PRESENTATIONS OF THE BRAID GROUP OF THE COMPLEX REFLECTION GROUP $G(d,d,n)$

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ABSTRACT. We show that the braid group associated to the complex reflection group $G(d,d,n)$ is an index d subgroup of the braid group of the orbifold quotient of the complex numbers by a cyclic group of order d. We also give a compatible presentation of $G(d,d,n)$ and its braid group for each tagged triangulation of the disk with n marked points on its boundary and an interior marked point (interpreted as a cone point of degree d) in such a way that the presentations of Broué-Malle-Rouquier correspond to a special tagged triangulation.

CONTENTS

1. Introduction

Our main aim is to give a family of presentations of the braid group $B(d, d, n)$ of the complex reflection group $G(d, d, n)$, for positive integers d, n with $d \geq 2$ (see [BMR98, §B]), with one presentation associated to each tagged triangulation (in the sense of [FST08, §7]) of an orbifold given by a disk with a single cone point of degree d. In addition, we show that $B(d, d, n)$ can be embedded in the n-strand braid group of the orbifold as a subgroup of index d (a result obtained independently in [Fle23]; see the comment after Theorem A below), generalising a result of Allock [All02, Theorem 1.1]. This allows us to give a geometric interpretation of the generators in each presentation in the family. This generalises a family of presentations of the Artin braid group of type D_n given in [GM17], which can be regarded as the case $d = 2$.

Recently, there have been a number of articles giving presentations of braid groups using the theory of cluster algebras, and these form part of the motivation for this paper. In [BM15], a family of presentations of finite Weyl groups was given, one for each seed in the corresponding cluster algebra; the subsequent article [GM17] gave alternative presentations in the simply-laced case, which lifted to the corresponding braid groups (see also [QZ20]). An independent proof of this was found by Alastair King and Qiu Yu (see [Qiu16, Prop. 10.3]). Braid group presentations (for all finite cases) were also given in [HHLP17]. Presentations for types H and I were given in [HHQ24, Thm. 3.5]. Presentations for affine Coxeter groups were given in [FT16a] and groups of a similar kind were associated to surfaces in [FT16a, FST25] (see also [FT16b, FLST21]). This article can be regarded as providing presentations similar in style to these cluster algebra-theoretic presentations, but we note that there is no cluster algebra associated to a complex braid group, and that the mutation considered here (see Section 3.2), although related, is not the same as Fomin-Zelevinsky mutation [FZ02, Defn. 4.2] (or the corresponding diagram mutation [FZ03, §8]); in particular, the quivers considered here have additional decorations that do not appear in the theory of cluster algebras.

It is also interesting to note the article [KQ20], which associates a groupoid, known as the cluster exchange groupoid, to a cluster algebra of Dynkin type, showing that the fundamental group is isomorphic to the corresponding Artin braid group [KQ20, Thm. 2.16], giving an alternative construction of the Artin braid group in these cases. There are also strong relationships with mapping class groups and groups generated by spherical twists and their presentations; see, for example [Qiu19] and the references therein, and [Qiu24]. In particular, Appendix C in [Qiu24] gives realisations of affine braid groups as fundamental groups of moduli spaces of framed quadratic differentials.

In the remainder of Section 1 we recall the relevant theory and background for real and complex reflection groups and braid groups, and state our main results in more detail. In Section 2 we give an orbifold realisation of the braid group of the complex reflection group $G(d, d, n)$. In Section 3 we give the promised family of presentations of $G(d, d, n)$ and the corresponding braid group, and in Section 4, we give a geometric interpretation of the generators in these presentations in terms of the geometric description in Section 2.

1.1. **(Real) reflection and braid groups.** In (a special case of) [BM15, Theorem 5.4], Barot and Marsh proved that if Q is a *mutation-Dynkin quiver*, i.e. a quiver that can be obtained by mutating a Dynkin quiver Δ of type ADE in the sense of [FZ02, Defn. 4.2] finitely many times, then the associated group $W(Q)$ is isomorphic to the Weyl group $W(\Delta)$. Let n be the number of vertices in Δ. As shown in [Bri71] (and recalled in [All02, Section 2]), the Artin braid group $\mathcal{A}(\Delta)$ of type Δ is isomorphic to the fundamental group

$$\pi_1\left(\left(V - \bigcup_{s \in \Sigma} H_s\right)/W(\Delta)\right),$$

where Σ is the set of reflections in $W(\Delta)$, V is the complexification of $\mathbb{R}^n$ and H_s the complexification of the set of fixed points of s in $\mathbb{R}^n$.

More abstractly, the Artin braid group $\mathcal{A}(\Delta)$ can be defined in terms of generators and relations associated to the corresponding graph of type Δ. For example, the Coxeter graph of type D_n:

$$(1)$$
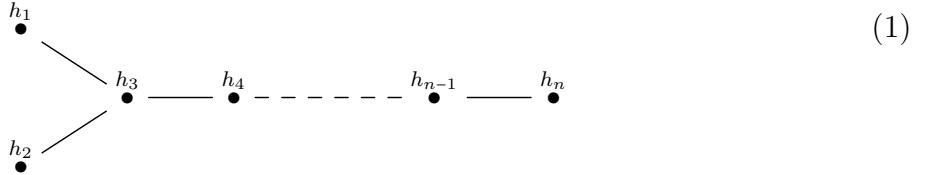
gives the standard presentation of $\mathcal{A}(D_n) = \langle h_1, h_2, \ldots, h_n \mid R \rangle$, where R is the set of relations $h_i h_j h_i = h_j h_i h_j$ if there is an edge between h_i and h_j and $h_i h_j = h_j h_i$ otherwise. The Weyl group $W(D_n)$ is then the quotient of $\mathcal{A}(D_n)$ obtained by adding the relations $h_i^2 = e$ for all i, where e is the identity element.

Allcock described the connection between some Artin braid groups and orbifold fundamental groups. In particular, in [All02, Theorem 1.1], he proved that $\mathcal{A}(D_n)$ is isomorphic to a subgroup of index 2 of the the orbifold fundamental group

$$Z_n(\mathcal{O}_2) = \pi_1((\mathcal{O}_2^n - \Delta_n)/S_n),$$

where $\mathcal{O}_2$ is the orbifold $\mathbb{C}/C_2$, $\Delta_n = \{(x_1, x_2, \ldots, x_n) \in \mathcal{O}_2^n : x_i = x_j \text{ for some } i \neq j\}$ and S_n is the symmetric group of degree n.

Subsequently, Grant and Marsh studied presentations of Artin braid groups of type ADE. In [GM17, Theorem A] they showed that if Q is a mutation-Dynkin quiver, then the associated braid group is isomorphic to the Artin braid group $\mathcal{A}(\Delta)$ of the corresponding Dynkin type. This way one obtains many presentations of the Artin braid groups of type ADE.

Moreover, they showed that an orientation of (1) coincides with the quiver Q_{T_0} associated with the *initial (tagged) triangulation* T_0 of (X, M), where X is the disk S with an interior marked point interpreted as a cone point of degree 2, and M is a set of n marked points on the boundary of X; see [GM17, page 91 and Figure 5]. See [FST08, §7] for the definition of tagged triangulations; see also [GM17, Section 3]. Note that the interior of X is isomorphic as an orbifold to $\mathcal{O}_2$.

Flipping a triangulation corresponds to mutating the quiver associated to it and, using the fact that the graph of flips of (tagged) triangulations of the disk is connected, it was shown in [GM17, Theorem A] that any (tagged) triangulation T of (X, M) gives a presentation of $\mathcal{A}(D_n)$. Moreover, T has an associated braid graph, the edges of which correspond to elements σ_i in $Z_n(\mathcal{O}_2)$. In [GM17, Theorem 3.6], Grant and Marsh proved that the subgroup

B_T of $Z_n(\mathcal{O}_2)$ generated by the elements σ_i is isomorphic to the group G_{Q_T} associated to the quiver corresponding to T. Hence the group presentation associated to the triangulation T gives a presentation of $\mathcal{A}(D_n)$ as a subgroup of index 2 of $Z_n(\mathcal{O}_2)$.

1.2. Complex reflection and braid groups.
In this paper, we are interested in studying a "complex" version of the above.

A *pseudo-reflection s* is a non-trivial element in the general linear group $\mathrm{GL}(\mathbb{C}^n)$ which fixes a hyperplane H_s pointwise, known as *the reflecting hyperplane of s*. A group generated by pseudo-reflections is known as a *complex reflection group*. The irreducible finite complex reflection groups were classified by Shepard and Todd in [ST54]. Broué, Malle and Rouquier provided presentations of all such groups using Coxeter-like diagrams; see [BMR98, Tables 1-4 in Appendix 2]. Here we focus on the complex reflection groups of the form $G(de, d, n)$ for positive integers d, n and e. We use the same notation as in [Shi05]. For $\sigma \in S_n$, denote by $[(x_1, x_2, \ldots, x_n) \mid \sigma]$ the $n \times n$ monomial matrix with non-zero entries x_i in the $i, \sigma(i)$ positions. Then

$$G(de, d, n) := \left\{ [(x_1, x_2, \ldots, x_n)|\sigma] : x_i \in \mathbb{C}^*, x_i^{de} = 1, \left(\prod_{j=1}^n x_j\right)^e = 1, \sigma \in S_n \right\}.$$

Note the close relationship to Weyl groups, which can be seen as a special case of the above. In particular, note that $W(A_n) = G(1, 1, n)$, $W(B_n) = G(2, 1, n)$ and $W(D_n) = G(2, 2, n)$.

Similarly to the real case above, one can construct the *braid group*, denoted by $B(de, d, n)$, associated to the complex reflection group $G(de, d, n)$. This is defined as the fundamental group

$$\pi_1\left(\left(\mathbb{C}^n - \bigcup_{s \in \Sigma} H_s\right)/G(de, d, n)\right),$$

where Σ is the set of pseudo-reflections in $G(de, d, n)$. See [BMR98, Tables 1,2 and 5] for presentations of both $G(de, d, n)$ and $B(de, d, n)$. Similarly to the real case, $G(de, d, n)$ is a quotient of $B(de, d, n)$, obtained by making all generators of finite order. For general d and e some generators have order larger than 2, while for $e = 1$ they all have order 2.

Moreover, note that $B(1, 1, n) = \mathcal{A}(A_n)$, $B(d, 1, n) = \mathcal{A}(B_n)$ for any $d \geq 2$, $B(2, 2, n) = \mathcal{A}(D_n)$, and $B(d, d, 2) = \mathcal{A}(I_2(d))$. In particular, [All02, Theorem 1.1] states that $B(2, 2, n)$ is isomorphic to a subgroup of order 2 of $Z_n(\mathcal{O}_2)$. Let $d \geq 2$ be an integer and $\mathcal{O}_d$ be the orbifold $\mathbb{C}/C_d$. In our first main result, we generalise Allcock's inclusion of groups to the case of arbitrary d, and fit it into a commutative diagram.

Theorem A. There is a commutative diagram of group homomorphisms

$$
\begin{array}{ccc}
N = \langle st_2s^{-1}, t_2, t_3, \ldots, t_n \rangle & \xrightarrow[\cong]{\varphi} & B(d, d, n) \\
\alpha \downarrow \text{ index } d & & \beta \downarrow \text{ index } d \\
\dfrac{\mathcal{A}(B_n)}{\langle s^d = e \rangle} & \xrightarrow[\cong]{\gamma} & Z_n(\mathcal{O}_d)
\end{array}
$$

FIGURE 1. Commutative diagram of group homomorphisms.

where γ, φ are isomorphisms and α, β are monomorphisms, and $\mathcal{A}(B_n)$ is the Artin braid group of type B_n with presentation by generators:

$$\underset{\bullet}{s} \overset{}{=\!=\!=\!=} \underset{\bullet}{\overset{t_2}{}} \overline{} \underset{\bullet}{\overset{t_3}{}} \overset{t_{n-1}}{\cdots\cdots} \underset{\bullet}{} \overline{} \underset{\bullet}{\overset{t_n}{}}$$

That is, $\mathcal{A}(B_n) = \langle s, t_2, t_3, \ldots, t_n \mid R \rangle$, where R is the set of relations $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ for $2 \le i \le n-1$, $t_i t_j = t_j t_i$ if $|i - j| > 1$ and $s t_2 s t_2 = t_2 s t_2 s$.

See Section 2 for more details of the groups and the morphisms, and the proof of the theorem.

We note that Theorem A also follows from [Fle25, Cor. 5.7] and [Fle23, Theorem A(2)]. The proof here was obtained independently. It is more direct (for this special case), avoiding use of the mapping class group and giving a construction specifically related to the approach of [BMR98] (i.e. more in the style of [All02]).

Our second main result generalises [BM15, Theorem 5.4] and [GM17, Theorem A] for the groups $G(d, d, n)$ and $B(d, d, n)$ with $d \ge 2$, where the case $d = 2$ recovers the classical results. Consider the marked surface (X, M), where X is the disk S with an interior marked point interpreted as a cone point of degree d. Note that the interior of X is isomorphic to $\mathcal{O}_d$ as an orbifold. Let M be a set of n marked points on the boundary of X.

In Section 3.1, we associate a decorated quiver Q_T to any tagged triangulation T of (X, M) and a group G_{Q_T} to Q_T. In Section 3.2, we introduce a mutation rule for such a quiver with respect to a chosen vertex, which corresponds to flipping the associated triangulation.

In particular, the initial triangulation T_0, illustrated in Figure 12, has associated quiver Q_{T_0}, which is an orientation of the presentation of $B(d, d, n)$ from [BMR98, Table 5]. Proving that at each mutation step we obtain an isomorphic group, and using the fact that the flipping graph of (tagged) triangulations of (X, M) is connected, we obtain the following result, providing a family of new presentations of the groups $B(d, d, n)$ and $G(d, d, n)$.

Theorem B. (=Theorems 3.36 and 5.1). Let T be a tagged triangulation of (X, M) and let G'_{Q_T} be the group defined in the same way as G_{Q_T} with the additional relations that all generators square to the identity element. Then

- $G_{Q_T} \cong B(d, d, n)$ and G_{Q_T} gives a presentation of $B(d, d, n)$,
- $G'_{Q_T} \cong G(d, d, n)$ and G'_{Q_T} gives a presentation of $G(d, d, n)$.

Finally, we re-interpret Theorem B by assigning explicit elements to the abstract generators in the newfound presentations both for the braid and reflection groups.

We combine the two theorems above to give a geometric interpretation of the new presentations of $B(d, d, n)$. Similarly to the real case above, each tagged triangulation T of (X, M) has an associated braid graph D_T such that the edges of D_T, one for each vertex i in Q_T, correspond to braids σ_i in $Z_n(\mathcal{O}_d)$. Let B_T be the subgroup of $Z_n(\mathcal{O}_d)$ generated by these braids σ_i. The following result generalises [GM17, Theorem 3.6] to the case $d \ge 2$, concluding that the group presentation associated to the triangulation T gives a presentation of $B(d, d, n)$ as a subgroup of index d of $Z_n(\mathcal{O}_d)$.

Theorem C. (=Theorem 4.6.) Let T be a tagged triangulation of (X, M). Then there is an isomorphism from B_T to G_{Q_T} taking the braid σ_i to the generator s_i of G_{Q_T} corresponding to the vertex i in Q_T. Furthermore, B_T is a subgroup of index d of $Z_n(\mathcal{O}_d)$.

Finally, combining Theorem B with results from [Shi05], we assign explicit reflections to the generators of the new presentations of $G(d, d, n)$ as follows. For the definition of the reflections appearing in the following result, we refer the reader to Section 2.1.

Theorem D. (=Theorem 5.7.) Let T be a tagged triangulation of (X, M) and fix a numbering $1, 2, \ldots, n$ of the n vertices of D_T. Associate a reflection $s(e) = s(a, b; c(e))$ to each edge e between vertices a and b in D_T, where for the edges appearing in the unique cycle of D_T, the integers $c(e)$ have to obey the condition explained in Setup 5.5. Then, there is an isomorphism of groups $\nu : G'_{Q_T} \to G(d, d, n)$ sending the generator of G'_{Q_T} associated to vertex v in Q_T to the reflection associated to the edge in D_T that is the dual of v.

2. THE COMPLEX BRAID GROUP $B(d, d, n)$ AS A SUBGROUP OF $Z_n(\mathcal{O}_d)$: PROOF OF THEOREM A.

The aim of this section is to prove Theorem A, i.e. to construct the commutative diagram of group homomorphisms in Figure 1. Note that the maps α and φ are mainly due to [BMR98]. For the remaining two maps, we proceed with a geometric argument. Details of the groups and presentations appearing in the diagram are given in the following sections.

2.1. **Building β.** We follow [All02, §2], using the notation from [Shi05]. We set $V = \mathbb{C}^n$, and denote the fixed hyperplane of a reflection $s : V \to V$, by H_s.

We use the notation from [Shi05]. Let d, n be positive integers with $d \geq 2$. Let S_n denote the symmetric group of degree n, and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For $\sigma \in S_n$ and $(x_1, x_2, \ldots, x_n) \in (\mathbb{C}^*)^n$, let $[(x_1, x_2, \ldots, x_n)]$ denote the $n \times n$ monomial matrix with x_i in the $i, \sigma(i)$ position for $i = 1, 2, \ldots, n$. The entries in such a matrix are powers of $\omega_d = e^{2\pi i/d}$.

Let $\Gamma(d, n)$ denote the group of all such matrices, and let $G(d, d, n)$ denote the complex reflection group:

$$G(d, d, n) = \left\{ [(x_1, x_2, \ldots, x_n) | \sigma] : x_i \in \mathbb{C}^*, x_i^d = 1, \prod_{j=1}^n x_j = 1, \sigma \in S_n \right\},$$

which is a normal subgroup of $\Gamma(d, n)$ of index d. For $1 \leq a < b \leq n$ and $0 \leq c \leq d - 1$, set

$$s(a, b; c) = [(1, \ldots, 1, \omega_d^{-c}, 1, \ldots, 1, \omega_d^c, 1, \ldots, 1) | (a, b)],$$

and for $a > b$, set $s(a, b; c) = s(b, a, -c)$. Thus, for $a < b$,

$$s(a, b; c)(z_1, \ldots, z_n) = (z_1, \ldots, z_{i-1}, \omega_d^{-c} z_b, \ldots, \omega_d^c z_a, \ldots, z_n).$$

These elements all have order two and, as remarked in [Shi05, 1.5], they constitute the reflections in $G(d, d, n)$; we denote this set by Σ. Thus $G(d, d, n)$ is generated by Σ, the set of reflections it contains.

Remark 2.1. For $i < j$, the reflection $s(a, b; c)$ fixes the *hyperplane*

$$H_{s(a,b;c)} = H(a, b; c) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_a = \omega_d^{-k} z_b\} = \ker(s(a, b; c) - id_{\mathbb{C}^n}).$$

Moreover, $s(a, b; c)$ has associated *hyperline*

$$L_{s(a,b;c)} = L(a, b; c) = \operatorname{im}(s(a, b; c) - id_{\mathbb{C}^n}).$$

Note that $\mathbb{C}^n = H_{s(a,b;c)} \oplus L_{s(a,b;c)}$, and so each element $x \in \mathbb{C}^n$ can be written uniquely as $x = x_H + x_L$ with $x_H \in H_{s(a,b;c)}$ and $x_L \in L_{s(a,b;c)}$.

Let

$$V_0 = V \setminus \cup_{s \in \Sigma} H_s.$$

As noted in [All02, 2.1] for the D_n case, V_0 is connected since each H_s has real codimension 2 in V.

It is well-known that $G(d,d,n)$ acts freely on V_0 (see e.g. [Gar23]). Since $G(d,d,n)$ is finite, it acts properly discontinuously on V_0, and it is clear the action is smooth, so we can form the quotient manifold $V_0/G(d,d,n)$ (by e.g. [Hat02, 7.10]). Let $p : V_0 \to V_0/G(d,d,n)$ be the canonical surjection, which is a manifold covering map [Lee13, Thm. 21.13]. Choose $x_0 \in V_0$. Then the fundamental group $\pi_1(V_0/G(d,d,n), p(x_0))$ is known as the **braid group** of $G(d,d,n)$ and denoted $B(d,d,n)$ (see [BMR98, 2B]).

The cyclic group C_d acts on $\mathbb{C}$, with a generator sending z to $\omega_d z$. Let $\mathcal{O}_d$ be the orbifold $\mathbb{C}/C_d$. The underlying space of $\mathbb{C}/C_d$ is $\mathbb{C}$, and it has a single cone point of degree d at the origin. The n-strand **pure braid space** of $\mathcal{O}_d$ is $\mathcal{O}_d^n - \Delta_n$, where

$$\Delta_n = \{(z_1, z_2, \ldots, z_n) \in \mathcal{O}_d^n : z_i = z_j \text{ for some } i \neq j\}.$$

The symmetric group S_n of degree n acts freely on $\mathcal{O}_d^n - \Delta_n$ and we can form the quotient

$$X_n = (\mathcal{O}_d^n - \Delta_n)/S_n,$$

which is the n-strand **braid space** of $\mathcal{O}_d$. Then the n-strand **braid group** $Z_n(\mathcal{O}_d)$ of $\mathcal{O}_d$ is the orbifold fundamental group (in the sense of [Thu22, Defn. 13.2.5]) of X_n with respect to a choice of basepoint $b = (b_1, \ldots, b_n) \in X_n$ which does not lie on the orbifold locus. The n-strand **pure** braid group $P_n(\mathcal{O}_d)$ of $\mathcal{O}_d$ is the orbifold fundamental group of Δ_n.

Proposition 2.2. There is an isomorphism of orbifolds $V/C_d^n \cong \mathcal{O}_d^n$ given by $(x_1, x_2, \ldots, x_n) \mapsto (x_1^d, x_2^d, \ldots, x_n^d)$. This induces an isomorphism $\varphi : V_0/\Gamma(d,n) \cong X_n$ and hence a d-fold orbifold covering map $V_0/G(d,d,n) \to X_n$ and an embedding β of $B(d,d,n) = \pi_1(V_0/G(d,d,n))$ as a subgroup of index d in $Z_n(\mathcal{O}_d) = \pi_1(X_n)$.

Proof. Recall that

$$V_0 = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : z_i \neq \omega_d^k z_j, \quad \text{for all } i \neq j \text{ and } 0 \leq k \leq d - 1\}.$$

The group $G(d,d,n)$ has a normal subgroup isomorphic to C_d^{n-1} consisting of the elements where the permutation is the identity, that is, using the notation $\omega_d := e^{2\pi i/d}$, the elements of the form

$$[(\omega_d^{k_1}, \omega_d^{k_2}, \ldots, \omega_d^{k_n}) \mid id]$$

where $k_1, k_2, \ldots, k_n \in \{0, 1, \ldots, d - 1\}$ satisfy $k_1 + k_2 + \cdots + k_n = 0$ modulo d. Note that $k_1, k_2, \ldots, k_{n-1}$ can be chosen freely and they determine k_n. It is easy to see that $G(d,d,n) = C_d^{n-1} \rtimes S_n$. The rest of the argument goes through as in [All02, Proof of Thm. 1.1]. $\square$

We use the same generating set of reflections for $G(d,d,n)$ as [BMR98, pp 151-152] with the following notation. Note that these give a presentation of $G(d,d,n)$ by [BMR98, Proposition 3.2].

Notation 2.3. We set

$$t_2' := s(1,2;1) : (z_1, z_2, z_3, z_4 \ldots, z_n) \mapsto (e^{-2\pi i/d} z_2, e^{2\pi i/d} z_1, z_3, z_4 \ldots, z_{n-2}, z_{n-1}, z_n),$$

$$t_2 := s(1,2;0) : (z_1, z_2, z_3, z_4 \ldots, z_n) \mapsto (z_2, z_1, z_3, z_4 \ldots, z_{n-2}, z_{n-1}, z_n),$$

$$t_3 := s(2,3;0) : (z_1, z_2, z_3, z_4 \ldots, z_n) \mapsto (z_1, z_3, z_2, z_4 \ldots, z_{n-2}, z_{n-1}, z_n),$$

$$\vdots$$

$$t_n := s(n-1, n; 0) : (z_1, z_2, z_3, z_4 \ldots, z_n) \mapsto (z_1, z_2, z_3, z_4 \ldots, z_{n-2}, z_n, z_{n-1}).$$

In other words, we are taking n reflections: $s(a-1, a; 0)$ for $a \in \{1, 2, \ldots, n\}$ and $s(1, 2; 1)$ and renaming them as above. As pointed out in [Shi05], these are all reflections of type I and hence they have order 2 and they lie in $G(d, d, n)$.

We next need some paths in V_0 as defined in [BMR98].

