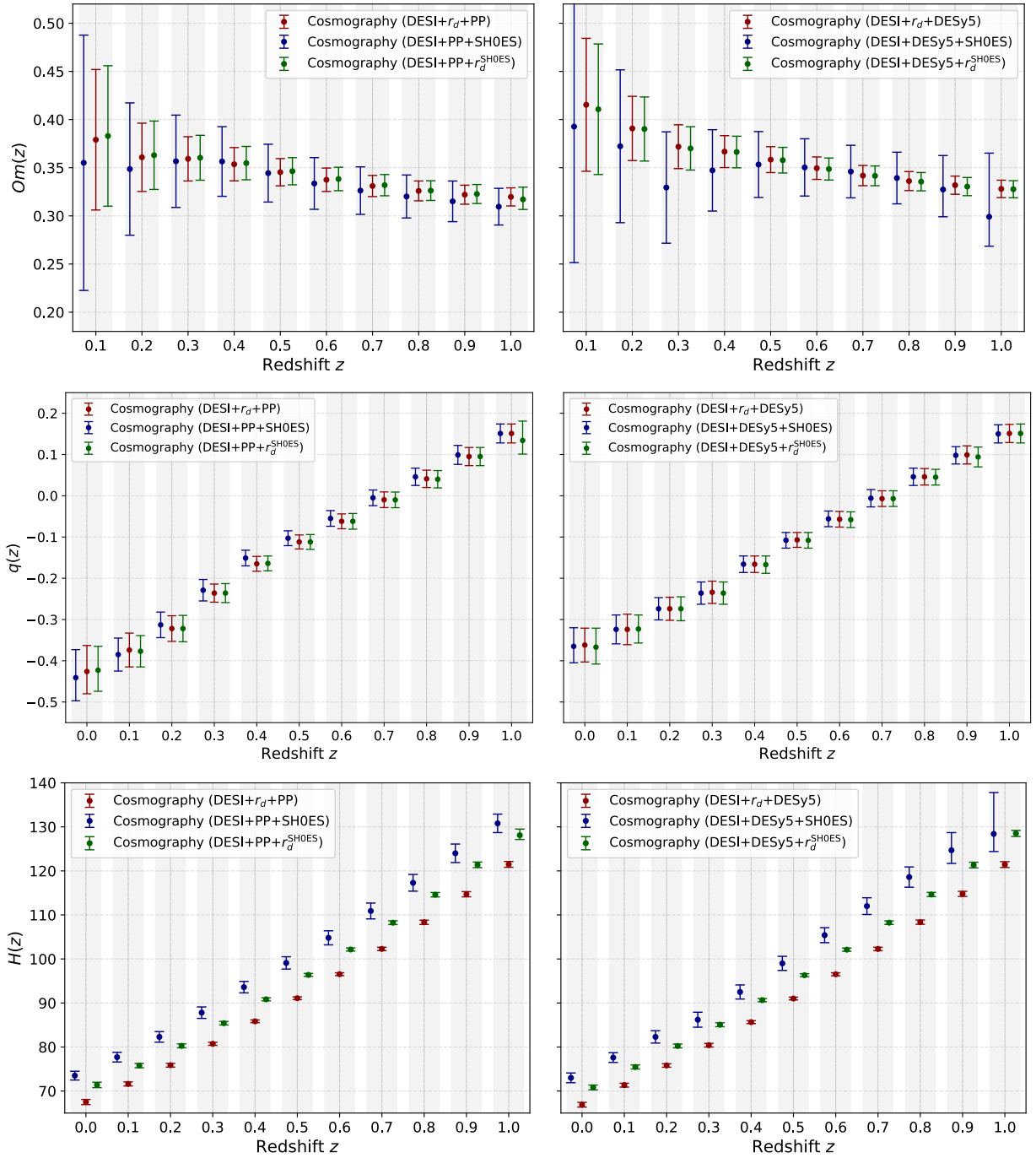


Figure 7. Whisker plot summarizing the 68% CL constraints on the cosmographic parameters $Om(z)$, $q(z)$, and $H(z)$ obtained by (i) calibrating BAO with the Planck sound horizon (red points), (ii) calibrating SN Ia with the SH0ES determination of H_0 while leaving BAO uncalibrated (blue points), and (iii) calibrating BAO with the reduced sound horizon $r_d^{\text{SH0ES}} = 139 \pm 0.26$ Mpc representative of early-time solutions (green points). As discussed in the text, these three calibrations induce an overall shift in the amplitude of $H(z)$ but leave the shape-dependent quantities $q(z)$ and $Om(z)$ unchanged, leading to mutually consistent constraints across all pivot redshifts.

definition

$$q(z) = \frac{d \ln H}{d \ln(1+z)} - 1 = \frac{1}{H(z)} \frac{dH}{d \ln(1+z)} - 1 \quad (\text{D2})$$

is invariant under the rescaling $H(z) \rightarrow \alpha H(z)$, as one can write

$$\frac{1}{H(z)^{\text{SH0ES}}} \frac{dH(z)^{\text{SH0ES}}}{d \ln(1+z)} = \frac{1}{\alpha H(z)^{\text{Planck}}} \frac{d(\alpha H(z)^{\text{Planck}})}{d \ln(1+z)} = \frac{1}{H(z)^{\text{Planck}}} \frac{dH(z)^{\text{Planck}}}{d \ln(1+z)}, \quad (\text{D3})$$

which shows explicitly that $q(z)$ is unchanged by such a shift. Similarly, the $Om(z)$ diagnostic is defined as

$$Om(z) \equiv \frac{\tilde{h}^2(z) - 1}{(1+z)^3 - 1}, \quad (\text{D4})$$

with $\tilde{h}(z) = H(z)/H_0$. Under a SH0ES-like rescaling one has

$$\tilde{h}(z)^{\text{SH0ES}} = \frac{H(z)^{\text{SH0ES}}}{H_0^{\text{SH0ES}}} = \frac{\alpha H(z)^{\text{Planck}}}{\alpha H_0^{\text{Planck}}} = \tilde{h}(z)^{\text{Planck}}, \quad (\text{D5})$$

and therefore $Om(z)$ is also invariant.

Altogether, this exercise shows that the cosmographic results for $q(z)$ and $Om(z)$ are essentially unaffected by new physics capable of shifting the absolute expansion scale, either through an early-time reduction of the sound horizon (as in EDE-like scenarios) or through a late-time recalibration consistent with SH0ES. As a result, the kinematic evidence for departures from Λ CDM reported in the main text is not tied to the particular calibration adopted and remains fully robust under the types of modifications typically invoked to address the Hubble tension.

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