Definition 2.4. [BMR98, §B, Eq. (2.13)]. Let $s \in G(d, d, n)$ be a reflection. Note that s has order 2. Let $x \in V_0$, with decomposition $x = x_H + x_L$ with $x_H \in H_s$ and $x_L \in L_s$, as in Remark 2.1. Then $y = s(x) = x_H - x_L$. Then the straight path $\tilde{p}_s : [0, 1] \to V$ sending t to $x_H + (1 - 2t)x_L$ does not lie in V_0, since $\tilde{p}_s(1/2) \in H$. So we take instead a version of the path which is perturbed close to H:

$$p_s(t) = \begin{cases} x + t(y - x) = x_H + (1 - 2t)x_L, & 0 \le t \le \frac{1}{3}; \\ x + (\frac{1}{2} - \frac{1}{6}e^{3i\pi(t-\frac{1}{3})})(y - x) = x_H + \frac{1}{3}e^{3i\pi(t-\frac{1}{3})}x_L, & \frac{1}{3} \le t \le \frac{2}{3}; \\ x + t(y - x) = x_H + (1 - 2t)x_L, & \frac{2}{3} \le t \le 1, \end{cases}$$

which is a special case of the construction in [BMR98, §B, Eq. (2.13)].

Remark 2.5. In Definition 2.4, the ith entry of $p_s(t)$ is either constant (if $x_i = y_i$) or a path from x_i to y_i which is the first third of the straight path in $\mathbb{C}$ from x_i to y_i ending at $x_i + \frac{1}{3}(y_i - x_i) = 2$, followed by an anticlockwise semicircle of radius $\frac{1}{6}|y_i - x_i|$ centred at the mid-point between x_i and y_i and ending at $x_i + \frac{2}{3}(y_i - x_i)$, followed by the last third of the straight path from x_i to y_i.

The following result is an instance of [BMR98, Proposition 3.2 and Theorem 2.27].

Proposition 2.6. The set $\{t_2', t_2, \ldots, t_n\}$ together with the relations described in [BMR98, Appendix 2 and Table 2] give a presentation by generators and relations of $G(d, d, n)$. Moreover, for s equal to, respectively, $t_2', t_2, t_3 \ldots, t_n$, the paths p_s, regarded as paths in $V_0/G(d, d, n)$, are s-generators of the monodromy, denoted respectively by $\tau_2', \tau_2, \tau_3, \ldots, \tau_n$, giving the presentation by generators and relations of $B(d, d, n)$ illustrated in Figure 2, with relations:

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad \tau_2' \tau_3 \tau_2' = \tau_3 \tau_2' \tau_3, \quad \tau_i \tau_j = \tau_j \tau_i \text{ for } |i-j| > 1, \quad \tau_2' \tau_i = \tau_i \tau_2' \text{ for } i > 3$$

$$\tau_2 \tau_2' \tau_3 \tau_2 \tau_2' \tau_3 = \tau_3 \tau_2 \tau_2' \tau_3 \tau_2 \tau_2', \quad \underbrace{\tau_2 \tau_2' \cdots}_{d \text{ terms}} = \underbrace{\tau_2' \tau_2 \cdots}_{d \text{ terms}}.$$
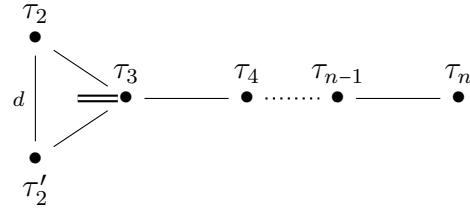
FIGURE 2. The presentation of $B(d,d,n)$ from [BMR98].

Remark 2.7. We recall part of the proof of Proposition 2.6 from [BMR98, Section 3]. Recall the standard presentation of the Artin braid group of type A_{n+1}, that is $\mathcal{A}(A_{n+1})$ or in [BMR98] notation $B(n+1)$, is:

By [BMR98, Thm. 3.6], the braid group $B(d,1,n)$ associated to the complex reflection group $G(d,1,n)$ is, for any $d > 1$, isomorphic to the subgroup of $\mathcal{A}(A_{n+1})$ generated by $\{\xi_1^2, \xi_2, \xi_3, \ldots, \xi_n\}$. By the discussion following the proof of [BMR98, Thm. 3.6], an application of the Reidemeister-Schreier method shows that this subgroup is isomorphic to the Artin braid group $\mathcal{A}(B_n)$, with generators associated to the vertices of its Dynkin diagram as follows:

Recall that, if $m > 1$, by [BMR98, Lemma 3.3], the complement in $\mathbb{C}^n$ of the union of the reflecting hyperplanes of $G(md,d,n)$ is

$$\mathcal{M}^{\#}(md,n) = \{(z_1, z_2, \ldots, z_n) \mid (\forall j, k, 1 \le j \ne k \le n)(\forall a \in \mathbb{Z})(z_j \ne 0)(z_j \ne e^{\frac{i\pi a}{md}} z_k)\},$$

while if $m = 1$, it is

$$\mathcal{M}(d,n) = \{(z_1, z_2, \ldots, z_n) \mid (\forall j, k, 1 \le j \ne k \le n)(\forall a \in \mathbb{Z})(z_j \ne e^{\frac{i\pi a}{d}} z_k)\}.$$

As remarked in [BMR98, Section 3C], [BMR98, Proposition 3.8] could be stated in a more general way, obtaining, by an application of the Reidemester-Schreier algorithm, an injective group homomorphism

$$\phi : \pi_1(\mathcal{M}^{\#}(md,n)/G(md,d,n)) \hookrightarrow \mathcal{A}(B_n),$$

where, letting $\xi_2' := \xi_1^2 \xi_2 \xi_1^{-2}$, the left hand side group has presentation

$$\tag{2}$$
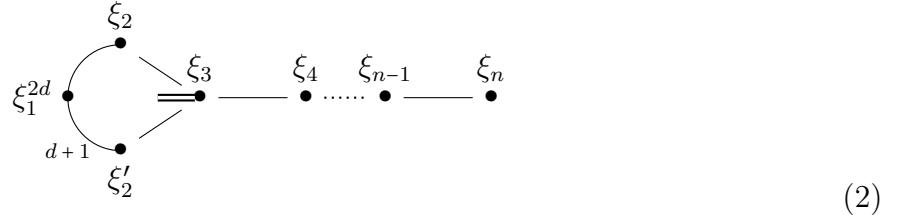
In particular, if $m > 1$, the left hand side is isomorphic to $B(md,d,n)$, but here we are interested in the case $m = 1$. By [BMR98, Section 3C], in this case there is an isomorphism

of groups

$$B(d,d,n) \cong \frac{\pi_1(\mathcal{M}^\#(d,n)/G(d,d,n))}{\langle \xi_1^{2d} = e \rangle}$$

and the presentation of $B(d,d,n)$ is obtained from (2) by suppressing the node corresponding to ξ_1^{2d} and adding an edge labelled d between ξ_2 and ξ_2'.

Remark 2.8. Note that [BMR98, Section 3] uses the base point $(x_1, x_2, x_3 \ldots, x_{n+1})$ with $x_1 < x_2 < x_3 < \cdots < x_{n+1}$ real numbers. Instead, we choose basepoint $(0, -1, 1, 2, \ldots, n-1)$. We also modify the $(2\ 3)$-generator of the monodromy ξ_2 (where $(2\ 3)$ is a generating transposition of S_{n+1}), taking the path as in Figure 6 rather than as in [BMR98, Section 3]: the arguments of [BMR98] go through unchanged with these choices. See Remark 2.10 for an explanation of this choice.

Proposition 2.9. The injective group homomorphism β from $B(d,d,n)$ to $Z_n(\mathcal{O}_d)$ from Proposition 2.2 is given by:

$$\beta : B(d,d,n) \to Z_n(\mathcal{O}_d) : \tau_2' \mapsto h_1, \tau_i \mapsto h_i \text{ for } 2 \le i \le n,$$

where τ_2', τ_2, $\tau_3, \ldots$, and τ_n are as in the presentation in Proposition 2.6 and the braids $h_1, h_2, \ldots, h_n$ are as illustrated in Figure 9.

Proof. Let us fix the basepoint in V_0 to be $b = (e^{-i\pi/d}, 1, 2, \ldots, n-1)$. We compute the paths p_s associated to each of the reflections s from Notation 2.3 following Definition 2.4.

Note that $t_2(b) = (1, e^{-i\pi/d}, 2, \ldots, n-1)$. By Remark 2.5, the first entry in the path p_{t_2} from Definition 2.4 is the first third of the straight path from $e^{-i\pi/d}$ to 1, followed by a semicircular path centred at the midpoint between $e^{-i\pi/d}$ and 1 of radius $\frac{1}{6}|1 - e^{-i\pi/d}|$, followed by the last third of the straight path from $e^{-i\pi/d}$ to 1. The second entry is a similarly-defined path going in the other direction, while all the other entries are constant. We sketch this, for $d = 3$, in Figure 3.

We have $t_2'(b) = (e^{-2\pi i/d}, e^{i\pi/d}, 2, \ldots, n-1)$, so the first entry of $p_{t_2'}$ is a path from $e^{-i\pi/d}$ to $e^{-2\pi i/d}$, while the second entry is a path from 1 to $e^{i\pi/d}$ and the other entries remain constant. We sketch this, for $d = 3$, in Figure 4.

For $3 \le j \le n$, we have $t_j(b) = (e^{-i\pi/d}, 1, \ldots, j-3, j-1, j-2, j, \ldots, n-1)$, so the $(j-1)$st entry of p_{t_j} is a path from $j-2$ to $j-1$ and the jth entry of p_{t_j} is a path from $j-1$ to $j-2$. We sketch this, for $d = 3$, in Figure 5.

By Proposition 2.6, for s equal to, respectively, $t_2', t_2, t_3 \ldots, t_n$, the paths p_s, regarded as paths in $V_0/G(d,d,n)$, are s-generators of the monodromy, denoted respectively by $\tau_2', \tau_2, \tau_3, \ldots, \tau_n$. Their images under the covering map $V_0/G(d,d,n) \to X_n$ in Proposition 2.2 are given by taking their dth powers, entry by entry.

These images are illustrated in Figures 7,6 and 8, for $d = 3$. Note also that the basepoint for $\mathcal{O}_d^n - \Delta_n$ is $(-1, 1, 2^d, \ldots, (n-1)^d)$. It is easy to see, by writing the paths in polar coordinates in $\mathbb{C}$, that the paths will be in the same half plane (i.e. above or below the real axis) as shown in these figures, for general d. Drawing these paths as braids using the same conventions as in [All02], that is placing the point of view from below, and recalling that the cone point of order d at 0 is interpreted as a pole of order d, we see that the elements $\tau_2', \tau_2, \tau_3, \ldots, \tau_n$ are sent respectively to the braids $h_1, h_2, h_3, \ldots, h_n$ illustrated in Figure 9.

Hence, the injective group homomorphism β from Proposition 2.2 sends $B(d,d,n)$ to the subgroup $\langle h_1, h_2, h_3, \ldots, h_n \rangle$ of $Z_n(\mathcal{O}_d)$ of index d, as claimed. □
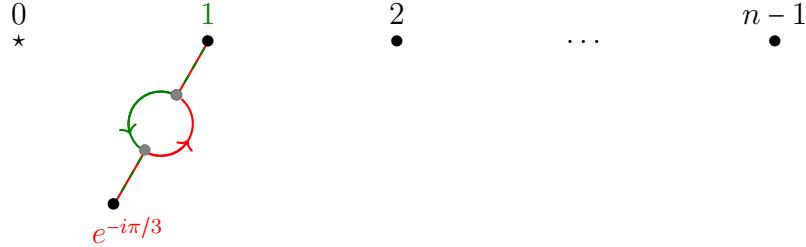
FIGURE 3. The path p_{t_2} associated to the reflection t_2. Note that only the first two entries of the n-tuple are non-constant paths. The drawing is for $d = 3$. For larger d, the endpoints of the paths are closer.
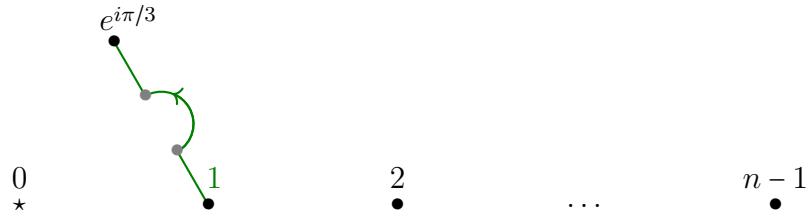

FIGURE 4. The path $p_{t_2'}$ associated to the reflection t_2'. Note that only the first two entries of the n-tuple are non-constant paths. The drawing is for $d = 3$. For larger d, the path starting at $e^{-i\pi/3}$ lies entirely within the fourth quadrant of the plane and the endpoints of both paths are closer.
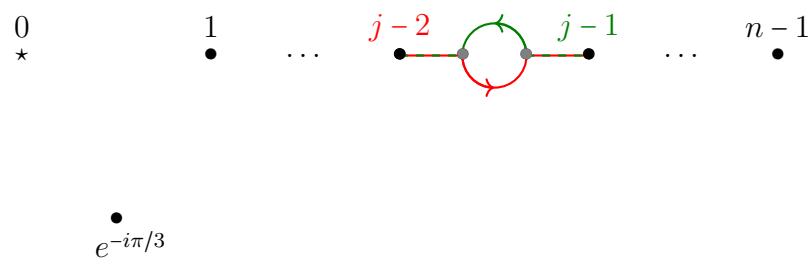
FIGURE 5. The path p_{t_j} associated to the reflection t_j for $3 \leq j \leq n$. Note that only the $(j-1)$st and jth entries of the n-tuple (with initial values $j-2$ and $j-1$ respectively) are non-constant paths. We draw the case $d = 3$.

Remark 2.10. Our choice of basepoint, together with the modified choice of $(2\ 3)$-generator for the monodromy for the braid group of type A (see Remark 2.8) allows to use the same

generating set of reflections for $G(d, d, n)$ as in [BMR98, Section 3A] (see Notation 2.3) while ensuring that the diagram in Figure 1 commutes and the images of the generators under β in Proposition 2.9 are the h_i.

If we did not modify the (2 3)- generator of the monodromy, we would need to replace the reflection $t_2' = s(1, 2; 1)$ with the reflection $t_2'' = s(1, 2; -1)$; by [Shi05, Theorem 2.19], we would still have a generating set of $G(d, d, n)$. We would also need to replace the basepoint $b = (e^{-i\pi/d}, 1, 2, \ldots, n-1)$ in the proof of Proposition 2.9 with $(e^{i\pi/d}, 1, 2, \ldots, n-1)$ and the element ξ_2' in Remark 2.7 with $\xi_1^{-2}\xi_2\xi_1^2$, also switching the labels ξ_2 and ξ_2' on the presentation of $B(d, d, n)$ given there. Modifying the morphisms α, ϕ and γ in Figure 1 appropriately (so that $\alpha(a_2) = t_2$, $\alpha(b_2) = s^{-1}t_2s$, $\varphi(a_2) = \tau_2$, $\varphi(b_2) = \tau_2'$, $\beta(\tau_2) = h_1$, $\beta(\tau_2') = h_2$, $\gamma(t_2) = h_1$ and $\gamma(s^{-1}t_2s) = h_2$), this would also be a valid construction. For $d = 2$, the paths p_s we consider here do not coincide with the paths $g_1, \ldots g_n$ from [All02, proof of Theorem 1.1] (even choosing $\epsilon = 1$). However, with this change, the basepoint would be $(i, 1, 2, \ldots, n-1)$ as in [All02], and we would recover exactly the same paths as in [All02].
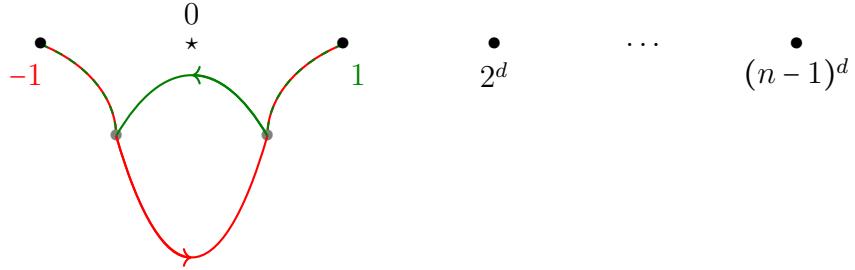

FIGURE 6. Path p_{t_2} to power d.
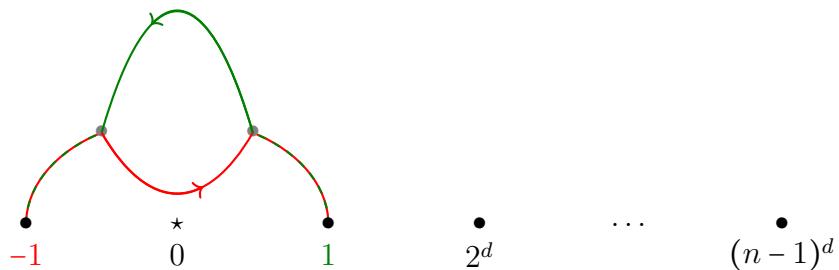
FIGURE 7. Path $p_{t_2'}$ to the power d.
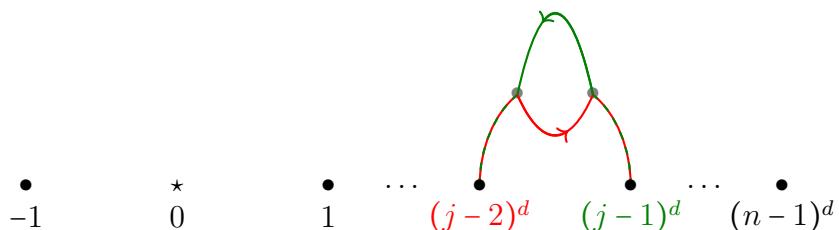

FIGURE 8. Path p_{t_j} to the power d for $3 \leq j \leq n$.

2.2. Building α and φ.

Lemma 2.11. The subgroup $N := \langle st_2s^{-1}, t_2, t_3, ..., t_n \rangle \subseteq \frac{\mathcal{A}(B_n)}{\langle s^d = e \rangle}$ is a normal subgroup of index d and

$$\frac{\mathcal{A}(B_n)}{\langle s^d = e \rangle} = N \uplus Ns \uplus Ns^2 \uplus \ldots Ns^{d-1}.$$

Proof. We first prove N is a normal subgroup by showing that its normaliser is the whole ambient group. Note that by construction $st_2s^{-1} \in N$ and for $i = 3, \ldots, n$ we have $st_is^{-1} = t_i \in N$. Moreover, using the relation $st_2st_2 = t_2st_2s$, or equivalently $st_2s^{-1} = t_2^{-1}s^{-1}t_2st_2$, we have that

$$s(st_2s^{-1})s^{-1} = st_2^{-1}s^{-1}t_2st_2s^{-1} = (st_2s^{-1})^{-1}(t_2)(st_2s^{-1}),$$

and this is an element in N since it is a product of three elements in N. Hence $sNs^{-1} \subseteq N$. Then the normaliser of N contains both s and all the elements of N. In particular it contains the generators $s, t_2, \ldots, t_n$ of the ambient group. Hence N is a normal subgroup of $\frac{\mathcal{A}(B_n)}{\langle s^d = e \rangle}$.

We now show that $\frac{\mathcal{A}(B_n)}{\langle s^d = e \rangle}$ is the semidirect product of $\langle s \rangle$ and N. First we show that $s \notin N$. Recall that $s^{-1} = s^{d-1}$ and the relations in the group preserve the sum of the exponents of copies of s modulo d in the expression of any element of the group. Then, we have that the sum of the exponents of copies of the element s in the expression for any element in N is always a multiple of d. Hence $s \notin N$ and $N \cap \langle s \rangle = \{e\}$, where e is the identity element. Now, we have

$$\frac{\mathcal{A}(B_n)}{\langle s^d = e \rangle} = N \uplus Ns \uplus Ns^2 \uplus \ldots Ns^{d-1}$$

and $\frac{\mathcal{A}(B_n)}{\langle s^d = e \rangle} = \langle N, \langle s \rangle \rangle$ and N is a normal subgroup, so

$$\frac{\mathcal{A}(B_n)}{\langle s^d = e \rangle} = N \langle s \rangle$$

and N is a subgroup of index d. $\qquad\square$

We define α to be the embedding of N into $\frac{\mathcal{A}(B_n)}{\langle s^d = e \rangle}$.

Theorem 2.12. *There is a group isomorphism*

$$\varphi : N \longrightarrow B(d,d,n) : st_2s^{-1} \mapsto \tau_2', \; t_i \mapsto \tau_i, \text{ for } 2 \leq i \leq n.$$

Proof. This can be seen by applying the Reidemester-Schreier algorithm to find a presentation of the subgroup N, using the set of coset representatives $\{e, s, s^2, \ldots, s^{d-1}\}$, and noting the presentation of $B(d,d,n)$ in Proposition 2.6 (from [BMR98]). This is very similar to the proof of Proposition 2.6 in [BMR98, Sections 3.7 and 3C]; see Remark 2.7. $\qquad\square$

2.3. Building γ.

Lemma 2.13. The element $l \in Z_n(\mathcal{O}_d)$ has order d.

Proof. By [Rou21, §4], $P_n(\mathcal{O}_d)$ is a subgroup of $Z_n(\mathcal{O}_d)$, embedded as the subgroup of braids in $Z_n(\mathcal{O}_d)$ in which each strand starts at ends at the same corresponding point in $\mathcal{O}_d$ (i.e. pure braids). By [Rou21, Rk. 2.15], there is a homomorphism $\xi_n : P_n(\mathcal{O}_d) \to P_{n-1}(\mathcal{O}_d)$ obtained by removing the nth strand. Iterating this homomorphism gives a homomorphism ξ from
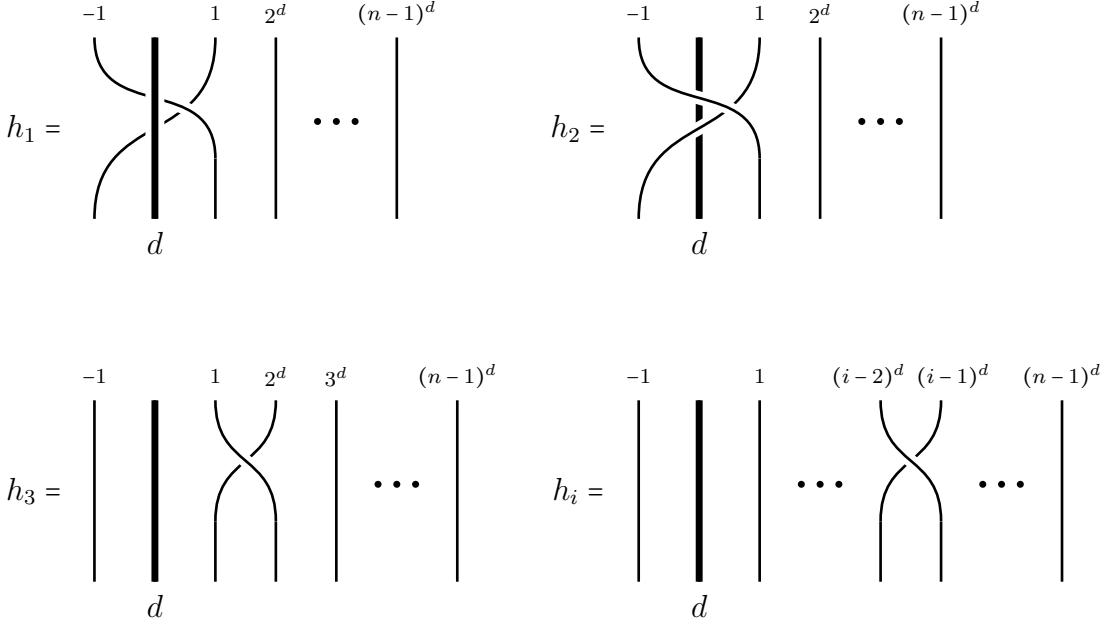
FIGURE 9. The braids $h_1, h_2, \ldots, h_n$. The thicker line (pole) represents the cone point of degree d.
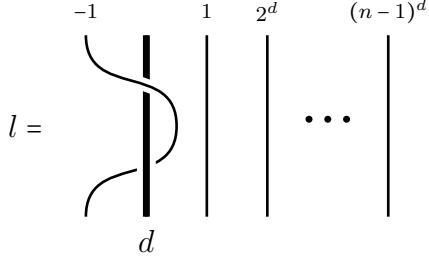

FIGURE 10. The loop l.

$P_n(\mathcal{O}_d)$ to $P_1(\mathcal{O}_d) = \pi_1(\mathcal{O}_d)$ removing all strands except the first. It follows from [Jr.19, Rk. 2.2.2] that $\xi(l^r)$ is not equal to the identity for $1 \le r \le d-1$, from which the result follows. $\square$

Lemma 2.14. Let $h_1, \ldots, h_n, l$ be the elements of $Z_n(\mathcal{O}_d)$ shown in Figures 9 and 10. We have that

$$Z_n(\mathcal{O}_d) = (\mathrm{im}\beta) \uplus (\mathrm{im}\beta)l \uplus (\mathrm{im}\beta)l^2 \uplus \cdots \uplus (\mathrm{im}\beta)l^{d-1} = \langle h_1, \ldots, h_n, l \rangle.$$

Proof. Note that $B(d, d, n)$ is torsion free by [Bes15, Thm. 0.4] (see discussion after the theorem), noting that $G(d, d, n)$ is well-generated (see, for example, [LM21, §2.2.2]). Hence, $\mathrm{im}\beta = \beta(B(d, d, n)) \subseteq Z_n(\mathcal{O}_d)$ is torsion-free. Hence, apart from the identity element, $\mathrm{im}\beta$ has no element of finite order. Since the element l has finite order d, we conclude that $l^q \notin \mathrm{im}\beta$ for any $q \notin d\mathbb{Z}$. Suppose now that $bl^q = b'l^p$ for some $b, b' \in \mathrm{im}\beta$ and non-negative integers q, p. Then $l^{q-p} = (b')^{-1}b \in \mathrm{im}\beta$ and so $l^{q-p} = e$. By Lemma 2.13, $q \equiv p$ modulo d and $l^p = l^q$. In other words, if $l^p \neq l^q$, that is $p \neq q$ modulo d, then $(\mathrm{im}\beta)l^p \cap (\mathrm{im}\beta)l^q = \varnothing$ and, since $\mathrm{im}\beta$ is a subgroup of index d in $Z_n(\mathcal{O}_d)$ by Propositions 2.2 and 2.9, the cosets of $\mathrm{im}\beta$ are $(\mathrm{im}\beta)l^p$ for $0 \le p \le d-1$. Since $\mathrm{im}\beta = \langle h_1, \ldots, h_n \rangle$ by Proposition 2.9, we conclude that $Z_n(\mathcal{O}_d) = \langle h_1, \ldots, h_n, l \rangle$. $\square$
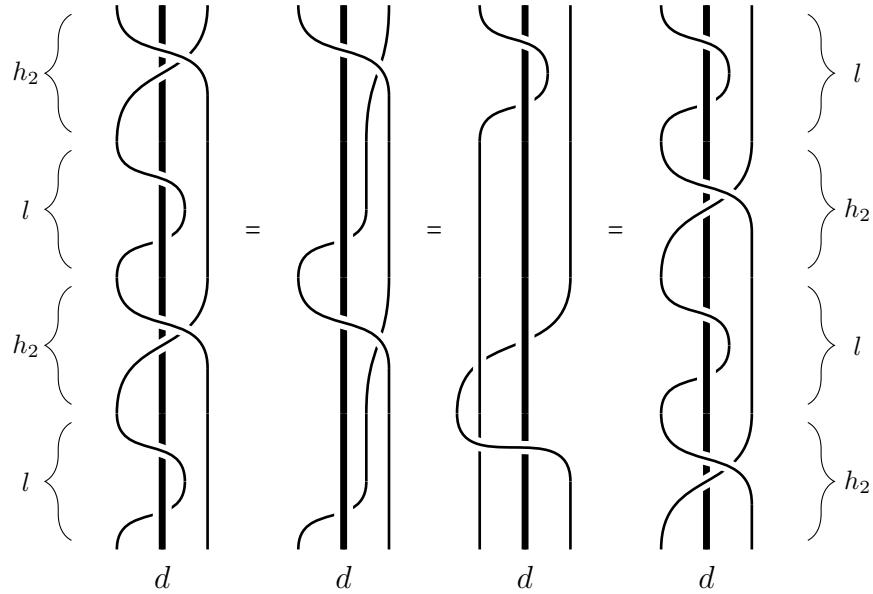
FIGURE 11. The braids lh_2lh_2 on the left and h_2lh_2l on the right are equal.

Theorem 2.15. *The following is a group isomorphism*

$$\gamma : \frac{\mathcal{A}(B_n)}{\langle s^d = e \rangle} \longrightarrow Z_n(\mathcal{O}_d) : s \mapsto l,\, t_2 \mapsto h_2,\, t_i \mapsto h_i \text{ for } 3 \le i \le n,$$

satisfying $\gamma(st_2s^{-1}) = h_1$. See Figures 9 and 10 for $h_1, \ldots . h_n, l$. Furthermore, the diagram in Figure 1 commutes.

Proof. We first show that the map γ preserves the relations and it is hence a well-defined group homomorphism. By composing the braids, it is easy to see that $\gamma(st_2s^{-1}) = lh_2l^{-1} = h_1$.

Composing the corresponding braids, it is immediate to see that the following relations are preserved by γ:

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \text{ for } 2 \le i \le n-1, \qquad t_i t_j = t_j t_i \text{ for } |i-j| \ge 2, i, j \ge 3,$$
$$t_2 t_i = t_i t_2 \text{ for } 4 \le i \le n, \qquad st_i = t_i s \text{ for } 3 \le i \le n.$$

Moreover, since the pole has degree d, it follows that l^d is the identity and the relation $s^d = e$ is preserved. It only remains to show that the relation $st_2st_2 = t_2st_2s$ is preserved. Note that $lh_2lh_2 = h_2lh_2l$ as braids, see Figure 11 and note that this relation does not depend on the order of the pole. Since only the first two strands and the pole are involved in this computation, we have omitted all the remaining strands from the pictures. Hence we conclude that γ is a well-defined group homomorphism.

Using the description of φ in Theorem 2.12 and the description of β in Proposition 2.9, we have $\gamma(\alpha(t_i)) = \gamma(t_i) = h_i$, and $\beta\varphi(t_i) = \beta(\tau_i) = h_i$. We also have $\gamma\alpha(st_2s^{-1}) = \gamma(st_2s^{-1}) = h_1$ (as noted above), while $\beta\varphi(st_2s^{-1}) = \beta(\tau_2') = h_1$. Hence the diagram in Figure 1 commutes.

We now prove that γ is a group isomorphism. First note that γ is surjective by Lemma 2.14, noting that $h_1, h_2, \ldots, h_n, l$ are all in the image of γ.

Recall that by Lemma 2.11,

$$\frac{\mathcal{A}(B_n)}{\langle s^d = e \rangle} = N \uplus Ns \uplus Ns^2 \uplus \ldots Ns^{d-1}.$$

Then, if $\gamma(ns^p) = \gamma(n's^q)$ for two elements ns^p and $n's^q$ in $\frac{\mathcal{A}(B_n)}{\langle s^d=e \rangle}$, where $n, n' \in N$ and $0 \leq p, q \leq d - 1$, we have that $\gamma(n)l^p = \gamma(n')l_q$. Since the diagram in Figure 1 commutes, we have $\beta(\varphi(n))l^p = \beta(\varphi(n'))l^q$. By Lemma 2.14, $\beta(\varphi(n)) = \beta(\varphi(n'))$ and $p = q$. Since β and φ are injective, $n = n'$ and we see that γ is injective and therefore an isomorphism as required.   □

This completes the proof of Theorem A.

3. PRESENTATIONS OF $B(d, d, n)$ AND $G(d, d, n)$

In this section, we give new presentations of the complex braid groups $B(d, d, n)$ and their corresponding complex reflection groups $G(d, d, n)$. In Definition 3.2, we associate a (decorated) quiver Q_T to an arbitrary tagged triangulation T of a disk with n marked points on the boundary and a cone point of degree $d \geq 2$ in its interior. (Note that we assume $d \geq 2$ throughout this section, except where otherwise stated, for some lemmas where it is convenient to allow the case $d = 1$). The quiver may have 2-cycles, which we consider to be unoriented edges. In Definition 3.4, we associate a group G_Q to such a quiver, given by generators and relations. A special case is Figure 12. The corresponding presentation is the known presentation of $B(d, d, n)$ from [BMR98, Thm. 2.27]. The associated quiver in this case is an orientation of the diagram associated to the presentation in [BMR98, Table 5] (drawn on the right in Figure 12); this formed part of the motivation for the approach we employ here. In Definition 3.9, we introduce a mutation rule for such a quiver which is compatible with flipping triangulations. We complete this section by showing that the group G_Q is invariant under mutation. Since the mutation graph of tagged triangulations of the disk is connected [FST08, Prop. 7.10], it follows that all the quivers constructed as above give presentations of the group $B(d, d, n)$. Adding the relations that the square of each generator is the identity gives a presentation of $G(d, d, n)$ by applying a result from [Ari95]; see Theorem 5.1.
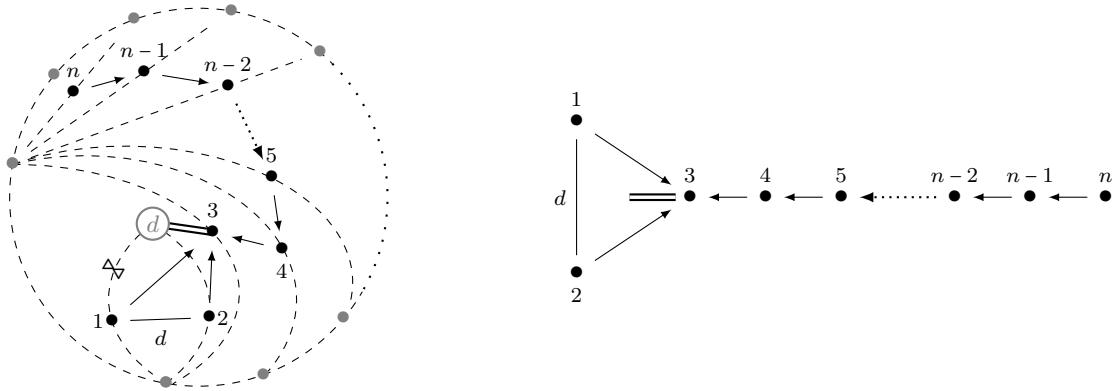

FIGURE 12. Original presentation from [BMR98] (shown on the right) embedded in a disk with n marked points on the boundary and a cone point of degree d in its interior.

3.1. **Quivers from triangulated surfaces and groups from quivers.** We now fix the surface (X, M) we will be working with: X is the disk S with an interior marked point interpreted a cone point of degree $d \geq 2$ denoted by ⓓ in figures, and M is a set of $n \geq 2$ marked points on its boundary.

Next, we give a way of associating a decorated quiver Q_T to each tagged triangulation of (X, M). We then give a way of associating a group G_{Q_T} to any such quiver.

Let T be a tagged triangulation of (X, M), regarding C as a marked point, as in [FST08, §7]. Note that C is the unique marked point in the interior of S.

Remark 3.1. By [FST08, Rk. 4.2], the tagged triangulation T can be built up by gluing puzzle pieces of the kind shown in Figure 13 by matching their boundary arcs (respecting the orientation).

Let B_T be the skew-symmetric matrix associated to T in [FST08, Rk. 4.2], and let $\widetilde{Q}_T$ be the corresponding quiver.

Definition 3.2. We associate a quiver Q_T to each tagged triangulation T as considered above by modifying $\widetilde{Q}_T$ in the following way.

If the triangulation T has precisely two arcs incident with the cone point then, from each of the two vertices in $\widetilde{Q}_T$ that have arrows to or from the corresponding vertices in $\widetilde{Q}$, we draw a double edge pointing towards the conepoint. Thus, we draw such a double edge from each vertex labelling an arc in the triangulation T bounding a region in the complement that has the cone point on its boundary. We also add an unoriented edge, labelled d, between these two vertices.

If the triangulation has at least 3 arcs adjacent to the cone point, there will be an oriented cycle between the corresponding vertices in $\widetilde{Q}_T$. We put the label ⓓ in the middle of such a cycle. Note that there is at most one such labelled cycle.

See Figure 12 for a complete example of Definition 3.2 and Figures 17, 18, 19, 20 and 21 for local portions of a triangulation and the associated quiver.
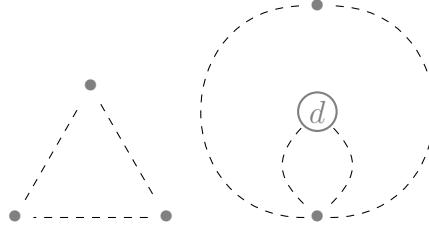
FIGURE 13. Puzzle pieces for a disk with a single interior marked point.

Remark 3.3. It follows from Remark 3.1 that the quiver Q_T is built up from individual portions associated to triangles (as on the left of Figure 13) and a portion associated to the union of the puzzle pieces incident with the cone-point as shown in Figure 14 (or with versions of the three right hand figures where the tags on all the arcs incident with the cone point have been flipped).
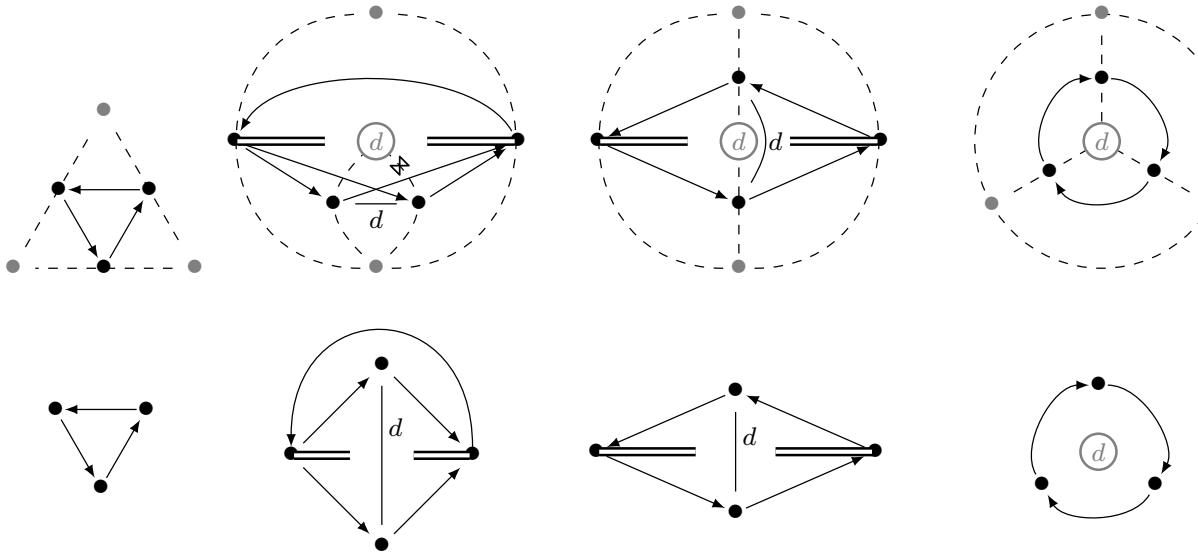
FIGURE 14. Building up the associated quiver for a disk with cone point. We show the corresponding quivers separately. Note that the cone point itself does not appear in the quiver in the second and third cases.

Definition 3.4. Let Q be a quiver as in Definition 3.2 and G_Q be the group with generators $S_Q = \{s_i\}_{i \in Q_0}$ subject to the following relations:

(1) $s_i s_j = s_j s_i$ if i and j are vertices with no arrows between them,

(2) $s_i s_j s_i = s_j s_i s_j$ if there is an arrow between i and j (in either direction).

(3) $s_i s_j s_k s_i = s_j s_k s_i s_j = s_k s_i s_j s_k$ if Q contains an oriented 3-cycle with no label $\textcircled{d}$ in the middle of the form
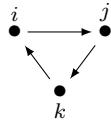
(4) $\underbrace{s_i s_j \cdots}_{d \text{ terms}} = \underbrace{s_j s_i \cdots}_{d \text{ terms}}$ if there is an (unoriented) edge labelled d between i and j,

(5) $\underbrace{s_1 s_2 \cdots s_r s_1 \ldots}_{d(r-1) \text{ terms}} = \underbrace{s_2 \cdots s_r s_1 \ldots}_{d(r-1) \text{ terms}} = \cdots = \underbrace{s_r s_1 \cdots s_r s_1 \ldots}_{d(r-1) \text{ terms}}$ if Q contains an oriented labelled chordless r-cycle, for $r \geq 3$, of the form
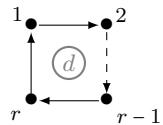
(6) $s_k s_i s_j s_k s_i s_j = s_i s_j s_k s_i s_j s_k$ if the vertices i, j, k appear in either of the following configurations in Q
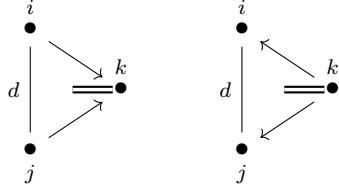
Note that s_i appears before s_j in the relation because i appears to the right of the double edge.

(7) $s_i s_j s_k s_l s_i s_j = s_l s_i s_j s_k s_l s_i$ and $s_j s_k s_l s_i s_j s_k = s_k s_l s_i s_j s_k s_l$ if the vertices i, j, k, l appear as follows in Q
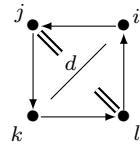
Let H_Q be the group defined as above, omitting the length d relations (4) and the cycle relations (5).

Remark 3.5. Note that when $n = 2$, the only possible tagged triangulations are the middle two figures in Figure 14, with the outer edges part of the boundary, or the same figures with tags flipped. In each case, the corresponding quiver is a pair of vertices connected by an unoriented edge labelled d, and the corresponding group has two generators, s_1 and s_2, with a relation of type (4) from Definition 3.4, i.e. it is the Artin braid group of type $I_2(d)$.

The construction of Grant and Marsh [GM17] becomes a special case of the above construction as follows. Our aim is to generalise their construction for larger values of d.

**Remark 3.6.** Note that when $d = 2$, the relations from Definition 3.4 simplify. In fact, arrows labelled 2 give commutation relations, so they can be omitted following rule (1). Moreover, we can see that double edges can be omitted as well. In fact, in the situation of relation (6), we have $s_i s_j = s_j s_i$, $s_i s_k s_i = s_k s_i s_k$ and $s_j s_k s_j = s_k s_j s_k$ and so

$$s_k s_i s_j s_k s_i s_j = s_k s_j s_i s_k s_i s_j = s_k s_j s_k s_i s_k s_j = s_j s_k s_j s_i s_k s_j = s_j s_k s_i s_j s_k s_j$$
$$= s_j s_k s_i s_k s_j s_k = s_j s_i s_k s_i s_j s_k = s_i s_j s_k s_i s_j s_k,$$

that is relation (6) becomes a consequence of the other relations. Note also that cycle relations as in (5) and (7) reduce to cycle relations as in [GM17, Defn. 2.2]. Hence in this case the defining relations coincide with those in [GM17, Defn. 2.2] and G_Q is isomorphic to $\mathcal{A}(D_n)$, that is the Artin braid group of type D_n, by [GM17, Remark 2.3 and Theorem 2.12]. Recall also that $\mathcal{A}(D_n) \cong B(2, 2, n)$; see for example [BMR98, pp 188].

Remark 3.7. Note that there are only two relations in (7) corresponding to the 4-cycle $i \to j \to k \to l \to i$, in contrast to the four relations appearing in [GM17, Defn. 2.2]:

$$s_i s_j s_k s_l s_i s_j = s_j s_k s_l s_i s_j s_k = s_k s_l s_i s_j s_k s_l = s_l s_i s_j s_k s_l s_i.$$

When $d = 2$, these relations are equivalent to those in (7) (see Remark 3.6), but when $d > 2$, we can see that the relations in [GM17, Defn. 2.2] would imply the unexpected

relation $s_i s_k = s_k s_i$. In fact, using one of the two equalities from (7), together with the other relations, we have

$$s_i s_j s_k s_l s_i s_j = s_l s_i s_j s_k s_l s_i \iff s_j s_k s_l s_i s_j = s_i^{-1} s_l s_i s_j s_k s_l s_i \iff s_j s_k s_l s_i s_j = s_l s_i s_i s_l^{-1} s_j s_k s_l s_i$$

$$\iff s_j s_k s_l s_i s_j = s_l s_i s_j s_l^{-1} s_k s_l s_i \iff s_j s_k s_l s_i s_j = s_l s_i s_j s_k s_l s_k^{-1} s_i$$

$$\iff s_j s_k s_l s_i s_j \underbrace{s_k s_i \cdots}_{d-2 \text{ terms}} = s_l s_i s_j s_k s_l s_k^{-1} s_i \underbrace{s_k s_i \cdots}_{d-2 \text{ terms}}$$

$$\iff s_j s_k s_l s_i s_j \underbrace{s_k s_i \cdots}_{d-2 \text{ terms}} = s_l s_i s_j s_k s_l s_i \underbrace{s_k s_i \cdots s_x^{-1}}_{d-1 \text{ terms}}$$

$$\iff s_j s_k s_l s_i s_j s_k \underbrace{s_i s_k \cdots s_x}_{d-2 \text{ terms}} = s_l s_i s_j s_k s_l s_i \underbrace{s_k s_i \cdots}_{d-2 \text{ terms}} \ ,$$

where $x = i$ or k depending on whether d is odd or even respectively. If the four relations from [GM17, Defn. 2.2] were true for $d > 2$, this would imply that $\underbrace{s_i s_k \cdots}_{d-2 \text{ terms}} = \underbrace{s_k s_i \cdots}_{d-2 \text{ terms}}$ and so by (4), that $s_k s_i = s_i s_k$. However, this statement is false for $d > 2$ (see Remark 4.7).

We can now use our construction to embed the presentation of the group $B(d, d, n)$ given by Broué, Malle and Rouquier [BMR98] $B(d, d, n)$ into a triangulation of the surface S as follows.

Remark 3.8. Consider the triangulated surface in Figure 12, where we have drawn the corresponding quiver following the rules in Definition 3.2. The associated group G_Q with generators and relations as in Definition 3.4 is exactly the presentation of the group $B(d, d, n)$ given by Broué, Malle and Rouquier in [BMR98, Table 5].

3.2. **Mutation of quivers and triangulations.** By [FST08, §7], given a tagged triangulation T of the disk S, and a choice of tagged or untagged arc, there is a unique tagged triangulation which coincides with T except for this arc, i.e. the flip of T at the given arc. We give a way of mutating the quivers constructed as in the previous section that agrees with flipping the triangulation. This will coincide with Fomin-Zelevinsky mutation, see [FZ03, Lemma 8.5], for the portion of the quiver "far from the cone point", but we need different rules for the double edges, labelled (unoriented) edges and labelled cycles.

Note that by the construction of Q, all 3-cycles in Q where arrows have no labels are oriented cyclically by [FZ03, Lemma 7.5]; see also [BM15, pp. 1948]. On the other hand, we sometimes have an unoriented edge labelled d creating "unoriented" cycles, see for example the situation of (7) in Definition 3.4.

Definition 3.9. Let Q be a quiver as in Definition 3.2 and k be a vertex in Q. We define *the mutation of Q at k*, denoted by $\mu_k(Q)$, as the following quiver on the same vertex set. See Figure 15 for a pictorial representation of the following rules.

   (1) Reverse the orientations of all (oriented) arrows in Q incident with k.
   (2) For any path of the form $j \to k \to i$ in Q:
      - if there is no arrow between i and j in Q, then there is an arrow $j \to i$ in $\mu_k(Q)$,
      - if there is an arrow $i \to j$ and j, k, i do not form a 3-cycle labelled $\textcircled{d}$ in Q, then there is no arrow between i and j in $\mu_k(Q)$,

- if j, k, i form a 3-cycle labelled $\textcircled{d}$ in Q, then there is an edge labelled d between i and j in $\mu_k(Q)$, the 3-cycle looses the label and the neighbours of i, j in $\mu_k(Q)$ acquire a double edge towards the labelled edge in $\mu_k(Q)$,
- if there is an edge labelled d between j and i in Q, then there is an arrow $j \to i$ in $\mu_k(Q)$, the 3-cycle j, i, k acquires the label $\textcircled{d}$ and all double edges are removed in $\mu_k(Q)$,

(3) If $j \to k \to i$ is part of an r-cycle labelled $\textcircled{d}$ in Q for $r \geq 4$, then the rules above apply and the label is kept in the $(r-1)$-cycle including i, j but not k in $\mu_k(Q)$.

(4) If there are arrows $j \to k \to i$ and i, j, but not k, are part of an r-cycle labelled $\textcircled{d}$ in Q for $r \geq 3$, then the rules above apply and the label is kept in the $(r+1)$-cycle which includes $i \to k \to j$ in $\mu_k(Q)$.

(5) If in Q there is one of the following configurations:

$$
\begin{array}{ccc}
\overset{j}{\circ}\ \underline{\ \ d\ \ }\ \overset{i}{\circ} & \quad\text{or}\quad & \overset{j}{\circ}\ \underline{\ \ d\ \ }\ \overset{i}{\circ}\\[2pt]
\searrow\ \|\ \swarrow & & \searrow\ \|\ \nearrow\\[2pt]
\underset{k}{\bullet} & & \underset{k}{\bullet}
\end{array}
$$
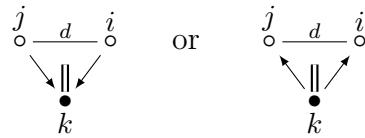
then follow the above rules (keeping also the double edge at k) and
- if there is a vertex l different from k with a double edge in Q, remove the double edge at l in $\mu_k(Q)$,
- if there is a vertex n different from l and k such that in Q

$$
\begin{array}{ccc}
\overset{j}{\circ}\ \underline{\ \ d\ \ }\ \overset{i}{\circ} & \quad\text{or}\quad & \overset{j}{\circ}\ \underline{\ \ d\ \ }\ \overset{i}{\circ}\\[2pt]
\searrow\ \|\ \swarrow & & \searrow\ \|\ \nearrow\\[2pt]
k\ \bullet & & k\ \bullet\\[2pt]
\downarrow & & \uparrow\\[2pt]
\underset{n}{\bullet} & & \underset{n}{\bullet}
\end{array}
$$
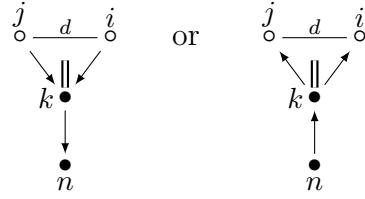
then add a double edge at n.

Remark 3.10. Note that when $d = 2$ the mutation rules simplify. In fact, as explained in Remark 3.6, arrows labelled 2, double edges and labels on cycles can be omitted and we simply recover Fomin-Zelevinsky mutation.

Lemma 3.11. Let T and T' be tagged triangulations of S such that T' is obtained from T by flipping the arc α as in [FST08, §7]. Let Q_T and $Q_{T'}$ be the corresponding quivers as in Definition 3.2. Then

(a) The flip of α is given locally by one of the mutations in Figures 17, 18, 19, 20 or 21 (from left to right or right to left), or by one of the mutations from Figure 18, 20 or 21 with all of the tags flipped.

(b) The quiver $Q_{T'}$ can be obtained from the quiver Q_T by applying the mutation rule in Definition 3.9.

Proof. Part (a) follows from Remarks 3.1 and 3.3 on consideration of which vertex is the conepoint when gluing together puzzle pieces. Part (b) follows from part (a) by computing the quiver in each case before and after mutation (see Figure 14). □
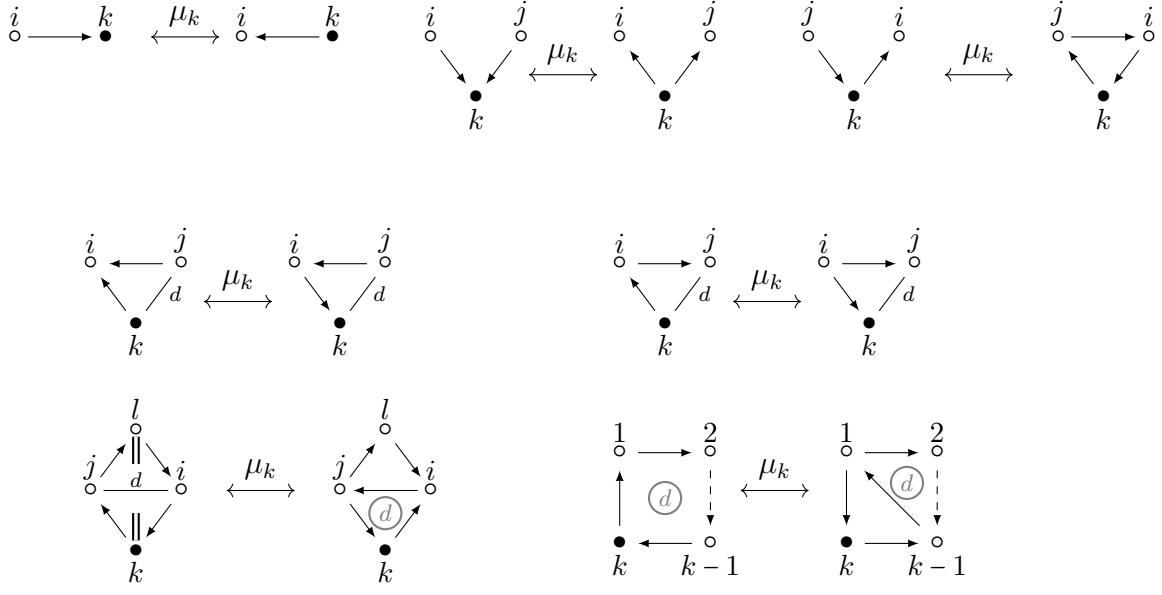
FIGURE 15. Local mutations following the rules from Definition 3.9. In the third line, in the first mutation vertex l might not exist (if it corresponds to a boundary segment of the disc); in the second mutation we are assuming k is at least 4.

3.3. **Mutation of groups.** In this section we will show that the group associated in Definition 3.4 to a quiver as in Definition 3.2 is invariant under the mutation introduced in Definition 3.9:

Theorem 3.12. *Let Q be a quiver as in Definition 3.2, G_Q its associated group as in Definition 3.4 with generators s_i, and k a vertex of Q. Let $\mu_k(Q)$ be the mutation of Q at k as in Definition 3.9, and let t_i be the generators of $G_{\mu_k(Q)}$. Then there is a group isomorphism $\varphi_k^Q : G_Q \cong G_{\mu_k(Q)}$ given by $\varphi_k^Q(s_i) = t_k t_i t_k^{-1}$ if $i \to k$ in Q or the vertices i and k correspond to the only two arcs incident with the conepoint in T and the arc corresponding to k is rotated anti-clockwise to the flipped arc; and $\varphi(s_i) = t_i$ otherwise.*

Note that, in the above theorem, the situation when there are only two arcs incident with the conepoint in T means that there is an unoriented edge between the corresponding vertices, labelled d.

We have already seen that the triangulation from Figure 12 satisfies $G_Q \cong B(d, d, n)$, so this will allow us to conclude that every tagged triangulation gives a presentation of the group $B(d, d, n)$.

Proposition 3.13. [GM17, Prop. 2.9] *Let Q be one of the quivers on the left or right of (a)–(f) in Figure 16. Let k be a vertex of Q. Let $Q' = \mu_k(Q)$ be the quiver obtained from Q by mutating at k. Suppose that the t_i are elements of a group satisfying the defining relations (1), (2) and (3) from Definition 3.4 for the quiver Q'. For $i \in Q_0$, let*

$$S_i = \begin{cases} t_k t_i t_k^{-1}, & i \to k \text{ in } Q; \\ t_i, & else. \end{cases}$$

Then, for each of the cases in Figure 16, the elements S_i satisfy the defining relations (1), (2) and (3) of Definition 3.4 for the quiver Q.
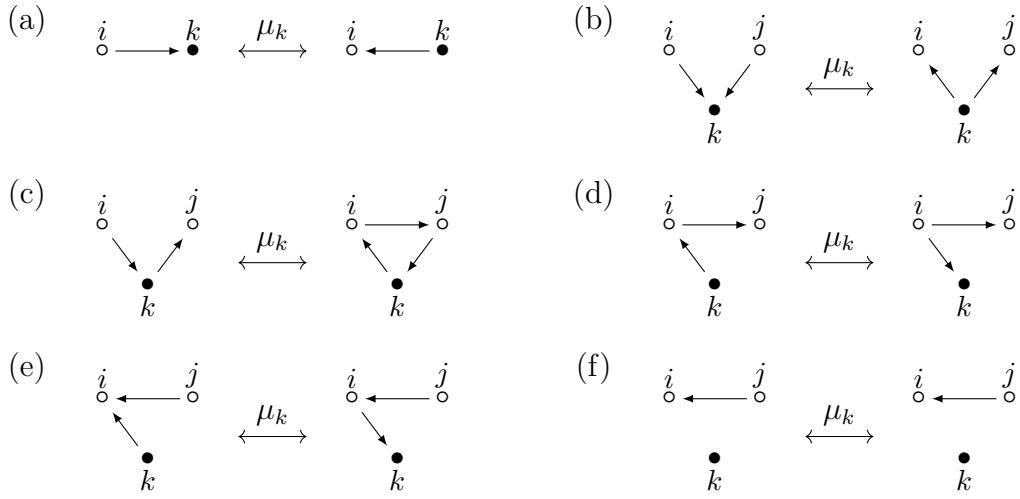
FIGURE 16. Cases for Proposition 3.13.
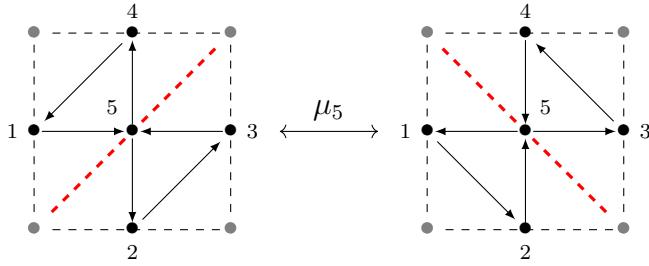
FIGURE 17. A mutation far from the conepoint.

Lemma 3.14. Let Q be the quiver on the left of Figure 17 and Q' the quiver on the right. Let G_Q (respectively $G_{Q'}$) be the group with generators s_i (respectively t_i) with $1 \le i \le 5$ satisfying the relations associated with Q (respectively Q'). Then, there are group homomorphisms:

- $\varphi_5^Q : G_Q \to G_{Q'}$ with $\varphi_5^Q(s_1) = S_1 = t_5 t_1 t_5^{-1}$, $\varphi_5^Q(s_3) = S_3 = t_5 t_3 t_5^{-1}$, $\varphi_5^Q(s_i) = S_i = t_i$ for $i = 2, 4, 5$;

- $\varphi_5^{Q'} : G_{Q'} \to G_Q$ with $\varphi_5(t_4) = T_4 = s_5 s_4 s_5^{-1}$, $\varphi_5(t_2) = T_2 = s_5 s_2 s_5^{-1}$, $\varphi_5(t_i) = T_i = s_i$ for $i = 1, 3, 5$.

Proof. For the first statement, it is enough to check that the elements S_i satisfy the defining relations of G_Q. This follows directly from Proposition 3.13. The proof of the second statement is similar. $\square$

Setup 3.15. Let Q be the quiver on the left of Figure 18(a) (respectively, the quiver on the left of Figure 18(b)) for $n \ge 3$ (respectively, for $n = 2$) and let Q' be the quiver on the right in each case. Let H_Q be the group defined in Definition 3.4, with generators s_i, and let $H_{Q'}$ be the group defined in Definition 3.4, with generators t_i. Let $S_1 = t_0 t_1 t_0^{-1}$, $S_c = t_0 t_c t_0^{-1}$, and $S_i = t_i$ for $i \ne n, c$. We regard the subscripts of the s_i and S_i for $1 \le i \le n$ to be taken modulo n (with representatives $\{1, 2 \dots, n\}$), and the subscripts of the t_i for $0 \le i \le n$ to be taken modulo $n + 1$ (with representatives $\{0, 1, 2, \dots, n\}$).

(a) Case $n \geq 3$.

(b) Case $n = 2$.
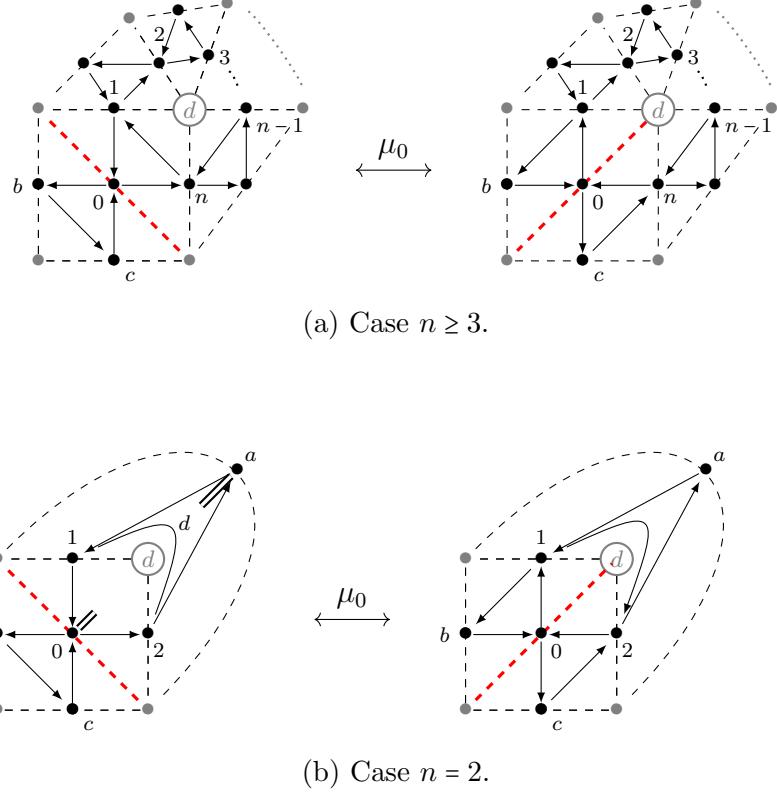
FIGURE 18. A mutation involving a cycle around the conepoint. In case (b), one of the vertices is labelled a to avoid confusion in arguments involving indices taken modulo n.

Lemma 3.16. Let $n \geq 2$ be an integer. In Setup 3.15, the elements S_i satisfy the defining relations of H_Q.

Proof. This follows from Proposition 3.13. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.17. Let $n \geq 2$ and $d \geq 1$ be integers. Suppose we are in Setup 3.15, and that $0 \leq i, r \leq n$ and $r \not\equiv i - 1, i \mod n + 1$. Then we have the following:

(a)
$$t_{r-n+1}t_{r-n+2}\cdots t_r t_i^{-1} = t_{i+1}^{-1}t_{r-n+1}t_{r-n+2}\cdots t_r.$$

(b)
$$t_{r-dn+1}\cdots t_{r-1}t_r t_i^{-1} = t_{d+i}^{-1}t_{r-dn+1}\cdots t_{r-1}t_r.$$

Proof. For (a), we have
$$\begin{aligned}
t_{r-n+1}t_{r-n+2}\cdots t_r t_i^{-1} &= t_{r-n+1}\cdots t_i t_{i+1}\cdots t_r t_i^{-1}\\
&= t_{r-n+1}\cdots t_i t_{i+1} t_i^{-1} t_{i+2}\cdots t_r\\
&= t_{r-n+1}\cdots t_{i+1}^{-1} t_i t_{i+1} t_{i+2}\cdots t_r\\
&= t_{i+1}^{-1}t_{r-n+1}t_{r-n+2}\cdots t_r.
\end{aligned}$$

For (b), we argue by induction on d. If $d = 1$, the result follows from part (a). Suppose the result holds for d. Then

$$t_{r-(d+1)n+1}\cdots t_{r-1}t_r t_i^{-1} = t_{r-(d+1)n+1}\cdots t_{r-dn}t_{r-dn+1}\cdots t_r t_i^{-1}$$

$$= t_{r-(d+1)n+1}\cdots t_{r-dn}t_{i+d}^{-1}t_{r-dn+1}\cdots t_r$$

$$= t_{i+d+1}^{-1}t_{r-(d+1)n+1}\cdots t_{r-1}t_r,$$

using the induction hypothesis and then noting that $r - dn \not\equiv i + d, i + d - 1 \mod n + 1$, since $r - dn \equiv r + d \mod n + 1$. $\qquad\square$

Lemma 3.18. Let $n \geq 2$ and $d \geq 1$ be integers. Then, in Setup 3.15, for any $1 \leq r \leq n$, we have:

$$t_d t_{d-1}\cdots t_1 \underbrace{S_{r-d(n-1)+1}\cdots S_{r-1}S_r}_{d(n-1) \text{ terms}} = \underbrace{t_{r-dn+1}\cdots t_{r-1}t_r}_{dn \text{ terms}}.$$

Proof. We prove the result by induction on d. For $d = 1$, we have (for the case $r = n$),

$$t_1 S_2 \cdots S_n = t_1 t_2 \cdots t_n,$$

as required. For $r \neq n$, we have

$$t_1 S_{r-(n-1)+1}\cdots S_r = t_1 S_{r+2}\cdots S_n S_1 \cdots S_r$$

$$= t_1 t_{r+2}\cdots t_n t_0 t_1 t_0^{-1}t_2 \cdots t_r$$

$$= t_1 t_{r+2}\cdots t_n t_1^{-1}t_0 t_1 t_2 \cdots t_r$$

$$= t_1 t_1^{-1}t_{r+2}\cdots t_n t_0 t_1 \cdots t_r$$

$$= t_{r+2}\cdots t_n t_0 t_1 \cdots t_r$$

$$= t_{r-(n+1)+2}\cdots t_{r-1}t_r$$

$$= t_{r-n+1}\cdots t_{r-1}t_r,$$

as required, giving the result for $d = 1$. Assume the result holds for an integer $d \geq 1$. Then, using the induction hypothesis, we have (for $r = n$):

$$t_{d+1}\cdots t_1 \underbrace{S_{n-(d+1)(n-1)+1}\cdots S_n}_{(d+1)(n-1)\text{terms}} = t_{d+1}(t_d\cdots t_1)(S_{n-(d+1)(n-1)+1}\cdots S_n S_1)S_2 \cdots S_n$$

$$= t_{d+1}(t_d\cdots t_1)(S_{1-d(n-1)+1}\cdots S_n S_1)S_2 \cdots S_n$$

$$= t_{d+1}(t_{1-dn+1}\cdots t_1)t_2 \cdots t_n,$$

as required, noting that $1 - dn + 1 \equiv d + 2 \mod n + 1$. For $r \neq n$, we have,

$$t_{d+1}\cdots t_1 \underbrace{S_{r-(d+1)(n-1)+1}\cdots S_r}_{(d+1)(n-1)\text{terms}} = t_{d+1}(t_d\cdots t_1) \underbrace{(S_{r-(d+1)(n-1)+1}\cdots S_{r-(n-1)}}_{d(n-1)\text{terms}} \underbrace{S_{r-(n-2)}\cdots S_r}_{n-1\text{ terms}}$$

$$= t_{d+1}(t_d\cdots t_1) \underbrace{(S_{r-(d+1)(n-1)+1}\cdots S_{r-(n-1)}}_{d(n-1)\text{terms}} \underbrace{S_{r+2}\cdots S_n S_1\cdots S_r}_{n-1\text{ terms}}$$

$$= t_{d+1}(t_d\cdots t_1) \underbrace{(S_{r+1-d(n-1)+1}\cdots S_{r+1})}_{d(n-1)\text{terms}} \underbrace{S_{r+2}\cdots S_n S_1\cdots S_r}_{n-1\text{ terms}}$$

$$= t_{d+1} \underbrace{t_{r+1-dn+1}\cdots t_{r+1}}_{dn\text{ terms}} t_{r+2}\cdots t_n t_0 t_1 t_0^{-1} t_2 \cdots t_r$$

$$= t_{d+1} \underbrace{t_{r+1-dn+1}\cdots t_{r+1}}_{dn\text{ terms}} t_{r+2}\cdots t_n t_1^{-1} t_0 t_1 t_2 \cdots t_r$$

$$= t_{d+1} \underbrace{t_{r+1-dn+1}\cdots t_{r+1}}_{dn\text{ terms}} t_1^{-1} t_{r+2}\cdots t_n t_0 t_1 t_2 \cdots t_r,$$

using the induction hypothesis. Note that the subscripts of the Ts are reduced mod n first before being applied to the Ts (and then reduced mod $n + 1$!). Since $1 \le r \le n-1$, $r+1 \not\equiv 0, 1 \mod n+1$, so by Lemma 3.17,

$$\underbrace{t_{r+1-dn+1}\cdots t_{r+1}}_{dn\text{ terms}} t_1^{-1} = t_{d+1}^{-1} \underbrace{t_{r+1-dn+1}\cdots t_{r+1}}_{dn\text{ terms}}$$

It follows that

$$t_{d+1}\cdots t_1 \underbrace{S_{r-(d+1)(n+1)+1}\cdots S_r}_{(d+1)(n-1)\text{terms}} = t_{d+1}t_{d+1}^{-1} \underbrace{t_{r+1-dn+1}\cdots t_{r+1}}_{dn\text{ terms}} t_{r+2}\cdots t_n t_0 t_1 t_2 \cdots t_r,$$

$$= \underbrace{t_{r+1-dn+1}\cdots t_{r+1}}_{dn\text{ terms}} t_{r+2}\cdots t_n t_0 t_1 t_2 \cdots t_r,$$

with $(d+1)n$ terms in total, as required, noting that $r+1-dn+1 \equiv r-(d+1)n+1 \mod n+1$. $\quad\square$

We have the following generalization of [GM17, Lem. 2.4].

Lemma 3.19. Let d be a positive integer and $n \ge 3$ be an integer. Let $g_0, g_1, \ldots, g_{n-1}$ be elements of a group G, with subscripts taken modulo n, satisfying the relations:

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad 0 \le i \le n-1$$
$$g_i g_j = g_j g_i, 0 \le i, j \le n-1, j \neq i, i+1 \mod n.$$

Then

(a) $g_{i+1}^{-1} g_i \cdots g_{i+n-2} = g_i \cdots g_{i+n-2} g_i^{-1}$, for $1 \le i \le n$.
(b) $g_1^{-1}(g_0 g_1 \cdots g_{(n-1)d-1}) = g_0 g_1 \cdots g_{(n-1)d-1} g_{(n-1)d+1}^{-1}$
(c) If the relation

$$g_r g_{r+1} \cdots g_{(n-1)d+r-1} = g_{r+1} g_{r+2} \cdots g_{((n-1)d+r)} \tag{3}$$

holds for some $r \in \{0, 1, \ldots, n-1\}$, then it holds for all r.

Proof. For (a), we have:

$$g_{i+1}^{-1}g_ig_{i+1}g_{i+2}\cdots g_{i+n-2} = g_ig_{i+1}g_i^{-1}g_{i+2}\cdots g_{i+n-2}$$
$$= g_ig_{i+1}\cdots g_{i+n-2}g_i^{-1},$$

as required, using commutations in the second step. For part (b), we have:

$$g_1^{-1}(g_0g_1\cdots g_{(n-1)d-1}) = g_1^{-1}(g_0\cdots g_{n-2})(g_{n-1}\cdots g_{2n-3})\cdots(g_{(d-1)(n-1)}\cdots g_{(n-1)d-1})$$
$$= (g_0\cdots g_{n-2})(g_{n-1}\cdots g_{2n-3})\cdots(g_{(d-1)(n-1)\cdots g_{(n-1)d-1}}))g_{1-d},$$

using part (a) d times. Finally, note that $1 - d \equiv (n-1)d + 1 \mod n$. For part (c), assume first that equation (3) holds for $r = 0$, so that

$$g_0g_1\cdots g_{(n-1)d+r-1} = g_1g_2\cdots g_{((n-1)d}.$$

Multiplying this on the left by g_1^{-1} and on the right by $g_{(n-1)d+1}$ gives

$$g_0g_1\cdots g_{(n-1)d-1} = g_2g_3\cdots g_{(n-1)d+1},$$

by part (b). Repeated application of this argument gives the desired result. □

Lemma 3.20. Let $n \geq 3$ be an integer. Let Q be the quiver on the left of Figure 18(a) and Q' the quiver on the right. Let G_Q (respectively $G_{Q'}$) be the group with generators s_i (respectively t_i), with i ranging through the vertices of the quivers, satisfying the relations associated with Q (respectively Q'). Then, there is a group homomorphism

$$\varphi_0^Q : G_Q \to G_{Q'} \text{ given by } \varphi_0^Q(s_1) = S_1 = t_0t_1t_0^{-1}, \ \varphi_0^Q(s_c) = S_c = t_0t_ct_0^{-1}, \ \varphi_0^Q(s_i) = S_i = t_i \text{ for } i \neq 1, c.$$

Proof. The defining relations for G_Q, apart from those corresponding to the n-cycle $1 \to 2 \to \cdots \to n \to 1$, hold by Lemma 3.16.

The defining relations for $G_{Q'}$ corresponding to the $n + 1$-cycle $0 \to 1 \to \cdots \to n + 1 \to 0$ can be written in the form:

$$\underbrace{t_{r-nd+1}\cdots t_{r-1}t_r}_{dn \text{ terms}} = \underbrace{t_{r+1-nd+1}\cdots t_rt_{r+1}}_{dn \text{ terms}},$$

for $0 \leq r \leq n$.

The defining relations for G_Q corresponding to the n-cycle $1 \to 2 \to \cdots \to n \to 1$ can be written in the form:

$$\underbrace{(S_{r-d(n-1)+1}\cdots S_{r-1}S_r)}_{d(n-1) \text{ terms}} = \underbrace{(S_{r+1-d(n-1)+1}\cdots S_rS_{r+1})}_{d(n-1) \text{ terms}},$$

for $1 \leq r \leq n$.

The cycle relations for G_Q for $1 \leq r \leq n - 1$ now follow from Lemma 3.18. The cycle relation for $r = n$ follows from Lemma 3.19(c), taking $g_0 = S_1$, $g_2 = S_2, \ldots$, $g_{n-1} = S_n$. This gives the required result. □

Lemma 3.21. Let $n = 2$. Let Q be the quiver on the left of Figure 18(b) and Q' the quiver on the right. Let G_Q (respectively $G_{Q'}$) be the group with generators s_i (respectively t_i), with i ranging through the vertices of the quivers, satisfying the relations associated with Q (respectively Q'). Then, there is a group homomorphism

$$\varphi_0^Q : G_Q \to G_{Q'} \text{ given by } \varphi_0^Q(s_1) = S_1 = t_0t_1t_0^{-1}, \ \varphi_0^Q(s_c) = S_c = t_0t_ct_0^{-1}, \ \varphi_0^Q(s_i) = S_i = t_i \text{ for } i \neq 1, c.$$

Proof. The defining relations for G_Q, apart from those corresponding to the 3-cycle $0 \to 2 \to 1 \to 0$, the 4-cycle $0 \to 2 \to a \to 1 \to 0$, the double edges and the unoriented edge labelled d in Q, all hold by Lemma 3.16.

We have $t_1 t_2 t_1 = t_2 t_1 t_2$, which gives:

$$S_0^{-1} S_1 S_0 S_2 S_0^{-1} S_1 S_0 = S_2 S_0^{-1} S_1 S_0 S_2.$$

Multiplying on the left by S_0 gives

$$S_1 S_0 S_2 \underline{S_0^{-1} S_1 S_0} = \underline{S_0 S_2 S_0^{-1}} S_1 S_0 S_2.$$

Applying the braid relations corresponding to the arrows $1 \to 0$ and $0 \to 2$ in Q we obtain:

$$S_1 S_0 S_2 S_1 S_0 S_1^{-1} = S_2^{-1} S_0 S_2 S_1 S_0 S_2,$$

and hence, multiplying on the left by S_2 and on the right by S_1:

$$S_2 S_1 S_0 S_2 S_1 S_0 = S_0 S_2 S_1 S_0 S_2 S_1,$$

which is the relation associated to the double edge incident with 0 in Q.

The relation associated to the double edge incident with the vertex a in Q is:

$$S_a S_1 S_2 S_a S_1 S_2 = S_1 S_2 S_a S_1 S_2 S_a. \tag{4}$$

We apply transformations to (4) to give a series of equivalent versions. Firstly, (4) is equivalent to

$$\underline{t_a t_0 t_1 t_0^{-1}} t_2 t_a \underline{t_0 t_1 t_0^{-1}} t_2 = t_0 t_1 t_0^{-1} t_2 t_a \underline{t_0 t_1 t_0^{-1}} t_2 t_a$$

Applying the commutation $t_a t_0 = t_0 t_a$ and the braid relation corresponding to the arrow $0 \to 1$ in Q' on both sides gives

$$t_0 t_a t_1 t_0^{-1} t_2 t_a t_1^{-1} t_0 t_1 t_2 = t_0 t_1 t_0^{-1} t_2 t_a t_1^{-1} t_0 t_1 t_2 t_a.$$

We multiply on the left by $t_1^{-1} t_0$ to obtain:

$$\underline{t_1^{-1} t_a t_1} t_0^{-1} t_2 t_a t_1^{-1} t_0 t_1 t_2 = t_0^{-1} t_2 t_a t_1^{-1} t_0 \underline{t_1 t_2 t_a}.$$

Since $t_2 t_a t_1 t_2 = t_a t_1 t_2 t_a$, we may substitute in

$$t_1 t_2 t_a = t_a^{-1} t_2 t_a t_1 t_2$$

on the right-hand side, while on the left-hand side we apply the braid relation for the arrow $3 \to 1$ in Q' to get:

$$t_a t_1 t_a^{-1} t_0^{-1} t_2 t_a t_1^{-1} t_0 t_1 t_2 = t_0^{-1} t_2 t_a t_1^{-1} t_0 t_a^{-1} t_2 t_a t_1 t_2.$$

Multiplying on the right by $t_2^{-1} t_1^{-1}$ and on the left by t_0 gives:

$$\underline{t_0 t_a} t_1 \underline{t_a^{-1} t_0^{-1}} t_2 t_a t_1^{-1} t_0 = t_2 t_a t_1^{-1} t_0 t_a^{-1} t_2 t_a.$$

Applying the commutation $t_0 t_a = t_a t_0$ twice on the left hand side, and multiplying on the right by t_2 gives:

$$t_a t_0 t_1 t_0^{-1} t_a^{-1} t_2 t_a t_1^{-1} t_0 = t_2 t_a t_1^{-1} t_0 \underline{t_a^{-1} t_2 t_a}.$$

We apply the braid relation corresponding to the arrow $2 \to 3$ in Q' to obtain:

$$t_a t_0 t_1 t_0^{-1} t_a^{-1} t_2 t_a t_1^{-1} t_0 = t_2 t_a t_1^{-1} t_0 t_2 t_a t_2^{-1},$$

and, multiplying on the right by t_2 we obtain:

$$t_a \underline{t_0 t_1 t_0^{-1}} t_a^{-1} t_2 t_a t_1^{-1} t_0 t_2 = t_2 t_a t_1^{-1} t_0 t_2 t_a.$$

Applying the braid relations corresponding to the arrows $0 \to 1$ and $2 \to a$ in Q' gives:

$$t_a t_1^{-1} t_0 \underline{t_1 t_2 t_a t_2^{-1} t_1^{-1}} t_0 t_2 = t_2 t_a t_1^{-1} t_0 t_2 t_a.$$

Since $t_2 t_a t_1 t_2 = t_a t_1 t_2 t_a$, we may substitute in

$$t_1 t_2 t_a t_2^{-1} t_1^{-1} = t_a^{-1} t_2 t_a$$

on the left-hand side to obtain:

$$t_a t_1^{-1} t_0 t_a^{-1} t_2 t_a t_0 t_2 = t_2 t_a t_1^{-1} t_0 t_2 t_a.$$

Applying the commutation $t_0 t_a = t_a t_0$ on the left hand side twice, we obtain:

$$\underline{t_a t_1^{-1} t_a^{-1}} t_0 t_2 t_0 t_a t_2 = t_2 t_a t_1^{-1} t_0 t_2 t_a.$$

Applying the braid relation corresponding to the arrow $a \to 1$ in Q' gives:

$$t_1^{-1} t_a^{-1} t_1 t_0 t_2 t_0 t_a t_2 = t_2 t_a t_1^{-1} t_0 t_2 t_a.$$

Multiplying on the left by $t_a t_1$ gives:

$$t_1 t_0 t_2 t_0 t_a t_2 = \underline{t_a t_1 t_2 t_a t_1^{-1}} t_0 t_2 t_a.$$

Since $t_a t_1 t_2 t_a = t_1 t_2 t_a t_1$, we may substitute in $t_a t_1 t_2 t_a t_1^{-1} = t_1 t_2 t_a$ on the right to get:

$$t_1 t_0 t_2 t_0 t_a t_2 = t_1 t_2 t_a t_0 t_2 t_a.$$

Applying the braid relation corresponding to the arrow $2 \to 0$ in Q' on the left-hand side, we obtain:

$$t_1 t_2 t_0 t_2 t_a t_2 = t_1 t_2 t_a t_0 t_2 t_a.$$

Multiplying both sides on the left by $t_2^{-1} t_1^{-1}$ gives

$$t_0 t_2 t_a t_2 = t_a t_0 t_2 t_a.$$

Since this final equivalent version holds, we see that (4) holds, as desired.

Next we check the two relations associated to the 4-cycle $0 \to 2 \to a \to 1 \to 0$ in Q. Since $t_a t_1 t_2 t_a = t_1 t_2 t_a t_1$, we have:

$$S_a S_0^{-1} S_1 S_0 S_2 S_a = S_0^{-1} S_1 S_0 S_2 S_a S_0^{-1} S_1 S_0.$$

Multiplying on the left by S_0 and applying the commutation $S_0 S_a = S_a S_0$ gives:

$$S_a S_1 S_0 S_2 S_a = S_1 S_0 S_2 S_a \underline{S_0^{-1} S_1 S_0}.$$

Applying the braid relation associated to the arrow $1 \to 0$ in Q gives:

$$S_a S_1 S_0 S_2 S_a = S_1 S_0 S_2 S_a S_1 S_0 S_1^{-1}.$$

Then, multiplying on the right by S_1 gives:

$$S_a S_1 S_0 S_2 S_a S_1 = S_1 S_0 S_2 S_a S_1 S_0,$$

which is one of the relations associated to the 4-cycle in Q.

Since $t_2 t_a t_1 t_2 = t_a t_1 t_2 t_a$ we have, multiplying on the left by $t_2 t_0$, that:

$$\underline{t_2 t_0 t_2} t_a t_1 t_2 = t_2 t_0 t_a t_1 t_2 t_a.$$

Applying the braid relation corresponding to the arrow $2 \to 0$ in Q' gives:

$$t_0 t_2 t_0 t_a t_1 t_2 = t_2 \underline{t_0 t_a} t_1 t_2 t_a.$$

Applying the commutation $t_0 t_a = t_a t_0$ on the right hand side gives:

$$t_0 t_2 t_a t_0 t_1 t_2 = t_2 t_a t_0 t_1 t_2 t_a.$$

We insert the product $t_0^{-1} t_0$ in two places to obtain:

$$t_0 t_2 t_a t_0 t_1 t_0^{-1} t_0 t_2 = t_2 t_a t_0 t_1 t_0^{-1} t_0 t_2 t_a,$$

which can be rewritten as:

$$S_0 S_2 S_a S_1 S_0 S_2 = S_2 S_a S_1 S_0 S_2 S_a,$$

which is the other relation associated to the 4-cycle in Q.

One of the relations corresponding to the 3-cycle $1 \to 2 \to 0 \to 1$ in Q' is

$$t_1 t_2 \cdots t_{2d} = t_2 t_3 \cdots t_{2d+1}.$$

By Lemma 3.18, taking $r = 1, 2$, we have

$$t_d t_{d-1} \cdots t_1 S_{1-d+1} \cdots S_0 S_1 = t_{1-2d+1} \cdots t_0 t_1$$

and

$$t_d t_{d-1} \cdots t_1 S_{2-d+1} \cdots S_1 S_2 = t_{2-2d+1} \cdots t_1 t_2.$$

so

$$S_{1-d+1} \cdots S_0 S_1 = S_{2-d+1} \cdots S_1 S_2,$$

which can be rewritten as

$$S_1 S_2 \cdots S_d = S_2 S_3 \cdots S_{d+1}$$

(switching the two sides of the equality if d is even). This is the remaining required defining relation of G_Q (corresponding to the unoriented edge labelled d in Q). Hence all the defining relations of G_Q hold, and the result is shown. $\qquad\square$

We consider the following setup:

Setup 3.22. Let Q be the quiver on the right of Figure 18(a) (respectively, the quiver on the right of Figure 18(b)) for $n \geq 3$ (respectively, for $n = 2$) and let Q' be the quiver on the left in each case. Let H_Q be the group defined in Definition 3.4, with generators s_i, and let $H_{Q'}$ be the group defined in Definition 3.4, with generators t_i. Let $S_n = t_0 t_n t_0^{-1}$, $S_b = t_0 t_b t_0^{-1}$, and $S_i = t_i$ for $i \neq n, b$. We regard the subscripts of the s_i and S_i for $0 \leq i \leq n$ to be taken modulo $n + 1$ (with representatives $\{0, 1, 2 \ldots, n\}$), and the subscripts of the t_i for $1 \leq i \leq n$ to be taken modulo n (with representatives $\{1, 2, \ldots, n\}$).

We note the following:

Lemma 3.23. Let $n \geq 2$ be an integer. In Setup 3.22, the elements S_i satisfy the defining relations of H_Q.

Proof. For $n \geq 3$, this follows from Proposition 3.13.

For $n = 2$, the braid and commutation relations and the relations for the 3-cycles all hold by Proposition 3.13, except for the relation for arrow from 1 to 2 and the relations for the 3-cycle $2 \to a \to 1 \to 2$.

The relation for the double edge incident with 0 in Q' is:

$$t_0 t_2 t_1 t_0 t_2 t_1 = t_2 t_1 t_0 t_2 t_1 t_0.$$

Substituting, this gives

$$S_0 S_0^{-1} S_2 S_0 S_1 S_0 S_0^{-1} S_2 S_0 S_1 = S_0^{-1} S_2 S_0 S_1 S_0 S_0^{-1} S_2 S_0 S_1 S_0,$$

which simplifies to

$$S_0 S_2 S_0 S_1 S_2 S_0 S_1 = S_2 S_0 S_1 S_2 S_0 S_1 S_0$$

Using the braid relations for the arrows $0 \to 2$ and $1 \to 0$ in Q:

$$\underline{S_0 S_2 S_0} S_1 S_2 \underline{S_0 S_1} = S_2 S_0 S_1 S_2 \underline{S_0 S_1 S_0}$$

gives:

$$S_2 S_0 S_2 S_1 S_2 S_0 S_1 = S_2 S_0 S_1 S_2 S_1 S_0 S_1,$$

which, after cancelling elements on the left and right, gives

$$S_2 S_1 S_2 = S_1 S_2 S_1.$$

There are two relations for the 4-cycle on vertices 2, a, 1 and 0 in Q'. One of these is:

$$t_0 t_2 t_a t_1 t_0 t_2 = t_2 t_a t_1 t_0 t_2 t_a,$$

which gives

$$S_0 S_0^{-1} S_2 S_0 S_a S_1 S_0 S_0^{-1} S_2 S_0 = S_0^{-1} S_2 S_0 S_a S_1 S_0 S_0^{-1} S_2 S_0 S_a,$$

and hence

$$\underline{S_0 S_2 S_0} S_a S_1 S_2 S_0 = S_2 S_0 S_a S_1 S_2 \underline{S_0 S_a}.$$

Applying the braid relation for the arrow $2 \to 0$ in Q and the commutation $S_0 S_a = S_a S_0$ gives

$$S_2 S_0 S_2 S_a S_1 S_2 S_0 = S_2 S_0 S_a S_1 S_2 S_a S_0.$$

Applying cancellations on the left and right gives

$$S_2 S_a S_1 S_2 = S_a S_1 S_2 S_a.$$

By [GM17, Lem. 2.4], we conclude that

$$S_2 S_a S_1 S_2 = S_a S_1 S_2 S_a = S_1 S_2 S_a S_1,$$

giving the result for $n = 2$. $\square$

Lemma 3.24. In Setup 3.22, let $d \geq 1$ be an integer. We have:

(a)
$$S_i^{-1} S_{i+2} \cdots S_{i+n+1} = S_{i+2} \cdots S_{i+n+1} S_{i-1}^{-1};$$

(b)
$$S_n^{-1}(S_1 S_2 \cdots S_{nd}) = (S_1 \cdots S_{nd}) S_{n-d}^{-1}.$$

Proof. We use Lemma 3.23 throughout. For (a), we have:

$$S_i^{-1} S_{i+2} S_{i+3} \cdots S_{i+n+1} = S_{i+2} S_{i+3} \cdots S_{i+n-1} S_i^{-1} S_{i+n} S_{i+n+1}$$
$$= S_{i+2} S_{i+3} \cdots S_{i+n-1} S_{i+n} S_{i+n+1} S_{i+n}^{-1},$$

as required, using commutations in the first step and noting that $i + n + 1 \equiv i$ and $i + n \equiv i - 1$ mod $n + 1$. For part (b), note that

$$S_1 \cdots S_{nd} = (S_1 \cdots S_n)(S_{n+1} \cdots S_{2n}) \cdots (S_d \cdots S_{d+n-1}),$$

and use part (a) d times. $\square$

Lemma 3.25. In Setup 3.22, let $d \geq 1$ be an integer. Then, for any $1 \leq r \leq n$, we have:

$$\underbrace{(t_r t_{r+1} \cdots t_{d(n-1)+r-1})}_{d(n-1) \text{ terms}} \underbrace{(S_n S_{n-1} \cdots S_{n-d+1})}_{d \text{ terms}} = \underbrace{S_r S_{r+1} \cdots S_{r+nd-1}}_{dn \text{ terms}}$$

Proof. We prove the result by induction on d, using Lemma 3.23 throughout. For $d = 1$, we have $t_1 t_2 \cdots t_{n-1} S_n = S_1 S_2 \cdots S_n$. For $r \geq 2$, we have (recalling that subscripts of the t_i are written modulo n):

$$
\begin{aligned}
t_r t_{r+1} \cdots t_{r+n-2} S_n &= t_r \cdots t_{n-1} t_n t_1 t_2 \cdots t_{r+n-2-n} S_n \\
&= (S_r \cdots S_{n-1})(S_0^{-1} S_n S_0)(S_1 S_2 \cdots S_{r-2}) S_n \\
&= (S_r \cdots S_{n-1})(S_n S_0 S_n^{-1})(S_1 S_2 \cdots S_{r-2}) S_n \\
&= (S_r \cdots S_{n-1}) S_n S_0 (S_1 \cdots S_{r-2}) S_n^{-1} S_n,
\end{aligned}
$$

and we see that the result holds for $d = 1$.

Assume the result holds for an integer $d \geq 1$. Then, using the induction hypothesis,

$$
\underbrace{(t_1 t_2 \cdots t_{(d+1)(n-1)})}_{(d+1)(n-1)\ \text{terms}} \underbrace{(S_n S_{n-1} \cdots S_{n-(d+1)-1})}_{d+1\ \text{terms}} = (t_1 t_2 \ldots t_{n-1}) \underbrace{(t_n t_{n+1} \cdots t_{(d+1)(n-1)})}_{d(n-1)\ \text{terms}} \underbrace{((S_n S_{n-1} \cdots S_{n-d+1})}_{d\ \text{terms}}) S_{n-d}
$$

$$
= (t_1 t_2 \cdots t_{n-1}) \underbrace{(S_n S_{n+1} \cdots S_{n+nd-1})}_{nd\ \text{terms}} S_{n-d}
$$

$$
= S_1 S_2 \cdots S_{n+nd-1} S_{n+dn},
$$

noting that $n - d \equiv n + dn \mod n + 1$. We also have, for $2 \leq r \leq n$, using the inductive hypothesis in the second step:

$$
\underbrace{(t_r t_{r+1} \cdots t_{(d+1)(n-1)+r-1})}_{(d+1)(n-1)\ \text{terms}} \underbrace{(S_n S_{n-1} \cdots S_{n-(d+1)+1})}_{d+1\ \text{terms}}
$$

$$
= \underbrace{(t_r t_{r+1} \cdots t_{r+n-2})}_{n-1\ \text{terms}} \underbrace{(t_{r+n-1} t_{r+n} \cdots t_{(d+1)(n-1)+r-1})}_{d(n-1)\ \text{terms}} \underbrace{(S_n S_{n-1} \cdots S_{n-d+1})}_{d\ \text{terms}} S_{n-d}
$$

$$
= \underbrace{(t_r t_{r+1} \cdots t_{r+n-2})}_{n-1\ \text{terms}} \underbrace{(t_{r-1} t_r \cdots t_{r-1+d(n-1)-1})}_{d(n-1)\ \text{terms}} \underbrace{(S_n S_{n-1} \cdots S_{n-d+1})}_{d\ \text{terms}} S_{n-d}
$$

$$
= \underbrace{(t_r t_{r+1} \cdots t_{r+n-2})}_{n-1\ \text{terms}} \underbrace{(S_{r-1} S_r \cdots S_{r-1+nd-1})}_{dn\ \text{terms}} S_{n-d}
$$

$$
= \underbrace{(S_r S_{r+1} \cdots S_{n-1})}_{n-r\ \text{terms}} S_0^{-1} S_n S_0 \underbrace{(S_1 S_2 \cdots S_{r-2})}_{r-2\ \text{terms}} \underbrace{(S_{r-1} S_r \cdots S_{nd+r-2})}_{dn\ \text{terms}} S_{n-d}
$$

$$
= \underbrace{(S_r S_{r+1} \cdots S_{n-1})}_{n-r\ \text{terms}} S_n S_0 S_n^{-1} \underbrace{(S_1 S_2 \cdots S_{dn})}_{dn\ \text{terms}} \underbrace{(S_{dn+1} S_{dn+2} \cdots S_{dn+r-2})}_{r-2\ \text{terms}} S_{n-d}
$$

$$
= \underbrace{(S_r S_{r+1} \cdots S_{n-1})}_{n-r\ \text{terms}} S_n S_0 S_n^{-1} \underbrace{(S_1 S_2 \cdots S_{dn})}_{dn\ \text{terms}} \underbrace{(S_{dn+1} S_{dn+2} \cdots S_{dn+r-2})}_{r-2\ \text{terms}} S_{n-d}
$$

$$
= \underbrace{(S_r S_{r+1} \cdots S_{n-1})}_{n-r\ \text{terms}} S_n S_0 \underbrace{(S_1 S_2 \cdots S_{dn})}_{dn\ \text{terms}} S_{n-d}^{-1} \underbrace{(S_{dn+1} S_{dn+2} \cdots S_{dn+r-2})}_{r-2\ \text{terms}} S_{n-d}
$$

as required, using Lemma 3.24 in the last but one step. Note that $dn+i \equiv n-d+i+1 \mod n+1$ for $1 \leq i \leq r-2$, so S_{n-d}^{-1} commutes with $S_{nd+1} = S_{n-d+2}, S_{nd+2} = S_{n-d+3}, \ldots, S_{nd+r-2} = S_{n-d+r-1}$, giving the required result for $d+1$, since $2 \leq r \leq n$. The result follows by induction on d. $\square$

Lemma 3.26. Let $n \geq 3$ be an integer. Let Q be the quiver on the right of Figure 18(a) and Q' the quiver on the left. Let G_Q (respectively $G_{Q'}$) be the group with generators s_i

(respectively t_i) with i ranging through the vertices of the quivers, satisfying the relations associated with Q (respectively Q'). Then, there is a group homomorphism, $\varphi_0^Q : G_Q \to G_{Q'}$ given by:

$$\varphi_0^Q(s_n) = S_n = t_0 t_n t_0^{-1}, \quad \varphi_0^Q(s_b) = S_b = t_0 t_b t_0^{-1}, \quad \varphi_0^Q(s_i) = S_i = t_i \text{ for } i \neq n, b.$$

Proof. The defining relations for G_Q, apart from the relations corresponding to the $(n+1)$-cycle $0 \to 1 \to \cdots \to n \to 0$, hold by Lemma 3.23. The relations for $G_{Q'}$ corresponding to the n-cycle $1 \to 2 \to \cdots \to n \to 1$ are:

$$\underbrace{(t_r t_{r+1} \cdots t_{d(n-1)+r-1})}_{d(n-1) \text{ terms}} = \underbrace{(t_{r+1} t_{r+2} \cdots t_{d(n-1)+r})}_{d(n-1) \text{ terms}},$$

for $1 \leq r \leq n$. The relations for G_Q corresponding to the $(n+1)$-cycle $0 \to 1 \to \cdots \to n + 1 \to 0$ are:

$$\underbrace{S_r S_{r+1} \cdots S_{r+nd-1}}_{dn \text{ terms}} = \underbrace{S_{r+1} S_{r+2} \cdots S_{dn+r}}_{dn \text{ terms}},$$

for $0 \leq r \leq n$. For $1 \leq r \leq n - 1$, these relations now follow from Lemma 3.25. The relations for $r = 0$ and $r = n$ follow from Lemma 3.19 applied to $g_0 = S_0$, $g_1 = S_1, \ldots, g_n = S_n$. $\qquad\square$

Lemma 3.27. Let $n = 2$. Let Q be the quiver on the right of Figure 18(b) and Q' the quiver on the left. Let G_Q (respectively $G_{Q'}$) be the group with generators s_i (respectively t_i), with i ranging through the vertices of the quivers, satisfying the relations associated with Q (respectively Q'). Then, there is a group homomorphism, $\varphi_0^Q : G_Q \to G_{Q'}$ given by:

$$\varphi_0^Q(s_n) = S_n = t_0 t_n t_0^{-1}, \quad \varphi_0^Q(s_b) = S_b = t_0 t_b t_0^{-1}, \quad \varphi_0^Q(s_i) = S_i = t_i \text{ for } i \neq n, b.$$

Proof. The defining relations for G_Q, apart from the relations corresponding to the 3-cycle $0 \to 1 \to 2 \to 0$, hold by Lemma 3.23.

The relations for the $G_{Q'}$ corresponding to the unoriented edge in Q' between vertices 1 and 2 labelled d are

$$t_1 t_2 \cdots t_d = t_2 t_3 \cdots t_{d+1}$$

By Lemma 3.25, taking $r = 1, 2$, we have

$$(t_1 t_2 \cdots t_d)(S_2 S_1 \cdots S_{2-d+1}) = S_1 S_2 \cdots S_{2d}$$

and

$$(t_2 t_3 \cdots t_{d+1})(S_2 S_1 \cdots S_{2-d+1}) = S_2 S_3 \cdots S_{2d+1},$$

giving

$$S_1 S_2 \cdots S_{2d} = S_2 S_3 \cdots S_{2d+1},$$

which is one of the cycle relations for the 3-cycle $0 \to 1 \to 2 \to 0$ in Q. The other cycle relations for this cycle follow from Lemma 3.19, taking $g_0 = S_0$, $g_1 = S_1$ and $g_2 = S_2$. Hence all the defining relations for G_Q hold. $\qquad\square$

Lemma 3.28. Let Q be the quiver on the left of Figure 19 and Q' the quiver on the right. Let G_Q (respectively $G_{Q'}$) be the group with generators s_i (respectively t_i) with $0 \leq i \leq 5$ satisfying the relations associated with Q (respectively Q'). Then, there is a group homomorphism:

$$\varphi_4^Q : G_Q \to G_{Q'} \text{ given by } \varphi_4^Q(s_0) = S_0 = t_4 t_0 t_4^{-1}, \quad \varphi_4^Q(s_2) = S_2 = t_4 t_2 t_4^{-1},$$

$$\varphi_4^Q(s_5) = S_5 = t_4 t_5 t_4^{-1}, \quad \varphi_4^Q(s_i) = S_i = t_i \text{ for } i = 1, 3, 4.$$
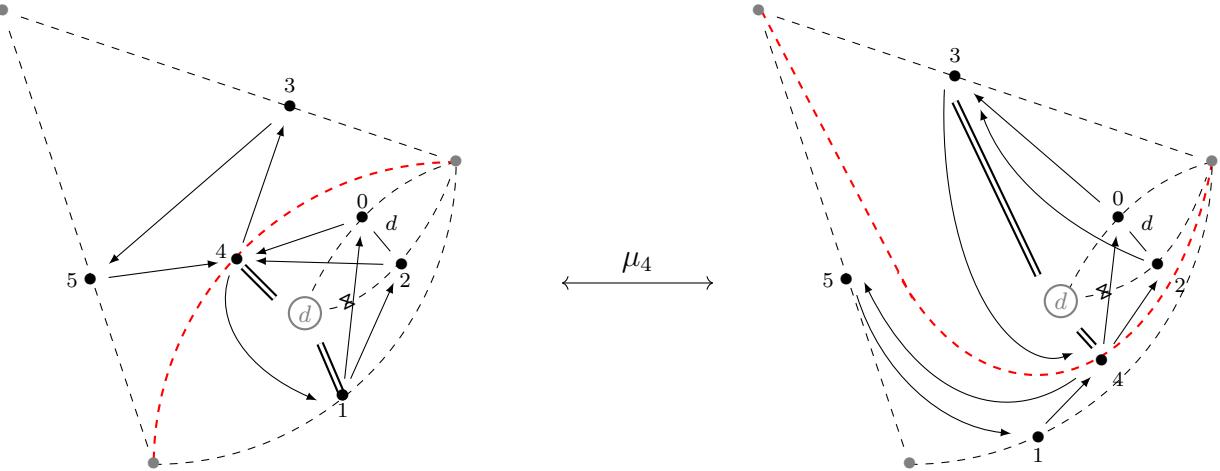
FIGURE 19. A mutation near the cone point.

Proof. In order to check that φ_4^Q is well-defined, it is enough to prove that the elements S_i, for $0 \leq i \leq 5$, satisfy the defining relations of G_Q.

All the braid and commutation relations for G_Q, as well as the cycle relations:

$$S_4 S_1 S_0 S_4 = S_1 S_0 S_4 S_1 = S_0 S_4 S_1 S_0,$$
$$S_4 S_1 S_2 S_4 = S_1 S_2 S_4 S_1 = S_2 S_4 S_1 S_2,$$
$$S_4 S_3 S_5 S_4 = S_3 S_5 S_4 S_3 = S_5 S_4 S_3 S_5,$$

hold by Proposition 3.13. Moreover, the relation

$$\underbrace{t_2 t_0 t_2 t_0 \cdots}_{d \text{ terms}} = \underbrace{t_0 t_2 t_0 t_2 \cdots}_{d \text{ terms}}$$

implies that

$$\underbrace{S_4^{-1} S_2 S_4 S_4^{-1} S_0 S_4 S_4^{-1} S_2 S_4 S_4^{-1} S_0 S_4 \cdots S_4}_{3d \text{ terms}} = \underbrace{S_4^{-1} S_0 S_4 S_4^{-1} S_2 S_4 S_4^{-1} S_0 S_4 S_4^{-1} S_2 S_4 \cdots S_4}_{3d \text{ terms}}$$

Since $S_4 S_4^{-1}$ is the identity, we can cancel the $d-1$ occurrences of this product on both sides, reducing the number of terms to $3d - (2d - 2) = d + 2$ on both sides. Finally, since the two sides start and end with the same element, these can be cancelled to obtain

$$\underbrace{S_2 S_0 S_2 S_0 \cdots}_{d \text{ terms}} = \underbrace{S_0 S_2 S_0 S_2 \cdots}_{d \text{ terms}}.$$

The double edge relation at vertex 4 in Q' is:

$$t_2 t_0 t_4 t_2 t_0 t_4 = t_4 t_2 t_0 t_4 t_2 t_0.$$

This, together with the definition of the S_i's, implies that

$$S_4^{-1} S_2 S_4 S_4^{-1} S_0 S_4 S_4 S_4^{-1} S_2 S_4 S_4^{-1} S_0 S_4 S_4 = S_4 S_4^{-1} S_2 S_4 S_4^{-1} S_0 S_4 S_4 S_4^{-1} S_2 S_4 S_4^{-1} S_0 S_4.$$

Multiplying both sides by S_4 on the left and by S_4^{-1} on the right and cancelling the occurrences of $S_4 S_4^{-1}$, we obtain $S_2 S_0 S_4 S_2 S_0 S_4 = S_4 S_2 S_0 S_4 S_2 S_0$. It only remains to show the double edge

relation at vertex 1 holds. Using the definition of the S_i's and the defining relations of $G_{Q'}$, we have

$$S_2 S_0 S_1 S_2 S_0 S_1 = t_4 t_2 t_4^{-1} \cancel{t_4} t_0 t_4^{-1} t_1 t_4 t_2 t_4^{-1} \cancel{t_4} t_0 t_4^{-1} t_1 = t_4 t_2 t_0 t_1 t_4 t_4^{-1} t_2 t_0 t_4^{-1} t_1$$

$$= t_4 t_1 t_2 t_0 t_4 t_2 t_0 t_1^{-1} t_4^{-1} t_1 = t_4 t_1 \underline{t_2 t_0 t_4 t_2 t_0 t_4} t_1^{-1} t_4^{-1}$$

$$= t_4 t_1 \underline{t_4 t_2 t_0 t_4 t_2 t_0} t_1^{-1} t_4^{-1} = t_1 t_4 t_1 t_2 t_0 t_4 t_2 t_0 t_1^{-1} t_4^{-1}$$

$$= t_1 t_4 t_2 t_0 t_1 t_4 t_1^{-1} t_2 t_0 t_4^{-1} = t_1 t_4 t_2 t_0 t_4^{-1} t_1 t_4 t_2 t_0 t_4^{-1}$$

$$= t_1 t_4 t_2 t_4^{-1} t_4 t_0 t_4^{-1} t_1 t_4 t_2 t_4^{-1} t_4 t_0 t_4^{-1} = S_1 S_2 S_0 S_1 S_2 S_0,$$

where the underlined relation is the relation for the double edge at vertex 4 in Q', while all the other equalities follow from braid relations, commutations or multiplying by, or simplifying, the identity $t_i^{-1} t_i$. Hence all the defining relations of G_Q hold. $\qquad\square$

Lemma 3.29. Let Q be the quiver on the right of Figure 19 and Q' the quiver on the left. Let G_Q (respectively $G_{Q'}$) be the group with generators s_i (respectively t_i) with $0 \leq i \leq 5$ satisfying the relations associated with Q (respectively Q'). Then, there is a group homomorphism

$$\varphi_4^Q : G_Q \to G_{Q'} \text{ given by } \varphi_4^Q(s_1) = S_1 = t_4 t_1 t_4^{-1}, \ \varphi_4^Q(s_3) = S_3 = t_4 t_3 t_4^{-1}, \ \varphi_4^Q(s_i) = S_i = t_i \text{ for } i \neq 1, 3.$$

Proof. In order to check that φ_4 is well-defined, it is enough to prove that the elements S_i, for $0 \leq i \leq 5$, satisfy the defining relations of G_Q.

First note that the relations

$$S_2 S_0 S_4 S_2 S_0 S_4 = S_4 S_2 S_0 S_4 S_2 S_0, \ \underbrace{S_0 S_2 \cdots}_{d \text{ terms}} = \underbrace{S_2 S_0 \cdots}_{d \text{ terms}}$$

trivially follow from the corresponding relations for Q'. Moreover, all the braid and commutation relations for Q, as well as the cycle relations

$$S_1 S_4 S_5 S_1 = S_4 S_5 S_1 S_4 = S_5 S_1 S_4 S_5,$$
$$S_2 S_3 S_4 S_2 = S_3 S_4 S_2 S_3 = S_4 S_2 S_3 S_4,$$
$$S_0 S_3 S_4 S_0 = S_3 S_4 S_0 S_1 = S_4 S_0 S_1 S_4$$

hold by Proposition 3.13. It only remains to show the second double edge relation holds. Using the definition of the elements S_i and the defining relations of $G_{Q'}$, we have

$$S_2 S_0 S_3 S_2 S_0 S_3 = t_2 t_0 t_4 t_3 t_4^{-1} t_2 t_0 t_4 t_3 t_4^{-1} = t_2 t_0 t_3^{-1} t_4 t_3 t_2 t_0 t_4 t_3 t_4^{-1} = t_3^{-1} t_2 t_0 t_4 t_2 t_0 t_3 t_4 t_3 t_4^{-1}$$

$$= t_3^{-1} \underline{t_2 t_0 t_4 t_2 t_0 t_4} t_3 t_4 t_4^{-1} = t_3^{-1} \underline{t_4 t_2 t_0 t_4 t_2 t_0} t_3 = t_4 t_4^{-1} t_3^{-1} t_4 t_2 t_0 t_4 t_3 t_2 t_0$$

$$= t_4 t_3 t_4^{-1} t_3^{-1} t_2 t_0 t_4 t_3 t_2 t_0 = t_4 t_3 t_4^{-1} t_2 t_0 t_3^{-1} t_4 t_3 t_2 t_0 = t_4 t_3 t_4^{-1} t_2 t_0 t_4 t_3 t_4^{-1} t_2 t_0$$

$$= S_3 S_2 S_0 S_3 S_2 S_0,$$

where the underlined relation is the relation for the double edge at vertex 4 in Q', while all the other equalities follow from braid relations, commutations or multiplying by, or simplifying, the identity $t_i t_i^{-1}$. Hence all the defining relations of G_Q hold as required. $\qquad\square$

Lemma 3.30. Let $d \geq 1$ be an integer and let G be a group containing elements a, b satisfying

$$\underbrace{ab \cdots}_{d \text{ terms}} = \underbrace{ab \cdots}_{d \text{ terms}} . \tag{5}$$
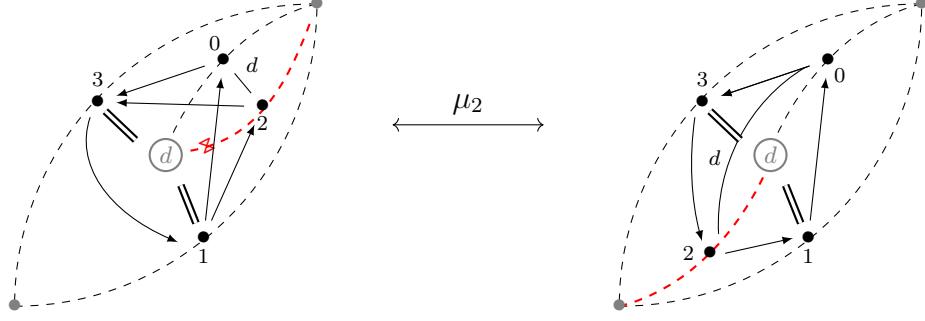
FIGURE 20. Mutation of a tagged to an untagged arc and viceversa, case 1.

Then the elements $A = a$ and $B = aba^{-1}$ satisfy the same relation:

$$\underbrace{AB\cdots}_{d \text{ terms}} = \underbrace{AB\cdots}_{d \text{ terms}}.$$

Proof. By relation (5), we have

$$\underbrace{A^{-1}BA\cancel{A}\cancel{A^{-1}}BA\cancel{A}\cdots A}_{3\cdot\lceil d/2\rceil + \lfloor d/2\rfloor \text{ terms}} = \underbrace{\cancel{A}\cancel{A^{-1}}BAA\cancel{A^{-1}}AB\cdots A}_{3\cdot\lfloor d/2\rfloor + \lceil d/2\rceil \text{ terms}},$$

where for a real number r, $\lfloor r \rfloor$ (respectively $\lceil r \rceil$) is the largest (respectively smallest) integer at most (respectively at least) equal to r. Cancelling the occurrences of AA^{-1}, on the left hand side we cancel $2 \cdot (\lceil d/2 \rceil - 1)$ terms, while on the right hand side we cancel $2 \cdot \lfloor d/2 \rfloor$ terms. Hence we have $d + 2$ terms on the left hand side and d terms on the right hand side. Since the last term on both sides is A, we can cancel it. Moreover, multiplying both sides by A on the left and simplifying the left hand side, we obtain

$$\underbrace{BA\cdots}_{d \text{ terms}} = \underbrace{AB\cdots}_{d \text{ terms}},$$

as required. $\square$

Lemma 3.31. Let Q be the quiver on the left of Figure 20 and Q' the quiver on the right. Let G_Q (respectively $G_{Q'}$) be the group with generators s_i (respectively t_i) with $0 \le i \le 3$ satisfying the relations associated with Q (respectively Q'). Then there is a group homomorphism as follows:

$$\varphi_2^Q : G_Q \to G_{Q'} \text{ defined as } \varphi_2^Q(s_1) = S_1 = t_2 t_1 t_2^{-1}, \ \varphi_2^Q(t_i) = S_i = T_i \text{ for } i = 0, 2, 3.$$

Proof. In order to check that φ_2^Q is well-defined, it is enough to prove that the elements S_i, for $0 \le i \le 3$, satisfy the defining relations of G_Q.

The relations $S_0 S_3 S_0 = S_3 S_0 S_3$, $S_2 S_0 S_3 S_2 S_0 S_3 = S_3 S_2 S_0 S_3 S_2 S_0$ and

$$\underbrace{S_0 S_2 \cdots}_{d \text{ terms}} = \underbrace{S_2 S_0 \cdots}_{d \text{ terms}}$$

trivially follow from the corresponding defining relations of $G_{Q'}$. Moreover, the relations $S_2 S_3 S_2 = S_3 S_2 S_3$, $S_1 S_3 S_1 = S_3 S_1 S_3$, $S_2 S_1 S_2 = S_1 S_2 S_1$ and $S_1 S_2 S_3 S_1 = S_2 S_3 S_1 S_2 = S_3 S_1 S_2 S_3$ hold by Proposition 3.13.

The relation for the double edge at 1 in Q':

$$t_0 t_2 t_1 t_0 t_2 t_1 = t_1 t_0 t_2 t_1 t_0 t_2$$

implies that

$$S_0 \cancel{S_2} \cancel{S_2^{-1}} S_1 S_2 S_0 \cancel{S_2} \cancel{S_2^{-1}} S_1 S_2 = S_2^{-1} S_1 S_2 S_0 \cancel{S_2} \cancel{S_2^{-1}} S_1 S_2 S_0 S_2 \iff S_2 S_0 S_1 S_2 S_0 S_1 = S_1 S_2 S_0 S_1 S_2 S_0,$$

that is the double edge relation corresponding to vertex 1 in Q.

The braid relation $t_1 t_0 t_1 = t_0 t_1 t_0$ together with the relations proved above imply that

$$S_2^{-1} S_1 S_2 S_0 S_2^{-1} S_1 S_2 = S_0 S_2^{-1} S_1 S_2 S_0 \iff S_1 S_2 S_0 S_1 S_2 S_1^{-1} = S_2 S_0 S_1 S_2 S_1^{-1} S_0$$
$$\iff \cancel{S_2 S_0 S_1 S_2} S_0 S_1 S_0^{-1} S_1^{-1} = \cancel{S_2 S_0 S_1 S_2} S_1^{-1} S_0$$
$$\iff S_1 S_0 S_1 = S_0 S_1 S_0.$$

The relation $t_2 t_1 t_0 t_3 t_2 t_1 = t_3 t_2 t_1 t_0 t_3 t_2$ together with the relations proved above imply

$$S_2 \cancel{S_2^{-1}} S_1 S_2 S_0 S_3 S_2 \cancel{S_2^{-1}} S_1 S_2 = S_3 \cancel{S_2} \cancel{S_2^{-1}} S_1 S_2 S_0 S_3 S_2 \iff S_1 S_2 S_0 S_3 S_1 = S_3 S_1 S_2 S_0 S_3$$
$$\iff \cancel{S_1 S_2 S_3} S_0 S_3 S_0^{-1} S_1 = \cancel{S_1 S_2 S_3} S_1 S_3^{-1} S_0 S_3$$
$$\iff S_3^{-1} S_0 S_3 S_1 = S_1 S_0 S_3 S_0^{-1}$$
$$\iff S_0 S_3 S_1 S_0 = S_3 S_1 S_0 S_3.$$

By [GM17, Lem. 2.4], we conclude that $S_0 S_3 S_1 S_0 = S_3 S_1 S_0 S_3 = S_1 S_0 S_3 S_1$. Hence all the defining relations of G_Q hold, as required. $\square$

Lemma 3.32. Let Q be the quiver on the right of Figure 20 and Q' the quiver on the left. Let G_Q (respectively $G_{Q'}$) be the group with generators s_i (respectively t_i) with $0 \leq i \leq 3$ satisfying the relations associated with Q (respectively Q'). Then there is a group homomorphism as follows:

$\varphi_2^Q : G_Q \to G_{Q'}$ defined as $\varphi_2^Q(s_3) = S_3 = t_2 t_3 t_2^{-1}$, $\varphi_2^Q(s_0) = S_0 = t_2 t_0 t_2^{-1}$, $\varphi_2^Q(s_i) = t_i$ for $i = 1, 2$.

Proof. In order to check that φ_2^Q is well-defined, it is enough to prove that the elements S_i, for $0 \leq i \leq 3$, satisfy the defining relations of G_Q.

The relations $S_2 S_3 S_2 = S_3 S_2 S_3$, $S_2 S_1 S_2 = S_1 S_2 S_1$ and $S_1 S_3 = S_3 S_1$ hold by Proposition 3.13, while

$$\underbrace{S_0 S_2 \cdots}_{d \text{ terms}} = \underbrace{S_2 S_0 \cdots}_{d \text{ terms}}.$$

holds by Lemma 3.30. The relation $t_3 t_0 t_3 = t_0 t_3 t_0$ implies that

$$S_2^{-1} S_3 S_2 \cancel{S_2^{-1}} S_0 S_2 \cancel{S_2^{-1}} S_3 S_2 = S_2^{-1} S_0 S_2 \cancel{S_2^{-1}} S_3 S_2 \cancel{S_2^{-1}} S_0 S_2.$$

Since the first and last term are equal on the two sides, we can cancel them and obtain $S_3 S_0 S_3 = S_0 S_3 S_0$. The relation $t_2 t_0 t_1 t_2 t_0 t_1 = t_1 t_2 t_0 t_1 t_2 t_0$ for the double edge at 1 in Q' implies that

$$S_2 \cancel{S_2^{-1}} S_0 S_2 S_1 S_2 \cancel{S_2^{-1}} S_0 S_2 S_1 = S_1 \cancel{S_2} \cancel{S_2^{-1}} S_0 S_2 S_1 S_2 \cancel{S_2^{-1}} S_0 S_2.$$

So the relation for the double edge at 1 in Q

$$S_0 S_2 S_1 S_0 S_2 S_1 = S_1 S_0 S_2 S_1 S_0 S_2$$

follows after cancellations. The relation $t_2 t_0 t_3 t_2 t_0 t_3 = t_3 t_2 t_0 t_3 t_2 t_0$ for the double edge at 3 in Q' implies that

$$\cancel{S_2}\cancel{S_2^{-1}}S_0 \cancel{S_2}\cancel{S_2^{-1}}S_3 S_2 S_2 \cancel{S_2^{-1}}S_0 \cancel{S_2}\cancel{S_2^{-1}}S_3 S_2 = S_2^{-1}S_3 S_2 \cancel{S_2}\cancel{S_2^{-1}}S_0 \cancel{S_2}\cancel{S_2^{-1}}S_3 S_2 S_2 \cancel{S_2^{-1}}S_0 S_2.$$

Multiplying both sides by S_2 on the left and by S_2^{-1} on the right and simplifying, the relation for the double edge at 3 in Q:

$$S_2 S_0 S_3 S_2 S_0 S_3 = S_3 S_2 S_0 S_3 S_2 S_0$$

follows.

Using one of the braid relations from the left diagram and the relations found above, we have that

$$t_1 t_0 t_1 = t_0 t_1 t_0 \iff S_1 S_2^{-1} S_0 S_2 S_1 = S_2^{-1} S_0 S_2 S_1 S_2^{-1} S_0 S_2 \iff S_1^{-1} S_2 S_1 S_0 S_2 S_1 = S_0 S_1^{-1} S_2 S_1 S_0 S_2$$

$$\iff \underline{S_2 S_1 S_0 S_2 S_1} = S_1 S_0 S_1^{-1} S_2 S_1 S_0 S_2 \iff \underline{S_0^{-1} S_1 S_0} \cancel{S_2 S_1 S_0 S_2} = S_1 S_0 S_1^{-1} \cancel{S_2 S_1 S_0 S_2}$$

$$\iff S_1 S_0 S_1 = S_0 S_1 S_0,$$

where the underlined relation follows from the double edge relation at vertex 1 in Q.

The last two relations left to prove correspond to the 4-cycle around the cone point in Q. The relation $t_1 t_0 t_3 t_1 = t_0 t_3 t_1 t_0 = t_3 t_1 t_0 t_3$ implies that

$$S_1 S_2^{-1} S_0 \cancel{S_2}\cancel{S_2^{-1}}S_3 S_2 S_1 = S_2^{-1} S_0 \cancel{S_2}\cancel{S_2^{-1}}S_3 S_2 S_1 S_2^{-1} S_0 S_2 = S_2^{-1} S_3 S_2 S_1 S_2^{-1} S_0 \cancel{S_2}\cancel{S_2^{-1}}S_3 S_2$$

Using the relations already proved, we see that the equality of the first and third expressions above is true if and only if

$$S_2 S_1 S_2^{-1} S_0 S_3 S_2 S_1 = S_3 S_1^{-1} S_2 S_1 S_0 S_3 S_2 \iff S_1^{-1} S_2 S_1 S_0 S_3 S_2 S_1 = S_1^{-1} S_3 S_2 S_1 S_0 S_3 S_2$$

$$\iff S_2 S_1 S_0 S_3 S_2 S_1 = S_3 S_2 S_1 S_0 S_3 S_2.$$

Similarly, the equality of the second and third expressions above is true if and only if

$$S_0 S_3 S_1^{-1} S_2 S_1 S_0 = S_3 S_3 S_1^{-1} S_2 S_1 S_0 S_3 \iff S_1 S_0 S_1^{-1} S_3 S_2 S_1 S_0 = S_3 S_2 S_1 S_0 S_3$$

$$\iff S_0^{-1} S_1 S_0 S_3 S_2 S_1 S_0 = S_3 S_2 S_1 S_0 S_3$$

$$\iff S_1 S_0 S_3 S_2 S_1 S_0 = S_0 S_3 S_2 S_1 S_0 S_3.$$

Hence all the relations corresponding to the quiver Q hold, as required.   □
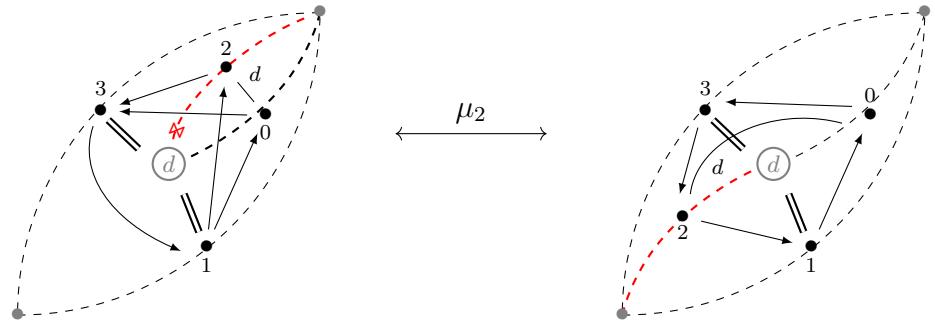
FIGURE 21. Mutation of a tagged to an untagged arc and viceversa, case 2.

Lemma 3.33. Let Q be the quiver on the left of Figure 21 and Q' the quiver on the right. Let G_Q (respectively $G_{Q'}$) be the group with generators s_i (respectively t_i) with $0 \le i \le 3$ satisfying the relations associated with Q (respectively Q'). Then there is a group homomorphism: $\varphi_2^Q : G_Q \to G_{Q'}$ defined as:

$$\varphi_2^Q(s_1) = S_1 = t_2 t_1 t_2^{-1}, \; \varphi_2^Q(s_0) = S_0 = t_2 t_0 t_2^{-1}, \; \varphi_2^Q(s_i) = S_i = t_i \text{ for } i = 2,3.$$

Proof. In order to check that φ_2^Q is well-defined, it is enough to prove that the elements S_i, for $0 \le i \le 3$, satisfy the defining relations of G_Q. The relations $S_2 S_3 S_2 = S_3 S_2 S_3$, $S_1 S_3 S_1 = S_3 S_1 S_3$, $S_2 S_1 S_2 = S_1 S_2 S_1$ and $S_1 S_2 S_3 S_1 = S_2 S_3 S_1 S_2 = S_3 S_1 S_2 S_3$ hold by Proposition 3.13, while the relation

$$\underbrace{S_2 S_0 \cdots}_{d \text{ terms}} = \underbrace{S_0 S_2 \cdots}_{d \text{ terms}}$$

holds by Lemma 3.30. The relation $t_1 t_0 t_1 = t_0 t_1 t_0$ implies that

$$S_2^{-1} S_1 \cancel{S_2} \cancel{S_2^{-1}} S_0 \cancel{S_2} \cancel{S_2^{-1}} S_1 S_2 = S_2^{-1} S_0 \cancel{S_2} \cancel{S_2^{-1}} S_1 \cancel{S_2} \cancel{S_2^{-1}} S_0 S_2.$$

Since the first and last term are equal on the two sides, we can cancel them and obtain $S_1 S_0 S_1 = S_0 S_1 S_0$. The relation for the double edge at 1 in Q'

$$t_0 t_2 t_1 t_0 t_2 t_1 = t_1 t_0 t_2 t_1 t_0 t_2$$

implies that

$$S_2^{-1} S_0 S_2 \cancel{S_2} \cancel{S_2^{-1}} S_1 \cancel{S_2} \cancel{S_2^{-1}} S_0 S_2 \cancel{S_2} \cancel{S_2^{-1}} S_1 S_2 = S_2^{-1} S_1 \cancel{S_2} \cancel{S_2^{-1}} S_0 S_2 \cancel{S_2} \cancel{S_2^{-1}} S_1 \cancel{S_2} \cancel{S_2^{-1}} S_0 S_2 S_2$$
$$\iff S_0 S_2 S_1 S_0 S_2 S_1 = S_1 S_0 S_2 S_1 S_0 S_2,$$

that is the double edge relation corresponding to vertex 1 in Q. Similarly, the relation for the double edge at 3 in Q':

$$t_2 t_0 t_3 t_2 t_0 t_3 = t_3 t_2 t_0 t_3 t_2 t_0$$

implies that

$$\cancel{S_2} \cancel{S_2^{-1}} S_0 S_2 S_3 \cancel{S_2} \cancel{S_2^{-1}} S_0 S_2 S_3 = S_3 \cancel{S_2} \cancel{S_2^{-1}} S_0 S_2 S_3 \cancel{S_2} \cancel{S_2^{-1}} S_0 S_2 \iff S_0 S_2 S_3 S_0 S_2 S_3 = S_3 S_0 S_2 S_3 S_0 S_2,$$

that is the double edge relation corresponding to vertex 3 in Q. Using the defining relations of Q', we have that

$$S_0 S_3 S_0 = t_2 t_0 t_2^{-1} t_3 t_2 t_0 t_2^{-1} = t_2 t_0 t_3 t_2 t_0 t_3 t_3^{-1} t_0^{-1} t_3^{-1} t_0 t_2^{-1}$$
$$= t_3 t_2 t_0 t_3 t_2 \cancel{t_0} \cancel{t_0^{-1}} t_3^{-1} \cancel{t_0^{-1}} \cancel{t_0} t_2^{-1} = t_3 t_2 t_0 t_2^{-1} t_3 \cancel{t_2} \cancel{t_2^{-1}} = S_3 S_0 S_3.$$

The relation $t_2 t_1 t_0 t_3 t_2 t_1 = t_3 t_2 t_1 t_0 t_3 t_2$, together with the relations proved above, implies that

$$\cancel{S_2} \cancel{S_2^{-1}} S_1 \cancel{S_2} \cancel{S_2^{-1}} S_0 S_2 S_3 \cancel{S_2} \cancel{S_2^{-1}} S_1 S_2 = S_3 \cancel{S_2} \cancel{S_2^{-1}} S_1 \cancel{S_2} \cancel{S_2^{-1}} S_0 S_2 S_3 S_2$$
$$\iff S_1 S_0 S_2 S_3 S_1 S_2 = S_3 S_1 S_0 S_2 S_3 S_2$$
$$\iff S_1 S_0 S_3 S_1 \cancel{S_2} \cancel{S_3} = S_3 S_1 S_0 S_3 \cancel{S_2} \cancel{S_3}.$$

By [GM17, Lem. 2.4], we conclude that $S_1 S_0 S_3 S_1 = S_3 S_1 S_0 S_3 = S_0 S_3 S_1 S_0$. Hence all the defining relations of G_Q hold, as required. $\qquad\square$

Lemma 3.34. Let Q be the quiver on the right of Figure 21 and Q' the quiver on the left. Let G_Q (respectively $G_{Q'}$) be the group with generators s_i (respectively t_i) with $0 \le i \le 3$ satisfying the relations associated with Q (respectively Q'). Then there is a group homomorphism as follows:

$$\varphi_2^Q : G_Q \to G_{Q'} \text{ defined as } \varphi_2^Q(s_3) = S_3 = t_2 t_3 t_2^{-1}, \ \varphi_2^Q(s_i) = S_i = t_i \text{ for } i = 0, 1, 2.$$

Proof. In order to check that φ_2^Q is well-defined, it is enough to prove that the elements S_i, for $0 \le i \le 3$, satisfy the defining relations of G_Q.

The relation $S_1 S_0 S_1 = S_0 S_1 S_0$ trivially follows from the relation $t_1 t_0 t_1 = t_0 t_1 t_0$. The relations $S_2 S_3 S_2 = S_3 S_2 S_3$, $S_2 S_1 S_2 = S_1 S_2 S_1$ and $S_1 S_3 = S_3 S_1$ hold by Proposition 3.13, while the relation

$$\underbrace{S_2 S_0 \cdots}_{d \text{ terms}} = \underbrace{S_0 S_2 \cdots}_{d \text{ terms}}.$$

is trivial to check. The relation $S_0 S_2 S_1 S_0 S_2 S_1 = S_1 S_0 S_2 S_1 S_0 S_2$ for the double edge at 1 follows trivially from the corresponding relation for Q'. Moreover, the relation $t_0 t_2 t_3 t_0 t_2 t_3 = t_3 t_0 t_2 t_3 t_0 t_2$ for the double edge at 3 in Q' implies that

$$S_0 \cancel{S_2 S_2^{-1}} S_3 S_2 S_0 \cancel{S_2 S_2^{-1}} S_3 S_2 = S_2^{-1} S_3 S_2 S_0 \cancel{S_2 S_2^{-1}} S_3 S_2 S_0 S_2 \iff S_2 S_0 S_3 S_2 S_0 S_3 = S_3 S_2 S_0 S_3 S_2 S_0,$$

that is, the relation for the double edge in Q follows.

Using the relations found above, we have that

$$t_0 t_3 t_0 = t_3 t_0 t_3 \iff S_0 S_2^{-1} S_3 S_2 S_0 = S_2^{-1} S_3 S_2 S_0 S_2^{-1} S_3 S_2 \iff S_0 S_2^{-1} S_3 S_2 S_0 = \underline{S_2^{-1} S_3 S_2 S_0 S_3 S_2 S_3^{-1}}$$

$$\iff S_0 S_2^{-1} S_3 S_2 S_0 = \underline{S_0 S_3 S_2 S_0 S_3 S_0^{-1} S_3^{-1}} \iff S_3 S_2 S_0 S_3 S_0 = S_2 S_3 S_2 S_0 S_3$$

$$\iff \cancel{S_3 S_2} S_0 S_3 S_0 = \cancel{S_3 S_2} S_3 S_0 S_3 \iff S_0 S_3 S_0 = S_3 S_0 S_3,$$

where the underlined relation follows from the double edge relation at vertex 3 in Q. The last two relations left to prove correspond to the 4-cycle around the cone point in Q. The relation $t_1 t_0 t_3 t_1 = t_0 t_3 t_1 t_0 = t_3 t_1 t_0 t_3$ implies that

$$S_1 S_0 S_2^{-1} S_3 S_2 S_1 = S_0 S_2^{-1} S_3 S_2 S_1 S_0 = S_2^{-1} S_3 S_2 S_1 S_0 S_2^{-1} S_3 S_2.$$

Using the relations already proved, we see that the equality of the first two expressions above is true if and only if

$$S_1 S_0 S_3 S_2 S_3^{-1} S_1 = S_0 S_3 S_2 S_3^{-1} S_1 S_0 \iff S_1 S_0 S_3 S_2 S_1 S_3^{-1} = S_0 S_3 S_2 S_1 S_3^{-1} S_0$$

$$\iff S_1 S_0 S_3 S_2 S_1 = S_0 S_3 S_2 S_1 S_0 S_3 S_0^{-1}$$

$$\iff S_1 S_0 S_3 S_2 S_1 S_0 = S_0 S_3 S_2 S_1 S_0 S_3.$$

Similarly, the equality of the first and third expression above is true if and only if

$$S_2 S_1 S_0 S_3 S_2 S_3^{-1} S_1 = S_3 S_2 S_1 S_0 S_3 S_2 S_3^{-1} \iff S_2 S_1 S_0 S_3 S_2 S_1 S_3^{-1} = S_3 S_2 S_1 S_0 S_3 S_2 S_3^{-1}$$

$$\iff S_2 S_1 S_0 S_3 S_2 S_1 = S_3 S_2 S_1 S_0 S_3 S_2.$$

Hence all the defining relations for Q hold, as required. $\qquad\square$

Remark 3.35. Flipping all of the tags in Figure 18, Figure 20 or 21 does not affect the corresponding quivers or groups, so it follows that Lemmas 3.20, 3.26, 3.21, 3.27, 3.31, 3.32, 3.33 and 3.34 hold for these cases too.

Proof of Theorem 3.12. Let T be a tagged triangulation of S, and Q_T the associated quiver as in Definition 3.2. Let G_{Q_T} be the associated group as in Definition 3.4, with generators s_i. Let α be an arc in T and k the corresponding vertex of Q. Let T' be the tagged triangulation obtained by flipping T at α, so that $Q_{T'} = \mu_k(Q)$ is the mutation of Q at k as in Definition 3.9, by Lemma 3.11(b). By Lemma 3.11(a), the flip of α is given locally by one of the mutations in Figures 17, 18, 19, 20 or 21 (from left to right or right to left), or by one of the mutations from Figure 18, 20 or 21 with all of the tags flipped.

It follows from Lemmas 3.14, 3.20, 3.26, 3.21, 3.27, 3.28, 3.29, 3.31, 3.32, 3.33, 3.34 and Remark 3.35 that there are group homomorphisms $\varphi_k^Q : G_Q \to G_{Q'}$ and $\varphi_k^{Q'} : G_{Q'} \to G_Q$.

Note that arcs appearing on the boundary in each figure could be on the actual boundary of S. Each defining relation in G_Q to be checked involves a certain collection of unmutated vertices, plus possibly the mutated vertex. It is easy to check that, in each case, the proof that this relation holds involves relations involving only the same collection of vertices. If we consider the same situation where one of the vertices corresponds to an arc on the boundary of S, the corresponding relation does not appear and therefore does not need to be shown. It follows that the corresponding results hold in the situation where some or all of the dashed diagonals on the boundary of the figure are on the boundary of the disk.

It is easy to check in each case that $\varphi_k^{Q'} \varphi_k^Q(s_i) = s_k^{-1} s_i s_k$ for all i, so $\varphi_k^{Q'} \varphi_k^Q$ is an isomorphism. By the same argument, the other composition $\varphi_k^Q \varphi_k^{Q'}$ is also an isomorphism, and hence so is φ_k^Q. $\qquad\square$

Theorem 3.36. *Let T be any tagged triangulation of (X, M), and Q_T the associated quiver. Then G_{Q_T} is isomorphic to the braid group $B(d, d, n)$, and thus gives a presentation for $B(d, d, n)$.*

Proof. There is a tagged triangulation of (X, M) for which the corresponding group G_Q is the presentation of $B(d, d, n)$ from [BMR98, Thm. 2.27] (see Figure 12). The result follows from combining this with Theorem 3.12 and the fact that there is a sequence of flips connecting any two tagged triangulations of (X, M) by [FST08, Prop. 7.10]. $\qquad\square$

4. GEOMETRIC INTERPRETATION OF THE NEW PRESENTATIONS

We work with the same surface (X, M) as in the previous section: X is the disk S with an interior marked point, interpreted as a cone point of degree $d \geq 2$, and M a set of $n \geq 2$ marked points on the boundary of X. In Section 3.1, we defined a way to associate a quiver Q_T to any tagged triangulation T of (X, M) and a group G_{Q_T}. As in [GM17, Defn 3.1], we associate another graph to T as follows.

Definition 4.1. Let T be a tagged triangulation of (X, M). We define the *braid graph* D_T of T to be the geometric dual of T regarded as a graph embedded in the disk. Thus D_T has a vertex in each connected component of the complement of T and, whenever two connected components share an edge of T, there is a corresponding edge in D_T between the two corresponding vertices. Note that, in general, D_T can have multiple edges between vertices.

If we regard T as a graph embedded in the plane, then D_T is the geometric dual in the plane with the vertex corresponding to the external face removed. Note that this geometric dual

is isomorphic to the combinatorial dual of T by [Whi32, Theorem 30] (see also [Har69, §11, page 115]), since T is a non-separable graph, so D_T also is well-defined as an abstract graph.

Moreover, note that the interior of X is isomorphic as an orbifold to $\mathcal{O}_d := \mathbb{C}/C_d$ and hence we identify the two orbifolds in our arguments.

Given a set of n vertices V in $\mathcal{O}_d^\circ$, that is $\mathcal{O}_d$ minus the cone point, one can define the corresponding *braid group* $Z_n(\mathcal{O}_d)$ (denoted $\Gamma(\mathcal{O}_d, V)$ in [GM17]). Each element of $Z_n(\mathcal{O}_d)$, also called *braid*, can be regarded as a permutation g of V together with a tuple $\gamma = (\gamma_v)_{v \in V}$ of paths $\gamma_v : [0, 1] \to \mathcal{O}_d^\circ$ with $\gamma_v(0) = v$ and $\gamma_v(1) = g(v)$ for each $v \in V$ and, for each $t \in [0, 1]$, the points $\gamma_v(t)$ for $v \in V$ are all distinct for all $v \in V$. See [All02] and [GM17, Section 3] for further details. Moreover, each path π in $\mathcal{O}_d^\circ$ with endpoints in V determines a braid σ_π in $Z_n(\mathcal{O}_d)$ (see e.g. [GM17, Defn 3.3]).

Let T be a tagged triangulation of (X, M), and note that this is a collection of n (tagged) arcs. Each (tagged) arc α_i in T corresponds to a vertex i in the quiver Q_T and to an edge π_i in the braid graph D_T. Following the same notation as [GM17], we let $\sigma_i := \sigma_{\pi_i}$ denote the corresponding braid in $Z_n(\mathcal{O}_d)$ and B_T be the subgroup of $Z_n(\mathcal{O}_d)$ generated by the braids σ_i, for i vertex in Q_T.

Proposition 4.2. Let T_0 be the triangulation of (X, M) shown in Figure 12. Then there is an isomorphism from B_{T_0} to $G_{Q_{T_0}}$ taking the braid σ_i to the generator s_i of $G_{Q_{T_0}}$. Furthermore, the subgroup B_{T_0} is a subgroup of $Z_n(\mathcal{O}_d)$ of index d.

Proof. Note that, via an isomorphism of the kind in [GM17, Remark 3.2], the element σ_i coincides with h_i^{-1} for $1 \le i \le n$, for the braids h_i in $Z_n(\mathcal{O}_d)$ illustrated in Figure 9. Hence, B_{T_0} coincides with the subgroup of $Z_n(\mathcal{O}_d)$ generated by the h_i and the map β from Proposition 2.9 gives an isomorphism from $B(d, d, n)$ to B_{T_0} taking τ_2' to h_1 and τ_i to h_i for $2 \le i \le n$. By Proposition 2.2, B_{T_0} is a subgroup of $Z_n(\mathcal{O}_d)$ of index d.

Since reversing the defining relations of $B(d, d, n)$ gives the same relations with τ_2 and τ_2' exchanged, there is an isomorphism from $B(d, d, n)$ to B_{T_0} taking τ_2 to $h_1^{-1} = \sigma_1$, τ_2' to $h_2^{-1} = \sigma_2$ and τ_i to $h_i = \sigma_i$ for $3 \le i \le n$. By Remark 3.8, there is an isomorphism $G_{Q_{T_0}} \cong B(d, d, n)$ (with the presentation from [BMR98]) taking s_1 to τ_2, s_2 to τ_2' and s_i to τ_i for $3 \le i \le n$. The composition of these two isomorphisms gives the required isomorphism. $\square$

Before stating and proving our final result, we recall a result from [Ser93, Théorème, part(iv)], see also [GM17, Lemma 3.5].

Lemma 4.3. Let A, B, C be three distinct points in $\mathcal{O}_d^\circ$ and suppose there is a topological disk in $\mathcal{O}_d^\circ$ with A, B, C lying in order clockwise around its boundary. Let AB denote the arc on its boundary between A and B, and define BC and CA similarly. Then $\sigma_{AB}\sigma_{BC} = \sigma_{BC}\sigma_{CA}$.

Definition 4.4. Let T be a tagged triangulation of (X, M). Suppose that there is an arrow $i \to k$ in Q_T. Then there are vertices X_i, Y_i and Z_i and edges X_iY_i and Y_iZ_i in D_T such that $\sigma_i = \sigma_{X_iY_i}$ and $\sigma_k = \sigma_{Y_iZ_i}$. We say that an embedding of D_T into (X, M) is *good* at k if, for every such i, the vertices X_i, Y_i and Z_i are in clockwise order.

Remark 4.5. Note that our convention for orienting the arrows of the quiver of a triangulation is opposite to the convention used in [GM17]. However, the proof of [GM17, Thm. 3.6] actually requires this opposite convention: then it goes through as stated (provided we
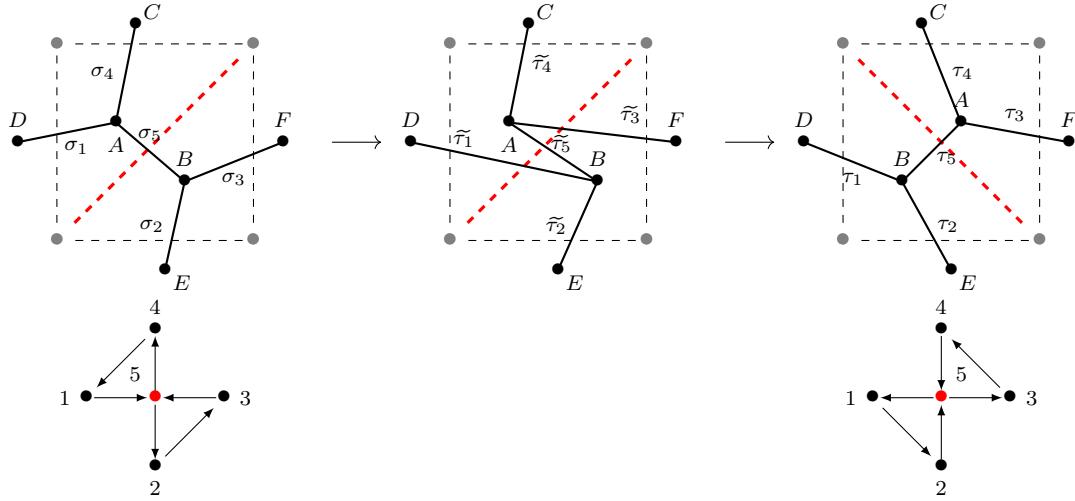
FIGURE 22. Flip of an arc: type A situation.

regard k as the mutation vertex). For example, in the notation used there (see [GM17, Fig. 6]), we have $\sigma_{\widetilde{\pi}_2} = \sigma_1^{-1}\sigma_2\sigma_1$, while $\tilde{\tau}_2 = \sigma_2$ as there is no arrow from 2 to 1 in the quiver used there. Thus the claim that $\sigma_{\widetilde{\pi}_2} = \tilde{\tau}_2$ does not hold, but this is resolved if the opposite convention for orienting the quiver is adopted, as we do here.

Theorem 4.6. *Let T be a tagged triangulation of (X, M). Then there is an isomorphism from B_T to G_{Q_T} taking the braid σ_i to the generator s_i of G_{Q_T}. Furthermore, B_T is a subgroup of index d of $Z_n(\mathcal{O}_d)$.*

Proof. By Proposition 4.2, the result holds for the triangulation $T = T_0$. Since any tagged triangulation can be obtained by flipping T_0 a finite number of times, it is enough to show that if the theorem holds for a tagged triangulation T, then it also holds for the flip of T at any of its tagged arcs.

Assume that the result holds for a tagged triangulation T, that is there is an isomorphism $\psi_T : B_T \to G_{Q_T}$ sending the braid σ_i to the generator s_i. Let the triangulation T' be obtained by flipping T at the arc α_k and the generators of $B_{T'}$ and $G_{Q_{T'}}$ be denoted by τ_i and t_i respectively. In the following arguments, we label the paths in $\mathcal{O}_d^\circ$ as the corresponding braids in B_T and $B_{T'}$, to avoid heavy notation.

Define $\widetilde{\tau}_i = \sigma_k^{-1}\sigma_i\sigma_k \in B_T$ if $i \to k$ in Q_T or the vertices i and k correspond to the only two arcs incident with the conepoint in T and the arc corresponding to k is rotated anti-clockwise to the flipped arc; and $\widetilde{\tau}_i = \sigma_i \in B_T$ otherwise. Note that B_T is generated by the $\widetilde{\tau}_i$, for i running through the vertices of Q_T.

The possible types of flips that can occur are given locally by one of the mutations in Figures 17, 18, 19, 20 or 21 (from left to right or right to left), or by one of the mutations from Figure 18, 20 or 21 with all of the tags flipped.

Consider first the flip in Figure 17, that is a flip far from the cone point. The left hand side of Figure 22 shows an embedding of D_T which is good at 5. Applying Lemma 4.3, the middle figure shows the paths corresponding to the braids $\widetilde{\tau}_i$. Rotating the vertices A and B clockwise, we get the diagram on the right of Figure 22, where we used the isomorphism $\iota : B_{T'} \to B_T$ given by $\tau_i \mapsto \widetilde{\tau}_i$ from [GM17, Remark 3.2]. Consider the composition of

isomorphisms $\varphi_5 \circ \psi_T \circ \iota$, where φ_5 is the isomorphism from Theorem 3.12. If $i \to 5$ in Q_T, then

$$\varphi_5 \circ \psi_T \circ \iota(\tau_i) = \varphi_5 \circ \psi_T(\sigma_5^{-1}\sigma_i\sigma_5) = \varphi_5(s_5^{-1}s_is_5) = t_5^{-1}t_5t_it_5^{-1}t_5 = t_i,$$

and otherwise

$$\varphi_5 \circ \psi_T \circ \phi(\tau_i) = \varphi_5 \circ \psi_T(\sigma_i) = \varphi_5(s_i) = t_i.$$

Hence we obtain an isomorphism $B_{T'} \to G_{Q'_T}$ sending τ_i to t_i as required. A symmetric argument works if we start with the triangulation on the right, with the shown embedding of the dual graph good at 5, and flip it to the triangulation on the left of Figure 22.

For the flips in Figures 18 and 19 (from left to right or right to left or with all the tags flipped), the result follows by arguments very similar to the above case, hence we omit the details. Since flipping the tags does not affect the arguments, it only remains to show that the result holds for the flips in Figures 20 and 21.
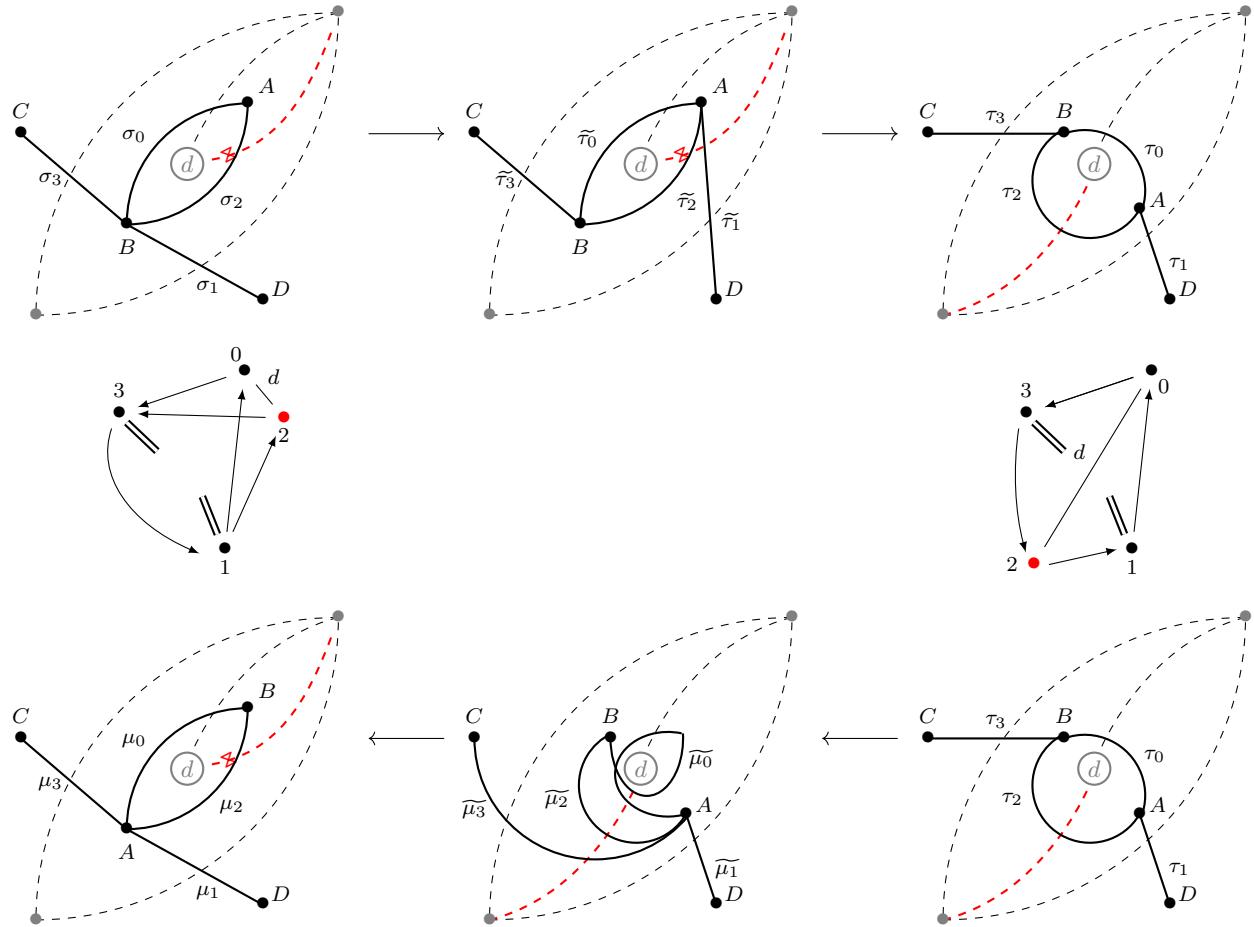
FIGURE 23. Mutation of a tagged to an untagged arc or vice versa, case 1.

Consider the flip in Figure 20 with T on the left. The left hand side of the top part of Figure 23 shows an embedding of D_T which is good at 2. Applying Lemma 4.3, the middle figure shows the paths corresponding to the braids $\widetilde{\tau}_i$. Note that, following the definition of $\widetilde{\tau}_i$, the only conjugated element is σ_1 in this case as the mutated arc α_2 is not rotated anti-clockwise to the flipped arc and so σ_0 is not conjugated. Rotating the vertices A and

B clockwise about the cone point, we get the diagram on the right of Figure 23, where we used the isomorphism $B_{T'} \to B_T$ given by $\tau_i \mapsto \widetilde{\tau}_i$ from [GM17, Remark 3.2]. Composing this with the isomorphism $\varphi_2 \circ \psi_T$, where φ_2 is the isomorphism from Theorem 3.12, we obtain an isomorphism $B_{T'} \to G_{Q'_T}$ sending τ_i to t_i as required.
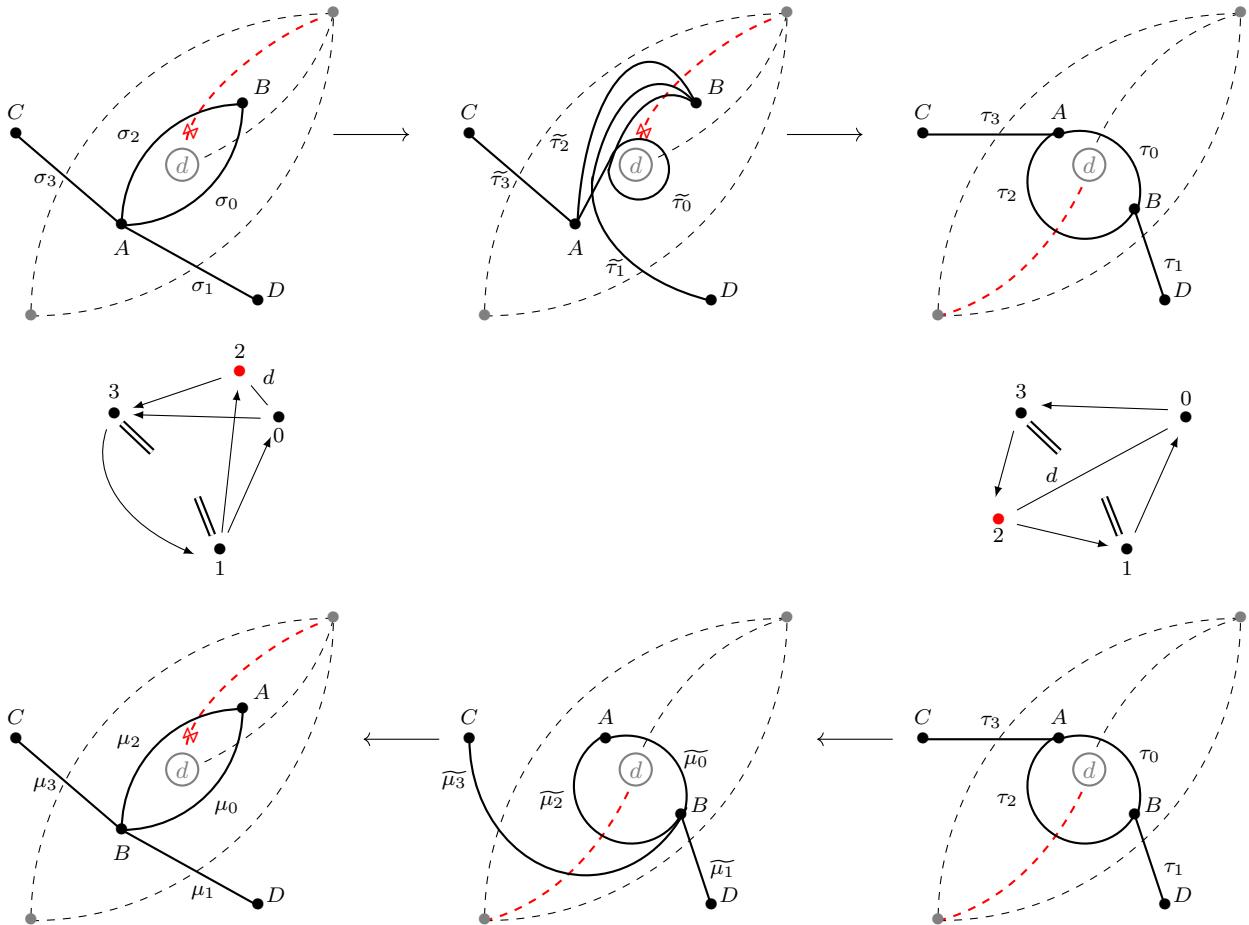

FIGURE 24. Mutation of a tagged to an untagged arc or viceversa, case 2.

Consider now the flip in Figure 21 with T on the left. The left hand side of the top of Figure 24 shows an embedding of the braid graph D_T which is good at 2. Applying Lemma 4.3, the middle figure shows the paths corresponding to the braids $\widetilde{\tau}_i$. Note that this time, the conjugated elements are σ_1, as there is an arrow $1 \to 2$, and σ_0, as the mutated arc α_2 is rotated anti-clockwise to the flipped arc and α_0 is also an arc at the cone point. Rotating the vertices A and B (with A moving clockwise and B anti-clockwise around the cone point) we obtain the diagram on the right of Figure 24, where we used the isomorphism $B_{T'} \to B_T$ given by $\tau_i \mapsto \widetilde{\tau}_i$ from [GM17, Remark 3.2]. Composing this with the isomorphism $\varphi_2 \circ \psi_T$, where φ_2 is the isomorphism from Theorem 3.12, we obtain an isomorphism $B_{T'} \to G_{Q'_T}$ sending τ_i to t_i as required. Note that not conjugating σ_0 would result in a final braid τ'_0 swirling around the conepoint. In the case $d = 2$, τ'_0 is isotopic to the braid τ_0 and hence conjugating σ_0 or not does not make a difference, see [GM17, proof of Thm 3.6]. However, for $d > 2$, τ'_0 is not isotopic to the braid τ_0 appearing in $B_{T'}$ and hence it is necessary to conjugate σ_0.

Note that the right hand side diagrams in Figures 23 and 24 coincide. Consider this as T and mutate the red arc. There is now a choice on whether rotating this arc anti-clockwise, see the bottom row of Figure 23, or clockwise, see the bottom row of Figure 24. The two options correspond respectively to conjugating or not conjugating the braid corresponding to vertex 0 in the associated quiver. Following arguments similar to the above two cases, one can check that in both cases the result holds.

As the above covers all the possible mutations, and the theorem follows. $\square$

We now have the ingredients we need to complete Remark 3.7.

Remark 4.7. In the situation of Remark 3.7 with $d > 2$, we have $s_i s_k \neq s_k s_i$. The situation described there is the (top) right hand diagram of Figure 23 with $i = 0$ and $k = 2$. The element $\sigma = \tau_2^{-1} \tau_0^{-1} \tau_2 \tau_0$ is a pure braid and $\xi(\sigma) \in \pi_1(\mathcal{O}_d)$ is a single strand winding around the pole twice, where ξ is the map from the proof of Lemma 2.13. Arguing as in the proof of Lemma 2.13, we have that σ is not equal to the identity, so $\tau_0 \tau_2 \neq \tau_2 \tau_0$ and hence $s_0 s_2 \neq s_2 s_0$ by Theorem 4.6.

5. Presentations of $G(d, d, n)$

Let T be a tagged triangulation of (X, M), and Q_T the associated quiver. Let G'_{Q_T} be the group defined in the same way as G_{Q_T} (see Definition 3.4) with the additional relations $s_i^2 = e$ for all i. Then we have:

Theorem 5.1. Let T be a tagged triangulation of (X, M). Then G'_{Q_T} is isomorphic to $G(d, d, n)$, thus giving a presentation of $G(d, d, n)$.

Proof. The fact that this is true for the tagged triangulation in Figure 12 follows from Theorem 3.36 and the presentation of $G(d, d, n)$ given in [Ari95] (see [BMR98, Prop. 3.2]). Note that the braid diagram that gives the presentation of $B(d, d, n)$ in [BMR98, Table 5] is the opposite of the diagram that gives the presentation of $G(d, d, n)$ in [BMR98, Table 2] (see [BMR98, Thm. 2.27]), but the presentation corresponding to the opposite diagram in this case is equivalent to that corresponding to the original diagram: passing to the opposite diagram amounts only to a relabelling, and therefore does not change the isomorphism class of the presented group.

The fact that it is true for an arbitrary tagged triangulation follows from iterated application of Theorem 3.12, noting that the proof of this goes through when adding relations stating that the generators have square equal to the identity, and the connectedness of the mutation graph [FST08, 7.10]. $\square$

Finally, we will show that, by applying a result from [Shi05], the generators can be regarded as reflections in $G(d, d, n)$, and explain how this can be done explicitly. Note that, although Shi assumes that $d > 2$ and $n > 2$, the results in [Shi05, §2,§3] still hold for $d = 2$ or $n = 2$ (replacing 'reflection of order m' with 'reflection of Type II of order m' throughout, e.g. in [Shi05, Lems. 2.1, 2.9, Thm. 2.8]).

We also note that if $n = 2$, the presentation from Theorem 5.1 is just given by the relation (4) from Definition 3.4, together with $s_1^2 = s_2^2 = e$: in this case $G(d, d, 2)$ is the real reflection group of type $I_2(d)$; see Remark 3.5.

We first recall the result of Shi [Shi05] that we need. For this we need the following definition of a graph from the paper (slightly modified in our discussion here).

Definition 5.2. Let R be a set of reflections in $G(d,d,n)$. Then $\overline{\Gamma}_R$ is the edge-labelled directed (multi-)graph with vertex set $\{1, 2, \ldots, n\}$. We take R as the set of directed edges. An element $r = s(a, b; c)$ in R (see Section 2.1) has start vertex a, end vertex b and is labelled c. We adopt the convention that such a directed edge is equivalent to a directed edge from b to a labelled $-c$. This convention ensures that $\overline{\Gamma}_R$ is well-defined, noting that $s(a, b; c) = s(b, a; -c)$. We define Γ_R to be the underlying unoriented graph of $\overline{\Gamma}_R$ with the labels removed: note that this graph is also well-defined.

Suppose that Γ_R is connected and contains precisely one cycle. By reversing some directed edges in $\overline{\Gamma}_R$ if necessary (and thus also negating their labels), we may assume that the corresponding directed edges in $\overline{\Gamma}_R$ form an oriented cycle, C. Set $\delta(R)$ to be the absolute value of the sum of the labels on the directed edges in C. Note that by taking the absolute value here, we ensure that $\delta(R)$ is well-defined. Then, we have the following:

Theorem 5.3 ([Shi05, Thm. 2.19])**.** *Let R be a set of reflections in $G(d,d,n)$ such that Γ_R is connected and contains precisely one cycle. Then R generates $G(d,d,n)$ if and only if $\delta(R)$ and d are coprime.*

Lemma 5.4. Let T be a tagged triangulation of (X, M), and let D_T be the corresponding braid graph (see Definition 4.1). Then D_T contains a unique cycle. In fact, it can be obtained from a cycle by adjoining a binary tree (possibly consisting of a single vertex) to each of its vertices.

Proof. By Remark 3.3, the induced subgraph of the braid graph on the vertices associated to the connected components of the complement of T incident with the cone point will be a cycle of length at least two. The vertices on the boundary of this union U of connected components must be on the boundary of X (since they are not the cone point), so the tagged triangulation must be built up from U by adding a triangulated polygon to each boundary edge of U intersecting U only in that edge (where we allow a degenerate case consisting of an edge only, i.e. where no polygon is attached).

It follows that the braid graph can be obtained from an oriented cycle by adjoining a binary tree (possibly consisting only of a single vertex) to each of the vertices of the cycle. In particular, it contains a unique cycle as claimed. □

For the rest of this section, we will work in the following setup.

Setup 5.5. Let T be a tagged triangulation of (X, M), and let D_T be the corresponding braid graph (see Definition 4.1). Fix a numbering $1, 2, \ldots, n$ of the n vertices of D_T. Let $\overline{D}_T$ be a directed graph with underlying unoriented graph D_T, chosen so that the unique cycle in D_T is an oriented cycle in $\overline{D}_T$.

We then associate a reflection $s(e) = s(a, b; c(e))$ to each edge e of D_T, where the corresponding directed edge in $\overline{D}_T$ has initial vertex a and end vertex b and $c(e)$ is an integer. We do this in such a way that, if the unique oriented cycle in $\overline{D}_T$ consists of vertices $a_1, a_2, \ldots, a_r$ with a directed edge f_m from a_m to a_{m+1} for all m (with a_{r+1} interpreted as a_1), then $|\sum_{m=1}^{r} c_m|$ and d are coprime, where $c_m = c(f_m)$. We define R to be the set of all reflections $s(e)$ for e an edge of D_T. Note that, by construction, Γ_{R_T} can be identified with D_T.

Proposition 5.6. Let T be a tagged triangulation of (X, M). Then the set R of reflections satisfies the defining relations of G'_{Q_T}, where each $s(e)$ is identified with the generator s_i associated to the vertex i of Q_T corresponding to the edge e of D_T.

Proof. Note first that each $s(e)$ squares to the identity by definition. The commuting and braid relations from Definition 3.4, parts (1) and (2) respectively, are then satisfied by [Shi05, Sec. 3.3(2),(3)]. Moreover, if the unique cycle in D_T has length $r = 2$ then, by [Shi05, Sec. 3.3(4)] we have that

$$\underbrace{s(a_1, a_2; c_1) s(a_2, a_1; c_2) s(a_1, a_2; c_1) \cdots}_{d \text{ terms}} = \underbrace{s(a_2, a_1; c_2) s(a_1, a_2; c_1) s(a_2, a_1; c_2) \cdots}_{d \text{ terms}}.$$

Hence, the relation from Definition 3.4(4) is satisfied.

Consider the situation of the left drawing in Figure 25, where s, t and u are the associated reflections. Then, by [Shi05, Sec. 4.5(v)], we have $sutu = utus$. Using the braid relations and the fact that the reflections square to the identity, we have that

$$sutu = utus \iff stut = tuts \iff stutts = tutsts \iff stus = tuttst \iff stus = tust.$$

Similarly, one can see that the third equality in the cycle relation is true and $stus = tust = ustu$, i.e. the relations from Definition 3.4(3) are satisfied.

As pointed out before, double edges in Q_T only appear if there are exactly two arcs at the conepoint in T, or equivalently the only cycle in D_T has length 2. Suppose that this is the case. Let j be a vertex connected to one of two vertices a_1 and a_2 in the 2-cycle, say a_2 via an edge g. In other words, there is a double edge at the vertex in Q_T corresponding to the edge g between a_2 and j in D_T. Denote the vector with a 1 in the ith-entry and 0 everywhere else by e_i and recall that $\omega_d = e^{2\pi i/d}$. By direct computation, one can check that

$$\left(s(a_2, j; c_g) s(a_1, a_2; c_1) s(a_2, a_1; c_2)\right)^2 = \left(s(a_1, a_2; c_1) s(a_2, a_1; c_2) s(a_2, j; c_g)\right)^2,$$

that is, the expected double edge relation from Definition 3.4(6) holds. In fact, both products clearly only affect the entries in a_1, a_2 and j positions and we compute that under both the left and the right hand side products of reflections, we have

$$e_{a_1} \mapsto \omega_d^{-2c_2 - 2c_1} e_{a_1}, \ e_{a_2} \mapsto \omega_d^{c_2 + c_1} e_{a_2}, \ e_j \mapsto \omega_d^{c_2 + c_1} e_j.$$

Similarly, it is easy to see by direct computation that when there are exactly two arcs at the conepoint, incident with two different vertices on the boundary, then the corresponding reflections satisfy the relations from Definition 3.4(7).

Consider now the situation on the right hand side of Figure 25, showing the unique cycle in D_T of length r, where $r \geq 3$. We show that the reflections satisfy the relations from Definition 3.4(5). In the following computations, subscripts are taken modulo r and addition of the elements c_j is carried out modulo d. For $1 \leq m \leq r$, consider the product of $d(r-1)$ reflections

$$s(a_m, a_{m+1}; c_m) s(a_{m+1}, a_{m+2}; c_{m+1}) \cdots s(a_{m+d(r-1)-1}, a_{m+d(r-1)}; c_{m+d(r-1)-1}). \tag{6}$$

Fix $j \in \{0, 1, \ldots, r - 2\}$. When computing the image of $e_{a_{m+d(r-1)-1-j}}$ under the product (6), only the reflections $s(a_{m+d(r-1)-i(r-1)}, a_{m+d(r-1)-i(r-1)}; c_{m+d(r-1)-i(r-1)})$ for $i = 0, 1, \ldots d - 1$ act
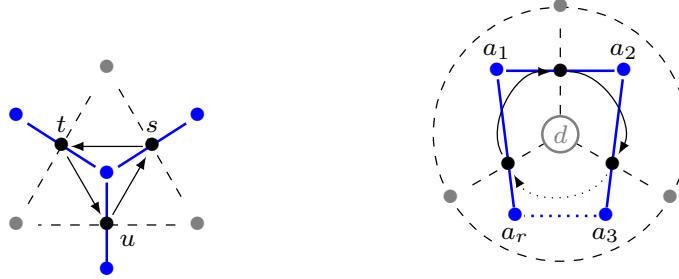
FIGURE 25. Two possible local behaviours appearing in T. The dashed arcs are arcs in T (in the right hand picture the arcs shown on the boundary are also allowed to be boundary segments). The solid edges are edges in D_T and edges in Q_T are oriented.

non-trivially. We compute that:

$$e_{a_{m+d(r-1)-1-j}} \mapsto \omega_d^{\sum_{i=0}^{d-1} c_{m+d(r-1)-1-j-i(r-1)}} e_{a_{m-1-j}} = \omega_d^{\sum_{i=0}^{d-1} c_{m+d(r-1)-1-j+i}} e_{a_{m-1-j}}.$$

When computing the image of $e_{a_{m+d(r-1)}}$ under the product (6), every reflection in the product acts non-trivially, and we compute that:

$$e_{a_{m+d(r-1)}} \mapsto \omega_d^{-\sum_{i=m}^{m+d(r-1)-1} c_i} e_{a_m}.$$

Note that

$$-\sum_{i=m}^{m+dr-1} c_i \equiv 0 \mod d,$$

since this involves adding up a multiple of d copies of each c_i. Hence, we have:

$$0 \equiv -\sum_{i=m}^{m+dr-1} c_i \iff -\sum_{i=m}^{m+d(r-1)-1} c_i \equiv \sum_{i=m+dr-d}^{m+dr-1} c_i \iff -\sum_{i=m}^{m+d(r-1)-1} c_i \equiv \sum_{i=0}^{d-1} c_{m+d(r-1)+i} \mod d.$$

It follows that

$$e_{a_{m+d(r-1)}} \mapsto \omega_d^{\sum_{i=0}^{d-1} c_{m+d(r-1)+i}} e_{a_m}.$$

Hence, for $p = 1, 2, \ldots, r$ we have:

$$e_{a_p} \mapsto \omega_d^{\sum_{i=0}^{d-1} c_{p+i}} e_{a_{p-d(r-1)}},$$

and we see that the products (6) are equal for all $1 \le m \le r$ and hence the reflections satisfy the relations from Definition 3.4(5). This completes the proof of the result. □

Theorem 5.7. *In the situation of Setup 5.5, there is an isomorphism of groups $\nu : G'_{Q_T} \to G(d,d,n)$ sending the generator of G'_{Q_T} associated to vertex v in Q_T to the reflection associated to the edge in D_T that is the dual of v.*

Proof. Let R be the set of reflections associated to the edges in D_T, as in Setup 5.5. By Proposition 5.6, these reflections satisfy the defining relations of G'_{Q_T}, so there is a group homomorphism $\nu : G'_{Q_T} \to G(d,d,n)$ sending the generator of G'_{Q_T} associated to vertex v in Q_T to the reflection associated to the edge in D_T that is the dual of v.

By Lemma 5.4, D_T has a unique cycle. By assumption (see Setup 5.5), $\delta(R)$ and d are coprime. Hence, by Theorem 5.3, R generates $G(d,d,n)$. Hence ν is surjective. By Theorem 5.1, $|G'_{Q_T}| = |G(d,d,n)|$, so ν is also injective, giving the required result. $\square$

Remark 5.8. Note that, in contrast to the braid group case, in the complex reflection group presentation the double edge relations can be made more symmetric. In fact, for reflections chosen as in Setup 5.5, by direct computations one can check that in the situation of Definition 3.4(6), we have

$$s_k s_i s_j s_k s_i s_j = s_i s_j s_k s_i s_j s_k = s_j s_k s_i s_j s_k s_i.$$

On the other hand, in the situation of Definition 3.4(7), for $d > 2$, we still get two separate relations also in the complex reflection group. To see this, let $s_i = s(a_1, a_2; c_1)$, $s_j = s(a_1, j; c(g))$, $s_k = s(a_2, a_1; c_2)$ and $s_l = s(a_2, l; c(h))$ be reflections satisfying Setup 5.5 and corresponding respectively to the vertices i, j, k and l in Definition 3.4(7). Then, it is easy to compute that

$$s_i s_j s_k s_l s_i s_j : e_{a_1} \mapsto \omega_d^{c_1} e_{a_2},$$
$$s_j s_k s_l s_i s_j s_k : e_{a_1} \mapsto \omega_d^{-2c_2-c_1} e_{a_2}.$$

As, by assumption, $c_1 + c_2$ is coprime to d, and hence invertible modulo d, we have that

$$-2c_2 - c_1 = c_1 \bmod d \iff 2(c_1 + c_2) = 0 \bmod d \iff d = 2.$$

Hence, for $d > 2$ we have that $s_i s_j s_k s_l s_i s_j \neq s_j s_k s_l s_i s_j s_k$.

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